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THE PLANCHEREL FORMULA FOR THE PSEUDO-RIEMANNIAN SPACE SL(n, $\mathbb{R}$)/GL($n-1$, $\mathbb{R}$)

G. van Dijk and M. Poel

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0. Introduction

The main result of this paper is a Plancherel formula for the rank one symmetric space $X = \mathrm{SL}(n, \mathbb{R})/\mathrm{GL}(n-1, \mathbb{R})$, $n \geqslant 3$. This means a desintegration of the left regular representation of G on $L^2(X)$ into irreducible unitary representations. One can also formulate it in terms of spherical distributions (cf. [7]). Then we are determining a desintegration of the δ-distribution at the origin of X into extremal positive-definite spherical distributions.

Section 1 and 2 are concerned with a precise definition. We also ask for uniqueness of the desintegration and introduce once more the notion of a generalized Gelfand pair. Special attention is paid to the relative discrete series. Section 3 contains the abstract theory, while Section 4 is devoted to an explicit determination of a parametrization of the relative discrete series for the space under consideration. The results we obtain are applied in Section 5 where the Plancherel formula is determined by a method previously used by Faraut [5]. This paper is a continuation of earlier work [7] and depends heavily on it. Recently Molčanov [9] has obtained the Plancherel formula for the case $n = 3$ by a quite different method. Our analysis of the relative discrete series seems to have some analogy with work of Kengmana [6].

1. Invariant Hilbert subspaces of $D'(G/H)$

Let G be a real Lie group and H a closed subgroup of G. Throughout this paper we assume both G and H to be unimodular. Let us fix Haar measures dg on G, dh on H and a G-invariant measure dx on G/H in such a way that $dg = dx\, dh$.

We shall take all scalar products anti-linear in the first and linear in the second factor.

Let π be a continuous unitary representation of G on a Hilbert space $\mathscr{H}$. A vector $v \in \mathscr{H}$ is said to be a C^∞-vector if the map $g \to \pi(g)v$ is in $C^\infty(G, \mathscr{H})$. The subspace $\mathscr{H}_\infty$ of C^∞-vectors in $\mathscr{H}$ can be endowed with a natural Sobolev-type topology (cf. [2], §1). Let us recall the definition. Let $\mathfrak{g}$ be the Lie algebra of G. For any $X \in \mathfrak{g}$ and $v \in \mathscr{H}_\infty$, put

$$\pi(X)v = \frac{d}{dt}\bigg|_{t=0} \pi(\exp tX)v.$$

Then $\pi(X)$ leaves $\mathscr{H}_\infty$ stable. The topology is defined by means of the set of norms $\|\cdot\|_m$ given by the following formula. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. Then

$$\|v\|_m^2 = \sum_{|\alpha| \leqslant m} \|\pi(X_1)^{\alpha_1} \ldots \pi(X_n)^{\alpha_n} v\|^2$$

with $|\alpha| = \alpha_1 + \ldots + \alpha_n$, α_i non-negative integers ($v \in \mathscr{H}_\infty$). $\mathscr{H}_\infty$ becomes a Frechet space in this manner.

The topology does not depend on the choice of the basis of $\mathfrak{g}$. The space $\mathscr{H}_\infty$ is G-invariant. The corresponding representation of G on $\mathscr{H}_\infty$ is called π_∞; the map $(g, v) \to \pi_\infty(g)v$ is continuous $G \times \mathscr{H}_\infty \to \mathscr{H}_\infty$.

Denote $\mathscr{H}_{-\infty}$ the anti-dual of $\mathscr{H}_\infty$, endowed with the strong topology. The inclusion $\mathscr{H}_\infty \subset \mathscr{H}$ and the isomorphism of the Hilbert space $\mathscr{H}$ with its anti-dual yield an inclusion $\mathscr{H} \subset \mathscr{H}_{-\infty}$, so $\mathscr{H}_\infty \subset \mathscr{H} \subset \mathscr{H}_{-\infty}$. The injections are continuous. G acts on $\mathscr{H}_{-\infty}$ and the corresponding representation is called $\pi_{-\infty}$. Denote by $D(G)$, $D(G/H)$ the space of C^∞-functions with compact support on G and G/H respectively, endowed with the usual topology. Let $D'(G)$, $D'(G/H)$ be the topological anti-dual of $D(G)$ and $D(G/H)$ respectively, provided with the strong topology.

For $v \in \mathscr{H}_\infty$, $a \in \mathscr{H}_{-\infty}$ we put $\langle v, a \rangle = a(v)$ and we write $\langle a, v \rangle$ instead of $\overline{\langle v, a \rangle}$. Similarly we put $\langle \phi, T \rangle = \overline{\langle T, \phi \rangle} = T(\phi)$ for $\phi \in D(G/H)$, $T \in D'(G/H)$. Denote $\phi_0 \to \phi$ the canonical projection map $D(G) \to D(G/H)$ given by

$$\phi(x) = \int_H \phi_0(gh)\, dh \quad (x \in G/H,\ x = gH).$$

For any $a \in \mathscr{H}_{-\infty}$ and $\phi_0 \in D(G)$, put

$$\pi_{-\infty}(\phi_0)a = \int_G \phi_0(g)(\pi_\infty(g)a)\, dg.$$

Then $\pi_{-\infty}(\phi_0)a \in \mathscr{H}_\infty$. A vector $a \in \mathscr{H}_{-\infty}$ is called *cyclic* if $\{\pi_{-\infty}(\phi_0)a\colon \phi_0 \in D(G)\}$ is a dense subspace of $\mathscr{H}$. Define

$$\mathscr{H}_{-\infty}^H = \{a \in \mathscr{H}_{-\infty}\colon \pi_{-\infty}(h)a = a \quad \text{for all } h \in H\}.$$

We say that π can be realized on a Hilbert subspace of $D'(G/H)$ if there is a continuous linear injection $j\colon \mathscr{H} \to D'(G/H)$ such that

$$j\pi(g) = L_g j$$

for all $g \in G$ (L_g denotes the left translation by g). The space $j(\mathscr{H})$ is said to be an invariant Hilbert subspace of $D'(G/H)$.

THEOREM 1.1: *π can be realized on a Hilbert subspace of $D'(G/H)$ if and only if $\mathscr{H}_{-\infty}^H$ contains non-zero cyclic elements. There is an one-to-one correspondence between the non-zero cyclic elements of $\mathscr{H}_{-\infty}^H$ and the continuous linear injections $j\colon \mathscr{H} \to D'(G/H)$ satisfying $j\pi(g) = L_g j$ ($g \in G$). To a cyclic vector $a \neq 0$ in $\mathscr{H}_{-\infty}^H$ corresponds j, such that $j^*\colon D(G/H) \to \mathscr{H}$ is given by $j^*(\phi) = \pi_{-\infty}(\phi_0)a$.*

The proof is quite similar to [2, Théorème 1.4].

Let π be a representation realized on $D'(G/H)$ and $j\colon \mathscr{H} \to D'(G/H)$ the corresponding injection. Denote by ξ_π the cyclic vector in $\mathscr{H}_{-\infty}^H$ defined by Theorem 1.1. Then we put

$$\langle T, \phi_0 \rangle = \langle \xi_\pi, \pi_{-\infty}(\phi_0)\xi_\pi \rangle \quad (\phi_0 \in D(G)).$$

T is a distribution on G which is left and right H-invariant. We call T the reproducing distribution of π (or $\mathscr{H}$). T is positive-definite, bi-H-invariant and

$$\| j^*\phi \|^2 = \langle T, \tilde{\phi}_0 * \phi_0 \rangle$$

for all $\phi_0 \in$ D(G). Here $\tilde{\phi}_0$ is given by $\tilde{\phi}_0(g) = \overline{\phi_0(g^{-1})}$ ($g \in G$). Given a postive-definite bi-H-invariant distribution T on G, the latter formula shows the way to define a G-invariant Hilbert subspace of $D'(G/H)$ with T as reproducing distribution. Indeed, let V be the space $D(G/H)$ provided with the inner product

$$(\phi, \psi) = \langle T, \tilde{\phi}_0 * \psi_0 \rangle.$$

Let V_0 be the subspace of V consisting of the elements of length zero and define $\mathscr{H}$ to be the completion of V/V_0 and j^* the natural projection $D(G/H) \to \mathscr{H}$. Then clearly

$$\| j^*\phi \|^2 = \langle T, \tilde{\phi}_0 * \phi_0 \rangle$$

for all $\phi_0 \in D(G)$.

Furthermore, an easy calculation shows that jv is a C^∞-function for all $v \in \mathscr{H}_\infty$. Actually

$$jv(x) = \langle \xi_\pi, \pi(g^{-1})v \rangle \quad (x = gH \in G/H).$$

Note that j can be defined on $\mathscr{H}_{-\infty}$ (as anti-dual of j^*: $D(G/H) \to \mathscr{H}_\infty$). Then $j(\xi_\pi)$ is precisely the reproducing distribution T, considered as an H-invariant element of $D'(G/H)$. One has

$$(jj^*)(\phi) = \phi_0 * T$$

for all $\phi_0 \in D(G)$.

Summarizing we have

PROPOSITION 1.2: *The correspondence $\mathscr{H} \to T$ which associates with each invariant Hilbert subspace of $D'(G/H)$ its reproducing distribution is a bijection between the set of G-invariant Hilbert subspaces of $D'(G/H)$ and the set of bi-H-invariant positive-definite distributions on G.*

Denote Γ_G the set of bi-H-invariant positive-definite distributions and ext(Γ_G) the subset of those distributions which correspond to minimal G-invariant Hilbert subspaces of $D'(G/H)$ (or: to irreducible unitary representations π realized on a Hilbert subspace of $D'(G/H)$). Choose an admissible *parametrization* $s \to T_s$ of ext(Γ_G) as in [12]. Here S is a topological Hausdorff space. Then one has

PROPOSITION 1.3 [12, Proposition 9]: *For every $T \in \Gamma_G$ there exists a (non-necessarily unique) Radon measure m on S such that*

$$\langle T, \phi_0 \rangle = \int_S \langle T_s, \phi_0 \rangle \, \mathrm{d}m(s)$$

for all $\phi_0 \in D(G)$.

This result, except for the fixed parametrization independent of T, has been obtained by L. Schwartz and K. Maurin. See [12] for references. The proof of Proposition 1.3 is obtained by diagonalising a maximal commutative C^*-algebra commuting with the action of G in the Hilbert subspace, associated with T. The fixed parametrization can then be obtained by the techniques of [12]. Clearly we are mainly interested in the decomposition of the distribution $T \in \Gamma_G$ given by

$$\langle T, \phi_0 \rangle = \int_H \phi_0(h) \, \mathrm{d}h \quad (\phi_0 \in D(G))$$

which corresponds to the δ-function at the origin of G/H.

This could be called a Plancherel formula for G/H.

Let G be a connected, non-compact, real semisimple Lie group with finite center and σ an involutive automorphism of G. Let H be an open subgroup of the group of fixed points of σ. The pair (G, H) is called a semisimple symmetric pair.

Let $\mathbb{D}(G/H)$ denote the algebra of G-invariant differential operators on G/H. It is known that $\mathbb{D}(G/H)$ is a commutative, finitely generated algebra. For any $D \in \mathbb{D}(G/H$, define ${}'D$ by

$$\int_{G/H} \overline{D\phi(x)}\psi(x)\,\mathrm{d}x = \int_{G/H} \overline{\phi(x)}{}'D\psi(x)\,\mathrm{d}x$$

for all ϕ, $\psi \in \mathbb{D}(G/H)$. Then ${}'D \in \mathbb{D}(G/H)$. So $\mathbb{D}(G/H)$ is generated by "self adjoint" elements. Let $D \in \mathbb{D}(G/H)$ be such that $D = {}'D$. Then, regarding D as a density defined linear operator on $L^2(G/H)$, D is *essentially self-adjoint*. The proof of this fact in [11, Lemma 9] is incomplete. E.P. van den Ban [13] has recently shown the following non-trivial fact: any $D \in \mathbb{D}(G/H)$ maps $L^2(X)_\infty$ into itself and

$$\int_{G/H} \overline{D\phi(x)}\psi(x)\,\mathrm{d}x = \int_{G/H} \overline{\phi(x)}{}'D\psi(x)\,\mathrm{d}x$$

for all ϕ, $\psi \in L^2(X)_\infty$. Now the reasoning of the proof of [11, Lemma 9] goes through, observing that $\phi_0 * \psi \in \mathrm{L}^2(X)_\infty$ for any $\phi_0 \in D(G)$ and $\psi \in L^2(X)$. Let $\mathscr{A}$ be the closed *-algebra (C^*-algebra) generated by the spectral projections of the closures of all self-adjoint $D \in \mathbb{D}(G/H)$. By a result of Nelson [10, p. 603] any two of such closures strongly commute, so this algebra $\mathscr{A}$ is abelian. As mentioned before, the main part of the proof of Proposition 1.3 is obtained by diagonalising a maximal commutative C^*-algebra commuting with the action of G in $\mathscr{H}$, $\mathscr{H}$ being the Hilbert subspace associated with T.

So, in our situation, with $\mathscr{H} = L^2(G/H)$ we only have to extend $\mathscr{A}$ to a maximal commutative C^*-algebra. The result is a desintegration of $L^2(G/H)$ into irreducible Hilbert subspaces, even a formula of the form

$$\phi(eH) = \int_S \langle T_s, \phi\rangle\,\mathrm{d}m(s) \quad (\phi \in D(G/H))$$

such that T_s is a *common eigendistribution* for all $D \in \mathbb{D}(G/H)$, for m-almost all $s \in S$. Here we regard T_s as an element of $D'(G/H)$. For details of the (abstract) theory, we refer to [8], [12].

PROPOSITION 1.4: *Let (G, H) be a semisimple symmetric pair. There exists a (non-necessarily unique) Radon measure m on S such that*

(*i*) $\phi(eH) = \int_S \langle T_s, \phi\rangle\,\mathrm{d}m(s) \quad (\phi \in D(G/H))$

(*ii*) *for m-almost all* $s \in S$, T_s *is a common eigendistribution for all* $D \in \mathbb{D}(G/H)$.

Would Proposition 1.4 answer a problem raised by Faraut [5, p. 371]?

2. (SL(n, $\mathbb{R}$), GL($n-1$, $\mathbb{R}$)) is a generalized Gelfand pair for $n \geqslant 3$

We keep to the notation of Section 1. Generalizing the classical notion of a Gelfand pair, we define

DEFINITION 2.1: *The pair* (G, H) *is called a generalized Gelfand pair if for each irreducible unitary representation* π *on a Hilbert space* $\mathscr{H}$, *one has* $\dim \mathscr{H}_{-\infty}^{H} \leqslant 1$.

The following result is proved in [12].

PROPOSITION 2.2. *The following statements are equivalent*:

(*i*) (G, H) *is a generalized Gelfand pair*

(*ii*) *For any unitary representation* π *which can be realized on a Hilbert subspace of* $D'(G/H)$, *the commutant of* $\pi(G) \subset \mathscr{L}(\mathscr{H})$ *is abelian.* [$\mathscr{L}(\mathscr{H})$: *the algebra of the continuous linear operators of* $\mathscr{H}$ *into itself*]

(*iii*) *For every* $T \in \Gamma_G$ *there exists a UNIQUE Radon measure m on S such that*

$$\langle T, \phi_0 \rangle = \int_S \langle T_s, \phi_0 \rangle \, \mathrm{d}m(s)$$

for all $\phi_0 \in D(G)$.

For a more detailed discussion of generalized Gelfand pairs, including examples, we refer to [14]. Most examples are connected with symmetric spaces.

Let G be a connected semisimple Lie group with finite center, σ an involutive automorphism of G and H an open subgroup of the group of fixed points of σ. Then it was recently shown by E.P. van den Ban [13] that for every irreducible unitary representation π of G on $\mathscr{H}$, $\dim \mathscr{H}_{-\infty}^{H} < \infty$. He actually shows the following. Choose a Cartan involution θ of G commuting with σ and let K be the group of fixed points of θ. Given a finite-dimensional irreducible representation δ of K and an infinitesimal character χ, we write $A(G/H; \chi)$ for the space of right H-invariant real analytic functions ϕ: $G \to \mathbb{C}$ satisfying $z \cdot \phi = \chi(z)\phi$ for all $z \in Z(\mathfrak{g})$ [center of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G], and $A_\delta(G/H; \chi)$ for the subspace of K-finite elements of type δ. Then $\dim_{\mathbb{C}} A_\delta(G/H; \chi) \leqslant |W(\phi)| \dim(\delta)^2$, where $W(\Phi)$ is the Weyl group of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$ with respect to a Cartan subalgebra. This result clearly implies that $\mathscr{H}^H_{-\infty}$ is of finite dimension. Indeed, let v be a non-zero K-finite element in $\mathscr{H}$ of type δ and let χ be the infinitesimal character of π. For any subset (ξ^i_π) of linearly independent elements in $\mathscr{H}^H_{-\infty}$, the set of functions $\phi_i(g) = \langle \xi^i, \pi(g^{-1})v \rangle$ $(g \in G)$ is also linearly independent. Clearly $\phi_i \in A_\delta(G/H; \chi)$. The proof of van den Ban's result is completely in the spirit of Harish-Chandra's work. We now come to the pair $(\mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n-1, \mathbb{R}))$.

THEOREM 2.3: *The semisimple symmetric pair* $(\mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n-1, \mathbb{R}))$ *is a generalized Gelfand pair for* $n \geqslant 3$.

To prove this theorem, we apply a very useful criterion, due to Thomas (see [12, Theorem E]).

PROPOSITION 2.4: *Let* J: $D'(G/H) \to D'(G/H)$ *be an anti-automorphism. If* $J\mathscr{H} = \mathscr{H}$ *(i.e.* $J \mid \mathscr{H}$ *anti-unitary) for all G-invariant or minimal G-invariant Hilbert subspaces of* $D'(G/H)$ *then* (G, H) *is a generalized Gelfand pair.*

The proof is rather easy and consists of showing (ii) of Proposition 2.2.

In our situation we take $JT = \bar{T}$. To show that J satisfies the conditions of Proposition 2.4, it suffices to show the following: any positive-definite bi-H-invariant distribution T on G satisfies $\check{T} = T$. Here $\langle T, \phi_0 \rangle = \langle T, \check{\phi}_0 \rangle$, $\check{\phi}_0(g) = \phi_0(g^{-1})$ $(g \in G,\ \phi_0 \in D(G))$. By desintegration (Proposition 1.3) one sees that T may even be assumed to be spherical. We shall use the notation of [7] from now on. There is a right H-invariant function Q on $G = \mathrm{SL}(n, \mathbb{R})$, defined by

$$Q(g) = [gx^0, x^0] = \text{trace } gx^0g^{-1}x^0$$

where

$$x^0 = \begin{pmatrix} 1 & & \\ & 0 & & \emptyset \\ & & \ddots & \\ & \emptyset & & 0 \end{pmatrix}$$

with the following property. Put $X = G/H$. For $\phi \in D(X)$ define $M\phi$ on $\mathbb{R}$ by

$$\int_X F(Q(x))\phi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} F(t)M\phi(t)\,\mathrm{d}t$$

for all $F \in C_c(\mathbb{R})$ (see [7, Section 7]). Call $\mathscr{K} = M(D(X))$ and put the usual topology on $\mathscr{K}$. (See also Section 4). Then by [7, section 8] any *spherical* T is of the form $T = M'S$ for some $S \in \mathscr{K}'$. So $\langle T, \phi \rangle = \langle S, M\phi \rangle$ for all $\phi \in D(X)$. Here T has to be regarded as an H-invariant distribution on X. More precisely, putting

$$f^{\#}(x) = \int_H f(gh)\,\mathrm{d}h \quad (x = gH,\ f \in D(G))$$

we have the equality $\langle T, f \rangle = \langle T, f^{\#} \rangle$, where on the right-hand-side T has to be regarded as an H-invariant distribution on X. The problem to be solved amounts to the relation $M(\check{f})^{\#} = Mf^{\#}$ for all $f \in D(G)$. For all $F \in C_c(\mathbb{R})$ one has

$$\begin{aligned}\int_{-\infty}^{\infty} F(t) M(\check{f})^{\#}(t)\,\mathrm{d}t &= \int_X F(Q(x))(\check{f})^{\#}(x)\,\mathrm{d}x \\ &= \int_G F(Q(g))\check{f}(g)\,\mathrm{d}g \\ &= \int_G F\big(Q(g^{-1})\big) f(g)\,\mathrm{d}g.\end{aligned}$$

Since $Q(g) = Q(g^{-1})$ we get the result and the proof of Theorem 2.3 is complete.

REMARK: (SL(2, $\mathbb{R}$), GL(1, $\mathbb{R}$)) is not a generalized Gelfand pair.

3. Invariant Hilbert subspaces of $L^2(G/H)$

We keep to the notations of Section 1. Let π be a unitary representation of G on a Hilbert space $\mathscr{H}$, which can be realized on a Hilbert subspace of $D'(G/H)$. Let j: $\mathscr{H} \to D'(G/H)$ be the corresponding injection. Define ξ_π, T, j^* as usual.

PROPOSITION 3.1: *The following conditions on π are equivalent*:

(*i*) $j(\mathscr{H}) \subset L^2(G/H)$

(*ii*) *There exists a constant $c > 0$ such that for all $\phi_0 \in D(G)$,* $|\langle T, \tilde{\phi}_0 * \phi \rangle| \leqslant c \|\phi\|_2^2$.

PROOF: (i) $\Rightarrow$ (ii): The map j: $\mathscr{H} \to L^2(G/H)$ is closed and everywhere defined on $\mathscr{H}$, hence continuous by the closed graph theorem. This implies that j^*: $D(G/H) \to \mathscr{H}$ is continuous in the L^2-topology of $D(G/H)$, so (ii) follows.

(ii) $\Rightarrow$ (i): Clearly (ii) implies that j^* is continuous with respect to the L^2-topology. Extend j^* to $L^2(G/H)$. Then clearly $j(\mathscr{H}) \subset L^2(G/H)$.

We shall say that π belongs to the *relative discrete series* of G (with respect to H) if π is irreducible and satisfies one of the conditions of Proposition 3.1. We shall occasionally use the terminology: π is *square-integrable* mod H.

PROPOSITION 3.2: *Let π be an irreducible unitary representation of G on $\mathscr{H}$, which can be realized on a Hilbert subspace of $D'(G/H)$. Let j: $\mathscr{H} \to D'(G/H)$ be the corresponding injection. The following statements are equivalent*:

(*i*) *π is square-integrable* mod H

(*ii*) *$j(\mathscr{H})$ is a closed linear subspace of $L^2(G/H)$*

(*iii*) *$j(v) \in L^2(G/H)$ for a non-zero element $v \in \mathscr{H}$.*

PROOF: It suffices to prove the implication (iii) $\to$ (ii). Let $P = \{w \in \mathscr{H}: j(w) \in L^2(G/H)\}$. Clearly P is a G-stable and non-zero linear subspace of $\mathscr{H}$, hence dense in $\mathscr{H}$. Now observe that j: $P \to L^2(G/H)$ is a closed linear operator: if $w_k \to w (w_k \in P, w \in \mathscr{H})$ and $jw_k \to f$ in $L^2(G/H)$, then obviously $j(w) \in D'(G/H)$ is equal to f as a distribution. Polar decomposition of j and applying Schur's Lemma yields: j can be extended to a continuous linear operator $\mathscr{H} \to L^2(G/H)$ with closed image (cf. [1, p. 48]).

REMARK: It also follows (see [1, p. 48]) that there is a constant $c > 0$ such that $\| jv \|_2 = c \| v \|$ for all $v \in \mathscr{H}$.

One has the following orthogonality relations.

PROPOSITION 3.3: *Let π, π' be irreducible unitary representations on $\mathscr{H}$, $\mathscr{H}'$, both belonging to the relative discrete series. Define T, T' and ξ_π, $\xi_{\pi'}$ as usual. Then one has*:

(*i*) $\int_{G/H} \langle \pi(x^{-1})v, \xi_\pi \rangle \overline{\langle \pi'(x^{-1})v', \xi_{\pi'} \rangle} \, dx = 0$ *for all* $v \in \mathscr{H}_\infty$, $v' \in \mathscr{H}'_\infty$ *if π is not equivalent to π'.*

(*ii*) *There exists a constant $d_\pi > 0$, only depending on T, such that*

$$\int_{G/H} \langle \pi(x^{-1})v, \xi_\pi \rangle \overline{\langle \pi(x^{-1})v', \xi_\pi \rangle} \, dx = d_\pi^{-1} \langle v, v' \rangle$$

for all $v, v' \in \mathscr{H}_\infty$.

To prove this proposition, one follows the well-known receipt to introduce the invariant hermitian form

$$(v, v') = \int_{G/H} \langle \pi(x^{-1})v, \xi_\pi \rangle \overline{\langle \pi'(x^{-1})v', \xi_{\pi'} \rangle} \, dx$$

on $\mathscr{H}_\infty \times \mathscr{H}'_\infty$. This form is continuous with respect to the topology on $\mathscr{H} \times \mathscr{H}'$. Schur's lemma now easily implies the result. The "only" dependency on T follows from the formula: $\| jj^*\phi_0 \|_2^2 = d_\pi^{-1} \| j^*\phi_0 \|^2$, so $\| \phi_0 * T \|_2^2 = d_\pi^{-1} \langle T, \tilde{\phi}_0 * \phi \rangle$ for all $\phi_0 \in D(G)$.

REMARK: Observe that $\| jv \|_2 = d_\pi^{-1/2} \| v \|$ for all $v \in \mathscr{H}_\infty$. So c, introduced before, is equal to $d_\pi^{-1/2}$.

The constant d_π is called the *formal degree* of π. It depends on the choice of j (or T). Once a canonical choice j (or T) is possible, d_π has a more realistic meaning.

EXAMPLE: Let G_1 be as usual. Let $G = G_1 \times G_1$ and $H = \mathrm{diag}(G)$. Let π be an irreducible unitary representation of G. π can be realized on a Hilbert subspace of $D'(G/H) \simeq D'(G_1)$ if π is of the form $\bar{\pi}_1 \otimes \pi_1$, where π_1 is an irreducible unitary representation of G_1 on $\mathscr{H}_1$ whose (distribution-) character θ_1 exists [12]. Actually, the reproducing distribution T associated with π, can be taken equal to θ_1. This is a canonical choice. The injection $j\colon \overline{\mathscr{H}_1} \hat{\otimes}_2 \mathscr{H}_1 \to D'(G_1)$ has the form

$$j(v \otimes w)(x) = \langle \pi(x^{-1})v, w \rangle \quad (x \in G_1,\ v,\ w \in \mathscr{H}_1).$$

In this case Propositions 3.2 and 3.3 yield the well-known properties of square-integrable representations of G_1 (cf. [1, 5.13–5.15]). Note that any square-integrable representation π_1 of G_1 has a distribution-character.

Let us assume that (G, H) is a generalized Gelfand pair. Denote by $E_2(G/H)$ the set of equivalence-classes of irreducible square-integrable representations mod H. Fix a representative π in each class, together with the realisation j_π on a Hilbert subspace of $L^2(G/H)$ and call this set of representatives S. Denote by T_π the reproducing distribution and by d_π the formal degree of π. Let $\mathscr{H}_\pi$ be the space of π. Define $\mathscr{H}_d = \oplus j_\pi(\mathscr{H}_\pi)$ and let E be the orthogonal projection of $L^2(G/H)$ ontol $\mathscr{H}_d$. Then one has the following (partial) Plancherel formula for the relative discrete series.

PROPOSITION 3.4: *For all* $\phi_0 \in D(G)$,

$$\| E\phi \|_2^2 = \sum_{\pi \in S} d_\pi \langle T_\pi, \tilde{\phi}_0 * \phi \rangle.$$

Notice that $E\phi \in C^\infty(G/H)$ for all $\phi_0 \in D(G/H)$. So the formula in Proposition 3.4 is equivalent to

$$(E\phi)(eH) = \sum_{\pi \in S} d_\pi \langle T_\pi, \phi_0 \rangle \quad (\phi_0 \in D(G)).$$

The above formulae do not depend on the choice of the set S: $d_\pi T_\pi$ is independent of the choice of π in its equivalence class and the choice of j_π. In fact, $d_\pi\langle T_\pi, \tilde{\phi}_0 * \phi\rangle = \| E_\pi\phi \|_2^2$, where E_π is the orthogonal projection of $L^2(G/H)$ onto $j_\pi(\mathscr{H}_\pi)$. Indeed, choose an orthogonal basis (e_i) in $\mathscr{H}_\pi$. Then $d_\pi^{1/2} j(e_i)$ is an orthogonal basis for $j_\pi(\mathscr{H}_\pi)$ and $\| E_\pi\phi \|_2^2 = \sum_i d_\pi |(je_i, \phi)|^2 = \sum_i d_\pi |(e_i, j^*\phi)|^2 = d_\pi \| j^*\phi \|^2 = d_\pi\langle T_\pi, \tilde{\phi}_0 * \phi\rangle$.

REMARK: The above proposition is easily extended to the case of finite multiplicity: $m_\pi = \dim \mathscr{H}_{-\infty}^H < \infty$ for all $\pi \in E_2(G/H)$. Indeed, we can choose for each π, $T_\pi^1, \ldots, T_\pi^{m(\pi)}$ such that the corresponding Hilbert subspaces are orthogonal (regarded as subspaces of $L^2(G/H)$) and the G-action is equivalent to π. Then the above formula reads;

$$\| E\phi \|_2^2 = \sum_{\pi \in S} \sum_{i=1}^{m(\pi)} d_\pi^i \langle T_\pi^i, \tilde{\phi}_0 * \phi\rangle \quad (\phi_0 \in D(G)).$$

Again $\sum_{i=1}^{m(\pi)} d_\pi^i \langle T_\pi^i, \tilde{\phi}_0 * \phi\rangle = \| E_\pi\phi \|_2^2$, where E_π is the ortogonal projection of $L^2(G/H)$ onto $\bigoplus_{i=1}^{m(\pi)} j_\pi^i(\mathscr{H}_\pi^i) = \mathrm{Cl}(\sum_{\pi' \sim \pi} j_{\pi'}(\mathscr{H}_{\pi'}))$. If (G, H) is a semisimple symmetric pair, then E.P. van den Ban [13] has recently shown that the T_π^i can be chosen in such a way that they are common eigendistributions of $\mathbb{D}(G/H)$, the algebra of G-invariant differential operators on G/H. However, different eigenvalues may occur.

4. The relative discrete series of $\mathrm{SL}(n, \mathbb{R})/\mathrm{GL}(n-1, \mathbb{R})$

We recall briefly some facts from [7].

Let $G = \mathrm{SL}(n, \mathbb{R})$, $H = \mathrm{S}(\mathrm{GL}(1, \mathbb{R}) \times \mathrm{GL}(n-1, \mathbb{R}))$, $n \geqslant 3$. (G, H) is a semisimple symmetric pair. Write $X = G/H$. Let x^0 be the $n \times n$ matrix given by $\begin{pmatrix} 1 & \\ & \emptyset \end{pmatrix}$. G acts on the space of real $n \times n$ matrices $M_n(\mathbb{R})$ by $g \cdot x = gxg^{-1}$ $(g \in G,\ x \in M_n(\mathbb{R}))$. X is naturally isomorphic to $G \cdot x^0 = \{x \in M_n(\mathbb{R})\colon \text{rank } x = \text{trace } x = 1\}$. We defined a function Q: $X \to \mathbb{R}$ by $Q(x) = [x, x^0]$, where $[x, y] = \text{trace } xy$ $(x, y \in M_n(\mathbb{R}))$. Q has the following properties:

a. Q is H-invariant.
b. x^0 is a non-degenerate critical point for Q. The Hessian of Q in this point has signature $(n-1, n-1)$.
c. Besides x^0, the set $\mathscr{S} \subset X$, consisting of elements of the form

$$x = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & T & \\ 0 & & \end{pmatrix}$$

with $T \in M_{n-1}(\mathbb{R})$, rank T = trace $T = 1$, is a critical set of Q. For each $x \in \mathscr{S}$ one can choose coordinates $x_1, \dots, x_{2n-2}$ near x such that $Q = x_1 x_2$ and $\mathscr{S}$ is given by $x_1 = x_2 = 0$.

d. If $x \neq x^0$ and $x \notin \mathscr{S}$ then x is not a critical point of Q.
e. Q is real analytic.
f. Q assumes all real values.
g. For $\lambda \neq 0, 1$, $Q(x) = \lambda$ is an H-orbit.
h. $Q(x) = 1$ consists of 4 H-orbits.
i. $Q(x) = 0$ consists of 3 H-orbits.

Define for $f \in D(X)$ the function Mf on $\mathbb{R}$ by the property

$$\int_X F(Q(x)) f(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} Mf(t) F(t) \, \mathrm{d}t$$

for all $F \in C_c^\infty(\mathbb{R})$. Put $\mathscr{K} = M(D(X))$. $\mathscr{K}$ consists of all functions of the form

$$\phi(t) = \phi_0(t) + \phi_1(t) \log|t| + \phi_2(t) \eta(t)$$

where

$$\phi_i \in D(\mathbb{R}) \quad \text{and} \quad \eta(t) = \begin{cases} Y(t-1)(t-1)^{n-2} & \text{if } n \text{ is odd,} \\ (t-1)^{n-2} \log|t-1| & \text{if } n \text{ is even,} \end{cases}$$

Y being the Heaviside function: $Y(t) = 1$ if $t \geqslant 0$, $Y(t) = 0$ if $t < 0$.

Let $\square_0$ be the Laplace-Beltrami operator on X and put $\square = 2\square_0$. One can topologize $\mathscr{K}$ in such a way that

a. M: $D(X) \to \mathscr{K}$ is continuous.
b. Any H-invariant eigendistribution T of $\square$ is of the form $T = M'S$ for some $S \in \mathscr{K}'$.
c. $\square \cdot M' = M' \cdot L$, where L is the second order differential operator on $\mathbb{R}$, given by $L = a(t)\mathrm{d}^2/\mathrm{d}t^2 + b(t)\mathrm{d}/\mathrm{d}t$, with $a(t) = 4t(t-1)$, $b(t) = 4(nt-1)$.

Denote $D'_{\lambda,H}(X)$ the space of H-invariant eigendistributions T of $\square$ on X with eigenvalue λ.

Proposition 4.1: dim $D'_{\lambda,H}(X) = 2$ *for all* $\lambda \in \mathbb{C}$.

This is shown in [7, Proposition 7.10]. Let $T \in D'_{\lambda,H}(X)$. Then there is a continuous linear form S on $\mathscr{K} = M(D(X))$ satisfying $LS = \lambda S$ such that

$$T(f) = S(Mf) \quad (f \in D(X)).$$

Since $x \to Q(x)$ is submersive on $X - \{x^0) - \mathscr{S}$, S is actually a distribution on $\mathbb{R} \setminus \{0, 1\}$. Since L is elliptic there, we see that S is an analytic function $\mathbb{R} \setminus \{0, 1\}$. By abuse of notation we shall also call it S. Now notice that $\mathrm{L}u = \lambda u$ is a hypergeometric differential equation. Let $\lambda = s^2 - \rho^2$ ($s \in \mathbb{C}$) with $\rho = n - 1$. In [7, section 8] we have given a basis $M'S_0$ and $M'S_2$ if n is odd, $M'S_0$ and $M'S_1$ if n is even, of $D'_{\lambda,H}(X)$, for all s satisfying $\operatorname{Im} s \neq 0$. It is not difficult to extend the results in a natural way to all s with $\operatorname{Re}(s) \geqslant 0$. If $s \in \mathbb{N}$, analytic continuation does not work, one has to construct the distributions S_0, S_1 and S_2 by the method of [7, Appendix 2 and Section 8]. Instead of giving full details we shall describe the asymptotics of S_0, S_1 and S_2 as $t \to \pm\infty$.
I. $s \notin \mathbb{Z}$, $\operatorname{Re}(s) > 0$. ([7, Lemma 8.1])

$$
\begin{aligned}
S_i(t) &\sim d_{i,+}(s) t^{\frac{1}{2}(s-\rho)} && (t \to \infty) \\
S_i(t) &\sim d_{i,-}(s)(-t)^{\frac{1}{2}(s-\rho)} && (t \to -\infty) \quad (i = 0, 1, 2)
\end{aligned}
$$

where

$$
d_{0,+}(s) = \frac{\Gamma(\rho)\Gamma(s)}{\Gamma\left(\frac{1}{2}(s+\rho)\right)^2},
$$

$$
d_{1,+}(s) = (-1)^{\rho-1} \frac{\Gamma(s)\Gamma\left(\frac{1}{2}(s-\rho)\right)}{\Gamma(\rho-1)\Gamma\left(\frac{1}{2}(s-\rho+2)\right)} \cdot \cos \pi\left(\tfrac{1}{2}(s-\rho)\right),
$$

$$
d_{2,+}(s) = 0
$$

$$
d_{0,-}(s) = \frac{\Gamma(\rho)\Gamma(s)}{\Gamma\left(\frac{1}{2}(s+\rho)\right)^2} \cos \pi\left(\tfrac{1}{2}(s-\rho)\right),
$$

$$
d_{1,-}(s) = (-1)^{\rho-1} \frac{\Gamma(s)\Gamma\left(\frac{1}{2}(\rho-s)\right)}{\Gamma(\rho-1)\Gamma\left(\frac{1}{2}(s-\rho+2)\right)},
$$

$$
d_{2,-}(s) = -\pi \frac{\Gamma\left(\frac{1}{2}(\rho-s)\right)\Gamma(s) \cos \pi\left(\frac{1}{2}(s-\rho)\right)}{\Gamma\left(\frac{1}{2}(\rho+s)\right)\Gamma(\rho-1)\Gamma\left(\frac{1}{2}(s-\rho+2)\right)\Gamma\left(\frac{1}{2}(-s-\rho+2)\right)}.
$$

II. $\operatorname{Re}(s) = 0$, $s \neq 0$

$$
\begin{aligned}
S_i(t) &\sim d_{i,+}(s) t^{\frac{1}{2}(s-\rho)} + d_{i,+}(-s) t^{\frac{1}{2}(-s-\rho)} && (t \to \infty) \\
S_i(t) &\sim d_{i,-}(s)(-t)^{\frac{1}{2}(s-\rho)} + d_{i,-}(-s)(-t)^{\frac{1}{2}(-s-\rho)} && (t \to -\infty)
\end{aligned}
$$

($i = 0, 1, 2$), where $d_{i,\pm}(s)$ are as before.

Here we apply [3, formula (36) on page 107].
We now split up the cases n odd and n even.

III. (odd) $s = \rho + 2k + 1$, $s > 0$ ($k \in \mathbb{Z}$)
Put $v = \rho + k + \frac{1}{2}$ then for $t \to \infty$

$$S_0(t) \sim \frac{\Gamma(\rho)\Gamma(2k+1)}{\Gamma(v)^2} t^{k+\frac{1}{2}}$$

$$S_0(-t) \sim \frac{\Gamma(\rho)\Gamma\left(k+\frac{3}{2}\right) \sin v\pi}{\Gamma(\rho+2k)\Gamma\left(-k-\frac{1}{2}\right)} t^{-v}$$

The latter formula follows from [3, p. 109, formula (7)].

$$S_2(t) = 0$$

$$S_2(-t) \sim \frac{-2\pi^2}{\Gamma(\rho-1)\Gamma(1-v)\Gamma(\rho+2k)} t^{-v}$$

IV (odd) s even, $s = \rho + 2l$.

a. $l \geqslant 0$

S_0 is a polynomial of degree l with leading coefficient

$$\frac{\Gamma(\rho+2l)\Gamma(\rho)}{\Gamma(l+\rho)^2}$$

$$S_2(t) = 0 \quad (t \to \infty)$$

$$S_2(t) = \pi\alpha S_0(t) \quad (t \to -\infty),$$

where

$$\alpha = -\frac{\Gamma(\rho+l)^2}{\Gamma(l+1)^2\Gamma(\rho-1)\Gamma(\rho)}$$

b. $s = \rho + 2l$, $0 < s < \rho$ (so $l < 0$)

$$S_0(t) \sim \frac{\Gamma(\rho+2l)\Gamma(\rho)}{\Gamma(l+\rho)^2} t^l \quad (|t| \to \infty)$$

$$S_2(t) = 0 \quad (|t| \to \infty)$$

c. $s = \rho + 2l = 0$.

By [3, p. 110, formula (11)] we get

$$S_0(t) \sim \frac{(-1)^{l+1}\Gamma(\rho)}{\Gamma(\frac{1}{2}\rho)^2} t^{-\frac{1}{2}\rho} \log|t| + O(|t|^{-\frac{1}{2}\rho}) \quad (|t| \to \infty)$$

$$S_2(t) = 0 \quad (|t| \to \infty).$$

III (even) a. $s = \rho + 2k + 1$, $s > 0$ ($k \in \mathbb{Z}$).
Put $v = \rho + k + \frac{1}{2}$ then for $t \to \infty$

$$s_0(t) \sim \frac{\Gamma(\rho)\Gamma(\rho + 2k + 1)}{\Gamma(v)^2} t^{k+\frac{1}{2}}$$

$$S_0(-t) \sim (-1)^{k+1} 2 \frac{\Gamma(\rho)\Gamma(k + \frac{3}{2}) \sin v\pi}{\Gamma(\rho + 2k)\Gamma(-k - \frac{1}{2})} t^{-v}$$

$$S_1(t) \sim \frac{(-1)^{k+1}\pi\Gamma(v)^2}{\Gamma(\rho - 1)\Gamma(-k - \frac{1}{2})\Gamma(v + k + \frac{1}{2})\Gamma(k + \frac{3}{2})} t^{-v}$$

(apply [3, p. 109, formula (7)])

$$S_1(-t) \sim \frac{\Gamma(\rho + 2k + 1)\Gamma(-k - \frac{1}{2})}{\Gamma(\rho - 1)\Gamma(k + \frac{3}{2})} t^{k+\frac{1}{2}}$$

b. $s = 0$

$$S_0(t) \sim \frac{\Gamma(\rho)}{\Gamma(\frac{1}{2}\rho)^2} t^{-\frac{1}{2}\rho} \log|t| + 0(t^{-\frac{1}{2}\rho}) \quad (t \to \infty)$$

$$S_0(t) \sim 2\pi \sin(\pi\tfrac{1}{2}\rho) \frac{\Gamma(\rho)}{\Gamma(\frac{1}{2}\rho)^2} (-t)^{-\frac{1}{2}\rho} \qquad (t \to -\infty)$$

$$S_1(t) \sim \frac{2\pi \sin \pi\frac{1}{2}\rho}{\Gamma(1 - \frac{1}{2}\rho)\Gamma(\rho - 1)} \cdot \Gamma(\tfrac{1}{2}\rho) t^{-\frac{1}{2}\rho} \quad (t \to \infty)$$

$$S_1(t) \sim \frac{\Gamma(\frac{1}{2}\rho)}{\Gamma(1 - \frac{1}{2}\rho)\Gamma(\rho - 1)} (-t)^{-\frac{1}{2}\rho} \log|t| + O(|t|^{-\frac{1}{2}\rho})$$

$(t \to -\infty)$

(apply [3, p. 109, formula (7)]).

IV (even) $s = \rho + 2l$.

a. $l \geqslant 0$.

S_0 is a polynomial of degree l with leading coefficient

$$\frac{\Gamma(\rho+2l)\Gamma(\rho)}{\Gamma(l+\rho)^2}$$

$$S_1(t) \sim \frac{\Gamma(l+\rho)^2}{\Gamma(\rho-1)\Gamma(2l+\rho+1)\Gamma(l+1)} |t|^{-l-\rho} \quad (|t| \to \infty).$$

Here we apply [4, p. 170, formula (18)].

b. $s = \rho + 2l$, $0 < s < \rho$ (so $l < 0$).

$$S_0(t) \sim \frac{\Gamma(\rho+2l)\Gamma(\rho)}{\Gamma(l+\rho)^2} t^l \quad (|t| \to \infty).$$

By [3, p. 110, formula (14)] we get

$$S_1(t) \sim \frac{\Gamma(\rho+l)^2}{\Gamma(\rho+2l+1)\Gamma(\rho-1)} t^{-\rho-l} \quad (|t| \to \infty).$$

Let $K = \mathrm{SO}(n, \mathbb{R})$ and

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & \sinh t & & & \emptyset \\ \sinh t & \cosh t & & & \\ & & 1 & & \\ & \emptyset & & \ddots & \\ & & & & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

We quote the following lemma [7, Lemma 6.1].

LEMMA 4.2: *Every element $x \in X$ can be written as $x = ka_t x^0$ with $t \geqslant 0$ and $k \in K$. Then t is uniquely determined, and if $t > 0$ then k is determined uniquely modulo $M \cap K$, M being the centralizer of A in H. We can normalize the invariant measure* $\mathrm{d}x$ *on X in such a way that*

$$\int_X \phi(x)\,\mathrm{d}x = \int_K \int_0^\infty \phi(ka_t x^0) A(t)\,\mathrm{d}k\,\mathrm{d}t$$

for all $\phi \in D(X)$. Here $A(t) = \sinh^{n-2}(2t)\cosh(2t)$.

Let π be a non-necessarily irreducible unitary representation of G which can be realized on a Hilbert subspace of $L^2(X)$. Let T be the reproducing distribution on X and ξ_π the cyclic distribution vector associated with π. So

$$\langle T, \phi\rangle = \langle \xi_\pi, \pi_{-\infty}(\phi_0)\xi_\pi\rangle \quad (\phi_0 \in D(G)).$$

For $g \in G$, $\phi \in D(X)$ define

$$\phi_g(x) = \phi(g\cdot x) \quad (x \in X).$$

Then $\langle T, \phi_g\rangle = \langle \xi_\pi, \pi(g^{-1})\pi_{-\infty}(\phi_0)\xi_\pi\rangle$, and $g \to \langle T, \phi_g\rangle$ actually is a C^∞-function on X, which belongs to $L^2(X)$. Now apply Lemma 4.2 to conclude: for almost all $k \in K$ the function

$$\tau \to |\langle T, (\phi_k)_{a_\tau}\rangle|^2 A(\tau)$$

is in $L^1(0, \infty)$.
Since $A(\tau) \sim 2^{-\rho} e^{2\rho\tau}$ $(\tau \to \infty)$ we get

LEMMA 4.3: *The function*

$$\tau \to e^{\rho\tau}\langle T, (\phi_k)_{a_\tau}\rangle$$

is in $L^2(0, \infty)$ *for almost all* $k \in K$.

We now assume, in addition, that $T \in D'_{\lambda,H}(X)$ for certain $\lambda \in \mathbb{C}$. (This is clearly so if π is irreducible).
Then we can write

$$T = \alpha M' S_0 + \beta M' S$$

for some $\alpha, \beta \in \mathbb{C}$, where $S = S_2$ if n is odd, $S = S_1$ if n is even. For any $\psi \in D(X)$ with Supp $\psi \subset \{x\colon [x, x^0] > 1\}$ one has

$$\langle M' S_0, \psi\rangle = \int_0^\infty S_0(t) M\psi(t)\, dt = \int_X S_0([x, x^0])\psi(x)\, dx.$$

Put, as usual, $\xi^0 = \begin{pmatrix} 1 & & -1 & \\ & & & \emptyset \\ 1 & & -1 & \\ & \emptyset & & \emptyset \end{pmatrix} \in M_n(\mathbb{R})$ and $P_0(x) = [x, \xi^0]$ $(x \in X)$. Because of $P_0(x) = 4 \lim_{\tau\to\infty} e^{-2\tau}[x, a_\tau x^0]$ for all $x \in X$, we have Supp $\psi_{a_\tau} \subset \{x \in X\colon [x, x^0] > 1\}$ for $\tau \to \infty$, provided Supp $\psi \subset$

$\{x\colon P_0(x)>0\}$. Hence $\langle M'S_0,\, \psi_{a_\tau}\rangle = \int_X S([x,\, a_\tau x^0])\psi(x)\,\mathrm{d}x\ (\tau\to\infty)$.
Put $\lambda = s^2-\rho^2$, $\mathrm{Re}(s)\geqslant 0$.
Applying the results on the asymptotic behaviour of S_0, derived before, we obtain

(i) $\mathrm{Re}(s)>0$

$$\langle M'S_0,\, \psi_{a_\tau}\rangle \sim \text{const.} \int_X P_0(x)^{\frac{1}{2}(s-\rho)}\psi(x)\,\mathrm{d}x\cdot \mathrm{e}^{(s-\rho)\tau}$$

(ii) $\mathrm{Re}(s)=0,\ s\neq 0$

$$\langle M'S_0,\, \psi_{a_\tau}\rangle \sim c_1\cdot \int_X P_0(x)^{\frac{1}{2}(s-\rho)}\psi(x)\,\mathrm{d}x\cdot \mathrm{e}^{(s-\rho)\tau}$$

$$+c_2\cdot \int_X P_0(x)^{\frac{1}{2}(-s-\rho)}\psi(x)\,\mathrm{d}x\cdot \mathrm{e}^{(-s-\rho)\tau}$$

(iii) $s=0$

$$\langle M'S_0,\, \psi_{a_\tau}\rangle \sim \text{const.} \int_X P_0(x)^{-\frac{1}{2}\rho}\psi(x)\,\mathrm{d}x\cdot \tau\mathrm{e}^{-\rho\tau}$$

where the constants are non-zero and $\tau\to\infty$.
We can clearly find a $\psi\in D(X)$ as above of the form ϕ_k, so that $\tau\to \mathrm{e}^{\rho\tau}\langle T,\,(\phi_k)_{a_\tau}\rangle$ is in $L^2(0,\,\infty)$ (Lemma 4.3)
If n is odd, it follows now easily from (i), (ii) and (iii) that $\alpha=0$, so $T=\beta M'S_2$.

If n is even then $\alpha=0$ if $s=\rho+2l$ ($l\in\mathbb{Z}$, $s>0$) or $s=\rho+2l+1$ ($l\in\mathbb{Z}$, $s\geqslant 0$). In the other cases we get a linear relation between α and β.

A similar analysis for $\psi\in D(X)$ with Supp $\psi\subset\{x\colon P_0(x)<0\}$, using the asymptotic behaviour of S_0, S_1, S_2 for $\tau\to-\infty$, yields extra conditions on s.

The results are as follows.

THEOREM 4.4: *Let π be a unitary representation of G, realized on a Hilbert subspace of $L^2(X)$. Let T be the reproducing distribution of π and assume $T\in D'_{\lambda,H}(X)$. Then T is uniquely determined up to scalar multiplication. More precisely, if $\lambda=s^2-\rho^2$ with* $\mathrm{Re}(s)\geqslant 0$, *then we have*:
n odd:
s is in the set

$$\left\{\begin{matrix} s\ \textit{odd}, & s=\rho+2k+1,\ s>0 \\ s\ \textit{even}, & s=\rho+2k,\ ,\vec{0}\leqslant s<\rho \end{matrix}\right\}.$$

Moreover T is a scalar multiple of $M'S_2$.
n even:
s is in the set

$$\{s \text{ odd},\ s = \rho + 2l,\ s > 0\}$$

Moreover T is a scalar multiple of $M'S_1$.

It will turn out that the s, specified above, give indeed rise to a parametrization of the relative discrete series representations, provided we take s *odd*. So in the case n odd, the even s, $0 \leqslant s < \rho$, do not contribute to the discrete spectrum. The proof of this fact is obtained by performing the spectral resolution of the Laplace-Beltrami operator $\square$. This is the contents of section 5.

5. The Plancherel formula for $SL(n, \mathbb{R})/GL(n-1, \mathbb{R})$

The Plancherel formula is obtained by determining the spectral resolution of the self-adjoint extension $\tilde{\square}$ of the Laplace-Beltrami operator $\square$ on X. Our method is similar to Faraut's [5, Première démonstration de la formule de Plancherel].

We shall only provide the calculations for n even; for n odd they are essentially the same. So from now on n is even, unless otherwise stated.

Let $\lambda = s^2 - \rho^2 (s \in \mathbb{C})$ and define functions $S_{0,s}$ and $S_{1,s}$ on $\mathbb{R} \setminus \{0, 1\}$ as follows:

$$S_{0,s}(t) = \begin{cases} {}_2F_1\left(\frac{1}{2}(\rho+s), \frac{1}{2}(\rho-s); \rho; 1-t\right) & (t>0) \\ \frac{1}{2}\left[{}_2F_1\left(\frac{1}{2}(\rho+s), \frac{1}{2}(\rho-s); \rho; 1-t-i0\right)\right. & \\ \quad \left. + {}_2F_1\left(\frac{1}{2}(\rho+s), \frac{1}{2}(\rho-s); \rho; 1-t+i0\right)\right] & (t<0) \end{cases}$$

$$S_{1,s}(t) = \begin{cases} {}_2F_1\left(\frac{1}{2}(\rho+s), \frac{1}{2}(\rho-s); 1; t\right) & (t<1) \\ \frac{1}{2}\left[{}_2F_1\left(\frac{1}{2}(\rho+s), \frac{1}{2}(\rho-s); 1; t+i0\right)\right. & \\ \quad \left. + {}_2F_1\left(\frac{1}{2}(\rho+s), \frac{1}{2}(\rho-s); 1; t-i0\right)\right] & (t>1). \end{cases}$$

Then $S_{0,s}$ and $S_{1,s}$ correspond to a basis of $D'_{\lambda,H}(X)$, provided $\lambda \neq 4r(r + \rho)$, $r \in \mathbb{N}$.
Observe that $S_{0,s}$ and $S_{1,s}$ are even in s.

The above definition of $S_{1,s}$ differs a factor $(-1)^{\rho-1} \dfrac{\Gamma(\frac{1}{2}(\rho+s))\Gamma(\frac{1}{2}(\rho-s))}{\Gamma(\rho-1)}$ from the definition in [7, Section 8], which we used in the previous section.

The new definition is more convenient to work with here.
If $s_r = \rho + 2r$ $(r \in \mathbb{N})$, then

$$S_{0,s_r} = (-1)^r \frac{\Gamma(r+1)\Gamma(\rho)}{\Gamma(\rho + r)} S_{1,r}.$$

For these s_r, define

$$U_r = \frac{\mathrm{d}}{\mathrm{d}s}\left[(-1)^{r-1} \frac{\Gamma(\frac{1}{2}(s+\rho))}{\Gamma(\frac{1}{2}(s-\rho+2))\Gamma(\rho)} S_{0,s} + S_{1,s}\right]_{s=s_r}.$$

Then S_{1,s_r} and U_r correspond to a basis of $D'_{\lambda,H}(X)$ with $\lambda = 4r(r+\rho)$. We shall use this basis, which differs by constants from the one in the previous section.

Let $\zeta_{0,s}$ and $\zeta_{1,s}$ be the H-invariant eigendistributions of $\square$ on X defined in [7, section 4].

Let S_s^0 and S_s^1 be the continuous linear forms on $\mathscr{K} = M(D(X))$ defined by $M'S_s^0 = \zeta_{0,s}$ and $M'S_s^1 = \zeta_{1,s}$.

PROPOSITION 5.1 [7, Theorem 8.5]: *If* Im $s \neq 0$ *then*

$$S_s^0 = A_{0,0}(s) S_{0,s} + A_{0,1}(s) S_{1,s}$$

$$S_s^1 = A_{1,0}(s) S_{0,s} + A_{1,1}(s) S_{1,s}$$

with

$$A_{0,0}(s) = \frac{2^{-\rho+1}}{\Gamma(\frac{1}{4}(s-\rho+2))^2 \Gamma(\frac{1}{4}(-s-\rho+2))^2 \cos \pi\frac{1}{2}(s)}$$

$$A_{0,1}(s) = A_{0,0}(s)\big[\Gamma(\rho)\Gamma(\tfrac{1}{2}(s-\rho+2))\Gamma(\tfrac{1}{2}(-s-\rho+2)) \times \sin \pi(\tfrac{1}{2}(s+\rho))\big]\pi^{-1}$$

$$A_{1,0}(s) = -2^{-1-\rho}[\cos \pi\tfrac{1}{2}s]\Big[\Gamma(\tfrac{1}{4}(s-\rho+4))^2\Gamma(\tfrac{1}{4}(-s-\rho+4))^2 \times \sin^2\pi(\tfrac{1}{4}(s+\rho))\sin^2\pi(\tfrac{1}{4}(s-\rho))\Big]^{-1}$$

$$A_{1,1}(s) = A_{1,0}(s)\big[\Gamma(\rho)\Gamma(\tfrac{1}{2}(s-\rho+2))\Gamma(\tfrac{1}{2}(-s-\rho+2)) \times \sin \pi(\tfrac{1}{2}(s+\rho))\big]\pi^{-1}.$$

By analytic continuation the above proposition remains true for all $s \in \mathbb{C}$, $s \neq \pm s_r$ ($s_r = \rho + 2r$, $r \in \mathbb{N}$). Taking the limit $s \to s_r$ in the above expressions yield:

LEMMA 5.2:
(1) *for r even*

$$S^0_{s_r} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}s} S^0_s \mid_{s=s_r} = \frac{(-1)^{\frac{1}{2}(\rho+1)} 2^{-\rho-1} \Gamma(r+1)\Gamma(\rho)\Gamma\left(\frac{1}{2}(\rho+r+1)\right)^2}{\pi\Gamma(\rho+r)\Gamma\left(\frac{1}{2}(r+1)\right)^2} \times S_{1,s_r}$$

$$S^1_{s_r} = \frac{(-1)^{\frac{1}{2}(\rho+1)} 2^{2-\rho} \Gamma(r+1)\Gamma(\rho)}{\pi\Gamma(\rho+r)\Gamma\left(\frac{1}{2}(r+2)\right)^2 \Gamma\left(\frac{1}{2}(-\rho-r+2)\right)^2} U_r$$

(2) *for r odd*

$$S^0_{s_r} = -\frac{(-1)^{\frac{1}{2}(\rho+1)} 2^{2-\rho} \Gamma(r+1)\Gamma(\rho)}{\pi\Gamma(\rho+r)\Gamma\left(\frac{1}{2}(r+1)\right)^2 \Gamma\left(\frac{1}{2}(-\rho-r+1)\right)^2} U_r$$

$$S^1_{s_r} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}s} S^1_s \mid_{s=s_r} = -\frac{(-1)^{\frac{1}{2}(\rho+1)} 2^{-\rho-1} \Gamma(r+1)\Gamma(\rho)\Gamma\left(\frac{1}{2}(\rho+2)\right)^2}{\pi\Gamma(\rho+r)\Gamma\left(\frac{1}{2}(r+2)\right)^2} \times S_{1,s_r}.$$

Let us denote by $E(\mathrm{d}\lambda)$ the spectral measure of $\tilde{\square}$. For every $h \in C_c(\mathbb{R})$ annd ϕ, $\psi \in D(X)$ one has

$$\int_{-\infty}^{\infty} h(\lambda)\langle E(\mathrm{d}\lambda)\phi, \psi\rangle = -\frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} h(\lambda)\langle (R_{\lambda+i\epsilon} - R_{\lambda-i\epsilon})\phi, \psi\rangle \, \mathrm{d}\lambda$$

where R_λ is the resolvent $(\lambda I - \tilde{\square})^{-1}$, $\operatorname{Im} \lambda \neq 0$.

This formula is the starting point towards the Plancherel formula. R_λ has the following properties:

(i) $\langle R_\lambda \Box\phi, \psi\rangle = \langle R_\lambda\phi, \Box\psi\rangle = \bar{\lambda}\langle R_\lambda\phi, \psi\rangle - \langle\phi, \psi\rangle$ for all $\phi, \psi \in D(X)$.

Since R_λ commutes with the G-action on $L^2(X)$ and because of the continuity of the injection $D(X) \to L^2(X)$, there is an unique H-invariant distribution T_λ on X satisfying

$$\langle R_\lambda\phi, \psi\rangle = \langle T_\lambda, \tilde{\phi}_0 * \psi\rangle \quad \text{for all } \phi_0, \psi_0 \in D(G)$$

T_λ satisfies:

(ii) $(\lambda - \Box)T_\lambda = \delta(x^0)$.

(iii) $|\langle T_\lambda, \tilde{\phi}_0 * \psi\rangle| \leqslant \frac{1}{|\text{Im}\,\lambda|}\|\phi\|_2\|\psi\|_2 \quad (\phi_0, \psi_0 \in D(G))$.

The following lemma is crucial, and is also valid for n odd.

LEMMA 5.3: *Let T_λ be an H-invariant distribution on X satisfying $(\lambda - \Box)T_\lambda = \delta(x^0)$. There is an unique element $K_\lambda \in \mathscr{K}'$ such that $M'K_\lambda = T_\lambda$.*

PROOF: Put $X_1 = \{x \in X: Q(x) < 1\}$, $X_2 = \{x \in X: Q(x) > 0\}$. On X_1, T_λ satisfies $(\lambda - \Box)T_\lambda = 0$. By [7, section 7] there exists an unique continuous linear form $\tilde{K}_\lambda$ on $M(D(X_1))$ such that $M'\tilde{K}_\lambda = T_\lambda$ on X_1. The restriction of T_λ to X_2 is H-invariant, hence $T_\lambda = M'K_\lambda$ for an unique continuous linear form on $M(D(X_2))$ (same reference). Clearly $K_\lambda = \tilde{K}_\lambda$ on (0, 1). This provides the extension of K_λ to an element of $\mathscr{K}'$ satisfying $M'K_\lambda = T_\lambda$. The uniqueness of K_λ is clear, since M is surjective.

Choose K_λ as in Lemma 5.3, such that $M'K_\lambda = T_\lambda$ for $\text{Im}\,\lambda \neq 0$. Then K_λ is a solution of the equation

$$\lambda K_\lambda - LK_\lambda = \frac{1}{c}B_0$$

with $c = (-1)^{\frac{1}{2}(\rho+1)}\pi^{\rho-1}4/\Gamma(\rho)$, $B_0 \in \mathscr{K}'$ defined by $B_0(\phi_0 + \phi_1 \log|t| + \phi_2\eta) = \overline{\phi_2(1)}$. Thus

$$K_\lambda = a(s)S_{0,s} + b(s)S_{1,s} - \frac{\alpha(s)}{c\mu}Y(t-1)S_{0,s} - \frac{1}{c\mu}E$$

with Y the Heaviside function, $\mu = \rho - 1$, $\lambda = s^2 - \rho^2$,

$$\alpha(s) = -\frac{\Gamma(\frac{1}{2}(\rho+s))\Gamma(\frac{1}{2}(\rho-s))}{\Gamma(\frac{1}{2}(s-\rho+2))\Gamma(\frac{1}{2}(-s-\rho+2))\Gamma(\rho-1)\Gamma(\rho)}$$

and $E \in \mathscr{K}'$ as in [7, section 7].

Since $M'K_\lambda$ satisfies the inequality (iii) for $\operatorname{Im}\lambda \neq 0$, the coefficients of $t^{\frac{1}{2}(s-\rho)}$ and $(-t)^{\frac{1}{2}(s-\rho)}$ in the asymptotic expansion of K_λ for $t \to \infty$ and $t \to -\infty$ respectively, vanish for $s \notin \mathbb{R}$, $s \notin \mathrm{i}\mathbb{R}$ (cf. section 4). This yields:

$$a(s) = -\left[\Gamma\left(\tfrac{1}{2}(\rho+s)\right)\Gamma\left(\tfrac{1}{2}(\rho-s)\right)\right] \times\left[c\Gamma\left(\tfrac{1}{2}(s-\rho+2)\right)\Gamma\left(\tfrac{1}{2}(-s-\rho+2)\right)\Gamma(\rho)^2 \times \sin^2\pi\left(\tfrac{1}{2}(s-\rho)\right)\right]^{-1}$$

$$b(s) = \frac{\Gamma\left(\tfrac{1}{2}(\rho-s)\right)\cos\pi\left(\tfrac{1}{2}(s-\rho)\right)}{c\Gamma\left(\tfrac{1}{2}(-s-\rho+2)\right)\Gamma(\rho)\sin^2\pi\left(\tfrac{1}{2}(s-\rho)\right)}$$

$(s \notin \mathbb{R}, \mathrm{i}\mathbb{R})$.

LEMMA 5.4: *Extending $a(s)$, $b(s)$ and $\alpha(s)$ to meromorphic functions on $\mathbb{C}$ we get, for $\mu \in \mathbb{R}$*

(*i*) $\operatorname{Im} a(\mathrm{i}\mu) = \operatorname{Im}\alpha(\mathrm{i}\mu) = 0$

(*ii*) $\operatorname{Im} b(\mathrm{i}\mu) = \Gamma(\tfrac{1}{2}(\rho-\mathrm{i}\mu))\Gamma(\tfrac{1}{2}(\rho+\mathrm{i}\mu)\tanh\tfrac{1}{2}\mu\pi/c\pi\Gamma(\rho)$

We are now prepared to calculate the spectral resolution of $\tilde{\square}$. This implies a special type of resolution of the identity operator on $L^2(X)$. The resolution contains a continuous and discrete part. The continuous part is given by:

$$\langle\phi,\phi\rangle_{c\cdot p} = \frac{(-1)^{\frac{1}{2}(\rho-1)}}{4}\pi^{1-\rho}\int_0^\infty \Gamma\left(\tfrac{1}{2}(\rho-\mathrm{i}\mu)\right)\Gamma\left(\tfrac{1}{2}(\rho+\mathrm{i}\mu)\right) \times\tanh\tfrac{1}{2}\mu\pi\langle M'S_{1,\mathrm{i}\mu},\tilde{\phi}_0*\phi\rangle\,\mathrm{d}\mu \quad (\phi_0\in D(G)).$$

The discrete part, denoted by $\langle\phi,\phi\rangle_{d\cdot p}$, corresponds to $\lambda \geqslant -\rho^2$ and consists of point-measures located at $s_r = \rho + 2r \geqslant 0$ $(r\in\mathbb{Z})$. An explicit calculation of

$$\lim_{\epsilon\to 0} -\frac{1}{2\pi\mathrm{i}}\langle M'(K_{\lambda+\mathrm{i}\epsilon}-K_{\lambda-\mathrm{i}\epsilon}),\tilde{\phi}_0*\phi\rangle \quad (\phi_0\in D(G))$$

shows

$$\langle\phi,\phi\rangle_{d\cdot p} = \sum_{\substack{r<0\\ s_r\geqslant 0}} (-1)^{\frac{1}{2}(\rho-1)}\pi^{-1-\rho}s_r\Gamma(-r)\Gamma(\rho+r)\langle M'S_{1,s_r},\tilde{\phi}_0*\phi\rangle + \sum_{r\geqslant 0}(-1)^{\frac{1}{2}(\rho+1+2r)}\pi^{-1-\rho}2s_r\frac{\Gamma(\rho+r)}{\Gamma(1+r)}\langle M'U_r,\tilde{\phi}_0*\phi\rangle.$$

Combining these formulae, transforming them to $\zeta_{0,s}$ and $\zeta_{1,s}$ with Proposition 5.1 and Lemma 5.2 and simplifying them afterwards, yields

$$\langle\phi,\phi\rangle = \frac{2^{3\rho-3}\pi^{-\rho}}{\Gamma(\rho)}\Big\{\int_0^\infty\big[\Gamma(\tfrac{1}{4}(\rho+\mathrm{i}\mu))^2\Gamma(\tfrac{1}{4}(\rho-\mathrm{i}\mu))^2\Gamma(\tfrac{1}{2}(1+\mathrm{i}\mu))$$

$$\times\Gamma(\tfrac{1}{2}(1-\mathrm{i}\mu))\big]\big[\Gamma(\tfrac{1}{2}\mathrm{i}\mu)\Gamma(-\tfrac{1}{2}\mathrm{i}\mu)\big]^{-1}\langle\zeta_{0,\mathrm{i}\mu},\tilde{\phi}_0*\phi\rangle\,\mathrm{d}\mu$$

$$+\int_0^\infty\big[\Gamma(\tfrac{1}{4}(\rho+\mathrm{i}\mu+2))^2\Gamma(\tfrac{1}{4}(\rho-\mathrm{i}\mu+2))^2\Gamma(\tfrac{1}{2}(1+\mathrm{i}\mu))$$

$$\times\Gamma(\tfrac{1}{2}(1-\mathrm{i}\mu))\big]\big[\Gamma(\tfrac{1}{2}\mathrm{i}\mu)\Gamma(-\tfrac{1}{2}\mathrm{i}\mu)\big]^{-1}\langle\zeta_{1,\mathrm{i}\mu},\tilde{\phi}_0*\phi\rangle\,\mathrm{d}\mu$$

$$+\sum_{\substack{s_r\geqslant 0\\ r\text{ odd}}} s_r\Gamma(\tfrac{1}{2}(\rho+r))^2\Gamma(-\tfrac{1}{2}r)^2\langle\zeta_{0,s_r},\tilde{\phi}_0*\phi\rangle$$

$$+\sum_{\substack{s_r\geqslant 0\\ r\text{ even}}} s_r\Gamma(\tfrac{1}{2}(\rho+r+1))^2\Gamma(\tfrac{1}{2}(1-r))^2\langle\zeta_{1,s_r},\tilde{\phi}_0*\phi\rangle\Big\}$$

$$(\phi_0\in D(G)).$$

We now come to the Plancherel formula for SL(n, $\mathbb{R}$)/GL($n-1$, $\mathbb{R}$), $n\geqslant 3$. First we have to introduce the analogues of the Harish-Chandra c-function, called c_0 and c_1.

By [7, section 8] we have for $\phi\in D(X)$ with Supp $\phi\subset\{x\in X\colon P(x,\xi^0)>0\}$

$$\lim_{t\to\infty} 2^{(s-\rho)}\,\mathrm{e}^{(-s+\rho)t}\zeta_{0,s}(\phi_{a_t}) = c_{0,+}\Gamma(\tfrac{1}{4}(s-\rho+2))^2{}_0\hat{\phi}(\xi^0,s)$$

and

$$\lim_{t\to\infty} 2^{(s-\rho)}\,\mathrm{e}^{(-s+\rho)t}\zeta_{1,s}(\phi_{a_t}) = c_{1,+}\Gamma(\tfrac{1}{4}(s-\rho+4))^2{}_1\hat{\phi}(\xi^0,s).$$

Put

$$c_0(s) = c_{0,+}\cdot\Gamma(\tfrac{1}{4}(s-\rho+2))^2\left(\frac{2^\rho\pi}{\Gamma(\tfrac{1}{2}\rho)\Gamma(\tfrac{1}{2}n)}\right)\frac{\Gamma(\tfrac{1}{2}(-s-\rho+n))}{\Gamma(\tfrac{1}{2}(s-\rho+n))}$$

and

$$c_1(s) = -c_{1,+}\Gamma(\tfrac{1}{4}(s-\rho+4))^2\left(\frac{2^{\rho}\pi}{\Gamma(\frac{1}{2}\rho)\Gamma(\frac{1}{2}n)}\right)\frac{\Gamma(\frac{1}{2}(-s-\rho+n))}{\Gamma(\frac{1}{2}(s-\rho+n))}.$$

So

$$c_0(s) = \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{4}(\rho+s))^2\Gamma(\frac{1}{2}(1+s))},$$

$$c_1(s) = \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{4}(\rho+s+2)^2\Gamma(\frac{1}{2}(1+s))}.$$

THEOREM 5.5: *Put* $s_r = \rho + 2r$ *if* n *is even,* $s_r = \rho + 2r + 1$ *if* n *is odd* $(r \in \mathbb{Z})$. *The Plancherel formula for* $\mathrm{SL}(n, \mathbb{R})/\mathrm{GL}(n-1, \mathbb{R})$ $(n \geqslant 3)$ *is given by*

$$c.\int_X |\phi(x)|^2 \,\mathrm{d}x = \frac{1}{2\pi}\int_0^\infty \langle \zeta_{0,\mathrm{i}\mu}, \tilde{\phi}_0 * \phi\rangle \frac{\mathrm{d}\mu}{|\bar{c}_0(\mathrm{i}\mu)|^2}$$

$$+\frac{1}{2\pi}\int_0^\infty \langle \zeta_{1,\mathrm{i}\mu}, \tilde{\phi}_0 * \phi\rangle \frac{\mathrm{d}\mu}{|c_1(\mathrm{i}\mu)|^2}$$

$$+\sum_{\substack{s_r \geqslant 0 \\ r\ \mathrm{odd}}} \langle \zeta_{0,s_r}, \tilde{\phi}_0 * \phi\rangle \operatorname{Res}\left[\frac{1}{c_0(s)c_0(-s)}, s = s_r\right]$$

$$+\sum_{\substack{s_r \geqslant 0 \\ r\ \mathrm{even}}} \langle \zeta_{1,s_r}, \tilde{\phi}_0 * \phi\rangle \operatorname{Res}\left[\frac{1}{c_1(s)c_1(-s)}, s = s_r\right]$$

for all $\phi_0 \in D(G)$, where $c = \dfrac{\Gamma(\rho)}{2^{3\rho-4}\pi^{1-\rho}}$.

PROOF: We have to show that the positive-definite spherical distributions $\zeta_{0,\mathrm{i}\mu}$, $\zeta_{1,\mathrm{i}\mu}(\mu \geqslant 0)$, ζ_{0,s_r} $(s_r \geqslant 0,\ r$ odd) and ζ_{1,s_r} $(s_r \geqslant 0,\ r$ even), are extremal. Otherwise stated: they correspond in a natural manner to irreducible unitary representations of G, as explained in Section 1. Let $\phi \to_0 \hat{\phi}$ and $\phi \to_1 \hat{\phi}$ be the Fourier transforms defined in [7, p. 22] Now consider $\zeta_{0,i\mu}(\mu \in \mathbb{R})$. One has $\zeta_{0,i\mu}(\tilde{\phi}_0 * \phi) = \int_B |{}_0\hat{\phi}(b, i\mu)|^2 \, db$ ($\phi_0 \in D(G)$). A similar formula holds for $\zeta_{1,i\mu}$. By [7, section 6] there is a K-invariant function $\phi \in D(X)$ such that the Fourier transform ${}_0\hat{\phi}(\cdot, i\mu) \neq 0$ for all $\mu \in \mathbb{R}$, but ${}_1\hat{\phi}(\cdot, i\mu) = 0$ for all $\mu \in \mathbb{R}$. A similar fact holds for ${}_1\hat{\phi}$: there is a K-finite function $\phi \in D(X)$ such that ${}_1\hat{\phi}(\cdot, i\mu) \neq 0$ for all $\mu \in \mathbb{R}$, but ${}_0\hat{\phi}(\cdot, i\mu) = 0$ for all $\mu \in \mathbb{R}$.

This implies that $\zeta_{0,i\mu}$ and $\zeta_{1,i\mu}$ form a basis for $D'_{\lambda,H}(X)$ for all $\mu \in \mathbb{R}$; here $\lambda = -\mu^2 - \rho^2$. Moreover, if $\zeta \in D'_{\lambda,H}(X)$ is positive definite (as an H-invariant distribution on G), and $\zeta = c_0\zeta_{0,i\mu} + c_1\zeta_{1,i\mu}$ then clearly $c_0 \geqslant 0$, $c_1 \geqslant 0$. This implies that $\zeta_{0,i\mu}$ and $\zeta_{1,i\mu}$ correspond to irreducible unitary representations for all $\mu \in \mathbb{R}$. For $\mu \neq 0$ we may take $\pi_{0,i\mu}$ and $\pi_{1,i\mu}$ for these representations [7, Proposition 3.2]. If $\mu = 0$ we can take the natural representation on the closure of $\{{}_j\hat{\phi}(\cdot, 0)\colon \phi \in D(X)\}$ in $L^2(B)$ for $j = 0, 1$ respectively. We now come to the ζ_{0,s_r} (r odd) and ζ_{1,s_r} (r even). They are extremal because the $T \in D'_{\lambda,H}(X)$ ($\lambda = s_r^2 - \rho^2$), which correspond to a unitary representation with a realization on $L^2(X)$, span a 1-dimensional subspace in $D'_{\lambda,H}(X)$ by Theorem 4.4 (It is possible to realize the corresponding irreducible unitary representations as a subquotient of π_{0,s_r} and π_{1,s_r} respectively.)

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