COMPOSITIO MATHEMATICA

ALEXANDRU DIMCA

On the homology and cohomology of complete intersections with isolated singularities

Compositio Mathematica, tome 58, nº 3 (1986), p. 321-339
<http://www.numdam.org/item?id=CM_1986__58_3_321_0>

[1] Compositio Mathematica 58 (1986) 321–339

ON THE HOMOLOGY AND COHOMOLOGY OF COMPLETE INTERSECTIONS WITH ISOLATED SINGULARITIES

Alexandru Dimca

Let V be a complex *projective* complete intersection having only isolated singularities at the points a_i for $i = 1, \dots, k$.

In this paper we investigate to what extent the *integral* homology and the cohomology algebra of V can be determined from the local information associated to the singularities (V, a_i). The relevant local properties of the singularities (V, a_i) turn out to be contained in their Milnor lattices L_i.

In the first section we recall some facts on the homology of a smooth complete intersection. The only new result here is maybe the observation that the middle homology group of such a complete intersection (regarded as a lattice with the intersection form) contains in a natural way a *reduced* Milnor lattice $\bar{L} = L/\text{Rad } L$ (see (1.4), (1.5)).

In the next section we prove the main result of the paper (Theorem 2.1) which says that the homology of V is determined by a morphism of lattices

$$\varphi_V : L_1 \oplus \dots \oplus L_k \to \bar{L}$$

more precisely, by its kernel and cokernel. (We denote by $\oplus$ orthogonal direct sums.)

In fact, the source and the target lattices depend only on the local information at the singular points, the dimension and the multidegree of V. The global unknown information on which the homology of V depends (e.g. the position of the singularities on V) is contained in the application φ_V itself.

The third section is arithmetic in nature. Assuming the lattices L_i nondegenerate, we show that the torsion part $T(V)$ of the homology of V belongs to a finite set of groups which is computable entirely in terms of the lattices L_i, using the formalism of discriminant (bilinear) forms associated to lattices.

In particular, we compute the discriminant forms associated to the hypersurface simple singularities A_m, D_m and E_m. Using these computations, we are able to prove that $T(V) = 0$ in a lot of concrete situations.

The results in this section were worked out jointly with my student A. Nemethi.

In the forth section we compute the homology of two classes of varieties: the cubic surfaces in $\mathbb{P}^3$ and some $2m$-dimensional complete intersections of two quadrics. In these favourable cases the arithmetic of the lattices L_t and $\bar{L}$ determines completely the application φ_V.

In the next section we consider two complete intersections V_0 and V_1 with isolated singularities at the points a_i such that the singularities (V_0, a_i) and (V_1, a_i) are isomorphic for $i = 1, \dots, k$. We give in this situation a sufficient condition such that V_0 and V_1 have the same homology (Proposition 5.1). Also we show how this result can be used to relate the morphisms φ_V to the adjacency of singularities. This device is basic when the lattices L_t and $\bar{L}$ give too little information on φ_V, as for instance in the simple case of an odd-dimensional A_1-singularity.

In the final section we point out how the morphism φ_V determines the cohomology algebra of V with coefficients in an unitary ring. This provides us with a finer tool for showing the non-homotopy equivalence of some varieties (see (6.3)). Conversely, in the case of surfaces, this result combined with Whitehead classification of simply connected 4-dimensional CW-complexes, can be used to show that certain surfaces are homotopy equivalent (see (6.4), (6.5)).

1. The homology of smooth complete intersections

In this section we recall some facts concerning the (integral) homology groups $H_i(V)$ of a smooth complete intersection $V \subset \mathbb{C}\mathbb{P}^{n+p}$ with dim $V = n$ and multidegree $(d_1, \dots, d_p)$.

It is well known that $H_i(V) = H_i(\mathbb{C}\mathbb{P}^n)$ except for $H_n(V)$ which is a free group of rank $b_n(V)$, computable in terms of $d_1, \dots, d_p$ [13].

If $P_1 = \dots = P_p = 0$ are the homogeneous equations of V, then there is a natural S^1-bundle $p: K \to V$, where $K = \{x \in \mathbb{C}^{n+p+1}; |x| = 1, P_1(x) = \dots = P_p(x) = 0\}$.

We call p the *Milnor S^1-bundle* of the variety V and note that it gives a Gysin sequence in homology

$$\dots \to H_i(K) \xrightarrow{p_*} H_i(V) \xrightarrow{u_i} H_{i-2}(V) \to H_{i-1}(K) \to \dots \tag{1.1}$$

For n odd, u_{n+1} is multiplication by $d = d_1 \cdot \ldots \cdot d_p = \deg V$. For n even, the same is true for $u_n \circ u_{n+2}$.

Take now a hyperplane H in $\mathbb{C}\mathbb{P}^{n+p}$ such that $W = V \cap H$ is smooth and let us denote by N a tubular neighborhood of W in V. It is easy to see that the associated S^1-bundle $\partial N \to W$ is equivalent to the Milnor S^1-bundle of the variety W (use the equivalence between S^1-bundles and complex line bundles).

Next we prove the following basic result.

LEMMA 1.2: *The affine variety $U = V \setminus W$ is homeomorphic to the Milnor fiber of the singularity $(X, 0) = $ (cone over W, vertex).*

PROOF: Assume that $x_0 = 0$ is an equation for the hyperplane H. Then $U \subset \mathbb{C}^{n+p}$ is given by equations $f_i(\bar{x}) = 0$ for $i = 1, \dots, p$, where $\bar{x} = (x_1, \dots, x_{n+p})$, $f_i(\bar{x}) = P_i(1, \bar{x}) = Q_i(\bar{x}) + Q_i'(\bar{x})$ with Q_i a homogeneous polynomial of degree d_i and Q_i' a polynomial of degree $< d_i$.

Note that $Q_1 = \dots = Q_p = 0$ are the equations of the singularity $(X, 0)$.

Next take $y = \bar{x} \cdot r^{-1}$ for a real number r and denote

$$U_r = \{ y \in \mathbb{C}^{n+p};\ f_i(ry) = 0 \qquad \text{for} \quad i = 1, \dots, p \}.$$

It follows that $U_r \cap B_\epsilon \underset{\sim}{\overset{m}{\rightarrow}} U \cap B_{\epsilon r}$, where $B_\epsilon = \{ y \in \mathbb{C}^{n+p};\ | y | < \epsilon \}$ and m is multiplication by r. We have also $y \in U_r$ if and only if

$$Q_i^r(y) = Q_i(y) + Q_i'(ry) \cdot r^{-d_i} = 0 \qquad \text{for} \quad i = 1, \dots, p.$$

For r big enough, the equation $Q_i^r = 0$ is nothing else but a small deformation of the equation $Q_i = 0$.

It follows that $U_r \cap B_\epsilon$ is the Milnor fiber of the isolated singularity of complete intersection $(X, 0)$ [16].

On the other hand, for any algebraic (possibly singular) set $Z \subset \mathbb{C}^N$, it is known that $Z \cap B_r$ is homeomorphic to Z for r big enough. □

The Mayer-Vietoris sequence corresponding to the cover $V = U \cup N$ contains the morphism

$$j = (j_1, j_2) : H_n(U \cap N) \rightarrow H_n(U) + H_n(N)$$

and we want to interpret the components j_i of this morphism after we replace $U \cap N$ by $\partial N \simeq K'$ = the total space of the Milnor S^1-bundle of W, U by $\bar{X}$ = the Milnor fiber of $(X, 0)$ and N by the variety W.

We denote by L the Milnor lattice $(H_n(\bar{X}), (\ ,\ ))$ where the intersection form $(\ ,\ )$ is symmetric for n even and skew-symmetric for n odd. For any such lattice L, we denote by Rad L the sublattice $\{ x \in L;\ (x, y) = 0$ for any $y \in L \}$ and call the quotient $\bar{L} = L/\text{Rad } L$ with the induced biliniar form the *reduced* Milnor lattice of the singularity $(X, 0)$.

With these notations, j_1 can be identified with the natural inclusion

$$H_n(K') = \text{Rad } L \rightarrow L$$

and j_2 can be read off the corresponding Gysin exact sequence (1.1).

We get thus the exact sequence

$$0 \rightarrow \bar{L} + H_n(W) \overset{s}{\rightarrow} H_n(V) \rightarrow H_{n-1}(K') \overset{p_*}{\rightarrow} H_{n-1}(W) \tag{1.3}$$

COROLLARY 1.4: *For n odd, the lattice* $(H_n(V), \langle\ ,\ \rangle)$, *where* $\langle\ ,\ \rangle$ *denotes the intersection form, is isomorphic to the reduced Milnor lattice* $\overline{L}$. *In particular,* $\overline{L}$ *is unimodular.*

PROOF: In this case $H_n(W) = 0 = \ker p_*$, as can be seen from the corresponding exact sequence (1.1). □

Note that by recent work of Chmutov [6], the fact that an odd-dimensional hypersurface singularity has unimodular reduced Milnor lattice has interesting consequences on the monodromy group.

For n even, let $h \in H_n(V)$ be the class of the cycle corresponding to the intersection of V with a generic $(p+n/2)$-plane. Alternatively, $h = u_{n+2}(g)$ where g is the canonical generator of $H_{n+2}(V)$.

Then it is well-known that $\langle h, h \rangle = d$ and let us denote $h^{\perp}$ the orthogonal complement of h with respect to the intersection form $\langle\ ,\ \rangle$.

COROLLARY 1.5: *With the notations above, there is a lattice isomorphism* $\overline{L} \simeq h^{\perp}$.

PROOF: When n is even, in the exact sequence (1.3) one has $H_n(W) = \mathbb{Z}$, $\ker p_* = \mathbb{Z}/d\mathbb{Z}$.

The inclusion $W \subset V$ gives an identification $s(H_n(W)) = \mathbb{Z}.h$. The geometric description of the cycle h shows that $s(\overline{L}) \subset h^{\perp}$. A simple computation proves that the lattice $\mathbb{Z}h + h^{\perp}$ has index d in $H_n(V)$ and hence $s(\overline{L}) = h^{\perp}$. □

Note that the obvious consequences of (1.5) that $h^{\perp}$ is an even lattice and that any cycle $c \in h^{\perp}$ can be represented by an embedded sphere $S^n \subset V$ have been proved by completely different means in [15] (2.1).

On the other hand, it follows from (1.5) that sign $\overline{L} =$ sign $V - 1$, $rk\overline{L} = b_n(V) - 1$ and these relations can be used to get upper bounds for the number of singularities which may occur on a complete intersection of given dimension and multidegree [7].

2. Complete intersections with isolated singularities

We assume from now on that V is a complete intersection in $\mathbb{CP}^{n+p}$ with dim $V = n$, multidegree $(d_1, \dots, d_p)$ and having only isolated singularities at the points a_i for $i = 1, \dots, k$. If H is a hyperplane such that $W = V \cap H$ is smooth, we denote again by $(X, 0)$ the singularity defined by the cone over W.

Then, exactly as in the proof of (1.1), one can show that $U = V \setminus W$ is homeomorphic to a *singular* fiber in a deformation of the singularity $(X, 0)$, arbitrary close to the special fiber X.

By a result in the new book of Looijenga [16] (7.13), we deduce that U has the homotopy type of a bouquet of n-spheres and there is a natural exact sequence

$$0 \to L_1 \oplus \ldots \oplus L_k \overset{i_V}{\to} L \to H_n(U) \to 0$$

where L_i (resp. L) is the Milnor lattice of the singularity (V, a_i) for $i = 1, \ldots, k$ (resp. $(X, 0)$) and the inclusion i_V respects the intersection forms i.e. it is a morphism of lattices.

Let N be a tubular neighbourhood of W in $V \setminus \{a_1, \ldots, a_k\}$. The Mayer-Vietoris sequence corresponding to the cover $V = U \cup N$ contains the morphism

$$j = (j_1, j_2) : H_n(U \cap N) \to H_n(U) + H_n(N).$$

As in the smooth case treated in section 1, we can identify $U \cap N \simeq \partial N$ with the total space K' of the Milnor S^1-bundle of W and then the component

$$j_2 = p_* : H_n(K') \to H_n(N) = H_n(W)$$

can be read off from the corresponding sequence (1.1).

The other component j_1 corresponds to the composition

$$H_n(K') = \text{Rad } L \hookrightarrow L \to H_n(U) = L/im i_V$$

In particular, j_1 has the same kernel and cokernel as the morphism φ_V defined by the composition of the inclusion i_V with the natural projection $L \to \bar{L} = L/\text{Rad } L$. The advantage of replacing the morphism j_1 by φ_V is that the latter is a morphism of lattices.

We can state now our main result.

THEOREM 2.1:

(i) $H_i(V) = H_i(\mathbb{CP}^n)$ *for* $i \neq n, n+1$

(ii) $H_n(V) = H_n(\mathbb{CP}^n) + \text{coker } \varphi_V$

$H_{n+1}(V) = H_{n+1}(\mathbb{CP}^n) + \ker \varphi_V$, *where "+" denotes direct sum of groups.*

PROOF: The point (i) is clear from the fact that U is a bouquet of n-spheres and from the Gysin sequence (1.1) applied to W.

The second part (ii) must be proved separately for n odd and for n even. We give the details only for the case n even, which is slightly more delicate than the other case.

Assuming n even, one has $H_{n+1}(W)=0$ and the Mayer-Vietoris sequence above gives $H_{n+1}(V)=\ker j$ and the exact sequence

$$0 \to \operatorname{coker} j \overset{s}{\to} H_n(V) \to H_{n-1}(K') \overset{p_*}{\to} H_{n-1}(W) \tag{2.2}$$

Now, $j_2=0$ and hence $\ker j=\ker j_1=\ker \varphi_V$, which proves the second part of (ii).

Moreover, $\operatorname{coker} j=\operatorname{coker}\varphi_V+H_n(W)$ and $\ker p_* = Z/dZ$.

There is a natural S^1-bundle $q: K \to V$ constructed as in section 1 and by a result of Hamm [11] K is $(n-1)$-connected. If $a=i^*b$, where $i: V \subset \mathbb{CP}^{n+p}$ and $b \in H^2(\mathbb{CP}^{n+p})$ is the standard generator, then the morphism $u_m: H_m(V) \to H_{m-2}(V)$ which occurs in the Gysin sequence of q is precisely the cap-product with a. In particular, from

$$H_n(V) \overset{u_n=u}{\to} H_{n-2}(V)=\mathbb{Z} \to H_{n-1}(K)=0$$

it follows that there is an element $h_0 \in H_n(V)$ such that $u(h_0)=1$. We deduce that $H_n(V)=\ker u+\mathbb{Z}h_0$.

Let $g \in H_n(W)$ be a generator and $h=i_{0*}(g)$ where $i_0: W \subset V$. Then, by comparing the Gysin sequences for W and V, it follows that $u(h)=\pm d$ and hence $h=\pm dh_0+h'$ for some $h' \in \ker u$.

On the other hand, it is clear that $s(\operatorname{coker}\varphi_V)$ is contained in $\ker u$ and hence

$$\operatorname{im} s \subset \ker u+\mathbb{Z}h$$

But these two inclusions must be equalities since the index of $\ker u+\mathbb{Z}h$ in $H_n(V)$ is equal to d, which is the index of $\operatorname{im} s$ in $H_n(V)$ by (2.2). □

As trivial consequences of the Theorem we have the following

COROLLARY 2.3:

(i) $H_{n+1}(V)$ *is torsion free*
(ii) $\chi(V)=\chi(V_0)+(-1)^{n+1}\sum_{i=1,k}\mu(V, a_i)$

where χ *denotes the Euler-Poincaré characteristic,* V_0 *is a smooth complete intersection with the same dimension and multidegree as* V *and* $\mu(V, a_i)=rkL_i=$ *the Milnor number of the singularity* (V, a_i).

The difficulty in applying Theorem (2.1) in concrete cases consists in the fact that one does not known how to identify the embedding i_V starting, let's say, from the equations of V. The rest of the paper is devoted to various devices to circumvent this difficulty.

REMARK 2.4: The reduced Milnor lattice $\bar{L}$ is the same for all the complete intersections V of given dimension and multidegree (and for all choices of a smooth hyperplane section W) by μ-constant deformations arguments. Sometimes, information about $\bar{L}$ can be obtained using (1.4) and (1.5).

3. Singularities with nondegenerate Milnor lattices

First we recall some basic definitions concerning lattices. By a *lattice* $(M, (\ , \ ))$ we mean a finitely generated free abelian group M together with a bilinear form $(\ , \ )$ on M which is either

(i) symmetric and even i.e. $(x, x) \equiv 0 \pmod 2$ for any $x \in M$, or

(ii) skew - symmetric.

We call the lattice M *nondegenerate* if Rad $M = 0$. In this case the natural homomorphism of groups

$$i_M : M \to M^* = \mathrm{Hom}(M, \mathbb{Z}), \quad i_M(x) = (x, \cdot)$$

is an embedding and the *discriminant group D(M)* of M is by definition the finite group coker i_M.

The natural number det M equal to the order of the group $D(M)$ is an important numerical invariant of the lattice M.

In particular, the lattice M is called *unimodular* if det $M = 1$. If M is a sublattice of another lattice N such that $rk\ M = rk\ N$ (equivalently, the index $[N: M]$ is finite) then one has the equality

$$\det M = [N: M]^2 \det N. \tag{3.1}$$

Now we come back to our main problem (and to the notations from the previous section).

LEMMA 3.2: (i) *If the Milnor lattices of all the singular points $a_i \in V$ are nondegenerate, then $H_{n+1}(V) = H_{n+1}(\mathbb{CP}^n)$ and the rank of $H_n(V)$ is $b_n(V) = b_n(V_0) - \sum_{i=1,k} \mu(V, a_i)$.*

(ii) *If the Milnor lattices of all the singular points $a_i \in V$ are unimodular, then in addition $H_n(V)$ is a free group.*

PROOF: This result can be easily derived either from (2.1) or using the simple observation that V is a $\mathbb{Q}$-homology (resp. a $\mathbb{Z}$-homology) manifold in case (i) (resp. in case (ii)). Then the Poincaré duality over $\mathbb{Q}$ (resp. over $\mathbb{Z}$) and the formula for $\chi(V)$ in (2.3) give the result.

EXAMPLES 3.3: If $n = \dim V$ is even, then the Milnor lattices of the simple hypersurface singularities A_m, D_m, E_6, E_7 and E_8 are nondegenerate. More precisely: det $A_m = m + 1$, det $D_m = 4$, det $E_m = 9 - m$ and hence, in particular, E_8 is unimodular. (Note that we use the same symbol for a singularity and its Milnor lattice!). These and many other examples of even-dimensional hypersurface singularities with nondegenerate Milnor lattices can be found in Ebeling paper [9].

In odd dimensions, we can use the stabilization of singularities (i.e. addition of a square to the defining equation of a hypersurface singularity) and the relation between the corresponding intersection matrices [10] to show that the lattices A_{2m}, E_6 and E_8 are unimodular, while the lattices corresponding to the other simple hypersurface singularities are degenerate.

The first examples in this range of dimensions of nondegenerate lattices which are not unimodular are those corresponding to the singularities $T_{p,q,r}(p,\ q,\ r$ odd) and $Q_{k,i}(i$ even) which have det $T_{p,q,r} =$ det $Q_{k,i} = 4$ [6].

Further examples can be found in the papers of Brieskorn [3] and Hamm [12].

We assume from now on in this section that all the singularities $(V,\ a_i)$ have nondegenerate Milnor lattices L_i. Then the morphism φ_V is an embedding and the only unknown part of the homology of V is the finite torsion group

$$T(V) = \text{Tors } H_n(V).$$

We show now that the lattices L_i put strong arithmetic restrictions on the group $T(V)$.

The arithmetic problem is the following: given a nondegenerate lattice M, to describe the set of finite groups

$$T(M) = \{\text{Tors } (N/M); \quad N \text{ is a supralattice of } M, \text{ i.e. } M \subset N\}.$$

It is a simple observation that in this definition we can take only supralattices N with $rkN = rkM$. Using this fact and the formula (3.1) we get the following simple, but useful result.

COROLLARY 3.4: *If $F \in T(M)$, then $|F|^2$ divides det M, where $|F|$ denotes the order of F. Moreover, if we are in the symmetric case and sign $(M) \not\equiv 0$ (mod 8), then $|F|^2 \neq$ det M.*

PROOF: Since $|F| = [N : M]$, the first assertion is clear. The second part follows from the fact that an even symmetric lattice N with sign$(N) \not\equiv 0$ (mod 8) cannot be unimodular [17]. □

To describe more accurately the set $T(M)$ we can use the bilinear discriminant form of M, which we define now. The bilinear form on M

can be extended in a natural way to a bilinear form $M^* \times M^* \to \mathbb{Q}$ and this induces a bilinear form $b: D(M) \times D(M) \to \mathbb{Q}/\mathbb{Z}$ which is called the *bilinear discriminant form* of M[8].

If N is a supralattice of M such that $F(N) = N/M$ is a finite group, then there is a natural inclusion of $F(N)$ in $D(M)$ as an isotropic subgroup i.e. $b \mid F(N) \times F(N) = 0$. Moreover, this correspondence defines a bijection between supralattices N of M with $F(N)$ finite and isotropic subgroups in $D(M)$ [8] (1.4.4). Hence $T(M)$ is the same as the set of isotropic subgroups in $D(M)$ and can be computed if the bilinear discriminant form of M is known. And this discriminant form can be computed in most of the cases using the obvious fact $D(M_1 \oplus M_2) = D(M_1) \oplus D(M_2)$.

In the skew-symmetric case, any nondegenerate lattice M is an orthogonal direct sum of elementary lattices $M_d = \mathbb{Z}\langle e_1, e_2 \rangle$ with $(e_i, e_i) = 0$ and $(e_1, e_2) = d$. Direct computations show that $D(M_d) = (\mathbb{Z}/d\mathbb{Z})^2$ and $b((\hat{a}_1, \hat{b}_1), (\hat{a}_2, \hat{b}_2)) = (a_1b_2 - a_2b_1)d^{-1} \in \mathbb{Q}/\mathbb{Z}$. In particular, it follows that $T(M) = 0$ if and only if M is unimodular.

In the symmetric case the results are much more interesting. The bilinear discriminant form b of an even lattice M is determined by the corresponding quadratic form $q: D(M) \to \mathbb{Q}/2\mathbb{Z}$, $q(x + M) = (x, x) + 2\mathbb{Z}$ and the isotropic subgroups $F \subset D(M)$ are the subgroups F for which $q \mid F = 0$.

In the next results we determine the quadratic forms corresponding to the singularities A_m, D_m and E_m and give some direct consequences.

The proofs consist of simple but tedious computations involving the corresponding intersection forms and are not given here.

PROPOSITION 3.5:

(i) $D(A_m) = \mathbb{Z}/(m+1)\mathbb{Z}$ *and* $q(\hat{1}) = -m(m+1)^{-1}$.

(ii) $T(A_m) = \{\mathbb{Z}/e\mathbb{Z};\ e \mid m+1$ *and* $m(m+1)e^{-2}$ *is an even integer*$\}$.

As an example, for $1 \leqslant m \leqslant 20$ the only nontrivial sets $T(A_m)$ are the following

$$T(A_7) = T(A_{15}) = \{0, \mathbb{Z}/2\mathbb{Z}\}$$

$$T(A_8) = T(A_{17}) = \{0, \mathbb{Z}/3\mathbb{Z}\}.$$

PROPOSITION 3.6:

(i) *For* m *even*, $D(D_m) = (\mathbb{Z}/2\mathbb{Z})^2$ *and the generators* u_1, u_2 *of* $D(D_m)$ *can be chosen such that* $q(u_1) = -1$ *and* $q(u_2) = q(u_1 + u_2) = 0$, $6/4$, 1, $1/2$ *according to the cases* $m \equiv 0, 2, 4, 6 \pmod 8$.

(ii) *For* m *odd*, $D(D_m) = \mathbb{Z}/4\mathbb{Z}$ *and* $q(\hat{1}) = 7/4$, $5/4$, $3/4$, $1/4$ *according to the cases* $m \equiv 1, 3, 5, 7 \pmod 8$.

In particular, $T(D_m) = \{0, \mathbb{Z}/2\,\mathbb{Z}\}$ *for* $m \equiv 0 \pmod 8$ *and* $T(D_m) = \{0\}$ *otherwise*.

PROPOSITION 3.7:
(i) $D(E_6) = \mathbb{Z}/3\mathbb{Z}$ *and* $q(\hat{1}) = 2/3$
(ii) $D(E_7) = \mathbb{Z}/2\mathbb{Z}$ *and* $q(\hat{1}) = 1/2$
(iii) $D(E_8) = 0$
In particular, $T(E_m) = \{0\}$.

These results give obviously information on $T(V)$ when the even-dimensional variety V has precisely one singular point of type A_m (resp. D_m or E_m) and some other singularities with unimodular Milnor lattices. These unimodular lattices and the corresponding singularities will not be mentioned at all in what follows, since their presence do not affect the group $T(V)$ as follows from (2.1). But we can also handle with these techniques the case of several singularities, as is shown in the next two examples.

EXAMPLE 3.8: Assume that V has p singularities of type A_1, q singularities of type D_{m_i} (for various even integers m_i) and r singularities of type E_7. Then $T(V) = (\mathbb{Z}/2\mathbb{Z})^S$, where $2S \leqslant p + 2q + r$. Moreover, if $p + \Sigma m_i + 7r \not\equiv 0 \pmod 8$, the inequality above is strict.

PROOF: Let M be the lattice $pA_1 \oplus rE_7 \oplus D_{m_1} \oplus \dots \oplus D_{m_q}$. Then $D(M) = (\mathbb{Z}/2\mathbb{Z})^t$, where $t = p + 2q + r$. Since any subgroup $F \subset D(M)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space, it follows that $F = (\mathbb{Z}/2\mathbb{Z})^S$. Then use (3.4). □

EXAMPLE 3.9: If the singularities on V are described by a symbol in the following list, then $T(V) = 0$:

$$2A_1,\ 3A_1,\ A_1A_3,\ A_2A_3,\ 2A_1A_2,\ 2A_2A_1$$

PROOF: Let us treat for example the case A_1A_3. First note that we cannot use only (3.4) to get the result! Using (3.5) we find out that $D(A_1A_3) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $q(\hat{a}, \hat{b}) = -\frac{1}{2}a^2 - \frac{3}{4}b^2$. A simple verification shows that $q(\hat{a}, \hat{b}) = 0$ in $\mathbb{Q}/2\mathbb{Z}$ if and only if $\hat{a} = \hat{b} = 0$. Hence the only isotropic subgroup in $D(A_1A_3)$ is the trivial group.

4. Two geometric examples

In this section we show that Theorem (2.1) can be used to determine exactly the (nontrivial) torsion group $T(V)$ for the cubic surfaces in $\mathbb{CP}^3$ and for some types of complete intersections of two quadrics.

The singularities which may occur on a cubic surface $V \subset \mathbb{CP}^3$ (with isolated singularities) are the following [4]

$$A_1,\ 2A_1,\ 3A_1,\ \underline{4A_1};\ A_2,\ 2A_2,\ \underline{3A_2};\ A_1A_2,\ A_1A_3,\ A_1A_4,$$

$$\underline{A_1A_5};\ 2A_1A_2,\ \underline{2A_1A_3};\ 2A_2A_1;\ A_3,\ A_4,\ A_5;\ D_4,\ D_5,\ E_6.$$

All the groups of singularities in this list, except the four underlined ones, can give no torsion by the results in the previous section. The next proposition shows that these four cases give indeed nontrivial torsion in homology.

PROPOSITION 4.1: *The homology of a cubic surface $V \subset \mathbb{CP}^3$ with isolated singularities has no torsion, except the following cases.*

Singularities on V	$T(V) = \text{Tors } H_2(V)$
$4A_1$	$\mathbb{Z}/2\mathbb{Z}$
$3A_2$	$\mathbb{Z}/3\mathbb{Z}$
A_1A_5	$\mathbb{Z}/2\mathbb{Z}$
$2A_1A_3$	$\mathbb{Z}/2\mathbb{Z}$

PROOF: We give the details only for the case $4A_1$, the other cases being similar.

The reduced Milnor lattice $\bar{L}$ is in this situation E_6. We prove that an embedding $\varphi: 4A_1 \to E_6$ is unique up to an automorphism u of the lattice E_6. Assume that $4A_1 = \mathbb{Z}\langle f_1, \ldots, f_4 \rangle$ with $f_i^2 = -2$, $(f_i, f_j) = 0$ for $i \neq j$ and that $E_6 = \mathbb{Z}\langle e_1, \ldots, e_6 \rangle$ with the products given by the usual Dynkin diagram in which $e_1, \ldots, e_5$ stay in a line and e_6 is below e_3 [2]. Since the Weyl group $\text{Aut}(E_6)$ is transitive on the set of vectors v with $v^2 = -2$, we can take $\varphi(f_1) = e_6$. It follows that $\mathbb{Z}\langle f_2, f_3, f_4 \rangle$ is embedded in $e_6^{\perp}$, the orthogonal complement of e_6.

A basis for $e_6^{\perp}$ is given by the vectors e_2, e_1, $\bar{e}_3$, e_5, e_4 where $\bar{e}_3 = 2e_3 + e_2 + e_4 + e_6$ and the corresponding Dynkin diagram is precisely A_5. Using the explicit description of the lattice (or *root system*) A_5 and of its symmetries [2], it follows that the embedding $\bar{\varphi}: 3A_1 \to A_5$ induced by φ is equivalent up to an automorphism $\bar{u} \in \text{Aut}(A_5)$ with the obvious embedding $f_2 \to e_2$, $f_3 \to \bar{e}_3$, $f_4 \to e_4$.

Moreover, $\bar{u}$ extends to an $u \in \text{Aut}(E_6)$ since $\bar{u}$ is a composition of reflexions and $u(e_6) = e_6$.

This gives us Tors (coker φ) $= \mathbb{Z}/2\mathbb{Z}$ as required. □

REMARK 4.2: The homology of some singular cubic surfaces have been determined by Barthel and Kaup (see pp. 136, 275-276 in [1]) using completely different methods (e.g. desingularisations and local homology sheaves).

Now we consider two classes of complete intersections V of two quadrics in $\mathbb{CP}^{2m+2}$.

PROPOSITION 4.3:

(i) *If the Segre symbol corresponding to V is* $[(1, 1), \ldots, (1, 1), 2(m-k)+3]$ $k = 0, 1, 2, \ldots, m+1$, *then on the variety V there are $2k$ singular* *points of type* A_1 *and a singular point of type* $A_{2(m-k)+2}$ *for* $k \neq m+1$. *Moreover*, $T(V) = (\mathbb{Z}/2\mathbb{Z})^{k-1}$ *for* $k \geqslant 2$ *and is trivial for* $k = 0, 1$.

(ii) *If the Segre symbol corresponding to* V *is*

$$[(1,1),\ldots,(1,1),(2(m-k)+2,1)] \quad k = 0,\ldots,m$$

then on the variety V *there are* $2k$ *singular points of type* A_1 *and a singular point of type* $D_{2(m-k)+3}$ *(with the convention* $D_3 = A_3$*). Moreover* $T(V) = (\mathbb{Z}/2\mathbb{Z})^k$.

PROOF: The connection between the Segre symbol of a complete intersection of 2 quadrics V and the type of the singularities on V is explained in [14].

The reduced Milnor lattice $\bar{L}$ is in this situation D_{2m+3} [14], [15] and the computation of the torsion group $T(V)$ goes essentially as the proof of (4.1), using the following simple observation.

Assume that the lattice D_{2m+3} is the lattice $\mathbb{Z}\langle e_1,\ldots,e_{2m+3}\rangle$ with the corresponding Dynkin diagram having $e_2, e_3,\ldots,e_{2m+3}$ in a line and e_1 sitting under e_3. Then note that $e_1^{\perp}$ is the orthogonal direct sum of $\mathbb{Z}e_2$ and $\mathbb{Z}\langle \bar{e}_3, e_4,\ldots,e_{2m+3}\rangle$ where $\bar{e}_3 = 2e_3 + e_1 + e_2 + e_4$. Moreover, this last lattice is obviously D_{2m+1} (recall that $D_3 = A_3$). This fact allows one to prove (4.3) by induction on k. □

The interested reader can check that similar arguments give the homology of some other types of complete intersections of two quadrics. The simplest case which cannot be decided with these methods is the case of a 4-fold ($m = 2$) with 4 singular points of type A_1.

5. Deformations of complete intersections

First we prove a technical result which gives a sufficient condition for the inclusion of Milnor lattices $i_V: L_1 \oplus \ldots \oplus L_k \to L$ to be independent of V in the following sense.

Consider two complete intersections V_0, $V_1 \subset \mathbb{CP}^{n+p}$ of the same dimension n and multidegree $(d_1,\ldots,d_p)$.

Moreover, assume that V_0 and V_1 have isomorphic isolated singularities at the points $a_1,\ldots,a_m$ and (possibly) some other isolated singularities.

Let us denote by i_0 (resp. i_1) the restriction of the inclusion i_{V_0} (resp. i_{V_1}) defined in section 2 to the direct sum of the Milnor lattices corresponding to the singularities $a_1,\ldots,a_m$.

It is natural to ask when $i_0 = i_1$ (under some identification of the corresponding Milnor lattices).

Consider the vector spaces of polynomials

$$F = \{ f = (f_1, \dots, f_p);\ f_i \in \mathbb{C}[x_0, \dots, x_{n+p}] \text{ homogeneous of degree } d_i \text{ for } i = 1, \dots, p \}$$

$$G = \{ g = (g_1, \dots, g_p);\ g_i \in \mathbb{C}[x_1, \dots, x_{n+p}], \deg g_i \leqslant d_i \text{ for } i = 1, \dots, p \}$$

and the natural isomorphism $h: F \to G$

$$h(f) = f(1, x_1, \dots, x_{n+p}).$$

We can choose the coordinates on CP^{n+p} such that $H \cap V_t$ are smooth for $t = 0, 1$ where H is the hyperplane $x_0 = 0$. Let $F_0 \subset F$ be the Zariski open set corresponding to the complete intersections W such that $H \cap W$ is smooth. Then a system of equations for V_t gives a point $f^t \in F_0$ $(t = 0, 1)$.

Let s_i be the order of $\mathcal{K}$-determinacy of the singularity (V_0, a_i) [18] for $i = 1, \dots, m$ and consider the product of jet spaces

$$J = J^{s_1}(n+p,\ p) \mathcal{K} \dots \mathcal{K} J^{s_m}(n+p,\ p)$$

and let $S \subset J$ denote the corresponding product of $\mathcal{K}$-orbits.

There is a linear map $\phi: F \to J$ given by the composition of h with the map $G \to J$ which associates to a polynomial map g its s_i-jet at the point $\bar{a}_i$ for $i = 1, \dots, m$. Here we assume that $a_i = (1, \bar{a}_i)$ for $i = 1, \dots, m$.

With these notations, we have the following.

PROPOSITION 5.1: *If the constructible set* $\phi(F) \cap S$ *is irreducible, then* $i_0 = i_1$.

PROOF: Since ϕ is a linear map, it follows that $F_1 = \phi^{-1}(\phi(F) \cap S)$ is irreducible. Hence the open Zariski set $F_{01} = F_0 \cap F_1$ is connected. Take a path $f^t \in F_{01}$ for $t \in [0, 1]$, joining the two points f^0 and f^1.

The complete intersection V_t corresponding to the equation $f^t = 0$ has only isolated singularities (since $V_t \cap H$ is smooth!) and its singularity at the point a_i is isomorphic to (V_0, a_i) for $i = 1, \dots, m$.

To this variety V_t corresponds an embedding of lattices i_t, defined similarly to i_0 and i_1.

In this way we get a *continuous* family of embeddings and since the set of all homomorphism between two lattices is countable, we must have $i_t = i_0$ for any $t \in [0, 1]$. □

REMARK 5.2: Here are two simple cases when the condition in (5.1) is fulfilled.

(i) If the multidegree $(d_1, \dots, d_p)$ is big enough compared to the determinacy orders $(s_1, \dots, s_m)$, then the map ϕ is surjective. Since S is a smooth connected submanifold in J, it follows that $\phi(F) \cap S = S$ is irreducible. For example, in the hypersurface case ($p = 1$) it is enough to have

$$d_i \geqslant \sum_{i=1,m} (s_i + 1) - 1$$

(ii) Assume that we are in the hypersurface case and that the singularities (V_0, a_i) are all of type A_1.

Then S is a Zariski open subset of a vector space in J, the same is true for $\phi(F) \cap S$ and hence this last set is irreducible.

Now we present two applications of (5.1). The first one is a direct consequence of Theorem (2.1). With the notation above, assume that all the singularities on V_0 and V_1 different from the points $a_1, \dots, a_m$ have unimodular Milnor lattices. If the condition in (5.1) is fulfilled, then it follows easily that V_0 and V_1 have the same Betti number b_{n+1} and the same torsion part $T(V_0) = T(V_1)$. Moreover, note that instead of starting with the set of points $a_1, \dots, a_m$ we can start with two sets of points $a_1, \dots, a_m$ (singular points on V_0) and $b_1, \dots, b_m$ (singular points on V_1) which are projectively equivalent i.e. there is an autormorphism u of $\mathbb{CP}^{n+p}$ such that $u(a_i) = b_i$ for $i = 1, \dots, m$.

The second application is more subtle and combines (5.1) and the adjacency of singularities [16] to obtain information on the homology in the presence of singularities with *degenerate* Milnor lattices. This method will become clear from the following two examples.

EXAMPLE 5.3: Assume that the odd dimensional complete intersection V has a singular point a_1 of type A_1 and that all the other singular points of V (if any) have unimodular Milnor lattices.

Then:

(i) $\ker \varphi_V = 0$, except the case when V is a hypersurface of degree 2, when $\ker \varphi_V = \mathbb{Z}$.

(ii) coker φ_V is torsion free.

PROOF: When V is a hypersurface of degree 2, one has $\bar{L} = 0$ and hence everything is clear.

Assume now that V is a hypersurface of degree $d \geqslant 3$. Then, by (5.1) essentially, $\ker \varphi_V$ and Tors coker φ_V are the same as for the hypersurface W with affine equation

$$x_1^d + \dots + x_{n+1}^d + \alpha_i x_1^2 + \dots + \alpha_{n+1} x_{n+1}^2 + \beta x_1^3 = 0$$

Indeed, for a suitable choice of the constants α_i, β the hypersurface W has only one singular point $(1:0:\ldots:0)$ which is of type A_1.

On the other hand, if we take $\alpha_1 = 0$ we can arrange that W acquires a single A_2 singularity. This shows that the inclusion $i_W: A_1 \to L$ factorizes as a composition $A_1 \to A_2 \to L$. Since A_2 is unimodular, it follows that its image in $\bar{L}$ is a direct summand and this ends the proof in this case.

The case of complete intersections can be treated similarly (see [16], (7.18)). □

EXAMPLE 5.4: Assume that on the n-dimensional ($n \geqslant 3$ odd) hypersurface V of degree $d \geqslant d(S)$, the singularities correspond to one of the following symbols

S	$2A_1$	A_1A_3	A_3	A_5	D_4	D_5
$d(S)$	3	7	4	6	3	4

Then $\ker \varphi_V = \text{Tors coker } \varphi_V = 0$.

PROOF: The value of $d(S)$ is computed by the formula in (5.2.i) so that we may apply Proposition (5.1).

Hence it is enough to computer $\ker \varphi_W$ and Tors coker φ_W, where W is any hypersurface of degree d with the singularities prescribed by one of the symbols S.

Note that the (affine) equation

$$x_1^d + \ldots + x_{n+1}^d + C(x_1, x_2, x_3) + \lambda_4 x_4^2 + \ldots + \lambda_{n+1} x_{n+1}^2 = 0$$

where C is a generic cubic form and the constants λ_i are chosen conveniently defines a hypersurface W_0 with a single singularity of type $\tilde{E}_6$.

A study of the versal deformation of $\tilde{E}_6$ [5] shows that the singularity $\tilde{E}_6$ deforms to the singularity E_6 which in turns deforms to any of the symbols S in the list above (i.e. for any such symbol S there is a fiber in the versal deformation of E_6 whose singularities are exactly those prescribed by S).

Moreover, all these deformations can be performed using only monomials of degree $\leqslant 3$.

It follows that the corresponding morphisms φ_W factorize through the unimodular lattice E_6 and this gives the result. □

REMARK 5.5: We have said nothing about the case of curves (i.e. when $\dim V = 1$). If V is irreducible, its homotopy type can be easily obtained from its normalization $\tilde{V}$ (the singularities of V corresponding essentially to the identification of some points in $\tilde{V}$). We can safely leave the details for the reader.

6. The cohomology algebra and applications

Let R be a commutative unitary ring. Assuming the integral homology $H_*(V)$ of our complete intersection V known, the additive structure of $H^*(V, R)$ follows at once by the Universal Coefficient Theorem.

As to the cup-product, the only difficult point is to identify the pairing induced by it on the middle cohomology group

$$\alpha: H^n(V, R) \times H^n(V, R) \to H^{2n}(V, R) = R.$$

Let U be a regular neighborhood of V in $\mathbb{CP}^{n+p}$ (such that V is a deformation retract of U). Let $\bar{V}$ denote a smooth complete intersection of the same dimension and multidegree as V, which is contained in U. The natural morphism

$$j: \bar{V} \subset U \overset{r}{\searrow} V$$

gives a homomorphism at homology level.

Using (1.4), (1.5) and (2.1) we see that via this homomorphism we can identify $H_n(V, R)$ with $H_n(\bar{V}, R)/I$, where

$$I = I(V) = \operatorname{im} \varphi_V \otimes R \subset H_n(\bar{V}) \otimes R = H_n(\bar{V}, R)$$

Note that the Poincaré isomorphism

$$D: H_n(\bar{V}, R) \to \operatorname{Hom}(H_n(\bar{V}, R), R) = H^n(\bar{V}, R)$$

is given by $D(u) = \langle u, \cdot \rangle$, where $\langle\ ,\ \rangle$ is the intersection form on $H_n(\bar{V}, R)$. Moreover, D is compatible with the pairings on the source and the target (intersection form and cup-product).

The above description of $H_n(V, R)$ and the equality $H^n(V, R) = \operatorname{Hom}(H_n(V, R), R)$ shows that we can identify $H^n(V, R)$ with

$$\{ v \in H^n(\bar{V}, R);\quad v \mid I = 0 \}.$$

Thus we get the following.

PROPOSITION 6.1: *The pairing α is isomorphic with the pairing induced by the intersection form $\langle\ ,\ \rangle$ on the orthogonal complement of I in $H_n(\bar{V}, R)$.*

In particular, $H^n(V) = H^n(V, \mathbb{Z})$ is torsion free and hence is a lattice as defined in section 3.

Some information on this lattice can be easily derived from (6.1), using some general facts on lattices:

COROLLARY 6.2: (i) *If the lattice* $(I, \langle\ ,\ \rangle \mid I)$ *is nondegenerate, then* $H^n(V)$ *is also nondegenerate, and*

$$\det H^n(V) = \det I \cdot |T(V)|^{-2}$$

(ii) *If moreover n is even, then*

$$\operatorname{sign} H^n(V) = \operatorname{sign} \overline{V} - \operatorname{sign} I.$$

When the Milnor lattices L_i are nondegenerate, then $I = L_1 \oplus \ldots \oplus L_k$ and hence $\det I = \prod \det L_i$ and $\operatorname{sign} I = \sum \operatorname{sign} L_i$.

In some simple cases (e.g. those of section 4, or when $T(V) = 0$ and one has uniqueness results as in Example (6.4) below) the Proposition 6.1 determines effectively the lattice $H^n(V)$.

As an application, we consider the problem of deciding if certain varieties are or not homotopy equivalent.

EXAMPLE 6.3: Consider the complete intersections V_1, V_2 and V_3 (of the even dimension n and same multidegree) having the following type of singularities: $3A_1$, A_1A_2 and respectively A_3. Then the varieties V_i have the same homology and even the same *real* cohomology algebras. But these varieties have distinct *integral* cohomology algebras, as follows from the table (use (6.2.i))

i	1	2	3
$\det H^n(V_i)$	8	6	4

Hence V_i is not homotopy equivalent to V_j for $i \neq j$.

EXAMPLE 6.4: Let X_1 and X_2 be complete intersections of even dimension n and same multidegree, having both the same type of singularities, namely $3A_1$ (or A_1A_2, or A_3). Then X_1 and X_2 have the same integral cohomology algebra. If moreover $n = 2$, X_1 is homotopy equivalent to X_2.

PROOF: First note that $T(X_1) = T(X_2) = 0$ by the results in section 4 and hence the inclusions $I(X_i) \subset H_n(\overline{X}_i)$ are primitive.

By Proposition 6.1, the discriminant form of the lattice $H_i = H^n(X_i)$ is equal to the discriminant form $(D, -q)$ of the lattice $I(X_i)$ with reversed sign ([17], 1.6.2). The lattices H_i are even iff $d = \deg X_i$ is even. By Nikulin uniqueness results ([17], 1.13.3 and 1.16.10), the lattices H_i are isomorphic if:

(a) H_i are indefinite.

(b) Either d is even and $rk\ H_i \geqslant l(D) + 2$, or d is odd and $rk\ H_i \geqslant l(D) + 3$, where $l(D)$ denotes the minimal number of generators of

D. These conditions are fulfilled in most of the cases, the exceptions being covered in section 4.

When $n = 2$ and since there is no torsion, Whitehead's Theorem on the homotopy classification of 4-complexes (to be found for instance in [1], Chap. 2 together with many examples) can be easily applied. □

EXAMPLE 6.5: Let V_1 be a cubic surface in $\mathbb{CP}^3$ with a single E_6 singularity. Let V_2 be the projective cone over the twisted cubic curve C in $\mathbb{CP}^3$ (i.e. C is the image of the Veronese embedding $v_3 : \mathbb{CP}^1 \to \mathbb{CP}^3$).

Then V_1 and V_2 are homotopy equivalent.

PROOF: By the Whitehead Theorem mentioned above, it is enough to show that V_1 and V_2 have the same integral (torsion free) cohomology algebra. The cohomology algebra for V_2 is computed in [1, p. 72] while the cohomology algebra for V_1 is easily derived from (6.1). □

It is interesting to note that V_1 and V_2 are not homeomorphic (compare the local fundamental groups at the singular points!)

Similarly, the projective cone over the image of the Veronese embedding $v_4 : \mathbb{CP}^1 \to \mathbb{CP}^4$ and a complete intersection of two quadrics in $\mathbb{CP}^4$ with a single D_5 singularity are homotopy equivalent.

References

[1] G. BARTHEL and L. KAUP: Topologie des surfaces complexes compactes singulières. *Sém. Math. Sup.* 80, Les Presses de l'Université de Montréal, Montréal (1982).

[2] N. BOURBAKI: *Groupes et Algébre de Lie*. Chap. IV, V, VI, Paris (1968).

[3] E. BRIESKORN: Beispiele zur Differentialtopologie von Singularitäten. *Invent. math.* 2 (1966) 1–14.

[4] J.W. BRUCE and C.T.C. WALL: On the classification of cubic surfaces. *J. London Math. Soc.* 19 (1979) 245–256.

[5] J.W. BRUCE: A stratification of the space of cubic surfaces. *Math. Proc. Camb. Phil. Soc.* 87 (1980) 427–441.

[6] S.V. CHMUTOV: Monodromy groups of critical points of functions. *Invent. math.* 67 (1982) 123–131.

[7] A. DIMCA: On isolated singularities of complete intersections. *J. London Math. Soc.* (2) 31, (1985) 401–406.

[8] I. DOLGACHEV: Integral quadratic forms: applications to algebraic geometry. *Sém. Bourbaki*, 35e année, 1982/1983, no. 611 (1983).

[9] W. EBELING: Quadratische Formen und Monodromiegruppen von Singularitäten. *Math. Ann.* 255 (1981) 463–498.

[10] S.M. GUSEIN-ZADE: The monodromy groups of isolated singularities of hypersurfaces. *Russian math. Surveys* 32 (1977) 23–69.

[11] H.A. HAMM: Lokale topologische Eigenschaften Komplexer Räume. *Math. Ann.* 191 (1971) 235–252.

[12] H.A. HAMM: Exotische Sphären als Umgebungsränder in speziellen Komplexen Räumen. *Math. Ann.* 197 (1972) 44–56.

[13] F. HIRZEBRUCH: *Topological Methods in Algebraic Geometry*. Berlin-Heidelberg-New York, Springer-Verlag (1966).

[14] H. Knörrer: Isolierte Singularitäten von Durchschnitten zweier Quadriken. *Bonner Math. Schriften*, No. 117 (1980).
[15] A.S. Libgober and J.W. Wood: On the topological structure of even-dimensional complete intersections. *Trans. Amer. Math. Soc.* 267 (1981) 637-660.
[16] E.J.N. Looijenga: Isolated singular points on complete intersections. *L.M.S. Lecture Note Series* 77, Cambridge, Cambridge University Press (1984).
[17] V. Nikulin: Integral symmetric bilinear forms and some of their applications, *Izv. Acad. Nauk.* USSR 43 (1979) 111–177.
[18] C.T.C. Wall: Finite determinacy of smooth map-germs. *Bull. London Math. Soc.* 13 (1981) 481–539.

(Oblatum 18-10-1984 & 6-2-1985)

Department of Mathematics,
National Institute for Scientific
and Technical Creation
Bd. Păcii 220
79622 Bucharest
Romania