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DEDEKIND SUMS AND POWER RESIDUE SYMBOLS

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1. The main intention of this report is to discuss a conjecture about a relation between quadratic residue symbols in imaginary quadratic fields and Dedekind sums built up from elliptic functions, which were introduced in [11]. This conjecture generalizes the known relation between the Legendre symbol and the classical Dedekind sums, and also has some relation to the well-known conjectures of Stark and Birch-Swinnerton-Dyer about special values of L-series. In the easiest case the conjecture is as follows. Let K be an imaginary quadratic field of discriminant $\equiv 1(8)$. There is a canonical homomorphism Ψ (defined in Section 5) of $SL_2\mathcal{O}_K$ with values in $\mathcal{O}_H$, the ring of integers in the Hilbert classfield H of K. The conjecture says that the image module $\Psi(SL_2\mathcal{O}_K)$ reduced $\bmod(8\mathcal{O}_H)$ is a cyclic subgroup of $\mathcal{O}_H/8\mathcal{O}_H$, and that the homomorphism $SL_2\mathcal{O}_K \to \frac{1}{4}\pi i(\mathbb{Z}/8\mathbb{Z})$ induced in this way is the logarithm of a theta multiplier which occurs in the transformation theory of the theta series used by Hecke [6] to establish the quadratic reciprocity law in K. There are more general statements of a similar type but involving cocycles rather homomorphisms on $SL_2\mathcal{O}_K$, and valid for arbitrary imaginary quadratic fields.

I begin with a detailed review of the classical situation.

2. Dedekind sums are usually introduced as a multiplier of a modular form, but following Kronecker [9] we can introduce them most naturally as a logarithm of the Legendre symbol. The key point is the Lemma of Gauss, which gives a multiplicative decomposition of the Legendre symbol:

$$\left(\frac{p}{q}\right) = \prod_{r=1}^{q^*} s\left(\frac{pr}{q}\right), \quad q^* = \frac{q-1}{2}$$

for two relatively prime integers p, q ($q > 0$, odd), and the periodic function $s\colon \mathbb{R}/\mathbb{Z} \to \{\pm 1, 0\}$, given by $s(x) = \operatorname{sign}(x)$ for $|x| < 1/2$ (Gauss proved this only for a prime number q; the first general proof was given by the German mathematician Schering). Taking logarithms and observing that

$$\log s \equiv \frac{\pi i}{2}(1 - s) \bmod 2\pi i$$

for $s = \pm 1$, we get

$$\log\left(\frac{p}{q}\right) \equiv \frac{\pi i}{2}\left(q^* - \sum_{r=1}^{q^*} s\left(\frac{pr}{q}\right)\right) \bmod 2\pi i.$$

To develop this expression further we use the Fourier expansion

$$s\left(\frac{pr}{q}\right) = \frac{2}{\pi} \sum_{n \equiv 1(2)} \frac{\sin(2\pi npr/q)}{n},$$

and change the order of summation. Using the elementary identity

$$2 \sum_{r=1}^{q^*} \sin(2\pi npr/q) = \cot \pi\left(\frac{p-1}{2} + \frac{pn}{2q}\right),$$

we get in this way

$$\sum_{r=1}^{q^*} s\left(\frac{pr}{q}\right) = \frac{1}{\pi} \sum_{n \equiv 1(2)} \frac{\cot \pi\left(\frac{p-1}{2} + \frac{pn}{2q}\right)}{n}$$

$$= \frac{1}{\pi} \sum_{r(q)} \cot \pi\left(\frac{p-1}{2} + \frac{p(2r+1)}{2q}\right) \sum_{n \equiv 2r+1(2q)} \frac{1}{n}$$

$$= \frac{1}{2q} \sum_{r(q)} \cot \pi\left(\frac{p-1}{2} + \frac{p(2r+1)}{2q}\right) \cot \pi\left(\frac{2r+1}{2q}\right).$$

The last expression is called a Dedekind sum; the general definition

$$d(a, c, u, v) = \frac{1}{c} \sum_{r(c)}{}' \cot \pi\left(\frac{a(r+u)}{c} + v\right) \cot \pi\left(\frac{r+u}{c}\right)$$

makes sense for any integers a, c ($c \neq 0$) and complex numbers u, v (as usual, we indicate the omission of the meaningless elements in a sum by writing Σ' instead of Σ). With this notation we have therefore proved

THEOREM 1:

$$\log\left(\frac{p}{q}\right) \equiv \frac{\pi i}{4}\left(q - 1 - d\left(p, q; \frac{1}{2}, \frac{p-1}{2}\right)\right) \bmod 2\pi i.$$

In other words, the expression in brackets is always $0 \pmod 4$, *but it is* $0 \pmod 8$ *iff* $\left(\frac{p}{q}\right) = +1$.

This fact, though interesting in itself, has a deeper meaning. In the theory of modular forms, the Legendre symbol occurs as a multiplier of a theta series, and Dedekind sums as the (additive) periods of certain Eisenstein series. Theorem 1 says, therefore, that the multiplier of a theta series can be written as the exponential of a period of an Eisenstein series. But actually more is true. As noticed by Hecke [5], the theta series itself can be written as the exponential of the integral of an Eisenstein series. This fact is merely a reinterpretation of the beautiful triple product identity of Jacobi. Looking back, we can therefore interpret the Lemma of Gauss as a miniature version of the Jacobi identity. To be more specific, put

$$\Theta_{xy}(\tau) = \sum_{n\in\mathbb{Z}} (-1)^{nx} \, e^{\pi i(n+y/2)^2\tau}$$

for $x,\ y \in \{0, 1\}$ and $\tau \in \mathscr{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. These are the four "Thetanullwerte" in the notation of Hermite, but only Θ_{00}, Θ_{01}, Θ_{10} are of interest because Θ_{11} vanishes identically. Under the action of the full modular group SL(2, $\mathbb{Z}$), these functions are permuted as follows:

$$\Theta_{01}(\tau+1) = e^{\frac{1}{4}\pi i}\Theta_{01}(\tau),\ \Theta_{10}(\tau+1) = \Theta_{00}(\tau),\ \Theta_{00}(\tau+1) = \Theta_{10}(\tau),$$

$$\Theta_{01}\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\Theta_{10}(\tau),\ \Theta_{10}\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{\mathrm{i}}}\,\Theta_{01}(\tau),\ \Theta_{00}\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{\mathrm{i}}}\,\Theta_{00}(\tau)$$

with the principal value of the square root. And for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, $A \equiv 1(2)$ with $c > 0$ we have the following theorem of Hermite [7].

THEOREM 2:

$$\Theta_{xy}\left(\frac{a\tau+b}{c\tau+d}\right) = \chi\sqrt{\frac{c\tau+d}{\mathrm{i}}}\,\Theta_{xy}(\tau)$$

with an eighth root of unity $\chi = \chi_{xy}(A)$,

$$\chi = \left(\frac{c}{a(1+cx)}\right) \exp\left[\frac{\pi \mathrm{i}}{4}(a(1+cx) - b(d+2)y)\right].$$

In particular, for $A \equiv 1(8)$ we have $\chi = \left(\frac{c}{a}\right) e^{\frac{1}{4}\pi i}$. To prove this theorem, one uses the Poisson summation formula and then obtains an expression for χ involving a Gaussian sum. This is how the Legendre symbol enters the picture.

On the other hand, given real numbers u, v, we set

$$H(\tau, u, v) = c_0(u)\left[\tau\pi^2 c_2(v) - 4\pi i \log 2\right] + \pi \sum_{m \in \mathbb{Z}+u}' \left[\frac{\cot \pi(m\tau + v)}{m} + \frac{i}{|m|}\right]$$

with

$$c_0(u) := \begin{cases} -1, & u \in \mathbb{Z} \\ 0, & u \in \mathbb{R}\setminus\mathbb{Z} \end{cases} \quad \textit{and} \quad c_2(v) := \begin{cases} 1/3, & v \in \mathbb{Z} \\ \sin^{-2}\pi v, & v \in \mathbb{R}\setminus\mathbb{Z} \end{cases}.$$

Up to a constant, $H(\tau, u, v)$ is the integral $\int_\tau E(z)\, dz$ of the Eisenstein series

$$E(z) = \sum_{m \in \mathbb{Z}+u}' \sum_{n \in \mathbb{Z}+v}' (mz + n)^{-2}.$$

Taking the principal value of the logarithm, we have the following connection to the non-vanishing theta functions Θ_{00}, Θ_{10}, Θ_{01}:

THEOREM 3:

$$4\pi i \log \Theta_{xy}(\tau) = H\left(\tau, \frac{y+1}{2}, \frac{x+1}{2}\right).$$

This is a special case of a more general theorem of Hecke [5], and represents essentially a rewriting of Jacobi's triple product identity. By the way, in the excluded case $x = y = 1$ we have

$$4\pi i \log 2\eta(\tau) = H(\tau, 1, 1) = H(\tau, 0, 0)$$

with the well-known Dedekind eta-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}.$$

The behavior of $H(\tau, u, v)$ under the modular group is given by the following theorem [12].

THEOREM 4:

$$H\left(\frac{a\tau + b}{c\tau + d}, u, v\right) - H(\tau, u, v)$$
$$= 2\pi i \log \frac{c\tau + d}{i} + \pi^2\left(\frac{a+d}{c} c_0(u) c_2(v) + d(a, c; u, v)\right)$$

for $c > 0$ *and* $u \equiv au + cv$, $v \equiv bu + dv(1)$. The last condition imposes, of course, a restriction on the admissible values of u, v for a given matrix. A reciprocity formula for Dedekind sums can be derived from this theorem. We note here only the special case

$$d(a, c; \tfrac{1}{2}, \tfrac{1}{2}) + d(c, a; \tfrac{1}{2}, \tfrac{1}{2}) = -\operatorname{sign}(ac).$$

As a final application of all these formulas, we now give a second proof of Theorem 1. It follows from Theorem 2 that

$$\chi_{00}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{c}{a}\right) e^{\frac{1}{4}\pi i a}, \quad \textit{if} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1(2), \quad c > 0.$$

On the other hand, calculating χ with the help of Theorem 3 and 4, we get for $a, c > 0$,

$$\begin{aligned} \chi_{00}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \exp\left[\frac{\pi i}{4}\left(-d(a, c; \tfrac{1}{2}, \tfrac{1}{2})\right)\right] \\ &= \exp\left[\frac{\pi i}{4}\left(1 + d(c, a; \tfrac{1}{2}, \tfrac{1}{2})\right)\right], \end{aligned}$$

or

$$\left(\frac{c}{a}\right) = \exp\left[\frac{\pi i}{4}\left(a - 1 - d(c, a; \tfrac{1}{2}, \tfrac{1}{2})\right)\right].$$

This is the assertion of Theorem 1 for an even $p > 0$.

3. All the things we have discussed so far are connected with the modular group SL(2, $\mathbb{Z}$), and could be classified in modern terminology as part of the so-called Eisenstein cohomology of the group SL_2 over the field of rational numbers. Now we study Dedekind sums and theta series with respect to the group $\mathrm{SL}_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ denotes the integers of an imaginary quadratic field K. Though formally more complicated in this case, things become in a certain sense easier. One reason for this is that the transformation law of the theta series (suitably normalized) now can be written as

$$\Theta(A\tau) = \chi(A)\Theta(\tau)$$

with a fourth root of unity $\chi(A)$ for $A \in \Gamma(2)$,

$$\Gamma(2) = \{A \in \mathrm{SL}_2(\mathcal{O}_K) \mid A \equiv 1(2)\}.$$

Applying this law twice we conclude that

$$\chi(AB) = \chi(A)\chi(B),$$

if Θ does not vanish identically. In other words, Θ defines a homomorphism χ of $\Gamma(2)$ into the fourth roots of unity. For a matrix $A \equiv \pm 1(8)$ this homomorphism is given by

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{c}{a}\right),$$

where the symbol on the right hand side now denotes the Legendre symbol in K, defined for an integer $x \in \mathcal{O}_K$ and an odd prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ (i.e., $2 \notin \mathfrak{p}$) as usual by

$$\left(\frac{x}{\mathfrak{p}}\right) := -1 + \#\{y \bmod \mathfrak{p} \mid y^2 \equiv x(\mathfrak{p})\},$$

and extended to a multiplicative function $\left(\frac{x}{\mathfrak{a}}\right)$ in $\mathfrak{a}$. It is a highly remarkable fact, first noticed by Kubota [10], that the homomorphism property of χ is essentially equivalent to quadratic reciprocity in K. This implies, in particular, that the kernel of χ does not contain any congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$.

On the other hand, replacing the cotangent function in the definition of the Dedekind sum $d(a, c; u, v)$ by an appropriate elliptic function, we get Dedekind sums with respect to the group $\mathrm{SL}_2(\mathcal{O}_K)$; the exact definition will be given in the next section. The main point is that these new sums provide a supply of additive homomorphisms of a principal congruence subgroup $\Gamma(\mathfrak{a})$ into the complex numbers, $\mathfrak{a} \subset \mathcal{O}_K$ a nonzero ideal. We call them Eisenstein homomorphisms because they constitute the Eisenstein part in the usual decomposition

$$H^1(\Gamma(\mathfrak{a}), \mathbb{C}) = H^1_{\mathrm{Eis}}(\Gamma(\mathfrak{a}), \mathbb{C}) \oplus H^1_{\mathrm{cusp}}(\Gamma(\mathfrak{a}), \mathbb{C})$$

of the first cohomology group H^1 of $\Gamma(\mathfrak{a})$ (cf. [4]). The number of linearly inequivalent homomorphisms we get in this way equals the number of cusps of $\Gamma(\mathfrak{a})$, which is the class number of K in the case $\mathfrak{a} = \mathcal{O}_K$.

The basic question we are interested in can now be formulated as follows: is it possible to write every theta multiplier χ as $\chi = \exp \circ \Phi$ with a suitable Eisenstein homomorphism Φ: $\Gamma(2) \to \frac{1}{2}\pi i\mathbb{Z}$? Of course, we guess that the answer is always yes. We will discuss some numerical examples later, and this evidence will lead us to a much stronger conjecture.

After this survey we now give the definition of the theta series. The symmetric space for the group $\mathrm{SL}_2(\mathcal{O}_K)$ is the hyperbolic upper half-space H^3, which is most conveniently represented as the set of quaternions $\tau = z + jv$ ($z \in \mathbb{C}$, $v > 0$, $j^2 = -1$, $ij = -ji$) because the action of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathcal{O}_K)$ can be written in the familiar way

$$\tau \mapsto (a\tau + b)(c\tau + d)^{-1}.$$

Using this representation we define the theta series Θ_{xy} of the characteristic xy $(x,\ y \in \frac{1}{2}\mathcal{O}_K)$ by

$$\Theta_{xy}(\tau) := \sqrt{v} \sum_{\mu \in \mathcal{O}_K} \exp\left(-\frac{2\pi v \,|\, \mu + y \,|^2}{|\sqrt{D}\,|} + \pi \mathrm{i}\ \mathrm{tr}\left(\frac{(\mu + y)^2 z + 2\mu x}{\sqrt{D}} \right) \right),$$

where D is the discriminant of K, and tr denotes the trace map $\mathbb{C}/\mathbb{R}$. These functions were introduced by Hecke in his book [6] to prove the quadratic reciprocity law in K,but Hecke did not mention that they are automorphic functions for the group $\Gamma(2)$. The characteristic of a theta series is called odd (resp. even) if the number $\mathrm{tr}(4xy/\sqrt{D})$ is odd (resp. even). Shifting μ to $-\mu - 2y$ in the definition of Θ_{xy} we get

$$\Theta_{xy}(\tau) = \Theta_{xy}(\tau) \exp\left(-\pi \mathrm{i}\ \mathrm{tr}\left(\frac{4xy}{\sqrt{D}} \right) \right),$$

so Θ_{xy} vanishes identically if the characteristic is odd. For an even characteristic, Θ_{xy} is known to be a non-constant function. It is easy to check

$$\Theta_{x'y} = \Theta_{xy},\ \Theta_{xy'} = \Theta_{xy} \exp\left(-\pi \mathrm{i}\ \mathrm{tr}\left(\frac{2xw}{\sqrt{D}} \right) \right) \tag{7}$$

for $x' = x + w$, $y' = y + w$, and $w \in \mathcal{O}_K$. Therefore, we get by the definition above only 10 essentially different theta functions which do not vanish identically. To give the transformation law for these functions under the subgroup $\Gamma(2) \subset \mathrm{SL}_2(\mathcal{O}_K)$ we first note

$$\Theta_{xy}(\tau + b) = \Theta_{xy}(\tau) \exp \pi \mathrm{i}\ \mathrm{tr}\left(\frac{by^2}{\sqrt{D}} \right) \quad \text{for } b \in 2\mathcal{O}_K,$$

which follows immediately from the definition. For a general substitution $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ we have

THEOREM 6: $\Theta_{xy}(A\tau) = \chi_{xy}(A)\Theta_{xy}(\tau)$ *with* $\chi_{xy}(A) = \varphi\psi$ *given by*

$$\varphi := \exp\left(-\pi \mathrm{i}\ \mathrm{tr}\left(\frac{2xy(a+1)+acx^2+b(d+2)y^2}{\sqrt{D}}\right)\right),$$

$$\psi := \left(\frac{c}{a}\right)\begin{cases} 1, & D \equiv 0(4) \\ \left(\frac{2}{a}\right), & a \equiv \pm 1(4),\ D \equiv 1(4) \\ \pm \mathrm{i}\left(\frac{2}{a\sqrt{D}}\right), & a \equiv \pm\sqrt{D}\,(4),\ D \equiv 1(4). \end{cases}$$

Note that ψ does not depend on xy, and therefore φ and ψ are both homomorphisms of $\Gamma(2)$. In the case $D \equiv 1(8)$ we even can describe the action of the full group $SL_2(\mathcal{O}_K)$ on these theta functions.

THEOREM 7: *For* $D \equiv 1(8)$ *and* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *in* $SL_2(\mathcal{O}_K)$ *we have*

$$\Theta_{xy}(A\tau) = \Omega\Theta_{x'y'}(\tau)$$

with $x' = by + dx + bd/2$, $y' = ay + cx + ac/2$, *and an eighth root of unity* $\Omega = \Omega_{xy}(A)$, *given by*

$$\Omega = \exp \pi\mathrm{i}\ \mathrm{tr}\left(\frac{aby^2}{\sqrt{D}}\right) \text{ if } c = 0, \text{ resp. if } c \neq 0,$$

$$\Omega = G \exp \pi\mathrm{i}\ \mathrm{tr}\left(\frac{a^2cd}{4\sqrt{D}} + \frac{(by+dx)(ay+cx+ac)-xy}{\sqrt{D}}\right)$$

where G is the Gauss sum

$$G = \frac{1}{4|c|}\sum_{\nu(2c)} \exp \pi\mathrm{i}\ \mathrm{tr}\left(\frac{a(1+c)}{c\sqrt{D}}\nu^2\right).$$

The value of G can be determined explicitly. If $c \equiv 1(2)$, then by the theorems proved in Hecke's book [6],

$$G = \left(\frac{-2a}{c}\right)\cdot\begin{cases} 1, & c \equiv \pm 1(4) \\ \pm\mathrm{i}, & c \equiv \pm\sqrt{D}\,(4). \end{cases}$$

If $c \not\equiv 1(2)$, then $a \equiv 1(2)$ or $a + c \equiv 1(2)$; the best way to deal with these cases is perhaps to reduce them to the case $c \equiv 1(2)$ by applying Theorem 7 twice to the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -c & -d \\ a+c & b+d \end{pmatrix},$$

and using (7). Note that $x'y'$ is again an even characteristic because $\Theta_{x'y'}$ does not vanish identically. So we have an action of $\mathrm{SL}_2\mathcal{O}_K$ on the even characteristics, and one verifies easily that there are two orbits under this action; (1/2, 1/2) constitutes one orbit, while the other nine characteristics constitute the second orbit. There is a bijection between the last nine characteristics xy and the primitive 2-division points (u, v) in $(\frac{1}{2}\mathcal{O}_K/\mathcal{O}_K)^2$, given by the map

$$xy \mapsto (u, v) = (y + \tfrac{1}{2}, x + \tfrac{1}{2}). \tag{8}$$

The induced action of $\mathrm{SL}_2\mathcal{O}_K$ on the 2-division points is the usual linear action, i.e.

$$x'y' \mapsto (u, v)A.$$

A second corollary from the definition of $A(xy) := x'y'$ is that

$$AB(xy) \equiv A(B(xy)) \bmod \mathcal{O}_K$$

for all $A, B \in \mathrm{SL}_2\mathcal{O}_K$, or

$$\Omega_{xy}(AB) = \pm\Omega_{xy}(A)\Omega_{x'y'}(B).$$

To get rid of the disturbing $\pm$ sign (which comes from (7)), let R be a complete representative system of all even characteristics mod $\mathcal{O}_K$. Thus, for any even characteristic xy there is an unique $\pi(xy) \in R$ with $\pi(xy) \equiv xy(\mathcal{O}_K)$. The eight root of unity $\chi_{xy}(A)$ defined by

$$\Theta_{\pi(xy)}(A\tau) = \chi_{xy}(A)\Theta_{\pi(x'y')}(\tau)$$

then has the cocycle property

$$\chi_{xy}(AB) = \chi_{xy}(A)\chi_{x'y'}(B), \quad A, B \in \mathrm{SL}_2\mathcal{O}_K.$$

However, χ depends on R. To have a definite choice, we prescribe the coordinates x, y of $xy \in R$ to be elements of

$$\left\{0, \frac{1}{2}, \frac{3+\sqrt{D}}{4}, \frac{3-\sqrt{D}}{4}\right\}, \quad \text{if } D \equiv 9(16)$$

$$\text{resp. of} \quad \left\{0, \frac{1}{2}, \frac{1+\sqrt{D}}{4}, \frac{1-\sqrt{D}}{4}\right\}, \quad \text{if } D \equiv 1(16).$$

One reason for this choice is the property $\chi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 1$ it implies. We will refer to the cocycle defined this way as the theta cocycle. The value

$\chi(A)$ of $A \in \mathrm{SL}_2\mathcal{O}_K$ under χ is the map which assigns to the even characteristic xy the eighth root of unity $\chi_{xy}(A)$. Note that the particular root

$$\chi_{\frac{1}{2}\frac{1}{2}}\begin{pmatrix} 1 & \dfrac{1+\sqrt{D}}{2} \\ 0 & 1 \end{pmatrix} = e^{\frac{1}{4}\pi i}$$

is different from 1. This implies that the cohomology class of χ is not trivial.

The function $\Theta_{\frac{1}{2}\frac{1}{2}}$ deserves special attention. It is the only theta function which is a modular function for the full group $\mathrm{SL}_2(\mathcal{O}_K)$. For reference purposes we state its transformation law separately as

$$\Theta_{\frac{1}{2}\frac{1}{2}}(A\tau) = \chi(A)\Theta_{\frac{1}{2}\frac{1}{2}}(\tau) \quad \text{for } A \in \mathrm{SL}_2(\mathcal{O}_K),$$

where

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} = \chi\begin{pmatrix} -c & -d \\ a+c & b+d \end{pmatrix},$$

and at least one of these matrices has the property that the a_{21}-entry is $\equiv 1(2)$. Assuming $c \equiv 1(2)$, we have

THEOREM 8:

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma\left(\frac{-2a}{c}\right) \times \exp\left[\pi i \operatorname{tr}\left(\frac{ac(3+bc) + (b+d-2)(a+c+2ac)}{4\sqrt{D}}\right)\right]$$

with

$$\gamma = \begin{cases} 1, & c \equiv \pm 1(4) \\ \pm i, & c \equiv \pm\sqrt{D}\,(4). \end{cases}$$

Because it is difficult to find proofs of these transformation rules in the literature, we at least prove Theorem 7 here in the case $c \neq 0$. Following the original procedure of Hermite we write

$$\tau_1 = (a\tau + b)(c\tau + d)^{-1} = \tau_2 + a/c,$$

$$\tau_2 = -c^{-1}\tau_3^{-1}c^{-1}, \qquad \tau_3 = \tau + d/c.$$

This corresponds to the well-known Bruhat decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}.$$

Writing $\tau_k = z_k + jv_k$ we have $v_1 = v_2$, $v_3 = v$,

$$\tau_3 = z_3 + jv_3 = -\frac{1}{c}(z_2 + jv_2)^{-1}\frac{1}{c} = \frac{-\bar{c}^2\bar{z}_2 + j|c|^2 v_2}{|c^2 z_2|^2 + |c^2 v_2|^2},$$

in particular $\dfrac{\sqrt{v}}{|c|} = \sqrt{v_1}\sqrt{|\tau_3|}$. In the equation defining $\Theta_{xy}(\tau_1)$ we set $\mu = \nu + 2$ cm, and observe that

$$\operatorname{tr}\left(\frac{a(\mu+y)^2 + 2\mu xc}{c\sqrt{D}}\right) \equiv \operatorname{tr}\left(\frac{a(\nu+y)^2 + 2\nu xc}{c\sqrt{D}}\right) \bmod 2\mathbb{Z}.$$

This allows us to write $\Theta_{xy}(\tau_1)$ as

$$\sqrt{v_1} \sum_{\nu \in \mathcal{O}_K/2c\mathcal{O}_K} \exp \pi i \operatorname{tr}\left(\frac{a(\nu+y)^2 + 2\nu cx}{c\sqrt{D}}\right) \cdot$$

$$\sum_{\kappa \in \mathcal{O}_K + (\nu+y)/2c} \exp\left(-\frac{2\pi(4|c|^2 v_2)|\kappa|^2}{|\sqrt{D}|} + \pi i \operatorname{tr}\left(\frac{4c^2\kappa^2 z_2}{\sqrt{D}}\right)\right).$$

Applying the Poisson summation formula to the inner series we get the expression

$$\sqrt{\left|\frac{\tau_3}{4}\right|} \sum_{\mu \in \frac{1}{2}\mathcal{O}_K} \exp\left(-\frac{2\pi v_2|\mu|^2}{|\sqrt{D}|} + \pi i \operatorname{tr}\left(\frac{\mu^2 z_3}{\sqrt{D}} + 2\frac{\mu(\nu+y)}{c\sqrt{D}}\right)\right),$$

compare [6, p. 237]. Therefore $\Theta_{xy}(\tau_1)$ equals

$$\frac{\sqrt{v}}{4|c|} \sum_{\mu \in \frac{1}{2}\mathcal{O}_K} \exp\left(-\frac{2\pi v|\mu|^2}{|\sqrt{D}|} + \pi i \operatorname{tr}\left(\frac{\mu^2 z}{\sqrt{D}}\right)\right) \sum_{\nu(2c)} \exp \pi i E(\nu, \mu),$$

where

$$E(\nu, \mu) = \operatorname{tr}\left(\frac{a(\nu+y)^2 + 2\nu cx + \mu^2 d + 2\mu(\nu+y)}{c\sqrt{D}}\right).$$

Denote the inner sum over ν by T. Shifting $\nu \to \nu + c\ell$, $\ell \in \mathcal{O}_K$, and using $x^2 \equiv x(2)$ for $x \in \mathcal{O}_K$ (valid only for $D \equiv 1(8)$), we find

$$T = T \exp 2\pi i \operatorname{tr}\left(\ell(\mu + ay + cx + ac/2)/\sqrt{D}\right),$$

or $T = 0$ if $\mu \notin ay + cx + ac/2 + \mathcal{O}_K$. Therefore we may assume

$$\mu = \mu' + ay + cx + ac/2, \ \mu' \in \mathcal{O}_K.$$

Using this expression together with $(r+1)(s+1) \equiv 0(2)$ for relatively prime integers $r, s \in \mathcal{O}_K$ (again valid only for $D \equiv 1(8)$), we arrive after a short calculation at the equation

$$T = 4\,|\,c\,|\,\Omega \exp 2\pi i \operatorname{tr}\left(\mu'(by + dx + bd/2)/\sqrt{D}\right)$$

with Ω as in Theorem 7. This gives the desired result.

4. In this section we give the definition of the Eisenstein homomorphisms [11]. Let L be a nondegenerate lattice in the complex plane with the ring of multipliers $\mathcal{O}_L = \{m \in \mathbb{C} \mid mL \subset L\}$, and

$$E_k(u) = E_k(u, L) = \sum_{w \in L}{}' (w+u)^{-k} \,|\, w+u \,|^{-s} \big|_{s=0}, \quad k = 0, 1, 2,$$

where $\ldots |_{s=0}$ means the value defined by analytic continuation to $s = 0$. In addition to these periodic functions we need the function $E(u)$ given by

$$2E(u) = \begin{cases} 2E_2(0), & u \in L \\ \wp(u) - E_1(u)^2, & u \in \mathbb{C} \setminus L, \end{cases}$$

where $\wp(u)$ denotes the Weierstrass $\wp$-function. For every matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathcal{O}_L)$ we define a map $\Phi(A)$: $(\mathbb{C}/L)^2 \to \mathbb{C}$ as follows:

$$\begin{aligned}\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(u, v) := &-\left(\frac{\bar{a}}{c}\right) E(u) - \left(\frac{\bar{d}}{c}\right) E(u^*) - \frac{a}{c} E_0(u) E_2(v) \\ &- \frac{d}{c} E_0(u^*) E_2(v^*) \\ &- \frac{1}{c} \sum_{r \in L/cL} E_1\left(\frac{ar + u^*}{c}\right) E_1\left(\frac{r+u}{c}\right)\end{aligned}$$

if $c \neq 0$ ($u^* = au + cv$, $v^* = bu + dv$), and for $c = 0$ we set

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(u, v) = -\left(\frac{\bar{b}}{d}\right) E(u) - \frac{b}{d} E_0(u) E_2(v).$$

The function Φ has the property [11]

THEOREM 9: $\Phi(AB)(u, v) = \Phi(A)(u, v) + \Phi(B)((u, v)A)$ *for every* A, $B \in \mathrm{SL}_2(\mathcal{O}_L)$ *and* u, $v \in \mathbb{C}/L$. *In other words,* Φ *is a cocycle for the group* $SL_2(\mathcal{O}_L)$. *If the residue class* (u, v) *in* $(\mathbb{C}/L)^2$ *is fixed, then* Φ *becomes an additive homomorphism of the group*

$$\Gamma(u, v) := \{A \in \mathrm{SL}_2(\mathcal{O}_L) \mid (u, v)A = (u, v)\}.$$

In general this group is trivial, but taking for (u, v) a generic $\mathfrak{a}$-division point in $(\mathfrak{a}^{-1}L/L)^2$, where $\mathfrak{a} \subset \mathcal{O}_L$ is an $\mathcal{O}_L$-ideal, we have $\Gamma(u, v) = \Gamma(\mathfrak{a})$, the principal congruence subgroup of level $\mathfrak{a}$ in $\mathrm{SL}_2(\mathcal{O}_L)$. All this is valid for any period lattice L. Assuming L has complex multiplication (i.e., $\mathcal{O}_L \neq \mathbb{Z}$), we can say more. It is well-known that in this case L can be chosen in its similarity class so that the numbers $g_2(L)$, $g_3(L)$ in

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

both become algebraic. The homomorphism Φ with $(u, v) \in (\mathfrak{a}^{-1}L/L)^2$ then takes on values in the number field H generated by the different of $\mathcal{O}_L$, the numbers g_2, g_3, and the $\mathfrak{a}$-division values of $\wp$, $\wp'$. So the Eisenstein homomorphisms in the CM-case are essentially algebraic objects. Multiplying them by a non-zero number λ in H and taking the trace $\mathrm{tr}_{H/\mathbb{Q}}(\lambda\Phi)$ we get rational valued homomorphisms. Every such λ represents, of course, another choice of the period lattice L because of the homogeneity property

$$\Phi_{\alpha L} = \alpha^{-2}\Phi_L \quad \text{for } \alpha \in \mathbb{C}^*.$$

In the rest of this section we prove the following integrality theorem.

THEOREM 10: *Suppose that* g_2, g_3 *are algebraic integers, and* u, $v \in \frac{1}{2}L/L$. *Denote by* D *the discriminant of the multiplier ring* $\mathcal{O}_L$. *Then* $4\Phi(A)(u, v)$ *is an algebraic integer (in the field generated by* $\sqrt{D}$ *and the 2-division values of* $\wp$, $\wp'$) *for all* $A \in \mathrm{SL}_2(\mathcal{O}_L)$. *If* $A \in \Gamma(2)$ *or* u, $v \in L$, *then* $2\Phi(A)(u, v)$ *is integral.*

PROOF: We will deduce this theorem from the following result of Cassels [2]: If $my \in L$,then $2m^{1/2}\wp(y)$ is an algebraic integer. If m is not an odd prime power and y is a primitive m-division point, then $2\wp(y)$ is already an algebraic integer.

Now suppose that y is an n-division point, $n \geqslant 3$. We use the following trick of Swinnerton-Dyer (cf. [3]):

$$\begin{aligned} nE_1(y) &= (n-1)E_1(y) - E_1((n-1)y) \\ &= \sum_{k=1}^{n-2} \left(E_1(ky) + E_1(y) - E_1((k+1)y)\right). \end{aligned}$$

The identify (cf. [11])

$$E_1(x)+E_1(y)-E_1(x+y) = -\frac{1}{2}\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)}$$
$$= \pm\sqrt{\wp(x)+\wp(y)+\wp(x+y)} \qquad (9)$$

(valid for x, y, $x+y \in \mathbb{C}\setminus L$) and the integrality result just mentioned imply that $2^{1/2}n^{5/4}E_1(y)$ is an algebraic integer. If n is odd and we choose for x in (9) a primitive 2-division point, then we deduce from this that $2^{1/2}n^{5/4}E_1(x+y)$ is integral, where now $x+y$ is an arbitrary $(2n)$-division point. We have proved

LEMMA: *$2^{1/2}n^{5/4}E_1(y)$ is an algebraic integer if $ny \in L$ or $2ny \in L$ and n is odd.*

To get the corresponding result for $E_2(y)$ we use $\wp(y) = E_2(y) - E_2(0)$ for $y \in \mathbb{C}\setminus L$. This gives

$$\sum_{r\in L/\alpha L}{}' \wp\left(\frac{r}{\alpha}\right) = \sum_{r\in L/\alpha L} E_2\left(\frac{r}{a}\right) - \alpha\bar{\alpha}E_2(0) = \alpha(\alpha-\bar{\alpha})E_2(0)$$

for every non-zero $\alpha \in \mathcal{O}_L$, which implies that $2\sqrt{D}\,E_2(0)$ is an algebraic integer. Therefore $2\sqrt{D}\,E_2(y)$ is an algebraic integer if $ny \in L$ and n is not a power of an odd prime. Finally, if $y \notin L$ is a 2-division point, then $4E(y) = 2\wp(y)$ is an algebraic integer.

Now it is easy to prove Theorem 10. Consider first

$$4\Phi\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}(u, v) = -4\left(\frac{\bar{b}}{d}\right)E(u) - 4\frac{b}{d}E_0(u)E_2(v).$$

Using $\mathrm{E}_0(\mathbb{C}\setminus L) = 0$, $E_0(L) = -1$, we see in the case $u \notin L$ that the right hand side is an integer which is divisible by 2 if $2 \mid b$. If $u \in L$, then the right hand side may be written as

$$4\left(\frac{b}{d} - \left(\frac{\bar{b}}{d}\right)\right)E_2(0) + 4\frac{b}{d}(E_2(v) - E_2(0))$$

and this is clearly 2 times an algebraic integer. By the same kind of reasoning we conclude in the case $c \neq 0$ that

$$2^{7/2}|c|^7\Phi\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \quad \text{if } |c|^2 \text{ is even,}$$
$$4|c|^7\Phi\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \quad \text{if } |c|^2 \text{ is odd,}$$

are algebraic integers, divisible by 2 if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ or $u, v \in L$. But from Theorem 9 we have

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(u, v) = \Phi\begin{pmatrix} -c & -d \\ a & b \end{pmatrix}(-v, u).$$

If $|a|^2$, $|c|^2$ are relatively prime, then the statement of the theorem follows. If not, then we write

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(u, v) = \Phi\begin{pmatrix} a+cx & b+dx \\ c & d \end{pmatrix}(u, v-ux)$$
$$-\Phi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}(u, v-ux)$$

and choose $x \in \mathcal{O}_K$ such that $|a+cx|^2$ and $|c|^2$ are relatively prime. This finishes the proof.

5. The next question is: Given the similarity class of the period lattice L, how do we get a canonical choice for L such that $g_2(L)$, $g_3(L)$ become algebraic integers? To answer this question, we assume for the rest of the paper that $D \equiv 1(8)$ is a discriminant of an imaginary quadratic field K, and set

$$u = u(\tau) = -\frac{2^{12}\eta(2\tau)^{24}}{\eta(\tau)^{24}}, \quad \tau = \frac{1+\sqrt{D}}{2}.$$

Then u is a positive real number, and a unit in the Hilbert classfield $H = H(K)$ with is related by

$$(u-16)^3 = uj$$

to the j-invariant $j(\tau)$. Define

$$g_2 := 12 \cdot D \cdot (u-16), \quad g_3 := (2\sqrt{D})^3\sqrt{u(j-1728)},$$

where the square root is so chosen that g_3 is a positive real number. By results of Weber, compare [1], g_2, g_3 are algebraic integers in the Hilbert classfield H. The elliptic curve

$$y^2 = 4x^3 - g_2x - g_3$$

has discriminate $\Delta = 12^6 \cdot D^3 \cdot u$, the j-invariant $j(\tau)$, and period lattice

$L = L_\tau = \omega(\mathbb{Z} + \mathbb{Z}\tau)$, where

$$\omega = \frac{\pi}{d}\frac{\eta(\tau)^4}{\eta(2\tau)^2} = \frac{\pi}{d}\prod_{n=1}^{\infty}\left(\frac{1-e^{2\pi i n\tau}}{1+e^{2\pi i n\tau}}\right)^2, \quad d = (144\,|D|)^{1/4}.$$

Replacing τ in this construction by any quadratic irrationality τ of discriminant D,

$$\tau = \frac{b+\sqrt{D}}{2a}, \quad b^2 - D \equiv 0(4a),\ a \geqslant 1 \text{ such that } a \equiv 1(4),$$

we get the numbers g_2^σ, g_3^σ with $\sigma \in \mathrm{Gal}(H/K)$. The corresponding lattices L_τ represent the canonical choices in question for the $h(D)$ similarity classes of period lattices which admit complex multiplication by $\mathcal{O}_K$. Adopting Theorem 10 to this choice we get the corollary

COROLLARY: *For $L = L_\tau$ defined as above and u, $v \in \frac{1}{2}L/L$, the numbers $2D^{-1/2}\Phi(A)(u, v)$ are algebraic integers in the Hilbert classfield of $\mathbb{Q}(\sqrt{D})$ for all $A \in \mathrm{SL}_2\mathcal{O}_K$. If $A \in \Gamma(2)$ or u, $v \in L$, then $D^{-1/2}\Phi(A)(u, v)$ is integral.*

By a conjecture stated in [11], the values of

$$\Psi := \frac{1}{\sqrt{D}}\Phi(0, 0)$$

even belong to $F := \mathbb{Q}(j(\tau))$, but no direct proof is known so far [1)]. This conjecture is equivalent to $rk_{\mathbb{Z}} M = h(D)$ where $M = \Psi(\mathrm{SL}_2\mathcal{O}_K)$; in [11] it is shown only that $rkM \geqslant h(D)$. In any case, M is a canonically defined $\mathbb{Z}$-module in the Hilbert classfield, and one may ask the question, how can we characterize this module? We do not know how to answer this question, but we conjecture

$$M \equiv E_2(L)\mathbb{Z} \text{ modulo } 8\mathcal{O}_F.$$

In other words, reducing Ψ modulo 8 we get a homomorphism

$$\mathrm{SL}_2(\mathcal{O}_K) \to \mathbb{Z}/8\mathbb{Z}.$$

But in Section 3 we already met a homomorphism $\chi = \chi_{(1/2)(1/2)}$ (given by Theorem 8) of $\mathrm{SL}_2(\mathcal{O}_K)$ into the eighth roots of unity. Writing χ additively, as an homomorphism into $\mathbb{Z}/8\mathbb{Z}$, our main conjecture is

CONJECTURE 1: $\Psi \equiv E_2(L)\cdot\chi$ modulo $8\mathcal{O}_F$.

This conjecture can be proved for any fixed D by a finite calculation because $\mathrm{SL}_2(\mathcal{O}_K)$ is a finitely generated group. In the next section we will do this in the cases $D = -7, -15, -23, -31, -39, -55$.

An interesting consequence of this conjecture arises from the observation that every parabolic element A in $\Gamma(8) \subset \Gamma(1) = \mathrm{SL}_2(\mathcal{O}_K)$ is already an eighth power of an element in $\Gamma(1)$, so $\chi(A) = 0$. This means that the restriction $\chi|_{\Gamma(8)}$, given by

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2\left(1 - \left(\frac{c}{a}\right)\right),$$

represents a cohomology class in

$$H^1_{\mathrm{cusp}}(\Gamma(8), \mathbb{Z}/2\mathbb{Z}),$$

which cannot be trivial because the kernel of χ is not a congruence subgroup. Our conjecture therefore indicates a congruence between Eisenstein series and cusp forms.

We also investigated the relation between the other theta multipliers χ_{xy} and Eisenstein homomorphisms $\Phi(u, v)$, and were led so to two further conjectures. To explain them, recall that there are 9 even theta characteristics different from $(x, y) = (1/2, 1/2)$. On the other hand, there are exactly 9 primitive 2-division points in $(\frac{1}{2}L/L)^2$ (i.e. points which are not P- or $\bar{P}$-division points, P the ideal generated by 2 and $(1 + \sqrt{D})/2$), and every such point (u, v) can be written uniquely as the sum of a primitive P-division point (u_1, v_1) and a primitive $\bar{P}$-division point (u_2, v_2),

$$(u, v) = (u_1, v_1) + (u_2, v_2) \in \left(\tfrac{1}{2}L/L\right)^2.$$

We set

$$\Psi(u, v) := \frac{1}{\sqrt{D}}\left(\Phi(u, v) + \Phi(u_1, v_1) + \Phi(u_2, v_2)\right).$$

Then Ψ is a cocycle for $\mathrm{SL}_2\mathcal{O}_K$, and its restriction to $\Gamma(2)$ is a homomorphism. Moreover, the values of Ψ are always algebraic integers, although the three terms of which it is made up in general have a denominator 2. This is easily proved using the identity

$$\sum_{u \in \frac{1}{2}L/L} E(u) = E(0),$$

and the arguments in the proof of the Theorem 10. The map

$$(x, y) \mapsto (u, v) := \omega\left(y + \tfrac{1}{2}, x + \tfrac{1}{2}\right)$$

gives a bijection between the even theta characteristics $(x, y) \neq (\frac{1}{2}, \frac{1}{2})$ and the primitive 2-division points (u, v). Using this bijection, we can formulate our second conjecture:

CONJECTURE 2: $\Psi(u, v) \equiv 3E_2(L)\chi_{xy} \bmod 8\mathcal{O}_F$, *where* χ *is the theta cocycle defined in Section* 3.

Note that the relation between the xy and (u, v) is the same as in the classical case (cf. Theorem 3). Of course, it is enough to prove this conjecture for one particular even characteristic (x, y) $\neq (\frac{1}{2}, \frac{1}{2})$ because $SL_2\mathcal{O}_K$ acts transitively on the primitive 2-division points.

Finally, our last conjecture is

CONJECTURE 3: $\dfrac{1}{\sqrt{D}}(\Phi(0, \frac{1}{2}) + \Phi(\frac{1}{2}, 0) + \Phi(\frac{1}{2}, \frac{1}{2})) \equiv 0(8\mathcal{O}_F)$.

Again, the left hand side is an integer valued cocycle for $SL_2\mathcal{O}_K$, and the $SL_2\mathcal{O}_K$-action produces five further relations of this kind. It is tempting to associate these six relations with the six odd theta characteristics; then the fact that we have zero rather than χ_{xy} on the right-hand side would be related to the vanishing of the corresponding theta-series.

6. In this section we present some numerical evidence for our conjectures. First, by definition of χ and Φ, we have

$$\chi(S) = \Phi(S) = 0 \quad \text{for } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so the conjectures are true for this special matrix. For the values of

$$T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \; b \in \mathcal{O}_K$$

we have the following formulas:

$$\Psi(T)(0, 0) = \operatorname{tr}\left(\frac{b}{\sqrt{D}}\right)E_2(0), \quad \Psi(T)(0, \tfrac{1}{2}) = 3 \operatorname{tr}\left(\frac{b}{\sqrt{D}}\right)E_2(0),$$

$$\Psi(T)(u, v) = 0 \quad \text{for} \quad 2(u, v) = (1, 0), (1, 1), (1, z), (1, \bar{z}),$$

$$\Psi(T)(\tfrac{1}{2}z, \tfrac{1}{2}) = \Psi(T)(\tfrac{1}{2}z, \tfrac{1}{2}\bar{z}) = -\Psi(\bar{T})(\tfrac{1}{2}\bar{z}, \tfrac{1}{2})$$

$$= -\Psi(\bar{T})(\tfrac{1}{2}\bar{z}, \tfrac{1}{2}z) = \operatorname{tr}\left(\frac{bE_2(\omega\bar{z}/2)}{\sqrt{D}}\right),$$

where

$$z := \begin{cases} \dfrac{1+\sqrt{D}}{2}, & D \equiv 1(16) \\[2ex] \dfrac{3+\sqrt{D}}{2}, & D \equiv 9(16). \end{cases}$$

On the other hand, from Theorem 7 we have

$$\chi_{xy}(T) \equiv 4\ \mathrm{tr}\left(\frac{by^2}{\sqrt{D}}\right) \bmod 8 \quad \text{if } x,\ y \in \tfrac{1}{2}\{0,\ 1,\ z,\ \bar{z}\},$$

thus

$$\chi_{\frac{1}{2}\frac{1}{2}}(T) \equiv \chi_{0\frac{1}{2}}(T) \equiv \mathrm{tr}\left(\frac{b}{\sqrt{D}}\right) \bmod(8),$$

$$\chi_{xy}(T) \equiv 0 \quad \text{for } 2(x,\ y) = (1,0),\ (0,0),\ (\bar{z},0),\ (z,0),$$

$$\chi_{0\frac{1}{2}\bar{z}}(T) \equiv \chi_{\frac{1}{2}z\frac{1}{2}\bar{z}}(T) \equiv -\chi_{0\frac{1}{2}z}(\bar{T}) \equiv -\chi_{\frac{1}{2}\bar{z}\frac{1}{2}z}(\bar{T})$$

$$\equiv \mathrm{tr}\left(\frac{b\bar{z}^2}{\sqrt{D}}\right) \bmod(8).$$

These formulas show that Conjecture 1 is true for T; and to verify Conjecture 2 for T it is enough to check the congruence

$$E_2(\omega z/2) \equiv 3z^2 E_2(0) \bmod(8\mathcal{O}_H). \tag{9}$$

To test Conjecture 3 write Ψ_1 for the left hand side of this conjecture, and similarly

$$\Psi_2 : (0,\ \tfrac{1}{2}) + (\tfrac{1}{2},\ \tfrac{1}{2}z) + (\tfrac{1}{2},\ \tfrac{1}{2}\bar{z})$$

$$\Psi_3 : (\tfrac{1}{2},\ 0) + (\tfrac{1}{2}z,\ \tfrac{1}{2}) + (\tfrac{1}{2}\bar{z},\ \tfrac{1}{2})$$

$$\Psi_4 : (\tfrac{1}{2},\ \tfrac{1}{2}z) + (\tfrac{1}{2}\bar{z},\ \tfrac{1}{2}) + (\tfrac{1}{2}z,\ \tfrac{1}{2}\bar{z})$$

$$\Psi_5 : (\tfrac{1}{2},\ \tfrac{1}{2}\bar{z}) + (\tfrac{1}{2}z,\ \tfrac{1}{2}) + (\tfrac{1}{2}\bar{z},\ \tfrac{1}{2}z)$$

$$\Psi_6 : (\tfrac{1}{2},\ \tfrac{1}{2}) + (\tfrac{1}{2}\bar{z},\ \tfrac{1}{2}z) + (\tfrac{1}{2}z,\ \tfrac{1}{2}\bar{z})$$

for the five translations of Ψ_1 by $\mathrm{SL}_2\mathcal{O}_K$. Then we have

$$\Psi_1(T) = \Psi_2(T) = \mathrm{tr}\left(\frac{b}{\sqrt{D}}\right) E_2\left(\frac{\omega}{2}\right), \quad \Psi_k(T) = 0 \quad \text{for } k > 2$$

which shows that Conjecture 3 is true for T iff

$$E_2(\omega/2) \equiv 0\ (8\mathcal{O}_F). \tag{10}$$

In the following numerical examples we will list the 2-division values of

E_2, and leave to the reader the pleasure of checking the congruences (9) and (10).

$$D = -7{:}\quad E_2(0) = 3,\quad E_2\left(\frac{\omega}{2}\right) = 24$$

$$E_2\left(\frac{\omega z}{2}\right) = -\frac{15 - 3\sqrt{-7}}{2},\quad E_2\left(\frac{\omega \bar{z}}{2}\right) = \overline{E_2\left(\frac{\omega z}{2}\right)}.$$

These numbers can be found easily on a pocket calculator using classical formulas for E_2. Note that in this special case the ring $\mathcal{O}_K$ is euclidean, so S and T already generate $\mathrm{SL}_2\mathcal{O}_K$.

$$D = -15{:}\quad E_2(0) = 3\frac{3+\sqrt{5}}{2},\quad E_2\left(\frac{\omega}{2}\right) = 24\frac{1+\sqrt{5}}{2},$$

$$E_2\left(\frac{\omega z}{2}\right) = \frac{3 - 15\sqrt{5} - 21\sqrt{-15} + 45\sqrt{-3}}{4}.$$

The class number is 2, and the Hilbert classfield is $K(\sqrt{5})$. By calculations of Swan [13], the group $\mathrm{SL}_2\mathcal{O}_K$ is generated by the elements S, T, and

$$A = \begin{pmatrix} 4 & -\sqrt{-15} \\ \sqrt{-15} & 4 \end{pmatrix}.$$

Writing $2\Psi(A) = \alpha + \beta\sqrt{5}$, we found the following values for $\Psi(A)(u, v)$ and $\chi_{xy}(A)$:

$2u, 2v$	$1, 1$	$0, 1$	$\bar{z}, 1$	$z, 1$	$1, 0$	$1, \bar{z}$	$1, z$	$0, 0$	$z, \bar{z}$	$\bar{z}, z$
α	-30	-6	33	30	-6	30	30	-30	-54	-51
β	6	-18	-21	-6	-18	-6	-6	6	30	15
$\chi_{xy}(A)$	6	6	3	2	6	2	2	2	6	7
$2x, 2y$	$0, 0$	$0, 1$	$0, z$	$0, \bar{z}$	$1, 0$	$z, 0$	$\bar{z}, 0$	$1, 1$	$z, \bar{z}$	$\bar{z}, z$

The values $\Psi_k(A)$ are all $\equiv 0(8)$:

k	1	2	3	4	5	6
α	-72	24	24	-24	-24	-168
β	-24	-24	-24	24	24	$+72$

So the Conjectures 1, 2, 3 are true for $D = -7, -15$. In the following examples we restrict our attention to Conjecture 1, and calculate χ and Ψ only. We will do this for a finite set of matrices $M = \{A, B, C, \ldots\}$ found by N. Krämer [8]; this set has the property

$$\mathrm{SL}_2\mathcal{O}_K = \langle X, \bar{X}, \sigma(X), \sigma(\bar{X}) \mid X \in M \text{ or } X = S, T\rangle,$$

where

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

and bar denotes complex conjugation. It is easily verified that

$$\chi(\overline{X}) \equiv \chi(\sigma(X)) \equiv -\chi(X), \quad \Psi(\sigma(X)) = -\Psi(X)$$

for X in $\mathrm{SL}_2\mathcal{O}_K$, and we found in all examples that $\Psi\overline{X}) = -\Psi(X)$ holds too. So it is enough to list the values $\chi(X)$, $\Psi(X)$ for X in M.

$D = -23$: Here the class number is 3, and the Hilbert classfield is $H = K(\theta)$, where $\theta = -1.324\ldots$ is the real root of $\theta^3 - \theta + 1 = 0$. The E_2-values are:

$$E_2(0) = 8 - 4\theta - \theta^2, \quad E_2\left(\frac{\omega}{2}\right) = -24(1 + \theta - \theta^2),$$

$$2E_2\left(\frac{\omega z}{2}\right) = 48 + 12\theta - 27\theta^2 - (12\theta + 9\theta^2)\sqrt{-23}\,.$$

The norm of $E_2(0)$ is the prime number

$$N_{H/K}(E_2(0)) = 419.$$

This shows that our choice of the period lattice L is in general best possible (up to a unit). We have $M = \{A, B\}$ with

$$2A = \begin{pmatrix} 4 + 2\sqrt{D} & 11 - \sqrt{D} \\ 8 & -2\sqrt{D} \end{pmatrix}, \quad 2B = \begin{pmatrix} -7 + \sqrt{D} & 7 + \sqrt{D} \\ 3 + \sqrt{D} & 3 - \sqrt{D} \end{pmatrix},$$

and

$$\Psi(A) = 12\theta + 9\theta^2, \quad \Psi(B) = 8 - 4\theta - 7\theta^2,$$

$$\chi(A) \equiv 7, \qquad \chi(B) \equiv 7 \text{ modulo } 8.$$

$D = -31$: Then $h(D) = 3$, $H = K(\theta)$ where $\theta = -0.682\ldots$ is the real root of $\theta^3 + \theta + 1 = 0$, and

$$E_2(0) = 3(1 - 5\theta + \theta^2), \quad E_2\left(\frac{\omega}{2}\right) = 24(2 + \theta^2),$$

$$2E_2\left(\frac{\omega z}{2}\right) = -39 - 45\theta - 15\theta^2 + (9 + 3\theta - 15\theta^2)\sqrt{-31}\,.$$

We have $M=\{A, B, C, E\}$ with $E^2=-1$,

$$A=\begin{pmatrix}\sqrt{D} & -8\\ 4 & \sqrt{D}\end{pmatrix},\quad B=\begin{pmatrix}-2+\sqrt{D} & -9\\ 4 & 2+\sqrt{D}\end{pmatrix},$$

$$C=\begin{pmatrix}-5 & -1+\sqrt{D}\\ \dfrac{1+\sqrt{D}}{2} & 3\end{pmatrix},\quad E=\begin{pmatrix}\dfrac{3+\sqrt{D}}{2} & \dfrac{3-\sqrt{D}}{2}\\ 3 & -\dfrac{3+\sqrt{D}}{2}\end{pmatrix}$$

	A	B	C	E
Ψ	$-6-18\theta+18\theta^2$	$12(1-\theta-\theta^2)$	$-3-9\theta-3\theta^2$	0
χ	6	4	7	0

$D=-39$: Here $h(D)=4$, $H=K(\theta_2)$ where

$$\theta_1=\sqrt{13},\quad \theta_2=\sqrt{\frac{\sqrt{13}-1}{2}},\quad \theta_3=\theta_1\theta_2,\quad \text{and}$$

$$E_2(0)=\tfrac{1}{4}(-36+18\theta_1+3\theta_2+9\theta_3),$$

$$E_2\left(\frac{\omega}{2}\right)=\tfrac{24}{4}(5-\theta_1+5\theta_2+\theta_3),$$

$$E_2\left(\frac{\omega z}{2}\right)=\tfrac{3}{8}\big(-76+26\theta_1-37\theta_2$$

$$+\theta_3+(12-2\theta_1-15\theta_2+3\theta_3)\sqrt{-39}\big).$$

We have $M=\{A, B, C, E, F\}$ where $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in M$ is given by

	A	B	C	E	F
$2a$	$2\sqrt{D}$	$-4+2\sqrt{D}$	$11-\sqrt{D}$	$1-\sqrt{D}$	28
$2b$	-20	-22	$12+2\sqrt{D}$	6	$14+4\sqrt{D}$
$2c$	8	8	$3+\sqrt{D}$	6	$2\sqrt{D}$
$2d$	$2\sqrt{D}$	$4+2\sqrt{D}$	$-7+\sqrt{D}$	$1+\sqrt{D}$	$-11+\sqrt{D}$

Then $E^2=-E^{-1}$ or $E^6=1$, and therefore $\chi(E)=\Psi(E)=0$. For the other values we found the table

	A	B	C	F
χ	2	0	2	7
α	0	-72	24	60
β	12	24	-12	6
γ	-42	48	30	-51
δ	18	0	-6	-9

where

$$4\Psi = \alpha + \beta\theta_1 + \gamma\theta_2 + \delta\theta_3.$$

$D = -55$: In this case $h(D) = 4$, $H = K(\theta_2)$ with $\theta_1 = \sqrt{5}$, $\theta_2 = \sqrt{3 + 2\sqrt{5}}$, $\theta_3 = \theta_1\theta_2$, and

$$E_2(0) = \tfrac{3}{4}(47 - 15\theta_1 + 17\theta_2 - 5\theta_3),$$

$$E_2\left(\frac{\omega}{2}\right) = \tfrac{24}{4}(-1 + 5\theta_1 - \theta_2 + \theta_3),$$

$$E_2\left(\frac{\omega z}{2}\right) = \tfrac{3}{8}\big(149 - 85\theta_1 + 59\theta_2 - 23\theta_3 - (9 - 9\theta_1 - 25\theta_2 + 13\theta_3)\sqrt{-55}\,\big).$$

The set $M = \{A, B, C, D, E, F, G, H\}$ is given by

	A	B	C	D	E	F	G	H
$2a$	$-1+\theta$	$3-\theta$	2θ	$4+2\theta$	-14	$4+2\theta$	$8+2\theta$	$13+\theta$
$2b$	-10	10	-28	$-26+2\theta$	$-5-3\theta$	$-13+\theta$	$-10+2\theta$	$-11+3\theta$
$2c$	6	6	8	8	$3+\theta$	16	16	$5-\theta$
$2d$	$1+\theta$	$3+\theta$	2θ	$4+2\theta$	$-11+\theta$	$4+2\theta$	$8+2\theta$	$13+\theta$

where $\theta = \sqrt{-55}$. For these matrices we found the table

	A	B	C	D	E	F	G	H
χ	2	0	2	0	4	7	4	5
α	66	0	114	168	60	99	108	33
β	-18	0	-18	-72	-12	-51	36	15
γ	30	0	126	-24	-12	-3	36	63
δ	-6	0	-54	24	12	15	-36	-27

where

$$4\Psi = \alpha + \beta\theta_1 + \gamma\theta_2 + \delta\theta_3.$$

This last example is of special interest because $D = -55$ is the first discriminant $D \equiv 1(8)$ with

$$\operatorname{rank}_{\mathbb{Z}}(\mathrm{SL}_2\mathcal{O}_K)^{ab} > h(D).$$

The actual rank is 5 (cf. [8]), so there is a homomorphism of $\mathrm{SL}_2\mathcal{O}_K$ which cannot be represented by Dedekind sums.

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[1)] Note added in proof: a very simple proof was found by H. Ito (Nagoya).