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## COMPOSITION OF QUATERNARY QUADRATIC FORMS

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### Introduction

The study of composition of binary quadratic forms has a long history. However, the first deep results on composition of quaternary forms are relatively recent and are due to Brandt (see for example [Br 1], [Br 2] and [Br 3]). In contrast to the binary case, not every quaternary form admits of a composition. Brandt gave necessary and sufficient conditions for the existence of composition of integral quaternary forms. Further, he observed that, unlike in the binary case, equivalence classes of quaternary forms do not form a group and in this connection he introduced the notion of a groupoid.

Composition of binary forms over arbitrary commutative rings were considered recently in [Kn]. The aim of this paper is to study composition of quaternary forms over arbitrary commutative rings in the spirit of [Kn] and to present Brandt's results in this generality.

After introducing some notation and definitions, we describe in §2 the relations between quaternion algebras and composition of quaternary forms. In §3 we study Clifford algebras and the Arf invariant of quaternary forms. The next section gives a necessary and sufficient condition for the existence of composition in terms of the Clifford algebra. We give in §5 a generalization of Brandt's conditions for arbitrary commutative rings and show that they are necessary (but in general not sufficient) for the existence of a composition. We prove that under some restrictions on the ring, these conditions are sufficient. In the final section, we define certain groupoids of classes of quaternary forms and compare them with those of Brandt. Our main results are contained in Theorems 2.10, 4.1, and in Theorem 5.4 together with its corollaries.

Some other recent papers on related questions are [Brz 1], [Brz 2], [Ka], [Kö] and [P].

### 1. Notation and definitions

Throughout this paper,  $R$  denotes a commutative ring with 1. For any ring  $A$  we denote by  $A^\times$  its group of invertible elements. Unadorned tensor products are supposed to be taken over  $R$ . We say that a property holds *locally* if it holds for every localization  $R_p$ ,  $p \in \text{Spec } R$ . Let  $M$  be

a finitely generated projective  $R$ -module of rank  $m$  and  $q$  a quadratic form on  $M$  with the associated bilinear form  $b(x, y) = q(x + y) - q(x) - q(y)$ .

We call the pair  $(M, q)$  a *quadratic module*. We say that  $M$  (or  $q$ ) is *regular* (resp. *unimodular*) if the *adjoint homomorphism*  $\varphi: M \rightarrow \text{Hom}(M, R)$  defined by  $\varphi(x)(y) = b(x, y)$ ,  $x, y \in M$ , is injective (resp. bijective). We call  $M$  (or  $q$ ) *primitive* if  $Rq(M) = R$ . Two quadratic modules  $(M_1, q_1)$ ,  $(M_2, q_2)$  are said to be *similar* if there is an isomorphism  $\theta: M_1 \rightarrow M_2$  and a unit  $\lambda \in R$  such that  $q_2(\theta(x)) = \lambda q_1(x)$ ,  $x \in M_1$ . The map is called a *similitude* with *multiplier*  $\lambda$ . For  $x_1, \dots, x_m \in M$ , we set  $d(x_1, \dots, x_m) = \det(b(x_i, x_j))$  and define the *discriminant ideal*  $d(M)$  as the ideal of  $R$  generated by the elements  $d(x_1, \dots, x_m)$ , for all choices of  $x_i \in M$ . The ideal  $d(M)$  is locally principal and is invertible if  $M$  is regular.

Let  $(M_1, q_1)$ ,  $(M_2, q_2)$ ,  $(M, q)$  be quadratic modules of the same rank. A *composition*  $\mu: M_1 \times M_2 \rightarrow M$  is an  $R$ -bilinear map  $\mu: M_1 \times M_2 \rightarrow M$  such that  $q(\mu(x_1, x_2)) = q_1(x_1)q_2(x_2)$ ,  $x_i \in M_i$ . Given a composition of primitive quadratic modules, there exist locally principal ideals  $f_1, f_2$  of  $R$  such that  $d(M_i) = f_i^2 d(M)$ . The composition is called *proper* if  $q_1, q_2$  are regular and primitive and  $d(M) = d(M_1) = d(M_2)$ .

An associative  $R$ -algebra  $B$  with 1 is called a *quadratic* (resp. *quaternion*) *algebra* if the following conditions are satisfied:

- (1)  $B$  is a projective  $R$ -module of rank 2 (resp. 4).
- (2)  $B$  has an involution (i.e. an  $R$ -algebra antiautomorphism of order 2)  $x \mapsto \bar{x}$  such that the *trace*  $t(x) = x + \bar{x}$  and the *norm*  $n(x) = x\bar{x}$  take values in  $R$ .

The norm  $n$  is a quadratic form on  $B$ . Its discriminant ideal is called the *discriminant* of  $B$  and is denoted by  $d(B)$ . The algebra  $B$  is called *regular* if the norm  $n: B \rightarrow R$  is a regular quadratic form. A quadratic (resp. quaternion) algebra has locally a basis containing 1. Any element  $x$  with trace  $t$  and norm  $n$  satisfies the equation  $x^2 - tx + n = 0$ , so that the norm, the trace and the involution are uniquely determined by the algebra structure of  $B$ . Moreover, an  $R$ -algebra  $B$  is a quadratic (resp. quaternion) algebra if and only if it is so locally.

**REMARK 1.1:** An  $R$ -algebra which is projective of rank 2 as an  $R$ -module is in fact a quadratic algebra.

## 2. Quaternion algebras and composition of quaternary quadratic forms

In this section, we consider *quaternary* quadratic modules, i.e., modules of rank 4. Let  $B$  be a quaternion algebra over  $R$ . A quadratic module

$(M, q)$  is said to be of *type B* if there exists a right  $B$ -module structure on  $M$  such that

- (1)  $M$  is projective of rank 1, i.e.,  $M_p \cong B_p$  as  $B_p$ -modules for every  $p \in \text{Spec } R$ .
- (2)  $q(xb) = q(x)n(b)$  for  $x \in M, b \in B$ .

The right  $B$ -module structure can be converted into a left  $B$ -module structure on  $M$  through the involution on  $B$ . We thus get an equivalent definition if we require a left  $B$ -module structure on  $M$ . We say that a quadratic module is of *quaternionic type* if it is of type  $B$  for some quaternion algebra  $B$ .

For quaternion algebras  $A$  and  $B$ , we say that a quadratic module  $M$  is of *type  $(A, B)$*  if  $M$  is an  $A$ - $B$ -bimodule,  $A$  operating on the left,  $B$  on the right, such that

- (1)  $M$  is projective of rank 1 as a left  $A$ -module and as a right  $B$ -module,
- (2)  $q(axb) = n(a)q(x)n(b)$  for  $a \in A, x \in M, b \in B$ .

**REMARK 2.1:** If  $M$  is primitive,  $M$  is locally similar to the norm of  $A$  and to the norm of  $B$ .

**EXAMPLE 2.2:** If  $A$  is a quaternion algebra with norm  $n$  and  $c$  is a unit of  $R$ , then  $(A, cn)$  is of type  $(A, A)$ .

**EXAMPLE 2.3:** If  $M$  is of type  $(A, B)$  and  $\bar{M}$  is the quadratic module  $M$ , with  $A$  operating on the right and  $B$  on the left through their involutions, then  $\bar{M}$  is of type  $(B, A)$ . There is a bijection  $x \mapsto \bar{x}$  between  $M$  and  $\bar{M}$  satisfying  $ax\bar{b} = \bar{b}\bar{x}\bar{a}$  and  $q(\bar{x}) = q(x)$ .

**PROPOSITION 2.4:** *If  $M$  is of type  $B$ , there exists a quaternion algebra  $A$  and a left  $A$ -module structure on  $M$  such that  $M$  is of type  $(A, B)$ . The algebra  $A$  and its operation on  $M$  are determined uniquely up to isomorphism.*

**PROOF:** If  $A$  exists, there is a canonical homomorphism  $A \rightarrow \text{End}_B(M)$  which, by localization, is seen to be an isomorphism. This proves the uniqueness. The algebra  $\text{End}_B(M)$  is locally isomorphic to  $B$  so that  $\text{End}_B(M)$  is a quaternion algebra. For the existence, we set  $A = \text{End}_B(M)$  and define its action on  $M$  in the obvious way. The fact that  $M$  is projective of rank 1 over  $A$  and satisfies condition (2) is proved by localization.

**PROPOSITION 2.5:** *If  $(M_1, q_1)$  is of type  $(A, B)$  and  $(M_2, q_2)$  of type  $(B, C)$ , then there exists a unique quadratic form  $q = q_1 \otimes_B q_2$  on the*

module  $M = M_1 \otimes_B M_2$  so that the map  $\alpha: M_1 \times M_2 \rightarrow M$  given by  $\alpha(x_1, x_2) = x_1 \otimes x_2$  is a composition. Further,  $\alpha$  is proper if  $M_1$  and  $M_2$  are regular and primitive.

PROOF: The claim can be easily checked if  $M_1$  and  $M_2$  are free over  $B$ . In the general case, there is a covering of  $\text{Spec } R$  by affine open sets  $U$  such that  $M_1$  and  $M_2$ , restricted to  $U$ , are free over  $B|_U$ . Since there is a unique quadratic form  $q$  over  $U$  which satisfies the condition of the proposition, the existence of  $q$  over  $R$  follows by descent. The uniqueness of  $q$  is clear.

EXAMPLE 2.6: For  $(M, q)$  of type  $(A, B)$ , we have

$$(A, n) \otimes_A (M, q) \simeq (M, q) \simeq (M, q) \otimes_B (B, n).$$

EXAMPLE 2.7: With  $M, \bar{M}$  as in (2.3), we have  $(M, q) \otimes_B (\bar{M}, q) \simeq (A, n)$  and  $(\bar{M}, q) \otimes_A (M, q) \simeq (B, n)$ , the isomorphisms mapping  $x \otimes \bar{x}$ , resp.  $\bar{x} \otimes x$ , into  $q(x)$ . These isomorphisms are uniquely determined. For example, the second one maps  $b\bar{x} \otimes xb'$  to  $bb'q(x)$ ,  $b, b' \in B, x \in M$ . From this, the uniqueness follows by localizing and taking for  $x$  a  $B$ -generator  $e$  of  $M$ . Thus, this isomorphism is defined locally by  $b\bar{e} \otimes eb' \mapsto bb'q(e)$ .

LEMMA 2.8: Let  $(M, q)$  be a regular quaternary quadratic module with a composition  $\mu: M \times M \rightarrow M$  and let  $e \in M$  be such that  $\mu(x, e) = \mu(e, x) = x$  for any  $x \in M$ . Then  $\mu$  defines on  $M$  the structure of a quaternion algebra with  $q$  as the norm.

PROOF: By descent, we assume that  $M$  is free with a basis  $\{e_1 = e, e_2, e_3, e_4\}$ . Then  $d = d(e_1, e_2, e_3, e_4)$  is not a zero divisor in  $R$ . As in [BS], p. 408, we define the involution on  $M$  by  $\bar{x} = b(e, x) - x$ . By tensorizing with  $R[d^{-1}]$  and localizing, we assume that  $Re_1 + Re_2$  is unimodular,  $e_3$  is orthogonal to  $Re_1 + Re_2$  and  $q(e_3)$  is invertible. Then, denoting  $\mu(x, y)$  by  $xy$ ,  $e_3 = e_1e_3$  and  $e_2e_3$  are orthogonal to  $Re_1 + Re_2$  ([BS], p. 409). Therefore  $M = (Re_1 + Re_2) \perp (Re_1 + Re_2)e_3$  and it has an algebra structure with  $e_1$  as identity and with  $e_2, e_3$  as generators. By ([BS], p. 409) the multiplication is associative.

REMARK 2.9: For any regular quaternion algebra  $A$  over a domain  $R$ , a composition  $\mu: A \times A \rightarrow A$  such that  $\mu(x, 1) = \mu(1, x) = x$  for all  $x \in A$  is given by  $\mu(x, y) = xy$  or by  $\mu(x, y) = yx$ .

THEOREM 2.10: Let  $\mu: M_1 \times M_2 \rightarrow M$  be a proper composition of quaternary quadratic modules. Then there exists a quaternion algebra  $B$

operating on  $M_1$  on the right hand and on  $M_2$  on the left, making  $M_1, M_2$  into modules of quaternionic type, such that  $\mu(x_1b, x_2) = \mu(x_1, bx_2)$  for  $x_i \in M_i, b \in B$ . The algebra  $B$  and its operations on  $M_1, M_2$  are unique upto isomorphisms. There is an isomorphism  $v: M_1 \otimes_B M_2 \rightarrow M$  of quadratic modules such that  $v(x_1 \otimes x_2) = \mu(x_1, x_2)$ , i.e. the composition  $\mu$  is isomorphic to  $M_1 \times M_2 \rightarrow M_1 \otimes_B M_2, (x_1, x_2) \rightarrow x_1 \otimes x_2$ . If  $M_1$  is of type  $(A, B)$  and  $M_2$  is of type  $(B, C)$ , then  $M$  is of type  $(A, C)$ .

PROOF: The set  $B(\mu)$  of pairs  $(s_1, s_2)$  of similitudes of  $M_1$  and  $M_2$  respectively, satisfying  $\mu(s_1x_1, x_2) = \mu(x_1, s_2x_2)$  is an algebra with the multiplication given by  $(s_1, s_2)(s'_1, s'_2) = (s'_1 \circ s_1, s_2 \circ s'_2)$ . If the algebra  $B$  exists, we have a homomorphism  $s: B \rightarrow B(\mu)$  defined by  $b \mapsto (s_1(b), s_2(b))$  where  $s_1(b): x_1 \mapsto x_1b$ , resp.  $s_2(b): x_2 \mapsto bx_2$  define similitudes of  $M_1$ , resp.  $M_2$ , both with multiplier  $n(b)$ . The map  $s$  is injective since  $M_i$  are faithful over  $B, i = 1, 2$ . To prove the surjectivity of  $s$ , we may assume by localizing, that there exist  $e_i \in M_i$  with  $q_i(e_i)$  invertible,  $M_i$  being primitive. We then have  $M_1 = e_1B, M_2 = Be_2$ . The map  $\gamma_1: M_1 \rightarrow M$  defined by  $\gamma_1(x_1) = \mu(x_1, e_2)$  is a similitude with multiplier  $q_2(e_2) \in R^\times$ . Since  $M_1$  is regular,  $\gamma_1$  is injective. Let  $(s_1, s_2) \in B(\mu)$ . We define  $b_1, b_2 \in B$  by  $s_1e_1 = e_1b_1, s_2e_2 = b_2e_2$ . In view of the identities  $\gamma_1(e_1b_1) = \mu(s_1e_1, e_2) = \mu(e_1, s_2e_2)$ , we get  $e_1b_1 = e_1b_2$  so that  $b_1 = b_2 = b$  and  $(s_1, s_2) = s(b)$ .

To prove the existence, we set  $B = B(\mu)$  and define the action of  $b = (s_1, s_2) \in B$  on  $x_i \in M_i$  by  $x_ib = s_ix_i$ . Then  $\mu(x_1b, x_2) = \mu(x_1, bx_2)$  by definition. It remains to show that  $B$  is a quaternion algebra and that  $M_1, M_2$  are of quaternionic type. For this, we again localize and assume that there exist  $e_i \in M_i$  with  $q(e_i)$  invertible in  $R$ . Defining  $\gamma_1$  as above and  $\gamma_2: M_2 \rightarrow M$  by  $\gamma_2(x_2) = \mu(e_1, x_2)$ , we have similitudes  $\gamma_i: M_i \rightarrow M$  with multipliers in  $R^\times$ . Thus  $d(\gamma_iM_i) = d(M_i) = d(M)$  and hence  $\gamma_i$  are bijections. The map  $\mu' = \mu \circ (\gamma_1^{-1}, \gamma_2^{-1}): M \times M \rightarrow M$  is a composition for the quadratic form  $q' = q_1(e_1)^{-1}q_2(e_2)^{-1}q$  on  $M$ . Setting  $e = \mu(e_1, e_2) = \gamma_1(e_1)$ , we have  $\mu'(x, e) = \mu'(e, x) = x$  for every  $x \in M$ . By (2.8),  $M$  is a quaternion algebra for the multiplication  $\mu'$ .

We define a homomorphism  $\beta: M \rightarrow B$  by  $\beta(x) = (s_1, s_2)$ , where the pair  $(s_1, s_2)$  is defined by  $s_1x_1 = \gamma_1^{-1}(\mu'(\gamma_1x_1, x)), s_2x_2 = \gamma_2^{-1}(\mu'(x, \gamma_2x_2))$ . The map  $\beta$  is injective because  $(s_1, s_2) = 0$  implies  $x = \gamma_1(s_1e_1) = \gamma_2(s_2e_2) = 0$  and surjective because  $(s_1, s_2) \in B$  is the image of  $x = \gamma_1(s_1e_1) = \gamma_2(s_2e_2)$ . Thus  $\beta$  is an isomorphism and  $B$  is a quaternion algebra. Moreover,  $M_i$  is of quaternionic type because  $e_i$  is a  $B$ -generator of  $M_i$ . The remaining contentions are easily verified.

### 3. Clifford algebras and the Arf invariant

We say that a quadratic  $R$ -algebra  $S$  is *trivial* if there exists an  $R$ -algebra homomorphism  $\pi: S \rightarrow R$ . We call  $\pi$  a *supplementation*. A

separable quadratic  $R$ -algebra (i.e., a quadratic algebra with unimodular norm) is trivial if and only if  $S \simeq R \times R$ . If  $R$  is a domain, a regular quadratic algebra is trivial if and only if it is locally trivial. This is not true in general even if  $S$  is separable (see [KP]).

Let  $S$  be a trivial quadratic  $R$ -algebra with a supplementation  $\pi$ . Let  $D_\pi(S)$  be the ideal of  $R$  generated by all elements of the form  $t(x) - 2\pi(x)$ ;  $x \in S$ . The ideal  $D_\pi(S)$  is locally principal since if  $S$  is free with a basis  $\{1, z\}$ ,  $D_\pi(S)$  is generated by  $t(z) - 2\pi(z)$ .

**LEMMA 3.1:** *If  $S$  is a trivial quadratic algebra with a supplementation  $\pi$ , then  $d(S) = (D_\pi(S))^2$ . In particular, if  $S$  is regular,  $D_\pi(S)$  is an invertible ideal.*

**PROOF:** The claim follows from the formula  $(t(x) - 2\pi(x))^2 = -d(1, x)$ , for  $x \in S$ .

**REMARK 3.2:** Let  $K$  be the total quotient ring of  $R$  and let  $S$  be a quadratic  $R$ -algebra which is trivial over  $K$ . For any supplementation  $\pi$  of  $S \otimes K$ , we also denote by  $D_\pi$  the fractionary ideal of  $R$  generated by  $t(x) - 2\pi(x)$ ,  $x \in S$ . As in (3.1) the ideal  $D_\pi$  is invertible if  $S$  is regular and we have  $D_\pi^2 = d(S)$ . We note that if  $R$  is a domain, the ideal  $D_\pi$  does not depend on the choice of  $\pi$ .

Let  $(M, q)$  be a quaternary quadratic module and  $C(M) = C_0(M) \oplus C_1(M)$  its Clifford algebra. Let  $\alpha: C(M) \rightarrow C(M)$  be the canonical involution, i.e. the unique involution of  $C(M)$  which is the identity on  $M$ . The associated “trace map”  $T: C(M) \rightarrow C(M)$  given by  $T(x) = x + \alpha(x)$  maps  $C_0(M)$  into the centre of  $C_0(M)$ . To see this, we assume that  $M$  is free with a basis  $\{e_1, e_2, e_3, e_4\}$ . It suffices to check that  $T(e_1e_2e_3e_4)$  is in the centre. This is obvious if the basis is orthogonal and the general case follows by a specialization argument. For later purposes we need the following formulae. We set  $q(e_i) = a_i$ ,  $b(e_i, e_j) = a_{ij}$ ,  $1 \leq i, j \leq 4$ . Then

$$T(e_1e_2e_3e_4) = e_1e_2e_3e_4 + e_4e_3e_2e_1 = z - a,$$

where

$$\begin{aligned} z &= 2e_1e_2e_3e_4 - a_{34}e_1e_2 + a_{24}e_1e_3 \\ &\quad - a_{23}e_1e_4 - a_{14}e_2e_3 + a_{13}e_2e_4 - a_{12}e_3e_4, \end{aligned} \tag{3.3}$$

$$a = a_{13}a_{24} - a_{14}a_{23} - a_{12}a_{34}. \tag{3.4}$$

The element  $z$  satisfies the quadratic equation

$$z^2 = az + b,$$

where

$$b = 4a_1a_2a_3a_4 - a_1a_2a_{34}^2 - a_1a_3a_{24}^2 - a_1a_4a_{23}^2 - a_2a_3a_{14}^2 - a_2a_4a_{13}^2 - a_3a_4a_{12}^2.$$

From this we conclude that for any quaternary quadratic module  $(M, q)$ ,  $S(M) = R + T(C_0(M))$  is a quadratic  $R$ -algebra contained in the centre of  $C_0(M)$ . If  $M$  is free with basis  $\{e_1, e_2, e_3, e_4\}$ , then  $S(M)$  is generated by the element  $z$  of (3.3). We call  $S(M)$  the *Arf invariant* of  $(M, q)$ . If  $(M, q)$  is unimodular,  $S(M)$  coincides with the centre of  $C_0(M)$ . We say that a quaternary quadratic module  $(M, q)$  has *trivial Arf invariant* if the quadratic  $R$ -algebra  $S(M)$  is trivial.

LEMMA 3.5: *The discriminant  $d(S(M))$  of the quadratic algebra  $S(M)$  is the discriminant ideal of  $(M, q)$ .*

PROOF: We localize and note that  $-d(1, z) = a^2 + 4b = d(e_1, e_2, e_3, e_4)$  if  $\{e_1, e_2, e_3, e_4\}$  is a basis of  $M$ .

We now consider the Clifford algebra of a module of quaternionic type. Let  $M$  be of type  $(A, B)$  and let  $\bar{M}$  be as in Example (2.3). Using Examples (2.6) and (2.7) we have a multiplication on the set of matrices

$$\begin{pmatrix} A & M \\ \bar{M} & B \end{pmatrix}$$

which is associative and makes it into an  $R$ -algebra with an involution  $\beta: \begin{pmatrix} a & y \\ \bar{x} & b \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & x \\ \bar{y} & \bar{b} \end{pmatrix}$ . The map  $i_M: M \rightarrow \begin{pmatrix} A & M \\ \bar{M} & B \end{pmatrix}$  given by  $x \mapsto \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}$  satisfies  $[i_M(x)]^2 = q(x) \cdot 1$  and hence extends to a homomorphism

$$i_M: C(M) \rightarrow \begin{pmatrix} A & M \\ \bar{M} & B \end{pmatrix}. \tag{3.6}$$

The map  $i_M$  is  $Z/2Z$ -graded, if we put on  $\begin{pmatrix} A & M \\ \bar{M} & B \end{pmatrix}$  the chess-board gradation. Further,  $i_M\alpha = \beta i_M$ , where  $\alpha$  is the canonical involution of  $C(M)$  and  $\beta$  is as above.

REMARK 3.7: The map  $i_M$  is injective if  $(M, q)$  is regular. It is an isomorphism if  $(M, q)$  is unimodular.

PROPOSITION 3.8: *Any quadratic module of quaternionic type has trivial Arf invariant.*



PROOF: For any  $c \in C_0(M)$ ,  $i_M(T(c)) = i_M(c + \alpha(c)) = i_M(c) + \beta i_M(c)$  is contained in  $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ . Since the elements  $T(c)$  generate the Arf invariant  $S(M)$ , the projection onto one of the factors  $R$  induces a supplementation of  $S(M)$ .

REMARK 3.9: The tuple  $(A, B, M, \overline{M})$  with the maps  $M \otimes_B \overline{M} \rightarrow A$  and  $\overline{M} \otimes_A M \rightarrow B$  is a set of equivalence data in the sense of Bass [B, p. 62].

#### 4. The Bhaskara condition

In this section, we give a necessary and sufficient condition for a quaternary form to admit of a composition, in terms of its Clifford algebra.

THEOREM 4.1: *Let  $(M, q)$  be a regular primitive quaternary quadratic module over  $R$ . Let  $K$  be the total quotient ring of  $R$ . Then  $M$  is of quaternionic type if and only if there exists an idempotent  $e$  in the centre  $Z$  of  $C_0(M) \otimes K$  which generates  $Z$  and such that  $Me = C_1(M)e$ . If  $M$  is of type  $(A, B)$  then  $e$  can be chosen such that  $B \simeq C_0(M)e$  and  $A \simeq C_0(M)(1 - e)$ .*

PROOF: Suppose that  $M$  is of type  $(A, B)$ . By (3.7), the map  $i = i_M \otimes 1_K$  is an isomorphism, since  $M \otimes K$  is unimodular. If  $e = i^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in C_0(M) \otimes K$ , we have  $i(C_1(M)e) = i(Me) = \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}$ , so that  $C_1(M)e = Me$ . The idempotent  $e$  determines the gradation on  $C(M)$ . In fact, ([KP]):

$$\begin{aligned} C_0(M) &= \{x \in C(M) \mid xe = ex\}, \\ C_1(M) &= \{x \in C(M) \mid x(1 - e) = ex\}. \end{aligned} \tag{4.2}$$

It follows that  $M \simeq Me$  is a right  $C_0(M)e$ -module and a left  $C_0(M)(1 - e)$ -module. We see by descent that there are unique algebra isomorphisms  $\beta: C_0(M)e \rightarrow B$ ,  $\alpha: C_0(M)(1 - e) \rightarrow A$  such that  $(x\beta(ce))e = xce$  and  $(\alpha(c(1 - e))x)e = cxe$  for  $c \in C_0(M)$ .

Conversely, let  $Me = C_1(M)e$  for some idempotent  $e$  generating the centre of  $C_0(M \otimes K)$ . We set  $A = C_0(M)(1 - e)$ ,  $B = C_0(M)e$ . By (4.2), we have  $A = (1 - e)C(M)(1 - e)$ ,  $B = eC(M)e$ ,  $Me = C_1(M)e = (1 - e)C(M)e$ . Therefore  $M \cong Me$  has the obvious structure of a  $A$ - $B$ -bimodule. One can check that  $A$  and  $B$  are quaternion algebras with the involutions given by the restrictions of the canonical involution of  $C(M)$  and that  $M$  is of type  $(A, B)$ .

Let now  $M$  be a free  $R$ -module of rank 4 with a basis  $\{e_1, e_2, e_3, e_4\}$ . Let  $q$  be a primitive regular quadratic form on  $M$  such that  $q \otimes 1_K$  has trivial Arf invariant, where  $K$  is the total quotient ring of  $R$ . Let  $a_i = q(e_i)$ ,  $a_{ij} = b(e_i, e_j)$  and let  $z$  be defined as in (3.3). If  $\pi: S(M) \otimes K \rightarrow K$  is a supplementation, then the element  $D = t(z) - 2\pi(z)$  is such that  $D^2 = -d(1, z) = d(e_1, e_2, e_3, e_4)$  (see (3.5)) so that, since  $M$  is regular,  $D$  is a unit of  $K$ . The element  $e = e_\pi = (z - \pi(z))/t(z) - 2\pi(z)$  is an idempotent generating the centre of  $C_0(M \otimes K)$ . We note that  $e$  does not depend on the choice of the basis of  $M$ . The condition  $Me = C_1(M)e$  can be expressed in terms of the  $a_{ij}$ . Let

$$v_1 = e_2e_3e_4, \quad v_2 = -e_1e_3e_4, \quad v_3 = e_1e_2e_4 \quad \text{and} \quad v_4 = -e_1e_2e_3 \quad (4.3)$$

and

$$e_i z = \sum_j \gamma_{ij} e_j + \sum_j \varphi_{ij} v_j, \quad 1 \leq i \leq 4, \quad (4.4)$$

in  $C_1(M) \otimes K$ . One verifies that  $\varphi_{ij} = a_{ij}$ . Let  $\pi(z) = \delta$  and  $D = t(z) - 2\delta$  as above, so that  $e = (z - \delta)D^{-1}$ . From the relation  $De_i e = e_i(z - \delta) = e_i(z - \delta)e$  in  $C(M) \otimes K$ , we obtain

$$De_i e = \sum_j (\gamma_{ij} - \delta\delta_{ij}) e_j e + \sum_j \varphi_{ij} v_j e, \quad 1 \leq i \leq 4.$$

The matrix  $\Phi = (\varphi_{ij})$  belongs to  $GL_4(K)$ , since  $\det \Phi = d(e_1, e_2, e_3, e_4)$  is a unit of  $K$ . Denoting by  $\Gamma$  the matrix  $(\gamma_{ij} - \delta\delta_{ij})$ , we have

$$Me = C_1(M)e \Leftrightarrow \Phi^{-1}(D \cdot \mathbf{1} - \Gamma) \in M_4(R) \quad (4.5)$$

for  $e = e_\pi$  as above. We note that the L.H.S. of (4.5) is independent of the choice of basis. Therefore the same is true of the R.H.S.

We say that a quadratic  $R$ -module  $(M, q)$  satisfies the *Bhaskara \* condition* if  $M \otimes K$  has trivial Arf invariant, and for some supplementation  $\pi: S(M) \otimes K \rightarrow K$ , the R.H.S. of (4.5) holds locally,  $D, \delta$  being defined through  $\pi$ .

**PROPOSITION 4.6:** *A primitive regular quaternary quadratic module admits of a proper composition if and only if it satisfies the Bhaskara condition.*

**PROOF:** This is a consequence of (4.1), (4.5) and (3.8).

\* In honour of Bhaskara (12th century) for his contributions to quadratic problems.

The matrix  $D \cdot \mathbf{1} - \Gamma$  in (4.5) has the following explicit form

$$\begin{pmatrix} \delta + D & a_1 a_{34} & -a_1 a_{24} & a_1 a_{23} \\ -a_2 a_{34} & \delta + D + a_{12} a_{34} & -a_{12} a_{24} + a_{14} a_2 & a_{12} a_{23} - a_{13} a_2 \\ a_3 a_{24} & -a_{14} a_3 + a_{13} a_{34} & \delta + D - a_{13} a_{24} + a_{14} a_{23} & a_{12} a_3 \\ -a_4 a_{23} & a_{14} a_{34} + a_{13} a_4 & -a_{12} a_4 & c \end{pmatrix} \quad (4.7)$$

with  $c = \delta + D + a_{14} a_{23} - a_{13} a_{24} + a_{12} a_{34}$ .

REMARK 4.8: If  $(M, q)$  has trivial Arf invariant, then the entries of  $\Phi^{-1}(D \cdot \mathbf{1} - \Gamma)$  lie in  $R[d^{-1}]$ , where  $d = d(e_1, e_2, e_3, e_4)$ .

### 5. The Brandt condition

Let  $(M, q)$  be a quadratic  $R$ -module and let  $\varphi : M \rightarrow M^* = \text{Hom}(M, R)$  be the adjoint of  $q$ . Let  $K$  be the total quotient ring of  $R$ . If  $M$  is regular, then  $M \otimes K$  is unimodular and  $4 \otimes 1_K$  is an isomorphism. We then have a quadratic form

$$q^{-1} : M^* \otimes K \rightarrow K$$

defined by  $q^{-1}(f) = (q \otimes 1_K)((\varphi \otimes 1_K)^{-1}(f))$ ,  $f \in M^* \otimes K$ .

Let  $(M, q)$  be a regular quaternary quadratic module such that  $q \otimes 1_K$  has trivial Arf invariant. For any supplementation  $\pi$  of  $S(M) \otimes K$ , let, as in (3.2),  $D_\pi$  be the fractionary ideal of  $R$  generated by  $t(x) - 2\pi(x)$ ,  $x \in S(M)$ . We recall that  $D_\pi$  is an invertible ideal and that  $D_\pi^2 = d(S)$ . Following Brandt ([Br2]) we call  $(M, q)$  a  $K$ -form if  $D_\pi q^{-1}(M^*) \subset R$  for some supplementation  $\pi$  of  $S(M \otimes K)$ . If  $R$  is a domain and  $M$  is free with trivial Arf invariant over  $K$ , then the discriminant  $d$  of  $M$  is a square in  $K$ . In this case  $(M, q)$  is a  $K$ -form if and only if  $Dq^{-1}(M^*) \subset R$ , where  $D^2 = d$ .

We say that a regular quaternary quadratic module  $(M, q)$  satisfies the *Brandt condition* if

- (1)  $(M, q)$  is a  $K$ -form,
- (2)  $(M, q)$  has trivial Arf invariant.

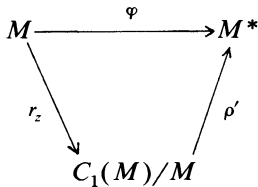
PROPOSITION 5.1. *Let  $(M, q)$  be a quaternary quadratic module which admits of a proper composition. Then  $(M, q)$  satisfies the Brandt condition.*

The proof of (5.1) needs some preliminaries. Let  $q$  be a quadratic form on a free module  $M$  with a basis  $\{e_1, e_2, e_3, e_4\}$  and let  $z$  be as in

(3.3); let  $T: C(M) \rightarrow C(M)$  be the trace map associated with the canonical involution of  $C(M)$ . For  $c \in C_1(M)$  and  $x \in M$  let  $T(xc) = \lambda + \mu z$ ,  $\lambda, \mu \in R$ . We define an  $R$ -linear map  $\rho: C_1(M) \rightarrow M^*$  by  $\rho(c)(x) = \mu$ . Since, for  $c \in M$ ,  $T(xc)$  belongs to  $R$ , we have  $\rho(M) = 0$ , so that  $\rho$  induces a linear map

$$\rho': C_1(M)/M \rightarrow M^*. \tag{5.2}$$

LEMMA 5.3: *The map  $\rho'$  is an isomorphism and the diagram*



is commutative, where  $\varphi$  is the adjoint of  $q$  and  $r_z$  is the right multiplication by the element  $z$  of (3.3).

PROOF: Let  $\{e_1^*, e_2^*, e_3^*, e_4^*\}$  be the basis of  $M^*$  dual to the basis  $\{e_1, e_2, e_3, e_4\}$  of  $M$ . A computation shows that  $\rho'(v_i) = e_i^*$ ,  $1 \leq i \leq 4$ , where the  $v_i$  are as in (4.3). Thus  $\rho'$  is surjective and is therefore an isomorphism,  $C_1(M)/M$  and  $M^*$  being both free of rank 4. From (4.4) we have  $r_z(e_i) = \sum_j \gamma_{ij} e_j + \sum_j \varphi_{ij} v_j$ ,  $\varphi_{ij} = b(e_i, e_j)$ , so that  $\rho' r_z(e_i) = \sum_j \varphi_{ij} \rho'(v_j) = \sum_j \varphi_{ij} e_j^* = \varphi(e_i)$ .

We now prove (5.1). By (2.10) and (2.4),  $M$  is of type  $(A, B)$ . Let  $i: S(M) \rightarrow R \times R$  be the restriction of  $i_M: C(M) \rightarrow \begin{pmatrix} A & M \\ \bar{M} & B \end{pmatrix}$  to  $S(M)$  (see the proof of (3.8)) and let  $p: R \times R \rightarrow R$  be the second projection. We have the supplementation  $\pi = p \circ i: S(M) \rightarrow R$  so that condition (2) is satisfied. We prove condition (1) for this supplementation  $\pi$ . We may assume  $R$  local and  $M$  free over  $B$  of rank one. Through the choice of a basis of  $M$  over  $B$ , we replace  $(M, q)$  by  $(B, n)$ , noting that condition (1) is unchanged by similarities. The algebra  $\begin{pmatrix} A & M \\ \bar{M} & B \end{pmatrix}$  is identified with the matrix ring  $M_2(B)$  and the map  $i_B: C(B) \rightarrow M_2(B)$  is induced by  $x \mapsto \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}$ ,  $x \in B$ , where  $x \mapsto \bar{x}$  is the involution on  $B$ . Since  $B$  is regular,  $i_B$  is injective and we use it to identify  $C(B)$  with a subalgebra of  $M_2(B)$ . As noted in §3, the canonical involution of  $C(B)$  then is the restriction of the involution of  $M_2(B)$  given by  $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \mapsto \begin{pmatrix} \bar{u} & \bar{x} \\ \bar{v} & \bar{y} \end{pmatrix}$ . We observe that the norm on  $C_0(B)$  induced by the canonical involution of

the Clifford algebra has values in  $S(M)$ . Let  $\{e_1, e_2, e_3, e_4\}$  be a basis of  $B$ . The element  $z$  given by (3.3) lies in  $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ . Let  $z = \begin{pmatrix} \epsilon & 0 \\ 0 & \delta \end{pmatrix}$ ,  $\epsilon, \delta \in R$ . By our choice of  $\pi$ , we have  $\pi(z) = \delta$  and the element  $D = t(z) - 2\pi(z) = \epsilon - \delta$  lies in  $R$  and generates the ideal  $D_\pi$ . We have  $D^2 = -d(1, z)$ , so that  $D$  is a unit in the total quotient ring  $K$  of  $R$ . Using (5.3) we compute  $q^{-1}$  on  $C_1(B)/B$ . Since  $r_z \otimes 1_K : B \otimes K \rightarrow (C_1(B)/B) \otimes K$  is an isomorphism, for a class  $[f] \in C_1(B)/B$ , we may choose  $f = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix} z \in C_1(B \otimes K)$ ,  $x \in B \otimes K$ , as a representative. There exists  $\begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix} \in B \otimes K$  such that  $f + \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix} \in C_1(B) \subset M_2(B)$  thus

$$\begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \delta \end{pmatrix} + \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix} \in M_2(B).$$

This shows that  $b = x\delta + t \in B$  and  $\epsilon\bar{x} + \bar{t} \in B$ . We have  $\epsilon\bar{x} + \bar{t} = \delta\bar{x} + \bar{t} + D\bar{x} = \bar{b} + D\bar{x}$ , so that  $f + \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ \bar{b} + D\bar{x} & 0 \end{pmatrix}$  with  $Dx \in B$ . The class  $[f]$  is the class of  $g = \begin{pmatrix} 0 & 0 \\ D\bar{x} & 0 \end{pmatrix} \in C_1(B)$  in  $C_1(B)/B$ . The element  $g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D\bar{x} \end{pmatrix}$  is in  $C_0(B)$ ; its norm  $\begin{pmatrix} 0 & 0 \\ 0 & x\bar{x}D^2 \end{pmatrix}$  belongs to  $S(M) = R + Rz$  as noted above. We write

$$\begin{pmatrix} 0 & 0 \\ 0 & x\bar{x}D^2 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} + s \begin{pmatrix} \epsilon & 0 \\ 0 & \delta \end{pmatrix}, \quad r, s \in R.$$

This implies  $r + s\epsilon = 0$ ,  $s\delta - s\epsilon = x\bar{x}D^2$ , that is  $x\bar{x}D = -s \in R$ . Since  $\varphi^{-1}([f]) = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}$ , we have  $Dq^{-1}([f]) = x\bar{x}D$ ,  $[f] \in C_1(M)/M \simeq M^*$ . Thus  $Dq^{-1}([f]) \in R$  for all  $[f] \in C_1(M)/M$ . This proves condition (1).

By (4.6) and (5.1), the Bhaskara condition implies the Brandt condition for primitive regular quaternary quadratic modules. The converse is however not true in general (see the examples at the end of this section). The following theorem asserts that, under some restrictions on the ring  $R$ , the Brandt condition implies the Bhaskara condition and hence the existence of a composition.

Following Bass [Ba, p. 210], for an invertible ideal  $I$  of  $R$  we write  $R[I^{-1}] = \bigcup_{n \geq 0} (I^n)^{-1}$  in the total quotient ring.

**THEOREM 5.4:** *Let  $R$  be a domain with quotient field  $K$ ,  $\bar{R}$  the integral closure of  $R$  in  $K$ . Let  $(M, q)$  be a primitive regular quaternary quadratic module over  $R$  which satisfies the Brandt condition. Let  $d = d(M)$  denote the discriminant ideal of  $(M, q)$ . let  $L = \bar{R} \cap R[d^{-1}] \cap R[1/2]$  if  $\text{char } K \neq 2$  and  $L = \bar{R} \cap R[d^{-1}]$  if  $\text{char } K = 2$ . Then  $(M, q) \otimes L$  satisfies the Bhaskara condition.*

PROOF: By finiteness arguments, we may assume that  $R$  is noetherian. Since both the Brandt and the Bhaskara conditions are local conditions for a domain, we assume that  $R$  is local and that  $M$  is free. Let  $\{e_1, e_2, e_3, e_4\}$  be a basis of  $M$  and let  $z$  be as in (3.3). Let  $\pi: S \rightarrow R$  be a supplementation of the Arf invariant  $S(M)$  of  $M$  such that  $D_\pi q^{-1}(M^*) \subset R$ . Let  $\delta = \pi(z)$  and  $D = t(z) - 2\pi(z)$ . Since  $D_\pi$  is the principal ideal generated by  $D$ , we have  $Dq^{-1}(M^*) \subset R$ . In particular,  $D\Phi^{-1} \in M_4(R)$ . It suffices to check that  $\Lambda = \Phi^{-1}\Gamma \in M_4(L)$ . We first show that  $\Lambda \in M_4(\bar{R})$ . By Mori's theorem ([N], p. 118),  $\bar{R}$  is a Krull ring and hence  $\bar{R} = \bigcap R_p$ , where the intersection is taken over all minimal primes  $p$  of  $R$ . Thus it is enough to check that if  $R$  is a discrete valuation ring, then  $\Lambda \in M_4(R)$ . Any regular quadratic module over a discrete valuation ring is an orthogonal sum of submodules of rank  $\leq 2$ . Since the Bhaskara condition and the condition  $D\Phi^{-1} \in M_4(R)$  are invariant under a change of basis, we may assume that for a suitable basis of  $M$ , the matrix  $\Gamma$  has the following form (see (4.7)):

$$\Gamma = \begin{pmatrix} -\delta & -a_1 a_{34} & 0 & 0 \\ a_2 a_{34} & -\delta - a_{12} a_{34} & 0 & 0 \\ 0 & 0 & -\delta & -a_{12} a_3 \\ 0 & 0 & a_{12} a_4 & -\delta - a_{12} a_{34} \end{pmatrix}.$$

We obtain the following expressions for the coefficients of the first  $2 \times 2$ -block of  $\Lambda$ :

$$\begin{aligned} \lambda_{11} &= -a_2(2\delta + a_{12}a_{34})d_1^{-1} \\ \lambda_{12} &= (-2a_1a_2a_{34} + a_{12}^2a_{34} + a_{12}\delta)d_1^{-1} \\ \lambda_{21} &= (2a_1a_2a_{34} + a_{12}\delta)d_1^{-1} \quad \lambda_{22} = -a_1(2\delta + a_{12}a_{34})d_1^{-1} \end{aligned}$$

with  $d_1 = 4a_1a_2 - a_{12}^2$ . The expression for the coefficients of the second  $2 \times 2$ -block are similar. Since  $2\delta + a_{12}a_{34} = 2\delta - a = -D$  by (3.4), we have  $\lambda_{ii} = Dq^{-1}(e_i^*)$ ,  $\{e_1^*, e_2^*, e_3^*, e_4^*\}$  denoting the basis of  $M^*$  dual to  $\{e_1, e_2, e_3, e_4\}$ . In view of the Brandt condition,  $\lambda_{ii} \in R$ . Further,  $\lambda_{21}$  satisfies the equation

$$X^2 - a_{34}X + a_3a_4 - a_1a_2d_2d_1^{-1} = 0.$$

Since  $a_1a_2d_2d_1^{-1} = D^2q^{-1}(e_1^*)q^{-1}(e_2^*) \in R$ , the equation has coefficients in  $R$ . The ring  $R$  being integrally closed,  $\lambda_{21}$  belongs to  $R$ . The element  $\lambda_{12}$  also lies in  $R$  since  $\lambda_{12} = \lambda_{21} - a_{34}$ . We thus have shown that  $\Lambda \in M_4(\bar{R})$ . By (4.8),  $\Lambda \in M_4(R[d^{-1}])$  so that  $\Lambda$  lies in  $M_4(L)$ ,

where  $L = \bar{R} \cap R[d^{-1}]$ . If  $\text{char } K \neq 2$ , and if  $L = \bar{R} \cap R[d^{-1}] \cap R[1/2]$ , the fact that  $\Lambda \in M_4(L)$  is a consequence of the following

LEMMA 5.5: *Let  $\Phi, \Gamma, \Lambda$  be as above. Then  $\Lambda = \frac{1}{2}(D\Phi^{-1} - \Sigma)$  where*

$$\Sigma = \begin{pmatrix} 0 & -a_{34} & a_{24} & -a_{23} \\ a_{34} & 0 & -a_{14} & a_{13} \\ -a_{24} & a_{14} & 0 & -a_{12} \\ a_{23} & -a_{13} & a_{12} & 0 \end{pmatrix}$$

PROOF: Let  $z$  be as in (3.3),  $\delta, D$  as above and  $e = (z - \delta)D^{-1}$ . Let  $w = eD$ . Then we have  $w^2 = wD$  and  $wx + xw = Dx$  for any  $x \in C_1(M) \otimes K$  by (4.2). Further, from (4.4) it follows that

$$e_i w = \sum_j (\gamma_{ij} - \delta \delta_{ij}) e_j + \sum \varphi_{ij} v_j.$$

Since the canonical involution  $\alpha$  on  $C(M)$  is the identity on the centre of  $C_0(M) \otimes K$ , we have  $\alpha(e_i w) = \alpha(w) \alpha(e_i) = w e_i$ . Hence

$$D e_i = e_i w + w e_i = 2 \sum_j (\gamma_{ij} - \delta \delta_{ij}) e_j + \sum_j \varphi_{ij} (v_j + \alpha(v_j)).$$

We verify that  $v_j + \alpha(v_j) = \sum_k s_{jk} e_k$ , where  $(s_{jk}) = \Sigma$ . Thus we have  $D \cdot \mathbf{1} = 2\Gamma + \Phi\Sigma$  and  $\Lambda = \Phi^{-1}\Gamma = \frac{1}{2}(D\Phi^{-1} - \Sigma)$ .

COROLLARY 5.6: *Let  $(M, q)$  be a primitive regular quaternary quadratic module with trivial Arf invariant over a domain  $R$ . Let  $d = d(M)$  be the discriminant ideal of  $M$ . If  $R$  is integrally closed in  $R[d^{-1}]$ , then  $(M, q)$  admits of a proper composition if and only if it is a  $K$ -form.*

COROLLARY 5.7: *Let  $R$  be a domain with quotient field  $K$  of characteristic not 2. Let  $R$  be integrally closed in  $R[1/2]$ . Then a primitive regular quaternary quadratic module admits of a composition if and only if the discriminant of  $M \otimes K$  is a square and  $(M, q)$  is a  $K$ -form.*

PROOF: Since  $\text{char } K \neq 2$  and the discriminant of  $M \otimes K$  is a square,  $M \otimes K$  has trivial Arf invariant. We show that  $M$  has trivial Arf invariant. We assume that  $R$  is local and that  $M$  is free with basis  $\{e_1, e_2, e_3, e_4\}$ . By (5.5),  $M \otimes R[1/2]$  satisfies the Bhaskara condition and by (4.6) and (5.1) has trivial Arf invariant. Let  $\pi: S(M) \otimes R[1/2] \rightarrow R[1/2]$  be a supplementation. Since  $R$  is integrally closed in  $R[1/2]$ , we have  $\pi(S(M)) = R$  and  $M$  has trivial Arf invariant.

EXAMPLE 5.8: Let  $k$  be a field of characteristic 2,  $R = k[t^2, t^3]$  and  $(M, q)$  the quadratic  $R$ -submodule of  $(M_2(k[t]), \det)$  with basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} t & t^2 \\ 0 & t \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} t^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then  $q$  is a  $K$ -form with trivial Arf invariant and discriminant  $d = d(M) = t^8$ , but  $e_2 \cdot e_3 \notin M$ , so that, in view of remark (2.9),  $M$  does not admit of a composition. (It is also easy to check that the Bhaskara condition is not satisfied). This shows that (5.6) does not hold without the assumption that  $R$  is integrally closed in  $R[d^{-1}]$ .

EXAMPLE 5.9: Let  $R$  be a domain with quotient field  $K$ ,  $\text{char } K \neq 2$ . Assume that there exists an  $\epsilon \in K \setminus R$  such that  $2\epsilon \in R$  and  $\epsilon^2 = a\epsilon + b$  for some  $a, b \in R$ . Let  $(M, q)$  be the quadratic  $R$ -submodule of  $(M_2(K), \det)$  with basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \epsilon & 2\epsilon^2 \\ 0 & \bar{\epsilon} \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 2\epsilon^2 & 0 \\ 0 & 0 \end{pmatrix},$$

where  $\bar{\epsilon} = a - \epsilon$ . Then  $q$  is a  $K$ -form, the discriminant of  $q \otimes 1_K$  is a square, but  $q$  does not admit of a composition. This shows that (5.7) does not hold if  $R$  is not integrally closed in  $R[1/2]$ .

EXAMPLE 5.10: Let  $R$  be a domain with quotient field  $K$  of characteristic 2. Assume that there is an element  $\epsilon \in K \setminus R$  such that  $\epsilon^2 = \epsilon + 1$ . Let  $(M, q)$  be the submodule of  $(M_2(K), \det)$  with basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon + 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The form  $q$  is unimodular and has trivial Arf invariant over  $K$ , hence is a  $K$ -form. But its Arf invariant is not trivial. The form does not admit of a composition. This shows that in (5.6) it is not enough to assume that the Arf invariant of  $M \otimes K$  is trivial.

## 6. Brandt's groupoids

For any quaternion algebra  $A_0$  over a commutative ring  $R$ , we shall define in a natural manner a groupoid  $G(A_0)$  and compare it with the different groupoids defined by Brandt for the classical case of maximal orders in a rational quaternion algebra [Br 3]. Let  $\mathfrak{B}(A_0)$  be the class of quaternion algebras  $B$  such that there exists a quaternary quadratic module of type  $(A_0, B)$ . For each isomorphism class of such algebras we pick a representative  $A_i$ . Let  $G_{ij}$  be the set of equivalence classes of



quadratic modules of type  $(A_i, A_j)$ , equivalence being isometry preserving the bimodule structure. We set  $G(A_0) = \bigcup_{i,j} G_{i,j}$ . The set  $G(A_0)$  is a groupoid with tensor product as multiplication, the identity elements and inverses being given by Examples (2.6) and (2.7). The groupoid  $G(A_0)$  is, upto isomorphism, independent of the choice of  $A_i$ .

In order to compare  $G(A_0)$  with the constructions of Brandt, we specialize to the case where  $R$  is a domain. Then, the  $A_i$  can be chosen as  $R$ -subalgebras of  $A_K = A_0 \otimes K$ ,  $K$  denoting the quotient field of  $R$ . If  $M$  is of type  $(A_i, A_j)$ , then  $M_K = M \otimes K$  is of type  $(A_K, A_K)$  and hence isomorphic to  $A_K$ . Since  $A_K$  is absolutely irreducible as an  $A_K$ -bimodule, this isomorphism is uniquely determined upto a scalar from  $K^\times$ . In other words,  $M$  can be taken as a left  $A_i$ -, right  $A_j$ -submodule of  $A_K$  with the quadratic form  $c^{-1}n$ , where  $c \in K^\times$  is such that  $R \cdot n(M) = Rc$ . The only possible changes are to replace the pair  $(M, c)$  by  $(kM, k^2c)$  for some  $k \in K^\times$ .

We consider the set  $\mathfrak{B}'(A_0)$  of  $R$ -subalgebras of  $A_K$  which belong to the class  $\mathfrak{B}(A_0)$  as defined above, i.e. all conjugates  $uA_i^{-1}u \in A_K^\times$ , for all  $A_i$ . For  $A, B, C \in \mathfrak{B}'(A_0)$  and  $R$ -submodules  $M$  and  $N$  of  $A_K$  of type  $(A, B)$  and  $(B, C)$  respectively (with respect to the multiplication in  $A_K$ ), multiplication in  $A_K$  is a proper composition and the product  $MN$  is of type  $(A, C)$ . We thus get a groupoid which we denote by  $B(A_0)$ . In the case  $R = \mathbb{Z}$ ,  $c$  is uniquely determined by  $M$  upto sign and there is no loss of generality if we restrict our attention to pairs  $(M, c)$  with  $c > 0$ . If  $A_0$  is a maximal order in  $A_{\mathbb{Q}}$ , we recover Brandt's groupoid of normal ideals [Br 3].

By our choice of  $A_i$  and  $M$  in  $A_K$ , we have represented  $G(A_0)$  as a quotient of a subgroupoid of  $B(A_0)$  by the action of  $K^\times$ . However, this subgroupoid depends on the choice of  $A_i$ . A different choice replacing  $A_i$  by  $u_i A_i u_i^{-1}$ ,  $u_i \in A_K^\times$  changes the pair  $(M, c)$  of type  $(A_i, A_j)$  into  $(u_i M u_i^{-1}, c n(u_i) n(u_j)^{-1})$ . For convenience we omit  $c$  in the sequel. The question arises whether there is a natural equivalence relation  $\sim$  on  $B(A_0)$  such that the induced map  $G(A_0) \rightarrow B(A_0)/\sim$  is independent of the choice of the  $A_i$ . One then certainly should have

$$M \sim u M v, \quad u, v \in A_K^\times. \tag{6.1}$$

It is well known that the only similarities of  $(A_K, n)$  are  $x \rightarrow uxv$  and  $x \rightarrow u\bar{x}v$ ,  $u, v \in A_K^\times$ , the former being called *proper*. The problem therefore is closely related to the question whether composition of quaternary quadratic forms induces a composition of proper similarity classes. There are well known examples to show that this is not true in general [Br 2], [P]. However, there is one special case where it works.

**PROPOSITION 6.2:** *Suppose  $M, N \in B(A_0)$  are of types  $(A, B)$  and*

$(B, C)$  respectively,  $u, v \in A_K^\times$  are such that  $Mu$  and  $vN$  have a product in the groupoid  $B(A_0)$ . Suppose further that

(\*) any automorphism of  $B$  is inner.

Then  $Mu \cdot vN = kMN$  for some  $k \in K^\times$ . If (\*) hold for every  $B \in \mathfrak{B}'(A_0)$  then the equivalence relation  $\sim$  defined by (6.1) is compatible with the groupoid structure on  $B(A_0)$ .

PROOF: We have that  $Mu$  is of type  $(A, u^{-1}Bu)$  and  $vN$  is of type  $(vBv^{-1}, C)$ . By assumption,  $u^{-1}Bu = vBv^{-1}$ , so that there exists  $b \in B^\times$ ,  $k \in K^\times$  such that  $uw = kb$ .

We now turn to the special case considered by Brandt in [Br 3] where  $R = \mathbb{Z}$  and  $A_0$  is a maximal order in  $A_{\mathbb{Q}}$ . In order to overcome the difficulties mentioned above, Brandt defines a series of different equivalence relations on the infinite groupoid  $B(A_0)$  such that the quotients become some sort of "finite groupoid of ideal classes". We consider only the two extreme cases. One is to replace the equivalence relation (6.1) by a stronger one,  $M \approx uMv$  if and only if each prime divisor of the discriminant  $D(A_0)$  occurs in  $n(u)$  and  $n(v)$  to an even power. It is not hard to show that " $\approx$ " is compatible with composition and that the map  $G(A_0) \rightarrow B(A_0)/\approx$  is injective. But in general the map is not surjective and depends on the choice of the representatives  $A_i$ . At the other extreme, the coarsest equivalence relation considered by Brandt amounts to declaring two modules to be equivalent if and only if they are of the same type  $(A_i, A_j)$ . The map of  $G(A_0)$  into the corresponding quotient of  $B(A_0)$  is obviously surjective and is independent of the choice of the representatives but is not injective.

Finally there is one variation of the groupoid  $G(A_0)$  similar to constructing the group  $H(C)$  instead of  $G(C)$  in [Kn]. It consists in considering quadratic forms and compositions, not up to isometry, but upto similarity and in exchange admit primitive quadratic forms not only having values in  $R$  but in any invertible  $R$ -module  $N$ . By doing so, one obtains another groupoid  $H(A_0)$  and a homomorphism  $G(A_0) \rightarrow H(A_0)$  similar to the situation at the end of [Kn, §6].

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