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HYPERCOMPLEX ALGEBRAS, HYPERCOMPLEX ANALYSIS AND CONFORMAL INVARIANCE

John Ryan

Abstract

In this paper we use spherical harmonics to deduce conditions under which a generalized, first order, homogeneous Cauchy-Riemann equation possesses a homotopy invariant, Cauchy integral formula. We also deduce conditions under which the solutions to these equations are invariant under Möbius transforms in C^n.

Introduction

The study of function theories over Clifford algebras has been developed and applied by a number of authors [3–9,11,12,14,16–23]. These function theories contain nautural generalizations of many aspects of one variable complex analysis [1]. Each function theory involves the study of solutions to generalized Cauchy-Riemann equations, and contains a Cauchy theorem, Cauchy integral formula, and Laurent expansion theorem. Moreover, the classes of solutions to the generalized Cauchy-Riemann equations are invariant under generalized Möbius transforms [18]. The study of these function theories is referred to as Clifford analysis [3].

Many of the results obtained in Clifford analysis rely on the existence of the generalized Cauchy kernels, and associated integral formulae. However, in recent work [17] the author has shown that many results in Clifford analysis, not associated with the generalized Cauchy kernels, may also be obtained over arbitrary complex, finite dimensional, associative algebras with identity. This observation leads naturally to the question 'what properties does an algebra require to admit a hypercomplex function theory, together with a homotopically invariant Cauchy integral formula?' In this paper we use spherical harmonics to deduce that a complex, associative algebra with identity admits a hypercomplex function theory with a real analytic Cauchy kernel and associated homotopy invariant Cauchy integral formula if and only if it contains a subalgebra which is isomorphic to a Clifford algebra.

We refer to algebras admitting these types of hypercomplex function theories as hypercomplex algebras. As an example of such an algebra one may consider the k-fold symmetric tensor product of a complex Clifford algebra with itself. In this case the associated hypercomplex analysis reduces to a study of the half integer spin massless fields considered in [4,10] and elsewhere.

Despite the presence of an isomorphic copy of a Clifford algebra within each hypercomplex algebra, it does not automatically follow that the associated function theory is as rich as the ones studied so far in Clifford analysis. As an example we construct a hypercomplex algebra algebra which does not admit any non-trivial solutions to its generalized Cauchy-Riemann equations. We conclude by deducing that a hypercomplex algebra admits conformally invariant classes of hypercomplex functions if and only if the subalgebra generated by the algebraic elements arising in the generalized Cauchy-Riemann equations is isomorphic to a Clifford algebra. The paper includes a number of examples of algebras admitting generalized Cauchy integral formulae, and some of the relations between these theories is described.

Preliminaries

Suppose that E and F are two real, finite dimensional vector subspaces of a space K. Suppose also that E and F are of equal dimension, that there is a quadratic form $Q: E \times E \to R$ and that there is an isomorphism $L: E \to F$, such that on the intersection of E with F the map L is the identity map. Let A be the minimal subspace of K containing $E \cup F$ and let

$$T(A) = \sum_{i=0}^{\infty} T^i(A) = R \oplus A \oplus A \otimes A \oplus \cdots$$

be the tensor algebra over A. Let $I_1(Q)$ be the two sided ideal generated by the elements $x \otimes y - Q(x, x)$ where $x \in E$, $y \in F$ and $y = L(x)$.

Definition 1: The quotient algebra $T(A)/I_1(Q)$ is called a *pre-Clifford algebra,* and it is denoted by $P_1C(E, F)$.

In the case where $E = F$ the algebra is a *universal Clifford algebra* of dimension 2^n, and the above construction corresponds to the one given in [2]. We denote the universal Clifford algebra by $C(E)$. It follows from our construction that each pre-Clifford algebra is associative. We define $i_Q: A \to P_1C(E, F)$ to be the canonical map given by the composition $A \to T(A) \to P_1C(E, F)$. It is not difficult to verify that the linear map $i_Q: A \to P_1C(E, F)$ is an injection. In the cases where $E \neq F$ it may be observed that for each $x \in E - (E \cap F)$ and each $y \in F - (E \cap F)$ the elements $\{i_Q(x)^p, i_Q(y)^p\}_{p=1}^{\infty}$ are independent elements of the algebra $P_1C(E, F)$. It follows that in these cases the pre-Clifford algebra is infinite dimensional. Let $\phi: A \to A'$ be a linear map into an algebra A', with unit 1, such that for all $x \in E$ and $Y \in F$ with $y = L(x)$ the identity $\phi(x)\phi(y) = Q(x, x)1$ is satisfied. Then it follows from our construction that there exists a unique homomorphism $\tilde{\phi}: P_1C(E, F) \to A'$ such that $\tilde{\phi} \cdot i_Q = \phi$.

DEFINITION 2: [2] Suppose that $E = F$ and $\phi: E \to A'$ is a linear map into an algebra A', with identity 1, such that for all $x \in E$ the identity $\phi(x)^2 = Q(x, x)1$ is satisfied. Then the minimal subalgebra of A' containing the space $\phi(E)$ is called a *Clifford algebra.*

Clearly each universal Clifford algebra is a Clifford algebra.

Although we have that $i_Q(x)i_Q(L(x)) = Q(x, x)1$ for each $x \in E$, it does not follow that $i_Q(L(x)) \cdot i_Q(x)) = Q(x, x)1$ for each $x \in E$.

LEMMA 1: *Suppose that $E \cap F \neq E$, and that the quadratic form $Q: (E - (E \cap F)), (F - (E \cap F)) \to R$ is nondegenrate. Then for each non-zero $x \in E - (E \cap F)$ the element $L(x) \otimes x - Q(x, x) \in T(A)$ is not a member of the ideal $I_1(Q)$.*

PROOF: For simplicity we shall place $L(x) = y$. Suppose that $y \otimes x - Q(x, x) \in I_1(Q)$ then there exists a positive integer N, and elements $x_1, \ldots, x_N, y_1, \ldots, y_N \in T(A)$ such that

$$y \otimes x - Q(x, x) = \sum_{i=1}^{N} x_i \otimes (x \otimes y - Q(x, y)) \otimes y_i. \tag{1}$$

Now consider the two sided ideal, $I_2(Q)$ of $T(A)$, generated by the elements

(i) $u \otimes v - Q(u, u)$

with $u \in E$, $v \in F$ and $v = L(u)$,
and

(ii) $y \otimes x + Q(x, x)$,

with $x \in E - (E \cap F)$ and $y = L(x)$.

The quotient algebra $B(E, F) = T(A)/I_2(Q)$ is well defined, associative, and has an identity. Also, we have a canonical injection $j_Q: A \to B(E, F)$ given by the composition $A \to T(A) \to B(E, F)$.

It follows that

$$j_Q(y) j_Q(x) = -Q(x, x)1,$$

where 1 is the unit of the algebra $B(E, F)$. However, if the identity (1) is valid we also have

$$j_Q(y) j_Q(x) = Q(x, x)1.$$

Hence,

$$Q(x, x) = 0. \tag{2}$$

Equation (2) contradicts our assumption that the quadratic form Q is non-degenerate on the space $E - (E \cap F)$. The result follows.

We now introduce the following algebra:

DEFINITION 3: For the two sided ideal, $I_3(Q)$ of $T(A)$, generated by the elements $x \otimes x - Q(x, x)$ and $y \otimes x - Q(x, x)$, where $x \in E$, $y \in F$ and $y = L(x)$, the quotient algebra $PC(E, F) = T(A)/I_3(Q)$ is called a *pseudo-Clifford algebra.*

Again, in the case where $E = F$ the pseudo-Clifford algebra is the universal Clifford algebra $C(E)$. As in the case for the pre-Clifford algebras these algebras are associative, and there is a canonical injection $k_C: A \to PC(E, F)$ given by the composition

$$A \to T(A) \to PC(E, F).$$

In the cases where $E \neq F$ it may be observed that for each $x \in E - (E \cap F)$ and each $y \in F - (E \cap F)$ the elements $\{k_Q(x)^p, k_Q(y)^p\}_{p=1}^{\infty}$ are independent elements of the algebra $PC(E, F)$. It follows that in these cases the pseudo Clifford algebra is infinite dimensional. Let $\phi: A \to A'$ be a linear map into an algebra A', with unit 1, such that for all $x \in E$ and $y \in F$ with $y = L(x)$ the identity $\phi(x)\phi(y) = \phi(y)\phi(x) = Q(x, x)1$ is satisfied. Then it follows from our construction that there exists a unique homomorphism $\tilde{\tilde{\phi}}: PC(E, F) \to A'$ such that $\tilde{\tilde{\phi}} \cdot k_Q = \phi$.

For the rest of this paper we shall restrict our attention to the cases where the quadratic form, Q, is negative definite.

Suppose that the dimension of E is n and the dimension of $E \cap F$ is q, where $0 \leqslant q \leqslant n$. Then from our constructions we may choose bases

$$e_1, \ldots, e_{n-q}, \quad f_1, \ldots, f_n \tag{3}$$

and

$$e'_1, \ldots, e'_{n-q}, \quad f'_1, \ldots, f'_n \tag{4}$$

of the spaces $i_Q(A)$ and $k_Q(A)$ respectively, such that

$$\begin{aligned} e_r f_r &= -1 & 1 \leqslant r \leqslant n-q \\ f_r^2 &= -1 & n-q+1 \leqslant r \leqslant n, \\ e'_r f'_r &= -1 & 1 \leqslant r \leqslant n-q \\ f_r'^2 &= -1 & n-q+1 \leqslant r \leqslant n, \end{aligned}$$

where $e_r \in i_Q(E)$, $f_r \in i_Q(F-(E\cap F))$, $e_r' \in k_Q(E)$, $f_r' \in k_Q(F-(E\cap F))$.

In the case where $E=F$ the elements $1, e_1, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_1 \ldots e_n$ form a basis for the algebra $C(E)$. A general basis element of $C(E)$ is denoted by $e_{j_1} \ldots e_{j_r}$, where $1 \leqslant r \leqslant n$ and $j_1 < \cdots < j_r$. From this basis it may be observed that the vector space $C(E)$ is canonically isomorphic to the vector space $\Lambda(E)$, where $\Lambda(E)$ is the alternating algebra generated from E. As in [2] and [15, Chapter 13] we observe that there are two natural automorphisms acting on $C(E)$. First we have

$$\sim : C(E) \to C(E): e_{j_1} \ldots e_{j_r} \to e_{j_r} \ldots e_{j_1}$$

and second

$$- : C(E) \to C(E): e_{j_1} \ldots e_{j_r} \to (-1)^r e_{j_r} \ldots e_{j_1}$$

For each $u \in C(E)$ we denote $\sim(u)$ by $\tilde{u}$ and $-(u)$ by $\bar{u}$. Moreover, we have $\widetilde{uv} = \tilde{v}\tilde{u}$ and $\overline{uv} = \bar{v}\bar{u}$, for each $v \in C(E)$.

A general vector u of $C(E)$ may be written as

$$u_0e_0 + u_1e_1 + \cdots + u_ne_n + \cdots + u_{j_1 \ldots j_r}e_{j_1} \ldots e_{j_r} + \cdots + u_{1\ldots n}e_1 \ldots e_n$$

where $u_0, u_1, u_n, u_{j_1 \ldots j_r}, u_{1\ldots n} \in R$.

It may be observed that the vector space

$$C^+(E) = \left\{ u \in C(E) : u = \sum_{j_1 \cdots j_r} u_{j_1 \ldots j_r} e_{j_1} \ldots e_{j_r} \text{ and } u_{j_1 \ldots j_r} = 0 \text{ for } r = 1 \bmod 2 \right\}$$

is a subalgebra of $C(E)$. It is called the even subalgebra of $C(E)$. Each even subalgebra is isomorphic to a Clifford algebra. This may be seen from the following construction [2]:

CONSTRUCTION: Suppose E_1 and E_2 are real vector spaces with $E_1 \subseteq E_2$ and the dimension of $E_2 - E_1$ is one. Then the linear map $\phi: C(E_1) \to C^+(E_2)$ given by $e_i \to e_ie_{n+1}$ is an algebra isomorphism.

We now give some examples of Clifford algebras [2]:

A. In the case where the dimension of E is one the algebra $C(E)$ corresponds to the complex field C.

B. In the case where the dimension of E is two the algebra $C(E)$ is spanned by the vectors e_0, e_1, e_2, e_1e_2, and the multiplication of these vectors satisfy the same relations as the basis elements of the quaternion algebra H, as is illustrated in [17]. It follows that the algebra $C(E)$ is isomorphic to the quaternionic division algebra.
C. In the case where the dimension of E is three the algebra $C(E)$ is spanned by the vectors e_0, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, $e_1e_2e_3$. Moreover, the vectors $(e_0 + e_1e_2e_3)$ and $(e_0 - e_1e_2e_3)$ satisfy the relation $(e_0 + e_1e_2e_3)(e_0 - e_1e_2e_3) = 0$. It may be deduced [2] that the vectors $(e_0 \pm e_1e_2e_3)$, $e_j(e_0 \pm e_1e_2e_3)$, where $j = 1, 2, 3$, span two subalgebras of $C(E)$, and each of these subalgebras is isomorphic to the quaternion algebra. It follows that in this case

$$C(E) \cong H \oplus H.$$

From now on we shall consider the complex algebras

$$C(E) \underset{R}{\circledS} C, \quad P_1(CE, F) \underset{R}{\circledS} C$$

and

$$PC(E, F) \underset{R}{\circledS} C$$

obtained by taking the symmetric tensor product of C with the real algebras $C(E)$, $P_1C(E, F)$ and $PC(E, F)$ respectively. We shall denote these algebras by $C_C(E)$, $P_1C_C(E, F)$ and $PC_C(E, F)$ respectively. In the case where the dimension of E is two it may be observed, [22] by making the identifications

$$e_0 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \to \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 \to \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

that the complex quaternion algebra $(H(C) \cong C_C(E))$ is canonically isomorphic to the algebra $M(2, C)$ of two by two complex matrices. It follows that in the case where the dimension of E is three that

$$C_C(E) \cong M(2, C) \oplus M(2, C).$$

It is straightforward [2] to deduce that when the dimension of E is n, and $n = 3 \bmod 4$ the elements $e_0 \pm e_1 \dots e_n$ commute with each element of the algebra $C_C(E)$ and $(e_0 + e_1 \dots e_n)(e_0 - e_1 \dots e_n) = 0$. It follows that the sets $C_C(E)^+ = C_C(E)(e_0 + e_1 \dots e_n)$ and $C_C(E)^- = C_C(E)(e_0 - e_1 \dots e_n)$ are two mutually annihilating two sided ideas of the algebra $C_C(E)$ and $C_C(E) = C_C(E)^+ \oplus C_C(E)^-$. Also, it may be deduced that when the dimension of E is n, and $n = 1 \bmod 4$ the elements $e_0 \pm ie_1 \dots e_n$

commute with each element of the algebra $C_C(E)$, and $(e_0 + ie_1 \dots e_n)(e_0 - ie_1 \dots e_n) = 0$. It follows that the sets $C_C(E)^{+,i} = C_C(E)(e_0 + ie_1 \dots e_n)$ and $C_C(E)^{-,i} = C_C(E)(e_0 - ie_1 \dots e_n)$ are two mutually annihilating two sided ideals of the algebra $C_C(E)$, and $C_C(E) = C_C(E)^{+,i} + C_C(E)^{-,i}$. It may be observed that the ideals $C_C(E)^+$, $C_C(E)^-$, $C_C(E)^{+,i}$, $C_C(E)^{-,i}$ are examples of Clifford algebras.

In the cases where the dimension of E is n and n is even the Clifford algebras $C_C(E)$ are simple [2].

We shall denote the complex subspace of $C_C(E)$ spanned by the vectors $\{e_j\}_{j=1}^n$ by C^n, where n is the dimension of E, and a vector $z_1e_1 + \cdots + z_ne_n \in C^n$ shall be denoted by $\underline{z}$. Moreover, we shall denote the complex subspace of $C_C(E)$ spanned by the vectors $\{e_k\}_{k=0}^n$ by $Ce_0 + C^n$, and a vector $z_0e_0 + z_1e_1 + \cdots + z_ne_n \in Ce_0 + C^n$ shall be denoted by $\underline{\underline{z}}$. It may be observed that for each $\underline{z} \in C^n$ and each $\underline{\underline{z}} \in Ce_0 + C^n$ we have $\underline{z}^2 \in C$ and $\underline{\underline{z}}\bar{\underline{\underline{z}}} \in C$. We shall also denote the real space spanned by vectors $\{e_j\}_{j=1}^{n_1} \cup \{ie_k\}_{k=n_1+1}^n$ by R^{n_1,n_2}, where $0 \leqslant n_1$, $n_2 \leqslant n$ and $n_1 + n_2 = n$. The set $\{\underline{z} \in C^n \underline{z}^2 = 0\}$ is denoted by S, and is called the null cone of C^n. Similarly the set $\{\underline{\underline{z}} \in Ce_0 + C^n : \underline{\underline{z}} \cdot \bar{\underline{\underline{z}}} = 0\}$ is denoted by S', and is called the null cone of $Ce_0 + C^n$. It may be observed that each $\underline{z} \in C^n - S$ and each $\underline{\underline{z}} \in (Ce_0 + C^n) - S'$ is invertible in the algebra $C_C(E)$; their inverses are $-\underline{z}(\underline{z}^2)^{-1}$ and $\bar{\underline{\underline{z}}}(\underline{\underline{z}}\bar{\underline{\underline{z}}})^{-1}$, respectively.

Using these inverses we may introduce the following groups [18]:

A. $\mathrm{Pin}_C(C^n) = \{Z \in C_C(E) : Z = \underline{z}_1 \dots \underline{z}_p$, where $p \in Z^+$, and for $r \in Z^+$ with $1 \leqslant r \leqslant p$ the element $\underline{z}_r \in C^n - S\}$. This group is a complex Lie group, and its dimension is $\frac{1}{2}(n^2 - n + 2)$. Moreover, it may be deduced from the construction we gave of the Clifford algebra $C(E)$ that for each $Z \in \mathrm{Pin}_C(C^n)$ we have that

$$ZC^n\tilde{Z} = C^n. \tag{5}$$

(The Lie algebra $\mathrm{pin}_C(C^n)$ of $\mathrm{Pin}_C(C^n)$ is spanned by the vector e_0 and the bivectors $e_1e_n, \dots, e_{n-1}e_n$).

B. $\mathrm{Spin}_C(C^n) = \{Z \in \mathrm{Pin}(C^n) : Z = \underline{z}_1 \dots \underline{z}_p$ and p is even$\}$.

C. $\mathrm{Spoin}_C(C^n) = \{Z \in C(E) : Z = \underline{z}_1' \dots \underline{z}_p'$, where $p \in Z^+$, and for $r \in Z^+$ with $1 \leqslant r \leqslant p$ the element $\underline{\underline{z}}_r \in (C_{e_0} + C^n) - S'\}$.

It may be observed from construction 1 that

$$\phi(\mathrm{Spoin}_C(C^n)) = \mathrm{Spin}_C(C^{n+1}).$$

Hypercomplex functions and generalized Cauchy integral formulae

In [17] we introduce the following class of functions:

DEFINITION 4: Suppose that A' is a complex, associative algebra and W is a complex, finite dimensional subspace spanned by the vectors $\{k_j\}_{j=1}^p$.

Then, for U a domain in W, a holomorphic function $F: U \to A'$ is called *left regular with respect to the vectors* $\{k_j\}$ if it satisfies the equation

$$\sum_{j=1}^{p} k_j \frac{\partial}{\partial z_j} F(Z) = 0$$

for each $Z \in U$.

OBSERVATION 1: It may be deduced [17] that a holomorphic function $J'': U \to A'$ is left regular with respect to the vectors $\{k_j\}$ if and only if the differential form $DZJ''(Z)$ is closed where $DZ = \sum_{j=1}^{p} (-1)^j k_j \, dz_1 \wedge \ldots dz_{j-1} \wedge dz_{j+1} \ldots \wedge dz_p$. From observation 1 we have:

THEOREM A (*Cauchy theorem*) [17]: *Suppose $F: U \to A'$ is a left regular function with respect to the vectors $\{k_j\}$, and M is a real p-dimensional, compact manifold lying in U. Then*

$$\int_{\partial M} DzF(z) = 0.$$

From theorem A we observe that equation (6) may be regarded as a generalization of the Cauchy-Riemann equations [1].

We now give some examples of generalized Cauchy-Riemann equations:

1. From the basis elements e_0, e_1, e_2, e_1e_2 of the quaternions we say that a holomorphic function $f_1 : U_1 \to H(C)$, where U_1 is a domain in $H(C)$, is *complex quaternionic left regular* if it satisfies the equation

$$e_0 \frac{\partial}{\partial z_0} f_1(z) + e_1 \frac{\partial}{\partial z_1} f_1(z) + e_2 \frac{\partial}{\partial z_2} f_1(z) + e_1 e_2 \frac{\partial}{\partial z_{12}} f_1(z) = 0. \quad (7)$$

Equation (7) is a holomorphic extension of the Cauchy-Riemann_Fueter equation studied in [7,8,13,23]. Properties of solutions to equation (7) have previously been studied in [12,22].

2. for U_2 a domain in C^n we say that a holomorphic function $f_2 : U_2 \to C_C(E)$ is $\mathrm{Pin}_C(C^n)$ *left regular* if it satisfies the equation

$$\sum_{j=1}^{n} e_j \frac{\partial}{\partial z_j} f_2(\underline{z}) = 0. \quad (8)$$

for each $\underline{z} \in U_2$.

Equation (8) is a holomorphic extension of the homogeneous Dirac equation studied in [5].

3. For U_3 a domain in $Ce_0 + C^n$ we say that a holomorphic function $f_3 : U_3 \to C_C(E)$ is *Spoin*$_C(C^n)$ *left regular* if it satisfies the equation

$$\sum_{k=0}^{n} e_k \frac{\partial}{\partial z_k} f_3(\underline{\underline{z}}') = 0, \tag{9}$$

for each $\underline{\underline{z}}' \in U_3$.

In the case where $n = 3$ equation (9) is a holomorphic extension of the Weyl neutrino equation studied in [9] and elsewhere. Properties of solutions to equation (9) have previously been studied in [3,6,16,19–21].

Using notation used in construction 1 we may deduce the following relation between the spaces of $\text{Spoin}_C(C^n)$ and $\text{Pin}_C(C^{n+1})$ left regular functions.

PROPOSTION 1: *Suppose that* $\Gamma_A(U_2, C^+(E_2)Ⓢ_R C)$ *denotes the right module, over* $C^+(E_2)Ⓢ_R C$, *of* $\text{Pin}_C(C^{n+1})$ *left regular functions,* $f_2 : U_2 \to C^+(E_2)Ⓢ_R \mathbf{C} \subseteq C_C(E_2)$, *and* $\Gamma_B(U_3, C_C(E_1))$ *denotes the right module over* $C_C(E_1)$, *of* $\text{Spoin}_C(C^n)$ *left regular functions* $f_3 : U_3 \to C_C(E_1)$. *Suppose also that a point* $z_1e_1 + \cdots + z_ne_n + z_{n+1}e_{n-1} \in U_2$ *if and only if the point* $z_{n+1}e_0 + z_1e_1 + \cdots z_ne_n \in U_3$. *Then the modules* $\Gamma_A(U_2, C^+(E_2)Ⓢ_R C)$ *and* $\Gamma_B(U_3, C_C(E_1))$ *are canonically isomorphic.*

PROOF: Consider the linear map

$$P : C^{n+1} \to Ce_0 + C^n : z_1e_1 + \cdots + z_{n+1}e_{n+1} \to z_{n+1}e_0 + z_1e_1 + \cdots z_ne_n,$$

and suppose that $f_3 : U_3 \to C_C(E)$ is a $\text{Spoin}_C(C^n)$ left regular function. Then it follows from observation 1 that the form $D\underline{z}'f_3(P(\underline{z}))$ is closed. It now follows from construction 1 that the form

$$\phi(D\underline{z}'f_3(P(\underline{z})) = \phi(D\underline{z}')\phi(f_3(P(\underline{z})))$$

is also closed. Consequently the form $e_{n+1}\phi(D\underline{z}')\phi(f_3(P(\underline{z})))$ is closed. However, on replacing the variable z_0 by z_{n+1} the form $e_{n+1}\phi(D\underline{z}')$ now becomes $D\underline{z}$. It now follows that the function $f_4(\underline{z}) = \phi(f_3(P(\underline{z})))$ is an element of the module $\Gamma_A(U_2, C^+(E_2)Ⓢ_R C)$.

It is straightforward to verify that the above constructions yields a canonical isomorphism between the modules $\Gamma_A(U_2, C^+(E_2)Ⓢ_R C)$ and $\Gamma_B(U_3, C_C(E_1)$. $\square$

In the case where $n = 3$ there is also a relation between the spaces of $\text{Spoin}_C(C^n)$ left regular functions and spaces of complex quaternionic left regular functions.

PROPOSITION 2: *Suppose that $\Psi_A(U_1, H(C))$ denotes the complex vector space of pairs of complex quaternionic left regular functions (f^1, f^2) defined over the domain U_1 and $\Psi_B(U_3, C_C(E^3))$ denotes the complex vector space of* $\mathrm{Spoin}_C(C^3)$ *left regular functions defined over the domain U_3. Suppose also that a point $z_0e_0 + z_1e_1 + z_2e_2 + z_{12}e_1e_2$ lies in U_1 if and only if $z_0e_0 + z_1e_1 + z_2e_2 + z_{12}e_3$ is in U_3. Then the complex vector spaces $\Psi_A(U_1, H(C))$ and $\Psi_B(U_3, C_C(E))$ are canonically isomorphic.*

PROOF: As the two sided ideals $C_C(R^3)(e_0 \pm e_1e_2e_3)$ are canonically isomorphic to the algebra $H(C)$ and the elements $\frac{1}{2}(e_0 \pm e_1e_2e_3)$ are idempotents of the algebra $C_C(R^3)$ with

$$\tfrac{1}{2}(e_0 + e_1e_2e_3) + \tfrac{1}{2}(e_0 - e_1e_2e_3) = e_0,$$

it follows that for each $\mathrm{Spoin}_C(C^3)$ left regular function $f_3 : U_3 \to C_C(R^3)$ the functions

$$f_3^+(\underline{z}) = f_3(\underline{z}) \cdot \tfrac{1}{2}(e_0 + e_1e_2e_3)$$

and

$$f_3^-(\underline{z}) = f_3(\underline{z}) \cdot \tfrac{1}{2}(e_0 - e_1e_2e_3)$$

satisfy the equations

$$\tfrac{1}{2}(e_0 + e_1e_2e_3)\left(\sum_{j=0}^{3} e_j \frac{\partial}{\partial z_j} f_3^+(\underline{z}')\right) = 0, \tag{10}$$

$$\tfrac{1}{2}(e_0 - e_1e_2e_3)\left(\sum_{j=0}^{3} e_j \frac{\partial}{\partial z_j} f_3^-(\underline{z}')\right) = 0, \tag{11}$$

and

$$f_3^+(\underline{z}) + f_3^-(\underline{z}) = f_3(\underline{z}).$$

Moreover, equations (10) and (11) are equivalent to equation (7). It is now straightforward to deduce that the construction of the functions f_3^+ f_3^- given above yields a canonical isomorphism between the spaces $\Psi_A(U_1, H(C))$ and $\Psi_B(U_3, C_C(R^3))$. □

Besides the generalized Cauchy-Riemann equations given in examples 1, 2 and 3 we may also consider the Clifford algebra valued operators:

a $\quad (e_0 \pm e_1 \dots e_n)\left(\sum_{j=1}^{n} e_j \frac{\partial}{\partial z_j}\right)$

and

b $\quad (e_0 \pm e_1 \ldots e_n)\left(\sum_{k=0}^{n} e_k \frac{\partial}{\partial z_k}\right)$

for $n = 1 \bmod 4$
and

c $\quad (e_0 \pm ie_1 \ldots e_n)\left(\sum_{j=1}^{n} e_j \frac{\partial}{\partial z_j}\right)$

d $\quad (e_0 \pm ie_1 \ldots e_n)\left(\sum_{k=0}^{n} e_k \frac{\partial}{\partial z_j}\right)$

for $n = 3 \bmod 4$.

Each of the hypercomplex function theories associated with equations (7), (8) and (9) and expressions a, b, c and d has a generalized Cauchy integral formula [3–9]. We shall now give the following classification of generalized Cauchy-Riemann equations whose associated hypercomplex function theory admits a generalization of the Cauchy integral formula given in [1].

THEOREM 1: *Suppose that A is a complex, associative algebra with an identity* 1, *and V is a complex, finite dimensional subspace of A, spanned by the vectors $\{k_j\}_{j=1}^{n}$. Suppose also that V_R is the real subspace of V spanned by the vectors $\{k_j\}$. Then for each point $z_0 \in V$ there exists a unique real analytic function*

$$W_{\underline{z}_0} : U_{z_0} \subset (V_R + \underline{z}_0) - \{\underline{z}_0\} \to A$$

such that for each left regular function $f: U \subseteq V \to A$, with respect to the vectors $\{k_j\}$, we have

$$f(Z_0) = \int_{\partial M} W_{Z_0}(Z)\, DZ f(Z),$$

where M is a real n dimensional, compact manifold lying in $U_{\underline{z}_0}$, with $z_0 \in \mathring{M}$, if and only if there exist vectors $\{l_j\}_{j=1}^{n}$ lying in A and satisfying the relations

$$l_j k_j = 1 \tag{12}$$

and

$$l_j k_p + l_p k_j = 0 \text{ for } j \neq p. \tag{13}$$

PROOF: Suppose that there exist elements $\{l_j\}_{j=1}^n \in A$ which satisfy the relations (12) and (13) then the function

$$W(x)=\frac{1}{w_{n-1}}(x_1l_1+\cdots+x_nl_n)\left(x_1^2+\cdots+x_n^2\right)^{-n/2},$$

where w_{n-1} is the surface area of the unit sphere, S^{n-1}, lying in R^n, satisfies the equation

$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}W(x)k_j=0 \tag{14}$$

Moreover, the function $W(z-z_0)$ is well defined, and real valued, on the set $(V_R+z_0)-\{z_0\}$. It now follows from equation (14) that for each left regular function $f:U\subseteq V\to A$, with respect to $\{k_j\}$, and for each $z_0\in U$ and each real n-dimensional, compact manifold $M'\subseteq(V_R+z_0)\subseteq U$ with $z_0\in \mathring{M}'$ we have

$$\int_{\partial M'}W(\underline{z}-\underline{z}_0)Dzf(\underline{z})=f(\underline{z}_0).$$

This proves the first part of the theorem. Now suppose that we have a real analytic function

$$W_{z_0}:(V_R+z_0)-\{z_0\}\to A$$

such that for each left regular function $f:U\subseteq V\to A$, with respect to $\{k_j\}_{j=1}^n$, we have

$$f(z_0)=\int_{\partial M}W_{z_0}(z)Dzf(z),$$

for each real n-dimensional, compact manifold lying in $(V_R+z_0)\cap U$, with $z_0\in\mathring{M}$. It follows that

$$\int_{\partial M}W_{z_0}(z)Dz=1, \tag{15}$$

and consequently

$$\int_{\partial M'}W_{z_0}(z)Dz=0$$

for each real, n-dimensional, compact submanifold of $(V_R+\underline{z}_0)-\{z_0\}$. From Stokes' theorem we have

$$\int_{M'}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}W_{z_0}(z)k_j\,\mathrm{d}x^n=0, \tag{16}$$

where dx^n is the Lebesque measure of the manifold M'. As the identity (16) is valid for each real, n-dimensional, compact submanifold of $(V_R + z_0) - \{z_0\}$ it follows that

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} W_{z_0}(z) k_j = 0 \tag{17}$$

By considering the integral appearing on the left hand side of equation (15) to be taken over M and M_r, where $r \in R^+$ and for each point $\underline{z} \in M$ we have that $r(\underline{z} - \underline{z}_0) + \underline{z}_0 \in M_r$, it may be observed from the uniqueness of the function W_{z_0} that it is homogeneous of degree $-(n-1)$ with respect to the point z_0. It follows from [19] that on the unit sphere

$$S^{n-1}(z_0) = \{z + z_0 \in V_R + z_0 : |z| = 1\}$$

we have

$$W_{z_0}(z) \circledS_R e_0 = \sum_{p=0}^{\infty} P_p(z),$$

where each function $P_n(z)$ is an $A \circledS_R C_C(V_R)$ valued harmonic polynomial homogeneous of degree p with respect to the point z_0. If we now consider the homogeneous function

$$K : (V_R + z_0) + \{z_0\} \to A \circledS_R C_C(V_R) : K(z)$$

$$= \sum_{p=0}^{\infty} p_p(z) / |z - z_0|^{-(n-1+p)} \tag{18}$$

we may observe from the homogeneity of the function $W_{z_0}(z)$ and expression (18) that

$$K(z) = W_{z_0}(z) \circledS_R e_0$$

for each $z \in (V_R + z_0) - \{z_0\}$. It follows that

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} K(z) k_j \circledS_R e_0 = 0.$$

It follows from expression (18) that

$$K(z) = G(z) \cdot |z|^{-n},$$

where $G : (V_R + z_0) - \{z_0\} \to A \circledS_R C_C(V_R)$, and is homogeneous of degree one with respect to the point z_0. Thus, from equation (17) we have that the function $G(z)$ satisfies the equation

$$\frac{1}{n}|z-z_0|^2 \sum_{j=1}^{n} \frac{\partial}{\partial x_j} G(z) k_j = G(z) \sum_{j=1}^{n} x_j k_j \circledS_R e_0. \tag{19}$$

As the right hand side of equation (19) is homogeneous of degree two it follows that the function

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} G(z) k_j$$

is homogeneous of degree zero. Direct calculation now reveals that this is only possible if the polynomials $P_p(z)$ are identically zero for $p=0$ and $p>1$. Thus

$$W_0(x) = \left(\sum_{j=1}^{n} x_h l_j\right)\left(\sum_{j=1}^{n} x_j^2\right)^{-n/2}$$

for some elements $l_j \in A$. It now follows from the identity (19) that the elements $\{l_j\}$ and $\{k_j\}$ satisfy the relations (12) and (13). This completes the proof. □

OBSERVATION 2: It may be deduced from relations (12) and (13), and definition 2 that the minimal complex subalgebra of A containing the elements $\{l_1 k_p\}_{p=2}^{n}$ is a complex Clifford algebra.

From observation 2 we have the following refinement to theorem 1.

THEOREM 1′: *Suppose that A is a complex, associative algebra with an identity. Then A admits a generalized Cauchy-Riemann equation, of the type givien in definition 4, together with a generalized Cauchy integral formula, of the type given in the statement of theorem 1, if and only if the algebra A contains a complex subalgebra which is isomorphic to a finite dimensional Clifford algebra.*

EXAMPLES:

1. Consider the complex pre-Clifford algebra $P_1 C(E, F)^- \circledS_R C$, then it may be observed that the minimal complex subalgebra containing the elements $\{e_1 f_j\}_{j=2}^{n}$, where e_i and f_i are given in expression (3), is a complex, 2^{n-1} dimensional Clifford algebra. It follows that the generalized Cauchy-Riemann operator

$$\sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j} \tag{20}$$

has an associated Cauchy integral formula of the type described in Theorem 1.

2. Also, for the complex pseudo-Clifford algebra, $PC(E, F) \circledS_R C$, it may be observed that the minimal complex subalgebra containing the elements $\{e_1' f_j'\}_{j=2}^n$, where e_i' and f_i' are given in expression (4), is a complex, 2^{n-1} dimensional Clifford algebra. It follows that the generalized Cauchy-Riemann operator

$$\sum_{j=1}^{n} f_j' \frac{\partial}{\partial z_j} \tag{21}$$

has an associated Cauchy integral formula of the type described in theorem 1.

3. Suppose that $A_1, \ldots, A_k$ are complex associative algebras with identities $i_1, \ldots, i_k$. Then, for the complex associative algebra $C_C(E) \circledS_C A_1 \circledS_C \cdots \circledS_C A_k$ the generalized Cauchy-Riemann operators

$$\sum_{j=0}^{n} e_j \circledS_C i_1 \circledS_C \cdots \circledS_C i_k \frac{\partial}{\partial z_j}$$

and

$$\sum_{l=1}^{n} e_l \circledS_C i_1 \circledS_C \cdots \circledS_C i_k \frac{\partial}{\partial z_1}$$

have associated Cauchy integral formulae of the type given in theorem 1. A special example of this case has previously been discussed in [5].

Although a generalized Cauchy-Riemann operator might admit an associated Cauchy integral formula it does not necessarily follow that the solutions to the generalized Cauchy-Riemann equation are nontrivial.

THEOREM 2: *For the complex pre-Clifford algebra $P_1C(E, F) \circledS_R C$, with $E \cap F = \{0\}$, the only solutions to the generalized Cauchy-Riemann operator* (20) *are constants.*

PROOF: Suppose that $F(z)$ is a solution to the operator (20) then it follows that it also satisfies the equation

$$\frac{\partial F(z)}{\partial z_1} + \sum_{k=2}^{n} (-e_1) f_k \frac{\partial F(z)}{\partial z_k} = 0.$$

As the elements of $1, -e_1 f_2, \ldots, -e_1 f_n$ are elements of a Clifford algebra it follows from [5 theorem 13] that in a neighbourhood $U(Z_0)$ of a point z_0 there is a power series expansion

$$\sum_{m=0}^{\infty} F_{k_2 \ldots k_n}(z - z_0) a_{k_2 \ldots k_n}$$

which converges uniformly to $F(z)$, where $a_{k_2\ldots k_n} \in P_1C(E, F)Ⓢ_R C$ and

$$F_{k_2\ldots k_n}(z) = \sum_{\sigma(k_2,\ldots,k_n)} s_{k_2}(z)\ldots s_{k_n}(z) \tag{22}$$

where summation is taken over every permutation, without repetition of the s_{k_i}'s, and $s_{k_i}(z) = z_i + e_1 f_i z_0$.
Moreover, it follows from [17] that

$$\frac{\partial}{\partial z_1} F_{k_2\ldots k_n}(z) + \sum_{k=2}^{n} -e_1 f_k \frac{\partial}{\partial z_i} F_{k_2\ldots k_n}(z) = 0 \tag{23}$$

It also follows from the construction of the polynomials (22) and the relations (12) and (13) that these polynomials take their values in the space spanned by the vectors 1, $e_1f_2, \ldots, e_1f_n$.
Suppose that we write

$$F_{k_2\ldots k_n}(z) = P_1(z) + P_2(z)e_1f_2 + \cdots + P_n(z)e_1f_n,$$

then as $F_{k_2\ldots k_n}(z)$ satisfies the equation (23) it follows that

$$\frac{\partial p_1}{\partial z_1}(z) - e_1 f_2 \frac{\partial p_2}{\partial z_2}(z)e_1f_2 - \cdots - e_1f_2\frac{\partial p_n}{\partial z_n}(z)e_1f_n = \sum_l \frac{\partial P_l}{\partial z_l}(z)$$
$$= 0.$$

For the polynomial (22) to also satisfy the operator (20) it would follow that

$$f_1 \frac{\partial p_1}{\partial z_1}(z) + \sum_{k=2}^{n} f_k \frac{\partial P_k}{\partial z_k}(z) e_1 f_k = 0. \tag{24}$$

As $E \cap F = \{0\}$ it follows from lemma 1 that equation (24) is only possible if and only if the polynomial $F_{k_2\ldots k_m}(z)$ is a constant. The result follows. □

The polynomials (24) are called Feuter polynomials, and are introduced in a more general setting in [17].

The function theory associated with the complex pseudo-Cliford algebra $PC(E, F)Ⓢ_R C$, and the generalized Cauchy-Riemann operator (21), is much richer than that for the operator (20). For example the generalized Cauchy kernel

$$W(x) = \left(\sum_{j=1}^{n} x_j e_j'\right)\left(\sum_{j=1}^{n} x_j^2\right)^{-n/2}$$

satisfies the equation

$$\sum_{j=1}^{n} f_j' \frac{\partial}{\partial x_j} W(x) = 0. \tag{25}$$

As a consequence of equation (25) it may be deduced that many of the results previously obtained in both real and complex Clifford analysis [3,16] may also be obtained over real and complex pseudo-Clifford algebras. As an example we have:

THEOREM 3: (*i*) *Suppose that the $PC(E, F)\circledS_R C$ valued holomorphic function $f(z+z_0)$ is left regular with respect to the vectors $\{f_j'\}_{j=1}^n$, in the variable $z+z_0$, where z_0 is a fixed point. Then the function $f(z+z_0)$ is left regular with respect to the vectors $\{f_j'\}_{j=1}^n$, in the variable z.*

(*ii*) *Suppose that the $PC(E, F)\circledS_R C$ valued holomorphic function $f(az\tilde{a})$ is left regular with respect to the vectors $\{f_j'\}_{j=1}^n$, in the variable aza, where*

$$a = z_p \ldots z_1, \quad \tilde{a} = z_1 \ldots z_p$$

with $p = 0$ 2 mod and

$$z_{2i+1} = z_{1,2i+1} f_1' + \cdots + z_{n,2i+1} f_n' \quad \text{for } 0 \leqslant i \leqslant \frac{p}{2} - 1$$

and

$$z_{2i} = z_{1,2i} e_1' + \cdots + z_{n,2i} e_n' \quad \text{for } 1 \leqslant i \leqslant \frac{p}{2},$$

with $a\,\tilde{a} \neq 0$. Then the function $\tilde{a}f(az\tilde{a})$ is left regular with respect to the vectors $\{f_j')_{j=1}^n$, in the variable z.

(The proof follows the same lines as given in [16,18] and [23], so it is omitted). □

In fact the action $az\tilde{a}$ described in theorem 3 is equivalent to the action (5), and the group of all such elements, a, is isomorphic to the group $\mathrm{Spin}_C(C^n)$. It may be seen that the action (5) describes a complex dilation together with an action of the complex special orthogonal group of $n \times n$ matrices

$$SO(C^n) = \left\{(a_{ij}) : a_{ij} \in C \ \text{ and } (a_{ij})(a_{ij})^T = I\right\}.$$

The group composed of the group $SO(C^n)$ together with the set of translations in C^n and complex dilations is called the *complex special Poincaré group.* It follows from theorem 3 that the set of $CP(E, F)Ⓢ_R C$ values holomorphic functions, which satisfy the operator (21), is invariant under actions of the complex special Poincaré group. When $E \neq F$ the complex special Poincaré group is the maximal subgroup of the conformal group in C^n under which the set of solutions to the operator (21) is invariant, as may be seen by the following result:

THEOREM 4: (*i*) *Suppose that the* $PC(E, F)Ⓢ_R C$ *valued holomorphic function* $f(bz'\tilde{b})$ *is left regular with respect to the vectors* $\{f_j'\}_{j=1}^n$, *in the variable* $bz'\tilde{b}$, *where*

$$z' = z_1'f_1' + \cdots + z_n'f_1'$$

$$\tilde{b} = z_p \ldots z_1, \quad b = z_1 \ldots z_p,$$

$p = 1 \bmod 2$ *and*

$$z_{2i+1} = z_{1,wi+1}e_1' + \cdots + z_{n,2i+1}e_n' \quad \text{for } i = 1, \ldots, (p-1)/2$$

$$z_{2i} = z_{1,2i}f_1' + \cdots + z_{n,2i}f_n' \quad \text{for } i = 1, \ldots, (p-1)/2$$

Then the function $\tilde{b}f(bz'\tilde{b})$ *is left regular with respect to the vectors* $\{e_j'\}_{j=1}^n$.

(*ii*) *Suppose that the* $PC(E, F)Ⓢ_R C$ *valued holomorphic function* $f((\sum_{j=1}^n z_jf_j')(\sum_{j=1}^n z_j^2)^{-1})$ *is left regular with respect to the vectors* $\{f_j'\}_{j=1}^n$, *in the variable* $z^{-1} = (\sum_{j=1}^n z_jf_j')(\sum_{j=1}^n z_j^2)^{-1}$, *where* $n = 0 \bmod 2$ *and* $\sum_{j=1}^n z_j^2 \neq 0$. *Then the function* $(\sum_{j=1}^n z_jf_j')(\sum_{j=1}^n z_j^2)^{-n/2}f(z^{-1})$ *is left regular with respect to the vectors* $\{e_j'\}_{j=1}^n$ *in the variable* $\sum_{j=1}^n z_jf_j'$.

(The proof follows the same lines as arguments given in [18] and [23] so it is omitted). □

Following Theorem 4 and [18,23] we may now obtain:

THEOREM 5: *Suppose that A is a complex associative algebra with an identity, and that V is a complex finite dimensional subspace spanned by the vectors* $\{k_j\}_{j=1}^n$. *Suppose also that* $n = 0$ *mod 2 and that the generalized Cauchy-Riemann operator* $\sum_{j=1}^n k_j \frac{\partial}{\partial z_j}$ *has an associated Cauchy integral formula. Then the set of A valued holomorphic functions which satisfy this operator is invariant under the transforms*

(i) $z \to z + z_0$

for some point $z_0 V$

(ii) $z \to \lambda z$

for $\lambda \in C - \{0\}$

(iii) $z \to 0_n z$,

where 0_n *is an element of the complex orthogonal group acting over* V

(iv) $z_1 k_1 + \cdots + z_n k_n \to (z_1 k_1 + \cdots + z_n k_n)(z_1^2 + \cdots + z_n^2)^{-1}$

if and only if the minimal complex subalgebra of A *generated by the elements* $\{k_j\}_{j=1}^n$ *is a finite dimensional Clifford algebra.*

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