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NUMERICAL SECTIONS ON ELLIPTIC SURFACES

Ian Morrison * and Ulf Persson **

§0. Introduction

Throughout this paper we let f: $X \to B$ be a relatively minimal elliptic surface X over a smooth complete curve B of genus g. We assume that X has a distinguished section 0 and that the Euler characteristic $\chi = \chi(\mathcal{O}_X)$ is positive (so that X is not a product of B with a curve of genus one). If C is a divisor on X let $\overline{C}$ denote its class in $NS(X)$. Let F be a fiber of X and let $\Sigma(X)$ be the set of all sections; $\Sigma(X)$ is the set of all effective, irreducible divisors S on X which satisfy $S \cdot F = 1$. An examination of the Leray spectral sequence for f shows that a section S also satisfies $S^2 = -\chi$ (cf. [10]). From this it follows that a section cannot move so that we may identify the elements of $\Sigma(X)$ with their classes $\overline{\Sigma}(X)$ in $NS(X)$. We define the set of numerical sections of X to be the subset $\Phi(X) \subset NS(X)$ satisfying the numerical conditions $C \cdot F = 1$ and $C^2 = -\chi$. This paper is primarily devoted to the study of $\Phi(X)$. For simplicity, we work here over $\mathbb{C}$ but the main results hold, modulo a few locutions, over any field of characteristic zero.

By definition, $\Phi(X) \supset \overline{\Sigma}(X)$ and it is natural to ask when equality holds. We show that $\Phi(X) = \overline{\Sigma}(X)$ if and only if all fibers of X are irreducible (2.3 and 3.8) and then we say that X is f-irreducible. A second natural question is whether the classes in $\Phi(X)$ must be effective. The answer is yes and in Theorem 2.2 we give a precise description of the associated algebraic systems. The effectiveness of the classes in $\Phi(X)$ turns out to be equivalent to certain inequalities (1.3) on the restriction of the intersection form on $NS(X)$ to the span of the components of a reducible fibre of X. We give some consequences of these inequalities ((1.8) and (2.4)) which lead to a criterion for identifying $\overline{\Sigma}(X)$ as a subset of $\Phi(X)$ (2.5).

Our main focus of interest is in certain group laws on $\Sigma(X)$ and $\Phi(X)$ which we now explain. We first show that $\Phi(X)$ is a set of coset representatives for the sublattice $U(X)$ on $NS(X)$ spanned by the

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classes of the distinguished section O and of the fibre F. Similarly $\overline{\Sigma}(X)$ is a set of coset representatives for the sublattice $L(X)$ spanned by the classes of O and of all divisors supported in the fibers of F (Theorem 3.1). By transport of structure we obtain abelian group laws on $\Phi(X)$ and $\Sigma(X)$, which we then describe in terms of the addition on $NS(X)$ in (3.2) and (3.6). The set $\Sigma(X)$ of sections of X has another natural group structure: we may add sections fibre-by-fibre in the elliptic curve F_b obtained by using $O \cdot F_b$ as an origin on F_b. We show that this operation and the operation on $\overline{\Sigma}(X)$ described above agree (3.4). Caution: although $\overline{\Sigma}(X)$ is a subset of $\Phi(X)$ it is *not* a subgroup except when X is f-irreducible. In fact the inclusion of $U(X)$ in $L(X)$ yields a surjection σ: $\Phi(X) \to \Sigma(X)$ which we describe geometrically: if $C \in \Phi(X)$, there is a unique base section S in the base locus of the algebraic system $[\overline{C}]$ and σ sends $\overline{C}$ to S(3.7). Although the statements of these results make no reference to it, the proofs depend heavily on the earlier study of the systems $[\overline{C}]$ for $\overline{C} \in \Phi(X)$.

An immediate corollary of these results is a formula (3.7) due to Tate [12] and Shioda [11] which was one of our principal guides in this work: if k_b is the number of components in the fibre of f over $b \in B$, rank $NS(X) = 2 + \text{rank } \Sigma(X) + \Sigma_{b \in B}(k_b - 1)$. Our description of the operation $\oplus$ on $\Phi(X)$ gives a method for enumerating the classes in $\Phi(X)$ in terms of a prescribed basis of $NS(X)$. As promised in [9], we illustrate this in the case when the surface X is rational. In this case, $\Phi(X)$ was first enumerated by Manin [7]; we simplified his proof in [9]. Both methods require that one guess a formula for certain sums in $\Phi(X)$ which is then fairly easily verified by induction. A more conceptual proof of this formula was another of our goals; in §4, we show that it is a formal consequence of our theory. We also illustrate in §4 the criteria for identifying $\overline{\Sigma}(X)$ and the kernel of the map $\Phi(X) \to \Sigma(X)$ as subsets of $\Phi(X)$ with a rational example.

The rest of the paper is organized as follows. In §1, we lay the groundwork for the inequalities we require on the intersection form on X. In §2 we prove the effectiveness of the classes in $\Phi(X)$, describe the curves in them and give the related corollaries. The study of the group laws on $\Phi(X)$ and $\Sigma(X)$ is done in §3 and the applications to rational X are given in §4.

It is a pleasure to thank Robert Steinberg for pointing out Lemma 1.4 to us. The proof of Theorem 1.3 based on this lemma replaces a long and tedious case analysis. We would also like to thank David Mumford for showing us the simple argument used to show the existence of the base section in the proof of Theorem 2.2.

§1. Inequalities on the intersection form

Before proving the inequalities in which we are interested, we recall some standard facts about the structure of the reducible fibres of the map f:

$X \to B$. The most complete treatment is Kodaira [6]; [1] provides a more compact reference. We will denote by $\bar{C}$ the class in $NS(X)$ of a divisor C on X, but we will usually drop the bars when computing intersection numbers. We let $[C]$ be the algebraic system associated to $\bar{C}$, that is, the set of effective divisors in the class $\bar{C}$, and we say $\bar{C}$ is effective if $[C] \neq \emptyset$. We denote by F_b the fibre of f over a point $b \in B$ and by F a general fibre of X. The letter S (possibly subscripted etc.) will always denote a section of X and D will always indicate a divisor (not necessarily effective) supported on components of fibres of X. Let $\chi = \chi(\mathcal{O}_X)$. We recall

LEMMA 1.1 ([10, VII, §3]): *$\bar{K}_X = (\chi + (2g-2))\bar{F}$. In particular $K_X \cdot D = 0$ for any divisor D supported on the fibres of X.*

Let $F_b = \sum_{i=0}^{k} \lambda_i D_i$ be the decomposition of a reducible fibre F_b of f into its irreducible components with multiplicities $\lambda_i \geqslant 1$, let $\tilde{\Lambda} = \tilde{\Lambda}_b$ be the span of the $\bar{D}_i$ in $NS(X)$ and let $\langle\ ,\ \rangle_b$ be the restriction to $\tilde{\Lambda}$ of the intersection form on $NS(X)$. Fix i for a moment and observe that $0 = D_i \cdot F_b = \lambda_i D_i^2 + \sum_{j \neq i} \lambda_j (D_i \cdot D_j)$. Now each λ_j is positive and each intersection number $D_i \cdot D_j$ with $i \neq j$ is non-negative. The connectedness of F_b implies that $D_i \cdot D_j$ is positive for some $j \neq i$, hence that $D_i^2 < 0$. But then using the lemma $0 \leqslant p_a(D_i) = (D_i^2 + K_X \cdot D_i)/2 + 1 = (D_i^2/2) + 1 < 1$ so we must have $p_a(D_i) = 0$. Thus each D_i is a smooth rational curve of self-intersection (-2). Since all the curves D_i lie in a fibre of the map $f\colon X \to B$ the Hodge index theorem implies that $\langle\ ,\ \rangle_b$ is negative semi-definite. In fact since $F_b \cdot D_i = 0$ for all i, $\langle\ ,\ \rangle_b$ is strictly semi-definite.

To F_b we associate a weighted graph $\tilde{\Gamma}_b$; $\tilde{\Gamma}_b$ has a vertex for each component $\bar{D}_i$, and $\bar{D}_i$ and $\bar{D}_j$ are joined by an edge of weight $a_{ij} = D_i \cdot D_j \in \mathbb{N}$. Conversely to such a weighted graph $\tilde{\Gamma}$ we may associate the lattice $\tilde{\Lambda} = \oplus_{i=0}^{k} \mathbb{Z} \bar{D}_i$ with its distinguished basis $\{\bar{D}_i\}$ and the bilinear form $\langle\ ,\ \rangle_{\tilde{\Gamma}}$ on $\tilde{\Lambda} \otimes_{\mathbb{Z}} \mathbb{R}$ which is defined by the conditions $\langle D_i, D_i \rangle_{\tilde{\Gamma}} = -2$ and $\langle D_i, D_j \rangle_{\tilde{\Gamma}} = a_{ij}$ if $i \neq j$.

If $\langle\ ,\ \rangle_{\tilde{\Gamma}}$ is negative strictly semi-definite, then $\tilde{\Gamma}$ is one of the completed Coxeter graphs $\tilde{A}_n$, $n \geqslant 2$; $\tilde{D}_n$, $n \geqslant 4$; or $\tilde{E}_n$ $n = 6, 7, 8$. Moreover there is a unique indivisible positive integral element $\bar{F} = \sum_{i=0}^{k} \lambda_i \cdot \bar{D}_i$ in the kernel of $\langle\ ,\ \rangle_{\tilde{\Gamma}}$ Proofs of these assertions and a table of the $\tilde{\Gamma}$'s showing the coefficients λ_i may be found in [1, Lemma 2.12] or [5]. Each $\tilde{\Gamma}$ occurs as the graph $\tilde{\Gamma}_b$ associated to a reducible fibre F_b, in which case $\bar{F}$ gives the divisor class of F_b. (We will not need these existence results for which the reader may refer to Kodaira [6]).

Now suppose that D_0 is a component of multiplicity one in F_b. Let Λ be the span of $\bar{D}_1, \ldots, \bar{D}_k$ in $NS(X)$ and continue to denote by $\langle\ ,\ \rangle_b$ the restriction to Λ of the intersection pairing. Let Γ_b be the associated weighted graph and let us agree to continue to attach the ormer coefficients λ_i to each vertex $\overline{D}_i$. The graph Γ_b is the A–D–E Coxeter graph corresponding to Γ_b; in particular, Γ_b does not depend on the multiplicity one component D_0 chosen.

The main result of this section is:

THEOREM 1.3: *Let* Γ *be one of the graphs* Γ_b. *If* $\overline{D} = \Sigma_{i=1}^{k} a_i \overline{D}_i \in \Lambda$, *then for each* i, $\langle D, D \rangle \leqslant \dfrac{-2a_i}{\lambda_i}$.

PROOF: It will be convenient to set $[\ ,\ ] = -\langle\ ,\ \rangle$ and to prove that $[\overline{D}, \overline{D}] \geqslant 2a_i/\lambda_i$. We first translate this inequality into the language of root systems. Indeed we may identify Γ with the Coxeter graph of the simple root system of the same name so that Λ becomes the root lattice, the $\overline{D}_i$'s become a fundamental system of simple roots and $[\ ,\ ]$ becomes the Killing form on Λ. The basic fact about root systems which we will need is:

LEMMA 1.4: *Let* $\overline{D} = \Sigma_{j=1}^{l} R_j$ *express* $\overline{D}$ *as a sum of roots of* Γ (*not necessarily simple*) *with* l *minimal. Then* $[D, D] \geqslant \Sigma_{j=1}^{l} [R_j, R_j]$.

PROOF: We claim that if $j \neq j'$ then $[R_j, R_{j'}] \geqslant 0$ from which the lemma follows immediately by bilinearity. If $[R_j, R_{j'}]$ were negative then by [3, Ch. VI, §3, Th. 1] either $R_j = -R_{j'}$ or $R_j + R_{j'}$ is again a root; either possibility would contradict the minimality of the expression for $\overline{D}$.

COROLLARY 1.5: *The roots of* Γ *are exactly the elements of* Λ *of shortest length* $\sqrt{2}$.

PROOF: Inspection of the Planche for Γ in [3] shows that indeed all roots of Γ are of length $\sqrt{2}$. (This property, in fact, characterizes the Γ of types A, D and E). If $\overline{D}$ is not a root then writing $\overline{D} = \Sigma_{j=1}^{l} R_j$ with each R_j a root and l minimal, we must have $l \geqslant 2$. Then $[D, D] \geqslant \Sigma_{j=1}^{l} [R_j, R_j] = 2l$, so D has length at least 2.

Now let $R_b = \Sigma_{i=1}^{k} \lambda_i \overline{D}_i$. Comparing, the table in [1, Lemma 2.12] with the Planche for Γ in [3] shows that R_b is the highest root of Γ with respect to the $\overline{D}_i$'s. That is, if $R = \Sigma_{i=1}^{k} \mu_i D_i$ is any root, then $\lambda_i \geqslant \mu_i$ for each i. Since $[R_b, R_b] = 2$ by the corollary, we can restate the theorem as: for all $\overline{D} = \Sigma_{i=1}^{k} a_i \overline{D}_i$ in Λ and all i, $[\overline{D}, \overline{D}]/[R_b, R_b] \geqslant a_i/\lambda_i$. This is clear from the definition of R_b is $\overline{D}$ is a root. Otherwise write $\overline{D} = \Sigma_{j=1}^{l} R_j$ with each R_j a root and l minimal and write $R_j = \Sigma_{i=1}^{k} \mu_{ji} \overline{D}_i$. Using Lemma 1.4 again,

$$\frac{[D, D]}{[R_b, R_b]} \geqslant \sum_{j=1}^{l} \left(\frac{[R_j, R_j]}{[R_b, R_b]} \right) \geqslant \sum_{j=1}^{l} \left(\frac{\mu_{ji}}{\lambda_i} \right) = \frac{a_i}{\lambda_i}.$$

COROLLARY 1.6: *If S is a section of X and D is a divisor supported on a reducible fibre F_b of X such that $S \cdot D = 0$, then $\overline{D} - (D^2/2)\overline{F}$ is effective.*

PROOF: The section S must meet F_b in a point of a component of multiplicity one which we may take to be D_0. Then $\overline{D} = \sum_{i=1}^{n} a_i \overline{D}_i \in \Lambda_b$ since $S \cdot D = 0$. Hence $\overline{D} - (D^2/2)\overline{F}_b = (-D^2/2)\overline{D}_0 + \sum_{i=1}^{n} (a_i - \lambda_i \cdot D^2/2)\overline{D}_i$. The coefficient of D_0 is non-negative since $\langle\ ,\ \rangle_b$ is negative definite and the coefficient of D_i is non-negative by Theorem 1.3 applied to $\overline{D}$. Therefore $\overline{D} - (D^2/2)\overline{F}$ is effective.

We conclude this section with a few other consequences of Theorem 1.3. Any $\overline{D} \in \tilde{\Lambda}_b$ has the form $\overline{D} = \alpha\overline{F} + \overline{E}$ with $\overline{E} \in \tilde{\Lambda}_b^+$, the effective cone in $\tilde{\Lambda}_b \subset NS(X)$. We let $\alpha(\overline{D})$ be the greatest such α, $\overline{E}(\overline{D})$ be the corresponding $\overline{E}$ and define $\overline{\Pi}_b$ the set of barely effective elements of $\tilde{\Lambda}_b$ to be the set of $\overline{D}$ for which $\alpha(\overline{D}) = 0$ or equivalently $\overline{D} = E(\overline{D})$. Note that $\overline{\Pi}_b = \{\overline{D} \in \tilde{\Lambda}_b^+ \mid (\overline{D} - \overline{F}) \notin \tilde{\Lambda}^+\}$. Now let us fix a component D_0 of multiplicity one in F_b and write a general $\overline{D} = \sum_{i=0}^{k} a_i \overline{D}_i$.

LEMMA 1.7: (*i*) *There is an exact sequence of lattices*

$$0 \to \mathbb{Z}f \to \tilde{\Lambda}_b \overset{\varphi}{\to} \Lambda_b \to 0$$

given by $\varphi(\overline{D}) = \overline{D} - a_0\overline{F}$.

(*ii*) *The map φ restricts to a bijection from $\overline{\Pi}_b$ to Λ_b.*

(*iii*) *The group structure $\oplus$ induced on $\overline{\Pi}_b$ by transport of structure from Λ_b is given by $\overline{D} \oplus \overline{D}' = \overline{D} + \overline{D}' - \alpha(\overline{D} + \overline{D}')\overline{F}$, and the bilinear form induced on $\overline{\Pi}_b$ is the restriction of $\langle\ ,\ \rangle$ to $\overline{\Pi}_b$.*

PROOF: The first assertion is clear and the second follows from the fact that the map $\overline{D} \to \overline{E}(\overline{D})$ from $\Lambda_b \to \overline{\Pi}_b$ is an inverse to $\varphi\,|_{\overline{\Pi}_b}$. As for (iii), if $\overline{D}$ and $\overline{D}'$ are in $\overline{\Pi}_b$, then $\varphi(\overline{D} + \overline{D}' - \alpha(\overline{D} + \overline{D}')F) = (\overline{D} + \overline{D}' - \alpha(\overline{D} + \overline{D}')\overline{F}) - (a_0 + a_0' - \alpha(\overline{D} + \overline{D}'))F = (\overline{D} - a_0\overline{F}) + (\overline{D}' - a_0'\overline{F}) = \varphi(\overline{D}) + \varphi(\overline{D}')$, and $\varphi(\overline{D}) \cdot \varphi(\overline{D})' = (\overline{D} - a_0\overline{F})(\overline{D}' - a_0'\overline{F}') = \overline{D} \cdot \overline{D}'$.

COROLLARY 1.8: (*i*) *If $\overline{D} \in \overline{\Pi}_b$ and D_i is a component of multiplicity one in F_b (i.e. $\lambda_i = 1$), then $\langle \overline{D}, \overline{D} \rangle \leqslant -2a_i$.*

(*ii*) *Suppose D is an effective divisor on X whose support lies in a reducible fibre F_b but such that $D - F_b$ is not effective, and suppose S is a section of X. Then $2SD + D^2 \leqslant 0$.*

PROOF: (i) We may take $i = 0$ by the symmetry of $\tilde{\Gamma}_b$. Let $\overline{D}' = \varphi(\overline{D}) \in \Lambda$. Then for some i, $0 \leqslant a_i < \lambda_i$ since $\overline{D} \in \overline{\Pi}_b$. Thus $-a_0\lambda_i \leqslant a_i' = a_i - a_0\lambda_i < (1 - a_0)\lambda_i$. By Theorem 1.3, $\langle \overline{D}, \overline{D} \rangle = \langle \overline{D}', \overline{D}' \rangle \leqslant -2a_i/\lambda_i < -2(a_0 - 1)$. Since $\langle\ ,\ \rangle$ is even, this implies that $\langle \overline{D}, \overline{D} \rangle \leqslant -2a_0$.

(ii) We may suppose that S meets F_b in a point of D_0. Then $2S \cdot D + D^2 = 2a_0 + \langle \overline{D}, \overline{D} \rangle \leqslant 0$ by (i).

§2. Effectiveness of numerical sections

Now we let $\Sigma(X)$ be the set of sections f: $X \to B$. Recall that a divisor S is a section if and only if S is effective and irreducible and if $S \cdot F = 1$. Since a section satisfies $S^2 = -\chi$ and we are assuming $\chi > 0$, S is always the unique curve in $[S]$ and we can and do identify $\Sigma(X)$ and $\overline{\Sigma}(X) = \{\overline{S} \mid S \in \Sigma(X)\}$ its image in $NS(X)$. We let $\Phi(X)$ be the set of divisor classes which behave numerically like the classes of sections; that is, $\Phi(X) = \{\overline{C} \mid C \cdot F = 1,\ C^2 = -\chi\}$. We call $\Phi(X)$ the set of numerical sections of X. By definition, $\overline{\Sigma}(X) \subset \Phi(X)$. In fact $\overline{\Sigma}(X)$ is the set of classes $\overline{C} \in \Phi(X)$ for which $[C]$ contains an effective irreducible curve C. In this paragraph, we address ourselves to two questions: first, when does $\overline{\Sigma}(X) = \Phi(X)$?; and second, which classes $\overline{C}$ in $\Phi(X)$ are effective and what do the curves in $[C]$ look like? As a corollary, we give a numerical characterization of $\overline{\Sigma}(X)$ as a subset of $\Phi(X)$.

The answer to the first question is suggested by a formula due to Tate [12] and Shioda [11]. The set $\Sigma(X)$ is naturally a finitely generated abelian group-for details see §3. If k_b is the number of components in the fibre F_b, their formula is

$$\text{rank } NS(X) = 2 + \text{rank } \Sigma(X) + \sum_{b \in B} (k_b - 1).$$

We will prove this (Corollary 3.7) as a corollary of our study of $\Phi(X)$. For the present, it motivates defining X to be f-irreducible if every fibre of X is an irreducible curve. This condition is easily seen to be generic for elliptic surfaces over a fixed base curve B but we shall not need this fact. The basic property of f-irreducible X is expressed in:

LEMMA 2.1: *If X is f-irreducible and C is an effective curve on X satisfying $CF = 1$, then $C^2 = -\chi(X)$ if and only if C is irreducible.*

PROOF: Write $C = \Sigma_{i=0}^{k} C_i + \Sigma_{j=1}^{l} F_j$ with each F_j a fibre and each C_i an irreducible curve not equal to a fibre. Since X is f-irreducible, $C_i \cdot F > 0$ for each i. Thus $C \cdot F = 1$ implies $k = 0$ and $C_0 \cdot F = 1$. Since C_0 is irreducible, it is therefore a section; hence $C_0^2 = -\chi$. But then $C^2 = -\chi(X) + 2l$ so $C^2 = -\chi$ if and only if $l = 0$; this in turn is equivalent to the irreducibility of C.

Now let us turn to our second question for a moment: we claim that every divisor class in $\Phi(X)$ is effective. More precisely, let us define a divisor D to be unifibral if the support of D is contained in a single reducible fibre F_b and, following §1, let us call such a D barely effective if D is effective but $D - F_b$ is not. Then

THEOREM 2.2: *If* $\overline{C} \in \Phi(X)$, *then* $\overline{C}$ *is effective and* $[C] = S + \Sigma_{i \in I} D_i + [\alpha F]$ *where* S *is a section of* X, *the curves* D_i *are barely effective curves supported on distinct reducible fibres of* X *and* $\alpha = \Sigma_{i \in I}(D_i)^2/2$. *That is, the curve* S *and the curves* D_i *are base curves of the algebraic system* $[C]$ *and the moving part of* $[C]$ *consists of* α *fibres of* X.

COROLLARY 2.2. (i) *If* X *is irreducible*, $\overline{\Sigma}(X) = \Phi(X)$.

(*ii*) $\overline{\Sigma}(X)$ *is the set of classes* $\overline{C}$ *in* $\Phi(X)$ *such that* $[C]$ *contains an irreducible curve.*

In fact, the converse of part (i) is true but the proof of this must wait until §3.

PROOF OF 2.2: First observe that $h^2(X, C + kF) = h^0(X, K_X - C - kF) = 0$ for any k since $(K_X - C - kF) \cdot F = -1$. Thus by Riemann-Roch, $h^0(X, C + kF) \geqslant \frac{1}{2}(C + kF)^2 - K_X(C + kF)) + \chi = k - g + \chi$. (We used Lemma 1.1 and the hypothesis $\overline{C} \in \Phi(X)$.) If $k \geqslant g$, then $C + kF$ is effective. Pick a curve $C' = \Sigma_{i=0}^{l} C_i + \Sigma_{j=1}^{m} D_j \in [C + kF]$ where each C_i is irreducible and does not lie in a fibre of X and each D_j is a component of a fibre of X. Using $C' \cdot F = 1$, we see as in the proof of (2.1) that $l = 0$ and C_0 is a section of X. We write S for C_0.

There is a unique expression $\overline{C}' = \overline{S} + \Sigma_{i=1}^{l} \overline{D}_i' + \overline{\alpha F}$ such that

(i) the D_i' are supported on distinct reducible fibres of X
(ii) $S \cdot D_i' = 0$ for each i.

The first condition determines each D_i' up to a multiple of F and the second condition then specifies the D_i''s and α uniquely. Taking self-intersection on both sides of $\overline{C} + k\overline{F} = \overline{S} + \Sigma_{i=1}^{l} \overline{D}_i' + \alpha \overline{F}$ gives $2k = \Sigma_{i=1}^{l} D_i'^2 + 2\alpha$, so $\overline{C} = \overline{S} + \Sigma_{i=1}^{l}((\overline{D}_i' - (D_i'^2/2)\overline{F})$. (As in 1, all the $D_i'^2$ are even.) But by Corollary 1.6, each divisor class $\overline{D}_i' - (D_i'^2/2)\overline{F}$ is effective, hence $\overline{C}$ is effective. If C and C' are in $[\overline{C}]$, then the argument above shows that $C = S + D$ and $C' = S' + D'$ with S and S' sections and D and D' effective curves supported in fibres. If F is a smooth fibre of X and $\sim$ denotes linear equivalence on F, $S \cdot F \sim (\mathcal{O}_X(C) \mid_F \sim \mathcal{O}_X(C') \mid_F \sim S' \cdot F$. Linearly equivalent effective divisors of degree one on a curve of genus one are equal so $S \cdot F = S' \cdot F$. Since S and S' are sections, $S = S'$.

Any curve $C \in [\overline{C}]$ will now have a unique expression of the form $C = S + D + \alpha F$ where D is a sum of barely effective unifibral divisors. If D' is another such divisor, then in every fibre of X there is a component whose multiplicity in $D - D'$ is non-negative and one whose multiplicity is non-positive. Thus $\overline{D} - \overline{D}'$ can be a multiple of a fibre only if $D = D'$. If $C' = S + D' + \alpha F$ is another curve in $[\overline{C}]$ we find

$(\alpha - \alpha')\overline{F} = \overline{D}' - \overline{D}$ hence $D = D'$ and $\alpha = \alpha'$. This completes the proof. Note that unlike the classes in $\overline{\Sigma}(X)$ those in $\Phi(X)$ may move in arbitrarily large linear systems.

COROLLARY 2.4: (*i*) *If S is a section of X and D is any divisor supported on the fibres of X and satisfying $2S \cdot D + D^2 = 0$, then $\overline{D}$ is effective.*

(*ii*) *Let $(\{D_i\}, \tilde{\Lambda}, \langle\ ,\ \rangle)$ be the data of one of the graphs $\tilde{\Gamma}$ let D_0 be a component of multiplicity one (i.e. $\lambda_0 = 1$) and let $\overline{D} = \sum_{i=0}^{k} a_i \overline{D}_i$. If $\langle \overline{D}, \overline{D} \rangle = -2a_0$, then each a_i is non-negative.*

PROOF: (i) The class $\overline{C} = \overline{S} + \overline{D}$ lies in $\Phi(X)$. By twisting by fibres, one sees that S is the base section of the linear system $[\overline{C}]$. Since $\overline{C} - \overline{S}$ is effective so is $\overline{D}$. Now (ii) is just the translation of (i) into the language of §1.

We conclude this section by giving a numerical characterization of the classes $\overline{\Sigma}(X)$ in $\Phi(X)$.

PROPOSITION 2.5: *If $\overline{C} \in \Phi(X)$, then $\overline{C} \in \overline{\Sigma}(X)$ if and only if $C \cdot D \geqslant 0$ for every component D of each fibre of X.*

PROOF: If $\overline{C}$ is the class of a section, then each $C \cdot D \geqslant 0$ since C and D are distinct, effective and irreducible. If $\overline{C} \in \Phi(X)$ but $\overline{C} \notin \overline{\Sigma}(X)$, then by Theorem 2.2 there is a curve in $[C]$ of the form $S + \sum_{i \in I} D_i + \alpha F$ with each D_i barely effective. If I is empty, then $\alpha = 0$ so $C = S$; hence I is not empty. Now $C \cdot D_i = S \cdot D_i + (D_i)^2 \leqslant -(S \cdot D_i) \leqslant 0$ using Corollary 1.8 (ii). If $S \cdot D_i < 0$, then $C \cdot D_i < 0$ and if $S \cdot D_i = 0$ then $C \cdot D_i = D_i^2 \leqslant -2$. In either case, since D_i is an effective sum of components of a fibre of X, C must meet some component in the support of D_i negatively.

§3. Group laws on $\Sigma(X)$ and $\Phi(X)$

In this section, we let O be a distinguished section of X. We will first show that $\Phi(X)$ and $\Sigma(X)$ re a natural set of coset representatives for certain sublattices of $NS(X)$ depending on O. Then we describe the group structures so induced on $\Phi(X)$ and $\Sigma(X)$. In particular, the group structure on $\Sigma(X)$ is that given by fibrewise addition: taking $O \cdot F_b$ as origin on F_b we add sections by adding their intersections with each fibre in the corresponding elliptic curve.

The lattices which concern us are $U(X)$, the span in $NS(X)$ of $\overline{O}$ and $\overline{F}$ and $L(X)$, the span of $\overline{O}$ and of all divisor classes $\overline{D}$ supported in fibres of X. We let R be the set of all components D of fibres of X such that $D \cdot O = 0$. The set R is linearly independent in $NS(X)$ and if $K(X)$ denotes its span then $L(X) = U(X) \oplus K(X)$. Our first main result in this section is:

THEOREM 3.1: (1) $\Phi(X)$ *is a set of coset representatives for $U(X)$ in $NS(X)$.*

(2) $\Sigma(X)$ *is a set of coset representatives for $L(X)$ in $NS(X)$.*

PROOF: Given a class $\bar{C} \in NS(X)$, let $\bar{C}' = \bar{C} + \beta\bar{O}$. Then $C' \cdot F = C \cdot F + \beta$ so if we set $\beta = 1 - C \cdot F$ then $C' \cdot F = 1$. Now let $\bar{C}'' = \bar{C}' + \alpha F$. Then $C'' \cdot F = 1$ and $(C'')^2 = (C')^2 + 2\alpha$ so $(C'')^2 = -\chi$ if and only if $\alpha = (-\chi - (C')^2)/2$. To prove (1), we must show that this α is integral. We therefore compute

$$\begin{aligned}(C')^2 + \chi &\equiv (C')^2 + (C' \cdot F)\chi \\ &\equiv (C')^2 + C' \cdot F(\chi + 2g - 2) \\ &\equiv (C')^2 + C' \cdot K_X \\ &\equiv 2P_a(C') + 2 \\ &\equiv 0 (\mathrm{mod}\ 2).\end{aligned}$$

To see (2), fix $\bar{C} \in NS(X)$. Write the irreducible components of each fibre F_b as $D_0, D_1, \dots, D_k$ with D_0 chosen so that $O \cdot D_0 = 1$ and $D_1, \dots, D_k$ spanning the lattice Λ_b of §1. Since the intersection pairing on Λ_b is non degenerate there is a unique $D_b \in \Lambda_b$ such that $D_b \cdot D_i = C \cdot D_i$ for $i > 0$. Then $\bar{C}' = \bar{C} - \sum_{b \in B} \bar{D}_b$ has intersection number zero with any element of $K(X)$. This property is not affected by adding multiples of $\bar{O}$ or $\bar{F}$ so arguing as above we may find a unique $\bar{C}''$ congruent to $\bar{C}'$ modulo $U(X)$, hence to $\bar{C}$ modulo $L(X)$ which both lies in $\Phi(X)$ and is orthogonal to $K(X)$. Since $C'' \cdot F = 1$ but $\bar{C}''$ is orthogonal to $K(X)$, $\bar{C}''$ has non-negative intersection number with every component of every fibre of X. Since $\bar{C}'' \in \Phi(X)$, $\bar{C}'' \in \bar{\Sigma}(X)$ by Proposition 2.5.

By transport of structure, the additions in $NS(X)/U(X)$ and $NS(X)/L(X)$ induce abelian group laws on $\Phi(X)$ and $\bar{\Sigma}(X)$. We wish to denote by $\oplus$ both these operations. This risks some confusion since the operation on $\bar{\Sigma}(X)$ is *not* obtained by restriction from that on $\Phi(X)$ even though $\bar{\Sigma}(X) \subset \Phi(X)$. (In fact, since $U(X) \subset L(X)$, $\bar{\sigma}(X)$ is naturally a quotient of $\Phi(X)$). We resolve this by identifying the sections in $\Sigma(X)$ and their classes in $\bar{\Sigma}(X)$ as usual, and then using $\oplus$ to denote the transported operations on $\Sigma(X)$ and on $\Phi(X)$. Then $\bar{S}_1 \oplus \bar{S}_2$ is a sum in $\Phi(X)$, $\overline{S_1 \oplus S_2}$ the class of a sum in $\Sigma(X)$. Our next goal is to relate the operation $\oplus$ to the addition in $NS(X)$ itself. We begin with $\Phi(X)$.

PROPOSITION 3.2: *If $\bar{C}_1$ and $\bar{C}_2$ are in $\Phi(X)$, then $\bar{C}_1 \oplus \bar{C}_2 = \bar{C}_1 + \bar{C}_2 - \bar{O} + \alpha\bar{F}$ where $\alpha = (C_1 + C_2) \cdot O - C_1C_2 + \chi$.*

PROOF: By definition, $\bar{C}_1 \oplus \bar{C}_2$ is the unique element of $NS(X)$ which lies in $\Phi(X)$ and is congruent to $\bar{C}_1 + \bar{C}_2$ modulo $U(X)$. The class

$\overline{C} = \overline{C}_1 + \overline{C}_2 - \overline{O} + \alpha \overline{F}$ satisfies the congruence and the equation $C \cdot F = 1$ for any α. The remaining condition $C^2 = -\chi$ holds exactly for the given α.

Observe that the sublattice $U(X)$ is unimodular so that that $NS(X)/U$ and $U^{\perp}$ are canonically isomorphic. We let ψ denote the induced isomorphism ψ: $U^{\perp} \to \Phi(X)$, which we think of as linearizing the operation $\oplus$ on $\Phi(X)$.

COROLLARY 3.3: (1) $\Phi(X)$ *is torsion free of rank equal to* rank $NS(X) - 2$.
(2) $\psi(\overline{C}) = \overline{O} + \overline{C} - (C^2/2)\overline{F}$.

PROOF: Since $\Phi(X) \cong U^{\perp} \subset NS(X)$, all of (1) will follow if we show that $NS(X)$ is torsion free. To see this, suppose that $\overline{T}$ is a torsion class in $NS(X)$. Then $h^0(X, \overline{T}) = 0$ and $\overline{T} \cdot \overline{C} = 0$ for all $\overline{C} \in NS(X)$. Using Riemann Roch we find

$$h^0(X, \overline{T}) - h^1(X, \overline{T}) + h^2(X, \overline{T}) = \left(\overline{T}^2 - \overline{K}_X \cdot \overline{T}\right)/2 + \chi$$

$$= \chi > 0.$$

Therefore $h^0(X, \overline{K}_X - \overline{T}) = h^2(X, \overline{T}) > 0$. Pick a curve $C \in [\overline{K}_X - \overline{T}]$. Since $C \cdot F = (K_X - T) \cdot F = 0$, C is supported in the fibres of X. If D is any irreducible component of a fibre of F, then using Lemma 1.1, $C \cdot D = (K_X - T) \cdot D = 0$ hence $\overline{C} = \alpha \overline{F}$ for some α. If S is a section of X, then $C \cdot S = K_X \cdot S$ so $\alpha = (2g - 2) + \chi$. Therefore $\overline{C} = \overline{K}_X$ and $\overline{T} = 0$.

To see (2), define $\bar{\psi}(\overline{C}) = \overline{O} + \overline{C} - (C^2/2)\overline{F}$. A quick check shows that $\bar{\psi}(\overline{C}) \cdot F = 1$ and $\bar{\psi}(\overline{C})^2 = -\chi$, hence that $\bar{\psi}(\overline{C}) \in \Phi(X)$. Conversely, if $\overline{C} \in \Phi(X)$ then $\overline{C}' = \overline{C} - \overline{O} - (C \cdot O + \chi)\overline{F} \in U^{\perp}$ and $(C')^2 = -2(\chi + C \cdot O)$. Therefore $\psi(\overline{C}') = \overline{C}' + \overline{O} + (\chi + C \cdot O)\overline{F} = \overline{C}$, so $\bar{\psi}$ is onto. To check that $\bar{\psi} = \psi$, it remains only to observe that $\bar{\psi}(\overline{C}) \equiv \overline{C}$ modulo $U(X)$.

Now we turn on our attention to $\Sigma(X)$. We begin by recalling the definition of the fibre-by-fibre addition of of sections of X. The set $(F_b)_{n.s}$ of non-singular points of each fibre F_b is naturally a principal homogeneous space for a commutative algebraic group (cf [4]). The point $O_b = O \cdot F_b$ is non-singular on F_b since O is a section. Taking this point as origin fixes a group operation on $(F_b)_{n.s}$ which we denote $+_b$. If $\sim_b$ denotes linear equivalence on F_b and the usual $+$sign denotes addition of divisors, the operation $+_b$ is characterized by the requirement that $P +_b Q$ solve $(P +_b Q) - O_b \sim_b (P - O_b) + (Q - O_b)$ or equivalently $(P +_b Q) \sim_b P + Q - O_b$. Now we get a first characterization of $\oplus$ on $\Sigma(X)$ by

PROPOSITION 3.4: (1) $\overline{S_1 \oplus S_2} = \bar{S}_1 + \bar{S}_2 - \bar{O} + \bar{D}(S_1, S_2)$ *where* $\bar{D}(S_1, S_2)$ *is a divisor supported on the fibres of* X.

(2) $(S_1 \oplus S_2) \cdot F_b = (S_1 \cdot F_b) +_b (S_2 \cdot F_b)$. *That is on* $\Sigma(X)$, $\oplus$ *the operation of fibre-by fibre addition of sections with* O *as origin.*

PROOF: Since $\overline{S_1 \oplus S_2}$ is congruent to $\bar{S}_1 + \bar{S}_2$ modulo $L(X)$ we must have $\overline{S_1 \oplus S_2} = \bar{S}_1 + \bar{S}_2 + \alpha\bar{O} + \bar{D}$, with $\bar{D}$ supported on the fibres of X. Since $\overline{S_1 \oplus S_2} \in \bar{\Sigma}(X)$, $(S_1 \oplus S_2) \cdot F = 1$ so $\alpha = -1$ as in (1). Now using the adjunction formula and the triviality of K_X restricted to F_b, we see that (1) restricts to $(S_1 \oplus S) \cdot F_b \sim_b S_1 - F_b + S_2 \cdot F_b - O_b$ on F_b which proves (2).

COROLLARY 3.5: (*Tate-Shioda*) *If* k_b *is the number of components in the fibre* F_b, *then*

$$\rho = \text{rank } NS(X) = 2 + \text{rank } \Sigma(X) + \sum_{b \in B} (k_b - 1).$$

PROOF: Since $L(X) \cong U(X) \oplus K(X)$, rank $U(X) = 2$, and rank $K(X) = \sum_{b \in B} (k_b - 1)$ this follows immediately from the isomorphism $\Sigma(X) \cong NS(X)/L(X)$.

REMARK: Of course, in the original statements of this formula the implied operation on $\Sigma(X)$ was that of fibrewise addition. In this light, the isomorphism $\Sigma(X) \cong NS(X)/L(X)$ may be seen as a more precise version of the Tate-Shioda formula taking torsion into account. In fact, one of our primary motivations in this work was to better understand this beautiful formula.

To complete the description of $\oplus$ on $\bar{\Sigma}(X)$, we wish to to identify the divisor class $\bar{D}(S_1, S_1)$ of (3.4). Some preliminary definitions will be required. If $S \in \Sigma(X)$, write $D_b(S)$ for the component of the fibre F_b meeting S. Next for any component D of F_b we define a divisor $\Delta(D)$ as follows. If $D = D_b(O)$ then $\Delta(D) = 0$. Otherwise $\Delta(D)$ is the unique barely effective divisor supported on F_b for which $\Delta(D) \cdot D = 1$, $\Delta(D) \cdot D_b(O) = -1$ and $\Delta(D) \cdot D' = 0$ for all other irreducible components D' of F_b.

Now observe that $D_b(O)_{n.s.}$ is the connected component of the identity O_b on $(F_b)_{n.s.}$ Therefore there is a group law, which we denote $+_b$, on the components of F_b: $D +_b D' = D''$ if $P +_b P' \in D''$ whenever $P \in D_{n.s.}$ and $P' \in D'_{n.s.}$. The components of multiplicity one in F_b from a subgroup with respect to $+_b$. Observe that, since $(S_1 \oplus S_2) \cdot F_b = (S_1 \cdot F_b) +_b (S_2 \cdot F_b)$, $D_b(S_1 \oplus S_2) = D_b(S_1) +_b D_b(S_2)$. Therefore if we set $D_b(S_1, S_2) = \Delta(D_b(S_1) +_b D_b(S_2)) - \Delta(D_b(S_1)) - \Delta(D_b(S_2))$, then $\overline{S_1 \oplus S_2}$ and $\bar{S}_1 + \bar{S}_2 - \bar{O} + \bar{D}_b(S_1, S_2)$ have the same intersection number with every component of F_b. Next let $\overline{D}(S_1, S_2) = \sum_{b \in B} \overline{D}_b(S_1, S_2)$ and let $\overline{T} = \overline{S}_1 + \overline{S}_2 - \overline{O} + \overline{D}(S_1, S_2)$. Then $\overline{S_1 \oplus S_2}$ and $\overline{T}$ have the same intersection number with every component of every fibre of X. Finally let $\overline{T}' = \overline{T} - ((T^2 + \chi)/2)\overline{F}$; as in (3.1.1) one checks that $T^2 \equiv \chi (\text{mod } 2)$. Then $T' \cdot F = 1$ and $(T')^2 = -\chi$ so $T' \in \Phi(X)$. But if D is any component of a fibre, $T' \cdot D = T \cdot D = (S_1 \oplus S_2) \cdot D \geqslant 0$ so $\overline{T}' \in \overline{\Sigma}(X)$ by Proposition 2.5. Since $T' \equiv (S_1 + S_2)$ modulo $L(X)$, $T' = S_1 \oplus S_2$. To summarize,

PROPOSITION 3.6: $\overline{S_1 \oplus S_2} = \overline{S}_1 + \overline{S}_2 - \overline{O} + \overline{D}(S_1, S_2) + \beta \overline{F}$ *where*

(*i*) $D(S_1, S_2) = \sum_{b \in B} D_b(S_1, S_2)$
(*ii*) $D_b(S_1, S_2) = \Delta(D_b(S_1) +_b D_b(S_2)) - \Delta(D_b(S_1)) - \Delta(D_b(S_2))$
(*iii*) $\beta = O \cdot (S_1 + S_2) - S_1 S_2 - (S_1 + S_2 - O - D(S_1, S_2)/2) \cdot D(S_1, S_2) + \chi$.

We conclude this section with a few remarks on the relation between $\Phi(X)$ and $\Sigma(X)$. The inclusion $U(X) \subset L(X)$ induces a surjection τ: ${}^{NS(X)}/U(X) \to {}^{NS(X)}/L(X)$ and hence a surjection σ: $\Phi(X) \to \Sigma(X)$.

PROPOSITION 3.7: (1) *The map σ sends a class $\overline{C}$ to the unique section S in the base locus of the system* $[C]$.
(2) $\ker \sigma = \{\overline{O} + \overline{D} - (D^2/2)\overline{F} \mid \overline{D} \in K(X)\}$.

PROOF: By Theorem 2.3 any class $\overline{C} \in \Phi(X)$ has the form $\overline{C} = \overline{S} + \overline{D}'$ with S a section lying in the base locus of $[\overline{C}]$ and $\overline{D}'$ supported on the fibres of X. Since D' therefore lies on $L(X)$, $\tau(\overline{C}) = \tau(\overline{S})$ and hence $\sigma(\overline{C}) = S$. If we write $\overline{D}' = \overline{D} + \alpha \overline{F}$ with $\overline{D} \in K(X)$ then as above $\alpha = -(D^2/2)$. If $\overline{C} \in \ker \sigma$ then $S = O$ so $\overline{C} = \overline{O} + \overline{D} - (D^2/2)\overline{F}$ as claimed

COROLLARY 3.8: *X is f-irreducible if and only if $\overline{\Sigma}(X) = \Phi(X)$.*

PROOF: If $\overline{\Sigma}(X) = \Phi(X)$, then $\ker \sigma$ is trivial. Therefore so is $K(X)$ i.e. all fibres of X are irreducible. The reverse implication is proved in Corollary 2.2.1.

§4. Applications: The rational case

The results of §3 provide a uniform method for obtaining the coordinates of elements of $\Phi(X)$ in terms of a given basis of $NS(X)$. If in addition the types of the reducible fibres of X are known then we can also coordinatize $\Sigma(X)$ and describe its torsion subgroup. In this paragraph we wish to illustrate these methods in the case when X is rational.

We recall that a rational elliptic surface X with section is obtained by blowing up $\mathbb{P}^2$ at the nine base points (some possibly infinitely near) of a pencil of plane cubic curves whose generic member is smooth. Since $\chi(\mathcal{O}_X) = -1$, $\Phi(X)$ is the set of classes C satisfying $C \cdot F = 1$ and $C^2 = -1$ and $\Sigma(X)$ is the set of exceptional curves of the first kind on X. The group $NS(X)$ has as basis $\overline{H}$, the pullback to X of the class of a line in the plane and $\overline{E}_1, \dots, \overline{E}_9$, the classes of the nine exceptional divisors of the blow-ups (i.e. $\overline{E}_i^2 = -1$). The $\overline{E}_i$ are all in $\Phi(X)$ but they need not all be exceptional curves of the first kind; they will fail to be such exactly when the pencil of cubics has infinitely near base points. However, at least the exceptional divisor associated to the last blow-up will be of the first kind, and hence will be a section of X. To keep our notation consistent with that of [7] and [9] we suppose $\overline{E}_1$ is a section and take it as origin section O. As in §3, let $\overline{U}$ be the span of $\overline{O}$ and $\overline{F}$ in $NS(X)$ and let ψ: $\overline{U}^{\perp} \to \Phi(X)$ be the natural isomorphism. The inverse of ψ sends $\overline{E}_i$, $i \geqslant 2$ to $\overline{T}_i = -3\overline{H} + \overline{E}_i + \Sigma_{j \geqslant 2} \overline{E}_j$. Let Λ be the sublattice of $\overline{U}^{\perp}$ spanned by the $\overline{T}_i$.

LEMMA 4.1: *(i) The intersection form on Λ is given by $T_i^2 = -2$ and $T_i \cdot T_j = -1$ if $i \neq j$.*
(ii) $\overline{U}^{\perp}/\Lambda \cong \mathbb{Z}/3\mathbb{Z}$.

PROOF: We leave (i) to the reader. To see (ii), we compute the determinant of the intersection form on Λ. By (i), this form has matrix

$$\begin{bmatrix} -2 & -1 & \dots & \dots & -1 \\ -1 & -2 & & & -1 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & & & -2 & -1 \\ -1 & \dots & \dots & -1 & -2 \end{bmatrix}$$

First, subtract in succession each row from the one above proceeding from top to bottom. Then add each column to the one on its left proceeding from right to left. We arrive at $\begin{pmatrix} 0 & I_7 \\ 9 & 0 \end{pmatrix}$ whence the determinant is 9. Thus Λ^*/Λ has order 9. But $NS(X)$, $\overline{U}$ and $\overline{U}^{\perp}$ are unimodular so $\Lambda^*/\overline{U}^{\perp} \cong \overline{U}^{\perp}/\Lambda \cong \mathbb{Z}/3\mathbb{Z}$.

We can now immediately recover the enumeration of $\Phi(X)$ due to Manin [7]. If $\overline{C} \in \Phi(X)$ let $ⓐ\overline{C} = \underbrace{\overline{C} \oplus \dots \oplus \overline{C}}_{\text{a times}}$ and if $A = (a_2, \dots, a_9) \in \mathbb{Z}^8$ let $\overline{E}^A = \oplus_{i=2}^{9} ⓐ_i \overline{E}_i$ and let $\overline{T}_A = \Sigma_{i=2}^{9} a_i \overline{T}_i$. If $\xi = \psi(\Lambda)$, then ξ is the subgroup of $\Phi(X)$ spanned by $\overline{E}_2, \dots, \overline{E}_9$, and $\Phi(X)/\xi \cong \overline{U}^{\perp}/\Lambda \cong \mathbb{Z}/3\mathbb{Z}$. Moreover $\overline{E}^A = \psi(\overline{T}^A) = \overline{O} + \overline{T} - (T^2/2)\overline{F}$ by (3.3.2). Picking coset representatives for $\Phi(X)/\xi$ and using the addition formula (3.3) yields coordinates for all of $\Phi(X)$. We omit the actual formulae which may be found in [7] or [9]. We stress that this method does not require X to be rational. What is needed is a basis of $NS(X)$ and the coordinates of $\overline{F}$ and of one $\overline{O} \in \overline{\Sigma}(X)$ in this basis.

We turn now to the question of identifying $\overline{\Sigma}(X)$ in $\Phi(X)$. Here we need a description of the reducible fibres of X. We will restrict our attention to the case when $\Sigma(X)$ is finite. The complete list of such surfaces may be found in [6].

Let us sketch the details of the identification of $\Sigma(X)$ when X has four reducible fibres of type I_3. A pencil of cubics has 4 singular fibres of type I_3 if and only if the pencil is spanned by a non-singular cubic and its hessian or equivalently if the base points of the pencil are the nine flexes of a non-singular cubic (which is thus in the pencil). Such pencils have the beautiful property that the hessian of every member is again in the pencil, so their base points are flexes for every curve in the pencil. For more details see Beauville [2].

Let $G = \mathbb{P}^2(\mathbb{F}_3)$ so $|G| = 13$ and let I be the line at infinity in G. Let $J = G - I$, a set of 9 points, and let $\check{J} = \check{G} - \{I\}$, a set of 12 lines l. This $(9_4 : 12_3)$ configuration is also realized by the flexes of a cubic: there are bijections between J and the set of flexes P, and between $\check{J}$ and the set of lines L joining flexes so that $p \in l$ if and only if $P \in L$. For each of the four points $\delta \in I$, there are three lines d_{δ^1}, d_{δ^2}, and d_{δ^3} in J through δ and the corresponding triangle D_{δ^1}, D_{δ^2}, D_{δ^3} in $\mathbb{P}^2$ is one of the reducible fibres of the pencils of cubics with the P's as base points.

By Proposition 2.5, a section S of X must meet all components of the reducible fibres non-negatively. If, say, $S \cdot D_{\delta^1} = 1$, $S \cdot D_{\delta^2} = S \cdot D_{\delta^3} = 0$, then since S is effective it must blow down to a point not in D_{δ^2} or D_{δ^3} in $\mathbb{P}^2$. Since $S \cdot F = 1$, S is the exceptional divisor over one of the flexes on D_{δ^1}. On the other hand, since the 9 flexes are distinct, each of the exceptional divisors is a section. Picking, say, E_1 as origin in $\Sigma(X)$ corresponds to using the base point P_1 as origin on each cubic. Then if P_i and P_j are two other flexes $P_i +_b P_j$ is another flex P_k independent of b. Hence $\Sigma(X)$ is isomorphic to the group of flexes on any element of the pencil: $\Sigma(X) \cong (\mathbb{Z}/3\mathbb{Z})^2$.

We conclude this example by showing how to coordinatize $\Phi(X)$ when $\Sigma(X)$ is finite using Proposition 3.7. Let O be the origin section, let P be the corresponding flex and Let, say, D_{δ^1} be the component of each reducible fibre which O intersects. Let $V_\delta = \operatorname{span}\{\overline{D}_{\delta^2}, \overline{D}_{\delta^3}\}$ in $NS(X)$. The intersection form on each V_δ is given by $(a, b) \to -2a^2 - 2b^2 + 2ab$ in this basis. If σ: $\Phi(X) \to \underline{\Sigma(X)}$ is the map of (3.7), a typical element of $\ker \sigma$ has the form $\overline{O} + \overline{D} - (D^2/2)\overline{F}$ where $\overline{D} \in \overline{K}(X) = \oplus_\delta V_\delta$, hence looks like

$$\overline{O} + \sum_\delta \left(a_\delta \overline{D}_{\delta^2} + b_\delta \overline{D}_{\delta^3}\right) - \sum_\delta \left(-a_\delta^2 - b_\delta^2 + a_\delta b_\delta\right) \overline{F}.$$

If E_p is the exceptional divisor over a flex P, then $\overline{D}_{\delta'} = \overline{H} - \sum_{p \in d_{\delta'}} E_p$. We may therefore substitute for the D_i's and for F to get a coordinatization of ker σ. The other 8 cosets of ker σ in $\Phi(X)$ are obtained by varying the origin section S.

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