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On infinitesimal deformations of rational surface singularities

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0. Introduction

This paper is concerned with the computation of the space T_X^1 of first order infinitesimal deformations of a two-dimensional rational singularity $(X, 0)$. For cyclic resp. dihedral quotient singularities the dimension of this space was determined in [Riemenschneider, 1974] [Pinkham, 1977] resp. [Behnke and Riemenschneider, 1977, 1978]. In these cases one obtains the formula

$$\dim T_X^1 = \dim T_{\tilde{X}}^1 + \operatorname{emb}(X) - 4 \qquad (0.1)$$

unless X is a rational double point. Here $\operatorname{emb}(X)$ denotes the embedding dimension of X, $\pi: \tilde{X} \to X$ the minimal resolution of X, and $T_{\tilde{X}}^1 \cong H^1(\tilde{X}, \Theta_{\tilde{X}})$ is the space of first order infinitesimal deformations of $\tilde{X}$. The data at the right hand side of (0.1) can for many rational surface singularities be computed in terms of the resolution graph (see e.g. [Artin, 1966], Cor. 6 and [Laufer, 1973]).

For arbitrary two-dimensional quotient singularities C. Kahn recently gave a (computer-aided) proof of (0.1), based on invariant theoretic results worked out by Kahn, Riemenschneider and the authors (cf. [Behnke et al., in prep.] and [Kahn, 1984]).

On the other hand J. Wahl had found an example of a (non Gorenstein) rational surface singularity for which $\dim T_X^1 > \dim T_{\tilde{X}}^1 + \operatorname{emb}(X) - 4$ (see [Behnke and Riemenschneider, 1977, 1978], p. 4 and Example 4.21. below). In a letter he also gave a proof of the inequality

$$\dim T_X^1 \geqslant \dim T_{\tilde{X}}^1 + \operatorname{emb}(X) - 4 \qquad (0.2)$$

for all rational surface singularities. We give his proof in an appendix to our paper.

In this article we prove (0.1) for a large class of two-dimensional rational singularities (see Theorem 4.10. below). We briefly sketch the method applied.

From Schlessinger's description of T_X^1 (cf. [Schlessinger, 1971] or Theorem 1.1. below) one concludes by local duality that the dual space $(T_X^1)^*$ can be computed as follows:

Let: $i: X \hookrightarrow \mathbb{C}^n$ be a closed embedding of a Stein representative, and let Ω^1_X and $\Omega^1_{\mathbb{C}^n}$ be the sheaves of Kähler differentials and ω_X the canonical sheaf. By $X' = X - \{0\}$ we denote the smooth part of X. Then $(T^1_X)^*$ is isomorphic to the cokernel of the natural map

$$H^0\big(X', i^*\Omega^1_{\mathbb{C}^n} \otimes \omega_X\big) \overset{(\mu' \otimes 1)}{\to} H^0\big(X', \Omega^1_X \otimes \omega_X\big),$$

induced by the epimorphism $\mu': i^*\Omega^1_{\mathbb{C}^n} \to \Omega^1_X$.

Let $f_1, \dots, f_n$ be a system of generators for the maximal ideal of $\mathcal{O}_{X,0}$. For a suitable trivialization $i^*\Omega^1_{\mathbb{C}^n} \cong \mathcal{O}^n_X$, $\mu': \mathcal{O}^n_X \to \Omega^1_X$ is defined by $\mu'(g_1, \dots, g_n) = g_1\, df_1 + \cdots + g_n\, df_n$. This map can be studied using the resolution $\pi: \tilde{X} \to X$. Let $E = \pi^{-1}(0)$ be the exceptional set, Z the fundamental cycle, and let $\Omega^1_{\tilde{X}}\langle \log E\rangle$ be the sheaf of meromorphic 1-forms with at most logarithmic poles along E. As above we have a map

$$\tilde{\mu}: \mathcal{O}^{\oplus n}_{\tilde{X}} \to \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z),$$

$(g_1, \dots, g_n) \to g_1\, \mathrm{df}_1 + \cdots + g_n\, df_n$, where now $f_1, \dots, f_n$ are considered as holomorphic functions on $\tilde{X}$. As $(X, 0)$ is a rational singularity and ω_X is reflexive there is a natural isomorphism between $\mathrm{H}^0(\mathrm{X}', \mathcal{O}^{\oplus n}_{X'} \otimes \omega_{X'})$ and $H^0(\tilde{X}, \mathcal{O}^{\oplus n}_{\tilde{X}} \otimes \omega_{\tilde{X}})$. Using this isomorphism one sees that T^1_X is dual to the cokernel of the following composite map

$$H^0\big(\tilde{X}, \mathcal{O}^{\oplus n}_{\tilde{X}} \otimes \omega_{\tilde{X}}\big) \overset{(\tilde{\mu} \otimes 1)^*}{\to} H^0\big(\tilde{X}, \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z) \otimes \omega_{\tilde{X}}\big)$$

$$\hookrightarrow H^0\big(X', \Omega^1_{X'} \otimes \omega_{X'}\big).$$

The cokernel of the inclusion $H^0(\tilde{X}, \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z) \otimes \omega_{\tilde{X}}) \hookrightarrow H^0(X', \Omega^1_{X'} \otimes \omega_{X'})$ can be computed using results of J. Wahl [Wahl, 1975] (see Ch. 2). For the discussion of $(\tilde{\mu} \otimes 1)^*$ we have to make more restrictive assumptions (e.g. that the fundamental cycle is reduced) in order to be able to control the kernel and the cokernel of μ. This discussion is performed in Ch. 3 and Ch. 4 and leads to the proof of (0.1) for a large class of rational surface singularities. The precise results are stated in Theorem 4.8. and Example 4.13.

1. Schlessinger's description of $\mathbf{T}^1_X$ and duality

Let $(X, 0)$ be a normal surface singularity. We recall a result of M. Schlessinger [Schlessinger, 1971] which gives a cohomological description of the space T^1_X of infinitesimal deformations of X. Then we apply duality to obtain the description of $(T^1_X)^*$ which is basic for our paper.

Let $i: X \hookrightarrow \mathbb{C}^n$ be an embedding of a small Stein space representing the singularity $(X, 0)$. Denote by $X' = X - \{0\}$ the smooth part of X, by $\Omega^1_{\mathbb{C}^n}$ resp.

Ω^1_X the sheaves of Kähler differentials on $\mathbb{C}^n$ resp. X, and by $\Theta_{\mathbb{C}^n}$ resp. Θ_X their duals.

THEOREM 1.1.

([Schlessinger, 1971] §1, Lemma 2). *The module T^1_X of first order infinitesimal deformations of $(X, 0)$ is the kernel of the map*

$$H^1(X', \Theta_{X'}) \to H^1(X', \Theta_{\mathbb{C}^n|X'})$$

which is induced by the natural inclusion of tangent sheaves $\Theta_{X'} \hookrightarrow \Theta_{\mathbb{C}^n|X'}$.
To apply local duality we remark that $H^1(X', \Theta_{X'})$ is canonically isomorphic to $H^2_{\{0\}}(X, \Theta_X)$, the second local cohomology group with support in the singular point 0. Similarly $H^1(X', \Theta_{\mathbb{C}^n|X'})$ is canonically isomorphic to $H^2_{\{0\}}(X, \Theta_{\mathbb{C}^n|X})$. Then we see by local duality that T^1_X is dual to the cokernel of

$$\mathrm{Hom}_{\mathcal{O}_X}(\Theta_{\mathbb{C}^n|X'}\omega_X) \to \mathrm{Hom}_{\mathcal{O}_X}(\Theta_X, \omega_X).$$

As all these sheaves are reflexive we finally get

COROLLARY 1.2.

$(T^1_X)^*$ is isomorphic to the cokernel of the map

$$H^0(X', \Omega^1_{\mathbb{C}^n|X'} \otimes \omega_{X'}) \to H^0(X', \Omega^1_{X'} \otimes \omega_{X'})$$

induced by the restriction map $\Omega^1_{\mathbb{C}^n} \otimes \mathcal{O}_X \to \Omega^1_X$.

REMARK 1.3.

We can make this result a little more explicit: observe that the restriction $\Omega^1_{\mathbb{C}^n|X}$ is generated as an $\mathcal{O}_X$-module by the differentials $\mathrm{d}f_1, \dots, \mathrm{d}f_n$ of the coordinate functions f_i on $\mathbb{C}^n$. Equivalently we can take for $f_1, \dots, f_n$ any set of generators for the maximal ideal of $\mathcal{O}_{X,0}$. Let $\mu : \mathcal{O}^n_X \to \Omega^1_X$ be the surjection defined by $\mu(g_1, \dots, g_n) = \sum_{i=1}^{n} g_i \,\mathrm{d}f_i$. Then $(T^1_X)^*$ is isomorphic to the cokernel of the map

$$\mu \otimes 1 : H^0(X', \omega^{\oplus n}_{X'}) \to H^0(X', \Omega^1_{X'} \otimes \omega_{X'}).$$

In an invariant way the image of $(\mu' \otimes 1)$ can be characterized as the subspace of $H^0(X', \Omega^1_{X'} \otimes \omega_{X'})$ generated by all elements of the form $\Sigma g_i \otimes \mathrm{d}h_i$, $g_i \in H^0(X', \omega_{X'})$, $h_i \in H^0(X', \mathcal{O}_{X'})$.

2. The case of rational singularities

We keep our previous hypotheses and assume moreover that X is a rational singularity. Let $\pi: \tilde{X} \to X$ be the minimal good resolution of X, and let $E = \pi^{-1}(0)$ be the exceptional set. The irreducible components $E_1, \ldots, E_r$ of E are nonsingular rational curves of selfintersection number $-b_i = E_i . E_i \leqslant -2$.

Let $\Omega^1_{\tilde{X}}$ resp. $\omega_{\tilde{X}}$ be the sheaves of holomorphic 1- resp. 2-forms on X. Observe that by rationality $H^0(\tilde{X}, \omega_{\tilde{X}}) \cong H^0(X', \omega_{X'})$ (see e.g. [Pinkham, 1980], §15). We denote the pull backs to $\tilde{X}$ of the functions f_i of Remark 1.3. also by f_i. Their differentials are sections of $\Omega^1_{\tilde{X}}\langle\log E\rangle(-Z)$, where $\Omega^1_{\tilde{X}}\langle\log E\rangle$ denotes the sheaf of meromorphic 1-forms on $\tilde{X}$ with logarithmic poles along E, and Z is the fundamental cycle of $\tilde{X}$. Again we define a sheaf map

$$\tilde{\mu}: \mathcal{O}_{\tilde{X}}^{\oplus n} \to \Omega^1_{\tilde{X}}\langle\log E\rangle(-Z)$$

by $\tilde{\mu}(g_1, \ldots, g_n) = \sum_{i=1}^{n} g_i \, \mathrm{d}f_i$. This induces a map

$$(\tilde{\mu} \otimes 1)^*: H^0\big(\tilde{X}, \omega_{\tilde{X}}^{\oplus n}\big) \to H^0\big(\tilde{X}, \Omega^1_{\tilde{X}}\langle\log E\rangle(-Z) \otimes \omega_{\tilde{X}}\big).$$

Let ρ be the inclusion

$$\rho: H^0\big(\tilde{X}, \Omega^1_{\tilde{X}}\langle\log E\rangle(-Z) \otimes \omega_{\tilde{X}}\big) \to H^0\big(X', \Omega^1_{X'} \otimes \omega_{X'}\big).$$

By Remark 1.3. we get

LEMMA 2.1.

$(T^1_X)^*$ is isomorphic to the cokernel of the composite map

$$\rho \circ (\tilde{\mu} \otimes 1)^*: H^0\big(\tilde{X}, \omega_{\tilde{X}}^{\oplus n}\big) \to H^0\big(X', \Omega^1_{X'} \otimes \omega_{X'}\big). \qquad \blacksquare$$

In this section we compute the cokernel of the map ρ. Let $\mathrm{Der}_E(\tilde{X})$ be the locally free sheaf of logarithmic vectorfields on $\tilde{X}$ which is dual to $\Omega^1_{\tilde{X}}\langle\log E\rangle$. Our result is

PROPOSITION 2.2.

The cokernel of the inclusion map

$$\rho: H^0(\tilde{X}, \Omega^1_{\tilde{X}}\langle\log E\rangle(-Z)\otimes\omega_{\tilde{X}}) \to H^0(X', \Omega^1_{X'}, \otimes\omega'_{X'})$$

has dimension

$$\dim H^1(\tilde{X}, \mathrm{Der}_E(\tilde{X})) - 3\cdot Z\cdot Z + Z\cdot E - 4.$$

As ρ is an injection this – together with Lemma 2.1. – implies

COROLLARY 2.3.

Let $\mathscr{R}$ resp. $\mathscr{C}$ be the kernel resp. cokernel of $\tilde{\mu}: \mathscr{O}_{\tilde{X}}^{\oplus n} \to \Omega^1_{\tilde{X}}\langle\log E\rangle(-Z)$. Then

$$\dim T^1_X = \dim H^1(\tilde{X}, \mathrm{Der}_E(\tilde{X})) + \dim H^0(\tilde{X}, \mathscr{C}\otimes\omega_{\tilde{X}})$$
$$+\dim H^1(\tilde{X}, \mathscr{R}\otimes\omega_{\tilde{X}}) - 3\cdot Z\cdot Z + Z\cdot E - 4.$$

REMARK 2.4.

Note that in the case of a reduced fundamental divisor this simplifies to

$$\dim H^1(\tilde{X}, \mathrm{Der}_E(\tilde{X})) + \dim H^0(\tilde{X}, \mathscr{C}\otimes\omega_{\tilde{X}}) + \dim H^1(\tilde{X}, \mathscr{R}\otimes\omega_{\tilde{X}})$$
$$+2\cdot\sum(b_i-2),$$

where we used that $-E^2-2=\sum_{i=1}^{r}(b_i-2)$.

For the proof of Proposition 2.2. we need the following vanishing result from [Wahl, 1975]:

THEOREM 2.5. (J. Wahl)

Let $\tilde{X}$ be the minimal good resolution of a normal surface singularity X. Then $H^1_E(\tilde{X}, \mathrm{Der}_E(\tilde{X}))=0$. ■

COROLLARY 2.6.

i) *On the minimal good resolution* $\tilde{X}$ *of a normal surface singularity* X

$$H^1(\tilde{X}, \Omega^1_{\tilde{X}}\langle \log E\rangle \otimes \omega_{\tilde{X}}) = 0$$

ii) *If* L *is a positive cycle on the exceptional set of* $\tilde{X}$, *then* $H^0(|L|, \mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_L(L)) = 0$.

Proof

i) is deduced from 2.5. by Serre duality. For ii) observe that $H^0(|L|, \mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_L(L))$ injects into

$$H^1_E(\tilde{X}, \mathrm{Der}_E(\tilde{X})) = \varinjlim_L H^0(|L|, \mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_L(L))$$

(see e.g. [Wahl, 1975], Proposition 2.2). ■

Now consider the first piece of the exact sequence of local cohomology (a locally free sheaf has no sections supported on E):

$$0 \to H^0(\tilde{X}, \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z) \otimes \omega_{\tilde{X}}) \to H^0(X', \Omega^1_{\tilde{X}'}\langle \log E\rangle(-Z) \otimes \omega_{\tilde{X}'})$$

$$\to H^1_E(\tilde{X}, \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z) \otimes \omega_{\tilde{X}}) \to H^1(\tilde{X}, \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z) \otimes \omega_{\tilde{X}}).$$

The cohomology group on the right hand side vanishes by 2.6. i) and the fact that $\mathcal{O}_{\tilde{X}}(-Z)$ is generated by global sections: there is an epimorphism of a direct sum of finitely many copies of $\Omega^1_{\tilde{X}}\langle \log E\rangle \otimes \omega_{\tilde{X}}$ to $\Omega^1_{\tilde{X}}\langle \log E\rangle(-Z) \otimes \omega_{\tilde{X}}$, and the functor $H^1(\tilde{X}, -)$ is right exact, since H^2's vanish on $\tilde{X}$.

The restrictions to X' of the sheaves $\Omega^1_{\tilde{X}} \otimes \omega_{\tilde{X}}$ and $\Omega^1_{\tilde{X}}\langle \log E\rangle(-Z) \otimes \omega_{\tilde{X}}$ are isomorphic, hence the cokernel of ρ can be identified with $H^1_E(\tilde{X}, \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z) \otimes \omega_{\tilde{X}})$ which by Serre-duality has the same length as $H^1(\mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_{\tilde{X}}(Z))$.

From the exact sequence of sheaves

$$0 \to \mathrm{Der}_E(\tilde{X}) \to \mathrm{Der}_E(\tilde{X})(Z) \to \mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z) \to 0$$

and 2.6. ii) we get

$$\dim H^1(\tilde{X}, \mathrm{Der}_E(\tilde{X})(Z)) = \dim H^1(\tilde{X}, \mathrm{Der}_E(\tilde{X})) + \dim H^1(|Z|, \mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z)).$$

Again using 2.6. ii) we get the equality

$$\dim H^1(|Z|, \mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z)) = -\chi(\mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z)).$$

Let $Z_0 = E_{i_0}$, $Z_1 = Z_0 + E_{i_1}$, ... $Z_l = Z_{l-1} + E_{i_l} = Z$ be a sequence of effective divisors with $Z_{k-1} \cdot E_{i_k} = +1$ for all k. Such a sequence exists by rationality of X. From the exact sequences

$$0 \to \mathcal{O}_{Z_k}(Z_k) \to \mathcal{O}_{Z_{k+1}}(Z_{k+1}) \to \mathcal{O}_{E_{i_{k+1}}}(Z_{k+1}) \to 0$$

tensored with $\mathrm{Der}_E(\tilde{X})$, and the split exact sequences

$$0 \to \mathcal{O}_{E_i} \to \mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_{E_i} \to \Theta_{E_i}(-t_i) \to 0,$$

t_i the number of components which meet the curve E_i, we get

$$\chi(\mathrm{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z)) = \sum_{k=0}^{l} (6 - t_{i_k} - 2b_{i_k}) - 2.$$

On the other hand, if K is a canonical divisor for $\tilde{X}$, since $\chi(\mathcal{O}_Z) = 1$ for the fundamental divisor of a rational singularity, we have $Z^2 = -2 - K \cdot Z = -2 + \sum_{k=0}^{l} (2 - b_{i_k})$. Together with $Z \cdot E = \sum_{k=0}^{l} (-b_{i_k} + t_{i_k})$ this proves what we want. ■

3. Computation of $H^0(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}})$

Recall that $\mathcal{C}$ was defined as the cokernel of

$$\tilde{\mu}: \mathcal{O}_{\tilde{X}}^{\oplus n} \to \Omega^1_{\tilde{X}}\langle \log E \rangle(-Z)$$

Let $\mathcal{F}$ be the image of $\tilde{\mu}$. Then $\mathcal{F}$ is a torsion free sheaf, and $\mathcal{C}$ is concentrated on E.

In this section we assume that the fundamental cycle meets every irreducible component of E – except possibly (-2) – curves-strictly negatively.

In order to compute $\mathcal{C}$ we will construct holomorphic functions on $\tilde{X}$ with prescribed divisors. We use the following observation of M. Artin ([Artin, 1966], proof of Theorem 4):

Lemma 3.1.

Let π: $\tilde{X} \to X$ be the minimal good resolution of a rational surface singularity. Let D be an effective divisor on $\tilde{X}$ such that $D \cdot E_i = 0$ for every irreducible *component* E_i *of* E. *Then there is an open neigbourhood* U *of* E *in* $\tilde{X}$ *and a holomorphic function* f *on* U *such that* $(f) = D \cap U$.

COROLLARY 3.2.

Let $\tilde{X}$ *be the minimal resolution of a rational surface singularity with reduced fundamental cycle* E. *Let* $E = E' + E''$ *be a decomposition into effective divisors with connected* E'. *Denote by* F *the sum of irreducible components of* E' *which meet* E'', *and write* $E' = E_0' + F$. *Let* D' *be an effective divisor with support in* E', *and let* Δ *be an effective divisor on a small neighbourhood* U *of* E_0' *which has no components in common with* E. *Put* $D := D' + \Delta$. *Suppose that*

i) $D \cdot E_i = 0$ *for all components* E_i *of* E_0'

ii) *the multiplicity of a component* E_i *of* F *in* D *is greater or equal to*

$$D \cdot E_i / (b_i - E_i \cdot E'').$$

Then there exists a holomorphic function f *on* $\tilde{X}$ *such that* $(f) \cap U = D$.

Proof

Let $E_1'', \ldots, E_k''$ be the connected components of E'', let F_i be the component of F meeting E_i'', and let m_i be its multiplicity in D. We put $C := D + \sum_{i=1}^{k} m_i E_i''$. Since $E_i'' + F_i$ is the exceptional set of a rational singularity with reduced fundamental cycle it follows from ii) that $C \cdot E_i \leqslant 0$ for all irreducible

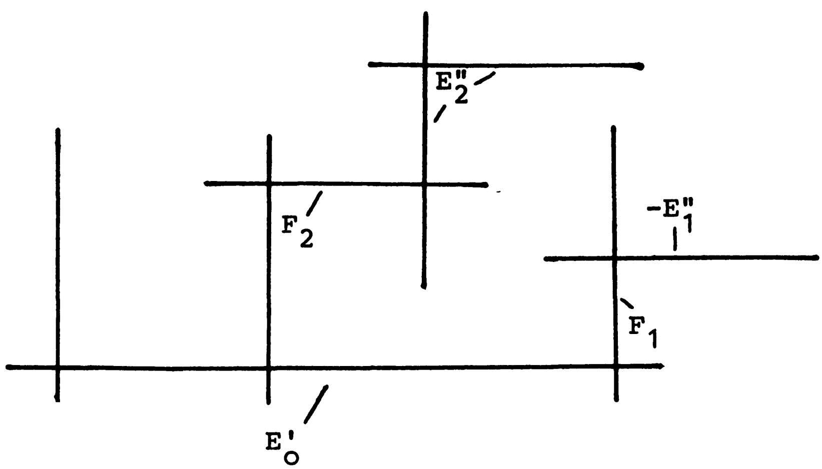


Fig. 1.

components of E. Obviously $C \cdot E_i = 0$ for all E_i contained in U, so we can modify C outside U to obtain an effective divisor $\tilde{C}$ with $\tilde{C} \cdot E_i = 0$ for all i. Applying (3.1) to $\tilde{C}$ we obtain the desired function f.

The next two lemmata give our description of $\mathscr{C}$. First we investigate $\mathscr{C}$ near curves with 'high self-intersection number'.

Lemma 3.3.

i) *Let p be a smooth point of E, and assume that $Z \cdot E_i < 0$ for the unique irreducible component E_i of E containing p. Then $\mathscr{C}_p = 0$.*
ii) *Let p be a point, where two components E_i, E_j of E intersect, and assume that $Z \cdot E_i < 0$ and $Z \cdot E_j < 0$. Then $\mathscr{C}$ is a skyscraper sheaf near p and* $\dim \mathscr{C}_p = 1$.

Proof

i) We can choose local coordinates (u, v) near p such that there is a holomorphic function f_1 on $\tilde{X}$ which in local coordinates is given by $f_1(u, v) = v^a$, a being the multiplicity of E_1 in the fundamental cycle Z. Locally $\Omega^1_{\tilde{X}}\langle \log E\rangle(-Z)$ is generated by $v^{a-1}\,\mathrm{d}v$ and $v^a\,\mathrm{d}u$.

Let Δ_0 be the curve $\{u = 0\}$. As $Z \cdot E_i < 0$ we can choose other curves $\Delta_1, \ldots, \Delta_l$ which are disjoint from Δ_0 and intersect the exceptional divisor transversally in smooth points such that

$$\left(Z + \sum_{k=0}^{l} \Delta_k\right) \cdot E_j = 0$$

for $j = 1, \ldots, r$. By Lemma 3.1. there is a holomorphic function f_2 on $\tilde{X}$ with divisor $Z + \sum_{k=0}^{l} \Delta_k$. After changing the u-coordinate, this function can be written locally as $f_2 = uv^a$. Obviously, $\mathrm{d}f_1$ and $\mathrm{d}f_2$ generate $\Omega^1_{\tilde{X}}\langle \log E\rangle(-Z)$ near p.

ii) We proceed as before and choose smooth curves Δ_1 and Δ_2 through p such that E_1, E_2, Δ_1, Δ_2 are pairwise transversal in p. There are local coordinates u, v with $E_i = \{v = 0\}$, $E_j = \{u = 0\}$, and holomorphic functions f, g_1, g_2 on $\tilde{X}$ such that $f = u^a v^b$, $g_k = u^a v^b(\alpha_k u + \beta_k v + \text{higher order terms})$ with $\alpha_1 : \beta_1 \neq \alpha_2 : \beta_2$. Again a, b are the multiplicities of E_j, E_i resp. in Z.

Locally at p the sheaf $\Omega^1_{\tilde{X}}\langle \log E\rangle(-Z)$ is generated by $u^{a-1}v^b\,\mathrm{d}u$ and $u^a v^{b-1}\,\mathrm{d}v$, while $\mathscr{F}$ is generated by $\mathrm{d}f, \mathrm{d}g_1, \mathrm{d}g_2$. A simple calculation now shows $\dim \mathscr{C}_p = 1$.

Now we restrict to the case $Z = E$, i.e. the fundamental divisor is reduced. We want to see, how $\mathscr{C}$ looks like on a linear chain of (-2)-curves which have intersection number 0 with E. So, let $E_0, \ldots, E_{t+1}$ be irreducible components of E such that

$$E_1 \cdot E_1 = \cdots = E_t \cdot E_t = -2$$

$$E_0 \cdot E_1 = E_1 \cdot E_2 = \cdots = E_t \cdot E_{t+1} = 1,$$

for $i = 1, \ldots, t$, E_i meets no other component but E_{i-1} and E_{i+1}, and $E \cdot E_0 < 0$, $E \cdot E_{t+1} < 0$. Let U be a small neighbourhood of $E_1 \cup \cdots \cup E_t$. Since $E \cap U$ intersects $E_1, \ldots, E_t$ trivially, E is a principal divisor on U (cf. [Artin, 1966]). The ideal sheaf $\mathscr{I}_{E|U}$ is generated by a single holomorphic function, say f_1. It vanishes to first order along $E \cap U$.

Blowing down $E_1 \cup \cdots \cup E_t$ yields a rational double point A_t. So f_1 can be extended to a minimal set f_1, f_2, f_3 of generators of the algebra of holomorphic functions on U. It is well-known that f_2 and f_3 can be chosen such that $f_1^{t+1} = f_2 f_3$ and such that they have the divisors

$$(f_2) = \sum_{i=1}^{t} i \cdot E_i + (t+1)(E_{t+1} \cap U)$$

$$(f_3) = \sum_{i=1}^{t} (t-i+1) \cdot E_i + (t+1)(E_0 \cap U).$$

Remark 3.4.

$\mathscr{F}|_U$ is generated by df_1, $d(f_1 f_2)$, $d(f_1 f_3)$.

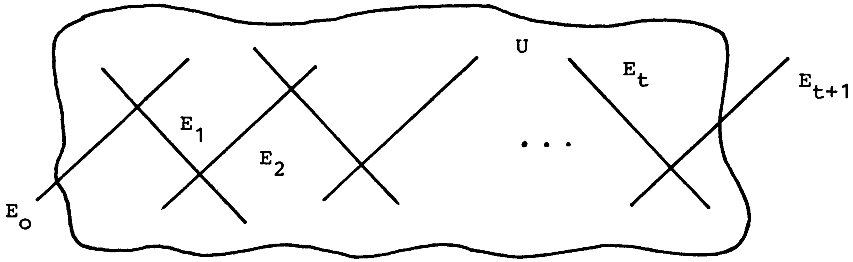


Fig. 2.

Proof

By Corollary 3.2. we see that f_1, $f_1 f_2$, $f_1 f_3$ can be chosen as restrictions of holomorphic functions on $\tilde{X}$. Conversely any holomorphic function on U which vanishes along $E \cap U$ is of the form $h \cdot f_1$, where h is in the ideal generated by f_1, f_2, f_3.

We put

$$D := \sum_{i=1}^{t} \max(i,\ t-i+1) E_i$$

Lemma 3.5.

i) *If t is odd, then* $\mathscr{C}\,|_U \cong \mathcal{O}_D$

ii) *If t is even, say $t = 2k$, then $\mathscr{C}\,|_U$ has a torsion subsheaf τ of lenght* 1, *concentrated at $E_k \cap E_{k+1}$, and there is an exact sequence*

$$0 \to \tau \to \mathscr{C}\,|_U \to \mathcal{O}_D \to 0$$

Proof

The sheaf $\Omega^1_{\tilde{X}}\langle\log E\rangle(-E)_{|U}$ is free with generators $f_1\ \mathrm{d}f_2/f_2$ and $f_1\ \mathrm{d}f_3/f_3$. This assertion is easily checked via an explicit resolution of the A_t-singularity. Since $(t+1)\ \mathrm{d}f_1 = f_1\ \mathrm{d}f_2/f_2 + f_1\ \mathrm{d}f_3/f_3$ we see that $\mathscr{C}\,|_U$ is cyclic with generator $f_1\ \mathrm{d}f_2/f_2 = -f_1\ \mathrm{d}f_3/f_3$. The claim now follows from (3.4) by a simple calculation in local coordinates. ■

For a later use we note

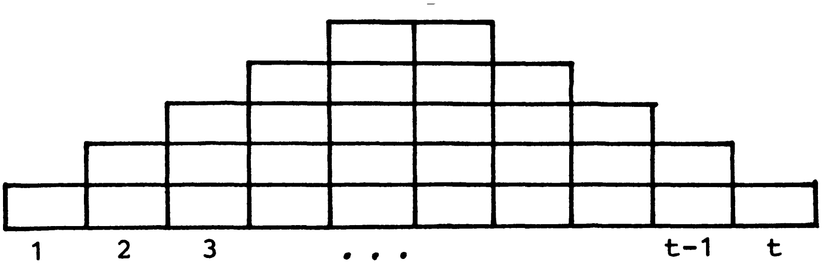


Fig. 3.

Lemma 3.6.

$$\dim H^0(D, \mathcal{O}_D) = [(t+1)/2]$$

Proof

For an effective cycle C supported on the exceptional locus of a rational surface singularity one has $H^1(|C|, \mathcal{O}_C) = 0$ (cf. [Artin, 1966]). So it is sufficient to compute the holomorphic Eulercharacteristic $\chi(\mathcal{O}_D)$ of $\mathcal{O}_D$.

Consider the sequence of divisors

$$D_1 = E_1, \quad D_2 = E_1 + E_2, \ \dots \ \dots, \ D_{t-1} = E_1 + \cdots + E_{t-1},$$
$$D_t = E_1 + \cdots + E_t \ D_{t+1} = E_1 + 2E_2 + E_3 + \ \dots, \ \dots$$

ending with D (cf. Fig. 3). Let E_{i_l} be the curve which is added to D_l to obtain D_{l+1}. Then the intersection number $D_l \cdot E_{i_l}$ is 1, if E_{i_l} does not start a new row, and it is 0 otherwise.

From the exact sequence

$$0 \to \mathcal{O}_{E_{i_l}}(-D_l) \to \mathcal{O}_{D_{l+1}} \to \mathcal{O}_{D_l} \to 0$$

we obtain

$$\chi(\mathcal{O}_{D_{l+1}}) = \chi(\mathcal{O}_{D_l}) + (1 - D_l \cdot E_{i_l}).$$

So

$$\chi(\mathcal{O}_D) = \sum_l (1 - D_l \cdot E_{i_l}).$$

By the discussion above this sum has precisely [(t + 1)/2] summands 1, and all other summands are zero. ■

4. Computation of $H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}})$

The most difficult part in formula (2.3) for dim T^1_X seems to be $H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}})$. Recall that we have the exact sequence

$$0 \to \mathcal{R} \to \mathcal{O}_{\tilde{X}}^{\oplus n} \xrightarrow{\tilde{\mu}} \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z) \to \mathcal{C} \to 0,$$

so by Hilbert's syzygy theorem $\mathcal{R}$ is locally free of rank $n-2$.

We will apply the results of Chapter 3, so we assume again that the fundamental cycle is reduced and meets all non-(-2)-curves strictly negatively. In other words: if an irreducible component E_i of E meets t_i other curves, then its self-intersection number $-b_i$ fulfills

$$b_i \geqslant t_i \quad \text{for } i = 1, \ldots, r$$

$$b_i \geqslant t_i + 1 \quad \text{if } b_i \neq 2.$$

The restriction of the locally free sheaf $\mathscr{R}$ to E_i is a direct sum of line bundles (cf. [Grauert and Remmert, 1977] VII, Satz 5). We now give estimates for the degrees of these bundles.

PROPOSITION 4.1.

Let E_i be an irreducible component of E.

i) *If $b_i \geqslant t_i + 2$, then $\mathscr{R} \otimes \mathcal{O}_{E_i}$ decomposes into line bundles of degree at least -2.*

ii) *If $b_i = t_i + 1$, then all direct summands of $\mathscr{R} \otimes \mathcal{O}_{E_i}$ have degree at least -1.*

iii) *If $b_i = 2$, $t_i = 1$ and E_i meets a (-2)-curve, then $\mathscr{R} \otimes \mathcal{O}_{E_i}$ is trivial.*

Proof

Consider the exact sequences

$$0 \to \mathscr{R} \to \mathcal{O}_{\tilde{X}}^{\oplus n} \xrightarrow{\tilde{\mu}} \mathscr{F} \to 0$$

$$0 \to \mathscr{F} \to \Omega^1_{\tilde{X}}\langle \log E \rangle(-E) \to \mathscr{C} \to 0$$

The first one remains exact, when restricted to E_i:

$$0 \to \mathscr{R} \otimes \mathcal{O}_{E_i} \to \mathcal{O}_{E_i}^{\oplus n} \xrightarrow{\tilde{\mu}} \mathscr{F} \otimes \mathcal{O}_{E_i} \to 0 \tag{4.2}$$

But $\mathscr{F} \otimes \mathcal{O}_{E_i}$ is no longer torsion free, the second sequence gives

$$0 \to \mathscr{T}\!or_1^{\mathcal{O}_{\tilde{X}}}(\mathscr{C}, \mathcal{O}_{E_i}) \to \mathscr{F} \otimes \mathcal{O}_{E_i} \to \Omega^1_{\tilde{X}}\langle \log E \rangle(-E) \otimes \mathcal{O}_{E_i} \to \mathscr{C} \otimes \mathcal{O}_{E_i} \to 0. \tag{4.3}$$

So the torsion subsheaf of $\mathscr{F} \otimes \mathcal{O}_{E_i}$ is concentrated in the points, where $\mathscr{C}$ is a skyscraper sheaf, and it has length 1 there (cf. (3.3) and (3.5)).
First we prove iii): In this case $\mathscr{T}\!or_1^{\mathcal{O}_{\tilde{X}}}(\mathscr{C}, \mathcal{O}_{E_i}) = 0$, while $\mathscr{C} \otimes \mathcal{O}_{E_i}$ is a

skyscraper sheaf of length 1 (see Lemma 3.5). Hence by (4.3) the Chern class of $\mathscr{F} \otimes \mathcal{O}_{E_i}$ is zero. By (4.2) we see that $\mathscr{R} \otimes \mathcal{O}_{E_i}$ has Chern class zero. But a subsheaf of $\mathcal{O}_{E_i}^{\oplus n}$ has trivial Chern class, if and only if it is trivial.

We now concentrate on i) and ii). If we want to show that $\mathscr{R} \otimes \mathcal{O}_{E_i}$ splits into direct summands of degree at least -1, it is sufficient to show the surjectivity of $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}) \xrightarrow{\tilde{\mu}} H^0(E_i, \mathscr{F} \otimes \mathcal{O}_{E_i})$. This follows from the cohomology sequence of (4.2) and the observation that $H^1(E_i, \mathscr{R} \otimes \mathcal{O}_{E_i})$ is never zero, if $\mathscr{R} \otimes \mathcal{O}_{E_i}$ has a line bundle summand of degree -2 or less. Similarly for the estimate -2 in (i) it suffices to prove the surjectivity of $H^0(E_i, \mathcal{O}_{E_i}(1)^{\oplus n}) \xrightarrow{\tilde{\mu}} H^0(E_i, \mathscr{F} \otimes \mathcal{O}_{E_i}(1))$.

We will discuss the torsion part and the non-torsion part of $\mathscr{F} \otimes \mathcal{O}_{E_i}$ separately. For the torsion part we use

LEMMA 4.4.

Let E_i, E_j be two components of E which meet in a point p and for which $E \cdot E_i < 0$, $E \cdot E_j < 0$. Let f be a holomorphic function on $\tilde{X}$ whose zero divisor contains E_i with multiplicity 2, E_j with multiplicity 1, and no other curve passing through p. Then $\mathrm{d}f$ represents a generator of the torsion part of $(\mathscr{F} \otimes \mathcal{O}_{E_i})_p$.

Proof

Let (u, v) be local coordinates around p with $E_i = \{v = 0\}$, $E_j = \{u = 0\}$. The computation in the proof of (3.3.ii)) shows that locally $\Omega^1_{\tilde{X}}\langle \log E \rangle(-E)$ is generated by $v\,\mathrm{d}u$ and $u\,\mathrm{d}v$, while $\mathscr{F}$ is generated by $v\,\mathrm{d}u + u\,\mathrm{d}v$, $u^2\,\mathrm{d}v$, $uv\,\mathrm{d}v$, $uv\,\mathrm{d}u$, $v^2\,\mathrm{d}u$. So the kernel of the map $\mathscr{F}/v\mathscr{F} \to \Omega^1_{\tilde{X}}\langle \log E \rangle(-E)/v \cdot \Omega^1_{\tilde{X}}\langle \log E \rangle(-E)$ is generated by $uv\,\mathrm{d}v$. □

COROLLARY 4.5.

Let E_i be a component of E such that $b_i \geqslant t_i + 1$. Then there are holomorphic functions on $\tilde{X}$ which vanish to order 2 along E_i and whose differentials generate the torsion of $\mathscr{F} \otimes \mathcal{O}_{E_i}$.

Proof

For each non-(-2)-curve E_j meeting E_i we find by (3.2) a holomorphic function on $\tilde{X}$ which vanishes to order 2 along E_i and all the curves $E_k \neq E_j$ that meet E_i.

The non-torsion part of $\mathscr{F} \otimes \mathcal{O}_{E_i}$ is the image $\tilde{\mathscr{F}}_i$ of $\mathscr{F} \otimes \mathcal{O}_{E_i}$ in $\Omega^1_{\tilde{X}}\langle \log E \rangle(-E) \otimes \mathcal{O}_{E_i}$. It is clear that the differential of a holomorphic function on $\tilde{X}$ has non-vanishing image in $\tilde{\mathscr{F}}_i$ only if it vanishes to order 1 along E_i. In view of (4.5) it suffices for the proof of i) resp. ii) to show that the maps $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}(1)) \xrightarrow{\tilde{\mu}} H^0(E_i, \tilde{\mathscr{F}}_i(1))$ resp. $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}) \xrightarrow{\tilde{\mu}} H^0(E_i, \tilde{\mathscr{F}}_i)$ are surjective. Before doing this we note

Lemma 4.6.

$\tilde{\mathscr{F}}_i$ has Chern class $2(b_i - t_i) - 2$ on E_i, and $H^1(E, \tilde{\mathscr{F}}_i) = 0$.

Proof

One can deduce from (2.6) that $\Omega^1_{\tilde{X}}\langle\log E\rangle(-E) \otimes \mathcal{O}_{E_i} \cong (\omega_{E_i}(t_i) \oplus \mathcal{O}_{E_i})(-E \cdot E_i)$. So the claim on the degree of $\tilde{\mathscr{F}}_i$ follows from the exact sequence

$$0 \to \tilde{\mathscr{F}}_i \to \Omega^1_{\tilde{X}}\langle\log E\rangle(-E) \otimes \mathcal{O}_{E_i} \to \mathscr{C} \otimes \mathcal{O}_{E_i} \to 0.$$

The sequence (4.2) shows that $H^1(E_i, \mathscr{F} \otimes \mathcal{O}_{E_i}) = 0$, hence also $H^1(E_i, \tilde{\mathscr{F}}_i) = 0$.

We now prove (4.1.i)): as mentioned above it suffices to prove the surjectivity of $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}(1)) \to H^0(E_i, \tilde{\mathscr{F}}_i(1))$. The latter space has dimension $2(b_i - t_i + 1)$ by Lemma 4.6. Now choose a small curve Δ transversal to E_i which does not meet any other component of E. By Corollary 3.2. we find for $0 \leqslant k < b_i - t_i$ holomorphic functions f_k on $\tilde{X}$ whose zero divisor contains E_i and all components of E adjacent to E_i with multiplicity 1, and Δ with multiplicity k.

Choose local coordinates (u, v) around the point of $E_i \cap \Delta$ such that $E_i = \{v = 0\}$, $\Delta = \{u = 0\}$. Then $f_k = \epsilon_k \cdot u^k \cdot v$ with some unit ϵ_k. So

$$\mathrm{d}f_k = k \cdot u^{k-1} v \,\mathrm{d}u + u^k \,\mathrm{d}v + \text{higher terms}.$$

If we take all linear combinations of $\mathrm{d}f_0, \ldots, \mathrm{d}f_{b_i - t_i}$ with coefficients in $H^0(E_i, \mathcal{O}_{E_i}(1))$ (which means that we allow constants and $\frac{1}{u}$ as coefficients), we get $2(b_i - t_i + 1)$ linearly independent sections of $\tilde{\mathscr{F}}_i(1)$.

Finally we prove (4.1.ii)): In this case $\dim H^0(E_i, \tilde{\mathscr{F}}_i) = 2$, and as above one constructs two independent holomorphic functions which vanish to first order along E_i. This shows that $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}) \xrightarrow{\tilde{\mu}} h^0(E_i, \tilde{\mathscr{F}}_i)$ is surjective. ■

As in Chapter 3 we also have to consider chains of (-2)-curves.

Proposition 4.7.

Let $E_0, E_1, \ldots, E_t, E_{t+1}$ be irreducible components of E such that $E_1, \ldots, E_t$ form a chain of (-2)-curves, E_0 meets E_1, E_{t+1} meets E_t, and there is no *intersection of $E_1, \dots, E_t$ with other components. Also assume that $E \cdot E_0 < 0$ and $E \cdot E_{t+1} < 0$.*

Then on a sufficiently small neighbourhood U of $E_1 \cup \cdots \cup E_t$ the vector bundle $\mathcal{R}$ splits into a trivial summand of rank $n-3$ and a line bundle $\mathcal{L}$. The restriction of $\mathcal{L}$ to the irreducible components are

$$\mathcal{L} \otimes \mathcal{O}_{E_i} \cong \begin{cases} \mathcal{O}_{E_i} & \text{if } 1 \leqslant i \leqslant t, \begin{array}{ll} i \neq k,\ k+1 & \text{for } t = 2k \text{ even} \\ i \neq k & \text{for } t = 2k-1 \text{ odd} \end{array} \\ \mathcal{O}_{E_k}(-2) & \text{if } i = k;\ t = 2k-1 \\ \mathcal{O}_{E_i}(-1) & \text{if } i = k,\ k+1;\ t = 2k \end{cases}$$

Proof

The splitting of $\mathcal{R}|_U$ into a trivial summand and a line bundle follows from Remark 3.4. It remains to compute the Chern classes of $\mathcal{R} \otimes \mathcal{O}_{E_i}$ ($1 \leqslant i \leqslant$ t). By (4.2) and (4.3) we have $c_1(\mathcal{R} \otimes \mathcal{O}_{E_i}) = -c_1(\mathcal{F} \otimes \mathcal{O}_{E_i}) = c_1(\mathcal{C} \otimes \mathcal{O}_{E_i}) - c_1(\mathcal{T}or_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{C}, \mathcal{O}_{E_i}))$. The claim is that this number is equal to $E_i \cdot D$, where D is the divisor of Lemma 3.5.

Let τ be the torsion subsheaf of $\mathcal{C}$. By Lemma 3.5. we have an exact sequence

$$0 \to \tau \to \mathcal{C} \to \mathcal{O}_D \to 0.$$

Tensoring this sequence with $\mathcal{O}_{E_i}$ we obtain

$$c_1(\mathcal{C} \otimes \mathcal{O}_{E_i}) - c_1\big(\mathcal{T}or_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{C}, \mathcal{O}_{E_i})\big) = c_1(\mathcal{O}_D \otimes \mathcal{O}_{E_i}) - c_1\big(\mathcal{T}or_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_D, \mathcal{O}_{E_i})\big).$$

But $\mathcal{O}_D \otimes \mathcal{O}_{E_i} \cong \mathcal{O}_{E_i}$, $\quad \mathcal{T}or_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_D, \mathcal{O}_{E_i}) \cong \mathcal{O}_{E_i}(-D)$.

The following theorem contains the main result of this paper:

THEOREM 4.8.

Let $\pi: \tilde{X} \to X$ be the minimal resolution of a rational surface singularity $(X, 0)$, let $E = \bigcup_{i=1}^{r} E_i$ be the decomposition of the exceptional set $E = \pi^{-1}(0)$ into irreducible components, and let $-b_i$ be the self-intersection number of E_i. Denote by t_i the number of components of E different from E_i which meet E_i, and by s_i the number of chains of curves of self-intersection number -2 and trivial intersection with E that meet E_i. Assume that

a) $b_i \geqslant t_i + 1$ *for* $b_i > 2$, $b_i \geqslant t_i$ *for* $b_i = 2$

b) $s_i \leqslant b_i - t_i - 2$ *if* $b_i - t_i \geqslant 2$

c) $s_i = 0$ *if* $b_i = t_i + 1$, $b_i \neq 2$.

Furthermore assume that inequality b) *is strict for at least one* E_i. *Then*

$$\dim T^1_X = \dim T^1_{\tilde{X}} + \operatorname{emb}(X) - 4.$$

Proof

From Corollary 2.3. we get

$$\dim T^1_X = \dim T^1_{\tilde{X}} + \sum_{i=1}^{r} (b_i - 3) + \dim H^0(\tilde{X}, \mathscr{C} \otimes \omega_{\tilde{X}})$$
$$+ \dim H^1(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}})$$

By our assumptions the formula for the embedding dimension in [Artin, 1966] gives $\operatorname{emb}(X) = 1 - E \cdot E = 1 + \sum_{i=1}^{r} (b_i - t_i)$. Hence

$$\dim T^1_X - \left(\dim T^1_{\tilde{X}} + \operatorname{emb} X - 4\right) = \dim H^0(\tilde{X}, \mathscr{C} \otimes \omega_{\tilde{X}})$$
$$+ \dim H^1(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}}) - (r-1) \tag{4.9}$$

Let $L_1, \ldots, L_p$ be the maximal chains of (-2)-curves $L_j = E_1^{(j)} \cup \cdots \cup E_{t_j}^{(j)}$ such that $E \cdot E_\tau^{(j)} = 0$ for $1 \leqslant \tau \leqslant t_j$. To each L_j we associate the divisor

$$D_j = \sum_{\tau=1}^{t_j} \max(\tau, t_j - \tau + 1) \cdot E_\tau^{(j)}$$

as in (3.5). Then we have the exact sequence

$$\cdots \to H^1\left(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}}(-D_1 - \cdots - D_p)\right) \to H^1(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}})$$
$$\cdots \to H^1\left(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1 + \cdots + D_p}\right) \to 0 \tag{4.10}$$

Then Theorem 4.8. follows from (4.9), (4.10) and

LEMMA 4.11.

Under the assumptions of Theorem 4.8. we have
i) $\dim H^0(\tilde{X}, \mathscr{C} \otimes \omega_{\tilde{X}}) + \dim H^1(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1 + \cdots + D_p}) = r - 1$
ii) $\dim H^1(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}}(-D_1 \cdots - D_p)) = 0$.

Proof

i) Every point, where two curves E_i, E_j with $b_i > t_i$, $b_j > t_j$ meet, gives a one-dimensional contribution to $H^0(\tilde{X}, \mathscr{C} \otimes \omega_{\tilde{X}})$, and all other contributions to the sum above come from the chains of (-2)-curves.

By Serre-duality and the adjunction formula $H^1(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1 + \cdots + D_p})$ has the same dimension as $\bigoplus_{j=1}^{p} H^0(|D_j|, \mathscr{R}^* \otimes \mathcal{O}_{D_j}(D_j))$. Recall from (4.7) that on D_j the bundle $\mathscr{R}^*$ decomposes into a trivial bundle and a line bundle, say $\mathscr{L}_j$, with $\mathscr{L}_j \otimes \mathcal{O}_{E_\tau^{(j)}} \cong \mathcal{O}_{E_\tau^{(j)}}(-D_j)$. By the negativity of the intersection matrix $\mathcal{O}_{D_j}(D_j)$ has no sections, hence

$$H^0\big(|D_j|, \mathscr{R}^* \otimes \mathcal{O}_{D_j}(D_j)\big) \cong H^0\big(|D_j|, \mathscr{L}_j \otimes \mathcal{O}_{D_j}(D_j)\big) \cong H^0\big(|D_j|, \mathcal{O}_{D_j}\big)$$

has dimension $[(t_j + 1)/2]$ by Lemma 3.6. On the other hand dim $H^0(|D_j|, \mathscr{C} \otimes \omega_{\tilde{X}}) = [(t_j + 2)/2]$ by (3.5) and (3.6). So each chain L_j contributes $t_j + 1$ to the sum on the right hand side of (4.11).

Using the fact that the resolution graph of X is a tree, one easily sees that the number of intersection points of curves not contained in $\bigcup_{j=1}^{p} L_j$ and the numbers $t_j + 1$ for every chain L_j sum up to r.

ii) We first check that $H^1(|E|, \mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-D_1 - \cdots - D_p)) = 0$. By Serre duality this means that $H^0(|E|, \mathscr{R}^* \otimes \mathcal{O}_E(E + D_1 + \cdots + D_p)) = 0$. Our hypotheses and the Propositions 4.1. and 4.7. imply that the restriction $\mathscr{R}^* \otimes \mathcal{O}_{E_i}(E + D_1 + \cdots + D_p)$ to E_i is a direct sum of line bundles of degree at most 0, and for one index i it is a direct sum of line bundles of degree at most -1. Hence $\mathscr{R}^* \otimes \mathcal{O}_E(E + D_1 + \cdots + D_p)$ has no nontrivial global sections.

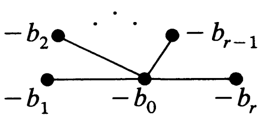
Since the fundamental cycle is reduced, the sheaves $\mathcal{O}_{\tilde{X}}(-nE)$ are generated by their global sections. This gives surjections of direct sums of copies of $\mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-D_1 - \cdots - D_p)$ to $\mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-D_1 - \cdots - D_p)(-nE)$. Hence $H^1(|E|, \mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-D_1 - \cdots - D_p)(-nE)) = 0$ for all positive integers n too, and from the exact sequences

$$H^1\big(|E|, \mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-nE - D_1 - \cdots - D_p)\big)$$
$$\to H^1\big(|E|, \mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{(n+1)E}(-D_1 - \cdots - D_p)\big)$$
$$\to H^1\big(|E|, \mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{nE}(-D_1 - \cdots - D_p)\big) \to 0$$

we get our vanishing result (see also [Wahl, 1975], Lemma 5.15.1). ∎

Example 4.12.

Consider the weighted dual graph



$\bullet \cong \mathbb{P}^1\mathbb{C},\ b_0, \dots, b_r \geqslant 2$
$1/b_1 + \cdots + 1/b_r < b_0$

If $b_0 \geqslant r-1$ this is the dual resolution graph of a rational surface singularity. Its embedding dimension is $\operatorname{emb}(X) = 3 + \sum_{i=0}^{r} (b_i - 2)$ for $r \geqslant b_0$, and $b_1 + \cdots + b_r - 3$ for $r = b_0 + 1$ (cf. [Artin, 1966]).

Theorem 4.8. gives $\dim T_X^1 = \dim T_{\tilde{X}}^1 + \sum_{i=0}^{r} (b_i - 2) - 1$ provided $b_0 \geqslant r+3$, or $b_0 = r+2$, and at least one of $b_1, \dots, b_r$ is greater than 3.

For the dimension of $T_{\tilde{X}}^1$ one computes from the exact sequence

$$0 \to \operatorname{Der}_E(\tilde{X}) \to \Theta_{\tilde{X}} \to \bigoplus_{i=0}^{r} \mathcal{O}_{E_i}(E_i) \to 0$$

that $\dim T_{\tilde{X}}^1 = \sum_{i=0}^{r} (b_i - 1) + \dim H^1(\tilde{X}, \operatorname{Der}_E(\tilde{X}))$.

The cohomology group $H^1(\tilde{X}, \operatorname{Der}_E(\tilde{X}))$ parametrizes the infinitesimal deformations of $\tilde{X}$ to which all the E_i lift. By Theorem 4.1. of [Laufer, 1973] the analytic type of $\tilde{X}$ (and of X) is determined by the location of the r intersection points on the central curve, hence $H^1(\tilde{X}, \operatorname{Der}_E(\tilde{X}))$ has dimension $r-3$.

Putting everything together, we get $\dim T_{\tilde{X}}^1 = \sum_{i=0}^{r} (b_i - 3)$ and

$$\dim T_X^1 = \sum_{i=0}^{r} (2b_i - 2) - 4$$

under the assumptions made above.

Example 4.13.

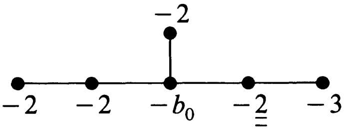
Let X be twodimensional quotient singularity of type T_m, O_m, I_m (cf. [Brieskorn, 1968] 2.9) and assume that the selfintersection number of the central curve of the exceptional set is at least $6+p$, where p denotes the number of chains of (-2)-curves E_i with $E \cdot E_i = 0$. Then the equality

$$\dim T_X^1 = \dim T_{\tilde{X}}^1 + \operatorname{emb}(X) - 4 = \sum_{i=0}^{r} (2b_i - 3) - 1$$

holds.

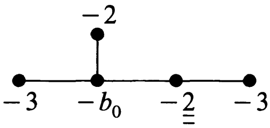
Proof

Theorem 4.8. applies to all cases of quotient singularities as listed in [Brieskorn, 1968] 2.11, apart from the following two types: I_m, $m = 30(b_0 - 2) + 7$ with resolution graph



$b_0 \geqslant 8$

I_m, $m = 30(b_0 - 2) + 17$ with resolution graph



, $b_0 \geqslant 7$

In both cases there is a chain (of length one) of (-2)-curves which meets a (-3)-curve. Let $L_1 = E_1$ be the (-2)-curve and E_2 the (-3)-curve in question. We replace the divisor D_1 in the proof of Theorem 4.8. by $D_1' := E_1 + E_2$. Put $D' := D_1' + D_2$ in the first case, and $D' := D_1$ in the second case. In analogy to Lemma 4.11. we have

Claim (4.14)

$H^1(\tilde{X}, \mathscr{R}^* \otimes \omega_{\tilde{X}}(-D')) = 0.$

Proof

As in 4.13. we have to show that $H^0(E, \mathscr{R}^* \otimes \mathcal{O}_E(E + D')) = 0$. The restriction of $\mathscr{R}^* \otimes \mathcal{O}_E(E + D')$ to the central curve E_0 and to E_2 is a direct sum of line bundles of negative degree (cf. 4.1), and it has degree $\leqslant 0$ on all components but E_1. On E_1 it is a direct sum of a line bundle of degree one and of line bundles of degree -1. This shows that the vectorbundle $\mathscr{R}^* \otimes \mathcal{O}_E(E + D')$ cannot have any global sections on E.

Claim (4.15)

$\dim H^1(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1'}) = 1.$

Proof

As in the proof of Lemma 4.11. i) it suffices to show that $\dim H^0(E_1 \cup E_2, \mathscr{R}^* \otimes \mathcal{O}_{E_1+E_2}(E_1+E_2)) = 1$.

Let g_1, g_2, g_3 be the global functions on $\tilde{X}$ of Remark 3.4., whose differentials generate $\mathscr{F}$ in a neighbourhood of E_1. We may assume that g_1 vanishes with multiplicity 1 along E_1 and E_2, g_2 vanishes with mulitiplicity 3 along E_1 and multiplicity 1 along E_2, and g_3 vanishes with mulitiplicity 3 both along E_1 and E_2.

Call $\mathscr{F}' \subset \Omega^1_{\tilde{X}}\langle \log E\rangle(-E)$ the subsheaf generated by $\mathrm{d}g_1, \mathrm{d}g_2, \mathrm{d}g_3$ and let $\mathscr{L}$ be the sheaf of relations between them:

$$0 \to \mathscr{L} \to \mathcal{O}_{\tilde{X}}^{\oplus 3} \to \mathscr{F}' \to 0. \tag{4.16}$$

One easily sees that $(\Omega^1_{\tilde{X}}\langle \log E\rangle(-E)/\mathscr{F}') \otimes \mathcal{O}_{E_2}$ is a torsion sheaf of length at least one, so $c_1(\mathscr{F}' \otimes \mathcal{O}_{E_2}) \leqslant 1$.

Hence by (4.16)

$\deg \mathscr{L} \mid_{E_2} \geqslant -1$, while by Proposition 4.7.

$\deg \mathscr{L} \mid_{E_1} = -2$.

Now by Proposition 4.7. the restriction of $\mathscr{R}^* \otimes \mathcal{O}_{E_1+E_2}(E_1+E_2)$ to E_1 is a sum of line bundles of negative degrees and one line bundle of degree one, namely $\mathscr{L}^* \otimes \mathcal{O}_{E_1}(E_1+E_2)$. By Proposition 4.1. and (4.6) the vectorbundle $\mathscr{R}^* \otimes \mathcal{O}_{E_2}(E_1+E_2)$ has at most one line bundle summand of non-negative degree, which then is trivial. This summand does not agree with $\mathscr{L}^* \otimes \mathcal{O}_{E_2}(E_1 + E_2)$ (which has degree $\leqslant -1$), so a holomorphic section of $\mathscr{R}^* \otimes \mathcal{O}_{E_1+E_2}(E_1 + E_2)$ has to vanish on E_2. This proves claim (4.15).

The rest of the proof for the equality $\dim T^1_X = \dim T^1_{\tilde{X}} + \mathrm{emb}(X) - 4$ for the singularities under consideration is analoguous to the proof of (4.8).

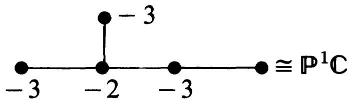
Remark 4.17.

There are 63 individual quotient singularities of type T, O, I that are not covered by Example 4.15.

Example 4.18.

Finally we want to give a partial analysis of the example of J. Wahl mentioned in the introduction.

Let X be the rational surface singularity with dual resolution graph



The fundamental cycle is $Z = 2E_0 + E_1 + E_2 + E_3$, where E_0 denotes the central curve. We have emb$(X) = 6$, dim $T^1_{\tilde{X}} = 7$, so formula (0.1) would give 9 for dim T^1_X. We want to show that dim $T^1_X \geqslant 10$. We apply Corollary 2.3.:

$$\dim T^1_X = \dim H^0(\tilde{X}, \mathscr{C} \otimes \omega_{\tilde{X}}) + \dim H^1(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}}) + 7.$$

By lemma 3.3. $\mathscr{C}$ is a skyscraper sheaf supported at the points of intersection of E_0 with the other components and with stalks of length 1 there. Hence, dim$T^1_X = 10 + \dim H^1(\tilde{X}, \mathscr{R} \otimes \omega_{\tilde{X}})$.

Remark 4.19.

In this example one can compute the map $(\mu' \otimes 1)$ from Section 2 quite explicitly using the canonical Gorenstein cover of X. One actually gets dim $T^1_X = 10$. For details see [Behnke et al., in prep.], Section 8.

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Appendix

In this section, which is entirely due to Jonathan Wahl, lower estimates for the dimension of T^1_X of rational and minimally elliptic surface singularities are given. Let X be a normal surface singularity with minimal good resolution $\pi: \tilde{X} \to X$, let E be the exceptional divisor for π, and let Z be an effective cycle supported on E. In [Wahl, 1979], §2, Wahl defines a deformation functor R_Z by

$R_Z(T) := \{$isomorphism classes of triples $(\tilde{\mathscr{X}}, \rho, \mathscr{Z})$, $\rho: \tilde{\mathscr{X}} \to T$ a deformation of $\tilde{X}$, and $\mathscr{Z}$ a relative effective Cartier divisor which lifts $Z\}$

Since the normal bundle of Z has no nontrivial sections, $\mathscr{X}$ is uniquely determined by $\tilde{\mathscr{X}}$, so that R_Z is a subfunctor of the deformation functor of the resolution. In 2.5.ii of [Wahl, 1979] it is checked that R_Z has a formally semiuniversal deformation space.

If X is rational or minimally elliptic, and Z is the fundamental cycle, then there is a natural blowing down map which maps R_Z finite to one to the deformation space of X ([Wahl, 1979], 2.7). Let $\sigma : \mathscr{X} \to V$ be the semiuniversal deformation of X, and let $\Phi(R_Z)$ be the image of R_Z in V. By Theorem 1 of [Karras, 1983] the fibre $\mathscr{X}_t$ over a general point of $\Phi(R_Z)$ has exactly one singular point x_t, and $(\mathscr{X}_t, x_t)$ is isomorphic to a cone over a rational (resp. elliptic) curve of degree $-Z^2$.

THEOREM *A.1.*

Let X be rational or minimally elliptic, $\pi : \tilde{X} \to X$ the minimal good resolution, and Z the fundamental cycle.

i) *If X is rational, then* $\dim T^1_X \geqslant \dim H^1(\tilde{X}, \Theta_{\tilde{X}}) - Z^2 - 3$.

ii) *If X is minimally elliptic, of degree $d = -Z^2 \geqslant 5$, then* $\dim T^1_X \geqslant \dim R_Z + d$.

Proof

By construction T^1_X is the tangent space of the base space V of the semiuniversal deformation of X at the special point 0. If t is a general point of $\Phi(R_Z) \subset V$, by standard semicontinuity $\dim \Theta_{V,t} \otimes \mathbb{C} \leqslant \dim \Theta_{V,0} \otimes \mathbb{C}$. Openness of versality (cf. [Pourcin, 1974]) shows that locally around t the base space V is the product of the base space of the semiuniversal deformation of $(\mathscr{X}_t, x_t)$ and of a smooth factor, over which the deformation of $\mathscr{X}_t$ induced by $\mathscr{X} \to V$ is trivial.

In the rational case $\Phi(R_Z)$ induces trivial deformations of $\mathscr{X}_t$, since the exceptional curve of a cone over a rational curve only lifts to trivial deformations. By Theorem 2.12. of [Wahl, 1979], $\Phi(R_Z)$ has dimension $\dim H^1(\tilde{X}, \Theta_{\tilde{X}}) - \dim H^1(|Z|, \mathscr{O}_Z(Z))$, which is $\dim H^1(\tilde{X}, \Theta_{\tilde{X}}) + Z^2 - 1$, as a little calculation shows.

It is well known, that the base space of the semiuniversal deformation of the cone over a rational curve of degree d has embedding dimension $2d-4$, $d \geqslant 3$, and $2d-3$, $d = 2$. Hence

$$\dim T^1_X \geqslant \dim(\Theta_{V,t} \otimes \mathbb{C}) \geqslant \dim H^1(\tilde{X}, \Theta_{\tilde{X}}) - Z^2 - 3$$

For X minimally elliptic, $d \geqslant 5$, we have a $(d+1)$ dimensional space of infinitesimal deformations for the simply elliptic singularity of the same degree. The space $\Phi(R_Z)$ on the other hand induces a nontrivial one parameter family in the versal deformation of the fibre singularity $(\mathscr{X}_t, x_t)$, Hence

$$\dim T_X^1 \geqslant (\dim \Phi(R_Z) - 1) + (d+1).$$

In the minimally elliptic case R_Z contains exactly those deformations of the resolution $\tilde{X}$, which blow down to deformations of the singularity. The dimension of this deformation space is computed in [Wahl, 1979], Corollary 5.7., 5.8..

PROPOSITION A.2.

Let X be a minimally elliptic singularity, let $\pi: \tilde{X} \to X$ be the minimal resolution, $E = \bigcup_{i=1}^{r} E_i$ the exceptional divisor, and let $\mathrm{Der}_E(\tilde{X})$ be the sheaf of vector fields, logarithmic along each E_i.

i) *If all irreducible components E_i are rational, and have normal crossings, then* $\dim \Phi(R_Z) = \dim H^1(\tilde{X}, \mathrm{Der}_E(\tilde{X})) + r + Z(Z-E)$.

ii) *If X is simply elliptic,* $\dim \Phi(R_Z) = 1$.

iii) *Otherwise $\Phi(R_Z)$ has dimension* $\dim H^1(\tilde{X}, \mathrm{Der}_E\tilde{X}) + \dim H^0(E, T_E^1)$.

REMARK

In case iii) of A.2. the curve E has exactly one singular point, and the second summand measures the space of infinitesimal deformations of that plane curve singularity.

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