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# THE CHARACTERIZATION OF DIFFERENTIAL OPERATORS BY LOCALITY: CLASSICAL FLOWS

Ola Bratteli, George A. Elliott and Derek W. Robinson

## Abstract

Let $C_0(X)$ denote the continuous functions over the locally compact Hausdorff space $X$ vanishing at infinity and $\tau$ an action of $\mathbb{R}^\nu$ as *-automorphisms of $C_0(X)$ and let $T$ denote the associated group of homeomorphisms of $X$. Further let $(\delta_1, \delta_2, \ldots, \delta_\nu)$ denote the infinitesimal generators of $\tau$ and $C_0^\infty(X)$ the continuous functions which vanish at infinity and which are in the common domain of all monomials in the $\delta_i$.

We prove that a linear operator $H$ from $C_0^\infty(X)$ into the bounded continuous functions $C_b(X)$ satisfies the locality condition

$$\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f), \quad f \in C_0^\infty(X),$$

if, and only if, it is a polynomial in the $\delta_i$. Moreover we characterize the boundedness and continuity properties of the coefficients of the polynomials which arise in this manner. For example if $T$ acts freely then the coefficients are in $C_b(X)$. If the action of $T$ is not free the coefficients can be unbounded. If $\nu = 1$ we prove that the coefficients are polynomially bounded in the frequencies of the orbits of $T$.

We also establish that $H$ is local and satisfies

$$H(\bar{f})f + \bar{f}H(f) - H(\bar{f}f) \geqslant 0, \quad f \in C_0^\infty(X),$$

if, and only if, it is quadratic in the $\delta_i$ and the coefficients satisfy certain positivity requirements.

## 1. Introduction

In 1960 Peetre [13] established that partial differential operators can be characterized by locality. Our version of Peetre's theorem states that if $H$ is a linear operator from $C_0^\infty(\mathbb{R}^\nu)$, the infinitely often differentiable functions over $\mathbb{R}^\nu$ which vanish at infinity, into $C_b(\mathbb{R}^\nu)$, the bounded continuous functions over $\mathbb{R}^\nu$, satisfying the locality conditions

$$\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f), \quad f \in C_0^\infty(\mathbb{R}^\nu),$$

then there exist a positive integer $n$ and bounded continuous functions $l_\alpha$ over $\mathbb{R}^\nu$ such that

$$(Hf)(x) = \sum_{|\alpha| \leqslant n} l_\alpha(x) D^\alpha f(x), \quad f \in C_0^\infty(\mathbb{R}^\nu),$$

where $x = (x_1, x_2, \ldots, x_\nu) \in \mathbb{R}^\nu$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\nu)$ consists of non-negative integers, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_\nu$, and

$$D^\alpha f = \frac{\partial^{\alpha_1 + \alpha_2 + \ldots + \alpha_\nu}}{\partial x_1{}^{\alpha_1} \partial x_2{}^{\alpha_2} \ldots \partial x_\nu{}^{\alpha_\nu}} f.$$

Peetre's original theorem draws the same conclusion on bounded open subsets of $\mathbb{R}^\nu$.

The primary purpose of this paper is to derive a similar description of a local operator defined on a domain associated with a flow on a topological space. (Other forms of locality have been considered in [2], [5], [6], [7].) We also characterize certain second-order elliptic operators in terms of locality and a dissipation property. In the above setting we establish that a linear operator $H: C_0^\infty(\mathbb{R}^\nu) \to C_b(\mathbb{R}^\nu)$ satisfies the conditions

1. $\operatorname{supp}(Hf) \subseteq \operatorname{supp} f$
2. $H(f\bar{f}) - \bar{f}(Hf) - (H\bar{f})f \leqslant 0$

for all $f \in C_0^\infty(\mathbb{R}^\nu)$ if, and only if, there exist bounded continuous functions $l_0$, $l_i$, $l_{ij}$ over $\mathbb{R}^\nu$ such that

$$(Hf)(x) = l_0(x) f(x) + \sum_{i=1}^{\nu} l_i(x) \frac{\partial f(x)}{\partial x_i} + \sum_{i,j=1}^{\nu} l_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

where $l_0 \geqslant 0$, and $(-l_{ij}(x))$ is a positive-definite real matrix for each $x \in \mathbb{R}^\nu$.

Related results on second-order elliptic operators have been given by Nelson [12], Theorem 5.3, Forst [4], [9], and Pulè and Verbeure [14]. Nelson considers positive contraction semigroups $S$ on $C_0(\mathbb{R}^\nu)$ with the property

$$\sup\{(S_t f)(x);\ 0 \leqslant f \leqslant 1,\ f \in C_0(\mathbb{R}^\nu)\} = 1$$

for all $t > 0$, $x \in \mathbb{R}^\nu$. By the Riesz representation theorem these are given by probability measures $\mu^t(x, \cdot)$ as

$$(S_t f)(x) = \int \mu^t(x, \mathrm{d}y) f(y).$$

Nelson assumes that the generator $H$ of $S$ contains $C_0^2(\mathbb{R}^\nu)$ in its domain, and then proves that $S$ has the property

$$\mu^t(x, \{y; |x-y| > \epsilon\}) = o(t),$$

for all $\epsilon > 0$, if, and only if, $H$ is an operator of the above kind with $l_0 = 0$ and the $l_i$ real. Forst establishes the same result for translation invariant semigroups, i.e. semigroups with $\mu^t(x, \cdot) = \nu^t(\cdot - x)$ for some convolution semigroup $\nu^t$ of measures over $\mathbb{R}^\nu$, and proves that then the conditions are equivalent to the locality properties $\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f)$. Pulè and Verbeure also derive an analogous result for dissipative operators in classical statistical mechanics [14]. (Lumer [10] considers local operators which generate diffusion semigroups more abstractly but the setting is too general to classify these generators.)

As a corollary of our characterization of second-order differential operators one can deduce that $H: C_0^\infty(\mathbb{R}^\nu) \to C_0(\mathbb{R}^\nu)$ is a derivation, i.e.

$$H(fg) = f(Hg) + (Hf)g, \; f, \; g \in C_0^\infty(\mathbb{R}^\nu),$$

if, and only if,

$$(Hf)(x) = \sum_{i=1}^{\nu} l_i(x) \frac{\partial f(x)}{\partial x_i}$$

where the functions $l_i$ are bounded and continuous.

Our aim is to derive similar results for local operators associated with a general dynamical system.

Throughout the sequel $(X, \mathbb{R}^\nu, T)$ denotes a dynamical system consisting of a continuous action $T$ of the group $\mathbb{R}^\nu$ as homeomorphisms of the locally compact Hausdorff space $X$. Moreover $(\mathfrak{A}, \mathbb{R}^\nu, \tau)$ denotes the associated $C^*$-dynamical system formed by the abelian $C^*$-algebra $\mathfrak{A} = C_0(X)$ and the strongly continuous action $\tau$ of $\mathbb{R}^\nu$ as *-automorphisms of $\mathfrak{A}$ defined by

$$(\tau_t f)(\omega) = f(T_t \omega)$$

where $f \in \mathfrak{A}$, $t = (t_1, t_2, \ldots, t_\nu) \in \mathbb{R}^\nu$, and $\omega \in X$. The generators $\delta_1, \delta_2, \ldots, \delta_\nu$ of $\tau$ now play the role of the partial differential operators, and the common domain

$$\mathfrak{A}_\infty = \bigcap_\alpha D(\delta^\alpha)$$

of all monomials in the $\delta_i$ replaces the $C^\infty$-functions. Here we have again used the notation

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\nu) \quad \text{and} \quad \delta^\alpha = \delta_1^{\alpha_1} \delta_2^{\alpha_2} \ldots \delta_\nu^{\alpha_\nu}.$$

THEOREM 1.1: *Assume $T$ is free, i.e. the stabilizer subgroups*

$$S(\omega) = \{ t \in \mathbb{R}^{\nu};\ T_t\omega = \omega \}$$

*are zero for all $\omega \in X$.*

*Let $H$ be a linear operator from $\mathfrak{A}_{\infty}$ into $\mathfrak{A}$.*

*A. The following four conditions are equivalent:*

1. $\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f)$, $f \in \mathfrak{A}_{\infty}$.
2. *If $f \in \mathfrak{A}_{\infty}$, $\omega \in X$, and $(\tau_t f)(\omega) = 0$ for all $t$ in a neighbourhood of the origin in $\mathbb{R}^{\nu}$, then $(Hf)(\omega) = 0$.*
3. *If $f \in \mathfrak{A}_{\infty}$ and $(\delta^{\alpha} f)(\omega) = 0$ for all $\alpha$ then $(Hf)(\omega) = 0$.*
4. *There exist a positive integer $n$ and a (unique) family of bounded continuous functions $l_{\alpha}$ over $X$ such that*

$$(Hf)(\omega) = \sum_{|\alpha| \leqslant n} l_{\alpha}(\omega)(\delta^{\alpha} f)(\omega), \quad f \in \mathfrak{A}_{\infty}.$$

*Moreover each finite family of bounded continuous functions $l_{\alpha}$ over $X$ determines a linear operator from $\mathfrak{A}_{\infty}$ into $\mathfrak{A}$ which satisfies these conditions.*

*B. The following two conditions are equivalent:*

1. *a.* $\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f)$, $f \in \mathfrak{A}_{\infty}$,
   *b.* $H(f\bar{f}) - \bar{f}(Hf) - (H\bar{f})f \leqslant 0$, $f \in \mathfrak{A}_{\infty}$.
2. *There exist bounded continuous functions $l_0$, $l_i$, $l_{ij}$ over $X$ such that*

$$(Hf)(\omega) = l_0(\omega)f(\omega) + \sum_{i=1}^{\nu} l_i(\omega)(\delta_i f)(\omega)$$

$$+ \sum_{i,j=1}^{\nu} l_{ij}(\omega)(\delta_i\delta_j f)(\omega)$$

*for all $f \in \mathfrak{A}_{\infty}$, where $l_0 \geqslant 0$, and $(-l_{ij}(\omega))$ is a positive-definite real matrix for each $\omega \in X$.*

*C. The following two conditions are equivalent:*

1. *$H(fg) = (Hf)g + f(Hg)$, $f$, $g \in \mathfrak{A}_{\infty}$.*
2. *There exist bounded continuous functions $l_i$ over $X$ such that*

$$(Hf)(\omega) = \sum_{i=1}^{\nu} l_i(\omega)(\delta_i f)(\omega).$$

If $T$ is not free then similar statements hold but three complications occur.

First, if the stabilizer subgroup $S(\omega)$ is not zero then some linear combinations of the $(\delta^{\alpha} f)(\omega)$ vanish for all $f \in \mathfrak{A}_{\infty}$. Therefore the local operator $H$ cannot have a unique representation as a polynomial in the $\delta^{\alpha}$. But this difficulty is principally one of formulation and is easily overcome. One establishes that there exists a unique element $l(\omega)$ of the universal enveloping algebra of the quotient Lie algebra $\mathbb{R}^{\nu}/s(\omega)$, where $s(\omega)$ denotes the Lie algebra of the stabilizer subgroup $S(\omega)$, such that

$$(Hf)(\omega) = l(\omega)(\tau f(\omega))(0), \quad f \in \mathfrak{A}_{\infty}.$$

Here $\tau f(\omega)$ denotes the function $t \mapsto (\tau_t f)(\omega)$ interpreted as a function on the quotient group $\mathbb{R}^{\nu}/S(\omega)$.

Second, since the stabilizer subgroups $S(\omega)$ can vary with $\omega$ the continuity properties of the map $l$: $\omega \in X \mapsto l(\omega)$ are more complex. One can identify the enveloping algebra of $\mathbb{R}^{\nu}/s(\omega)$ as the subalgebra of the enveloping algebra of $\mathbb{R}^{\nu}$ generated by the orthogonal complement $s(\omega)^{\perp}$ of $s(\omega)$ in $\mathbb{R}^{\nu}$. Then there is a unique homomorphism of the enveloping algebra of $\mathbb{R}^{\nu}$ onto this subalgebra which is the identity on this subalgebra and zero on $s(\omega)^{\perp}$ which we call the canonical projection. If $\omega$ converges to $\omega_0$ in $X$ then the canonical projection of $l(\omega)$ onto the subalgebra generated by $s(\omega_0)^{\perp} \subseteq \mathbb{R}$ converges to $l(\omega_0)$. If the dimension of $s(\omega)^{\perp}$ is ultimately equal to the dimension of $s(\omega_0)^{\perp}$ (it cannot be strictly less), then $l(\omega)$ itself converges to $l(\omega_0)$. (By convergence in the enveloping algebra of $\mathbb{R}^{\nu}$ we mean convergence of the coefficients with respect to the canonical basis.)

The third, and essential, complication is that $l$ is no longer necessarily bounded. Unboundedness of the coefficients of $l$ can occur at certain fixed points of the flow $T$, points which are enclosed by a local periodic flow of increasing frequency. Since this difficulty already occurs for $\nu = 1$ we will, for simplicity, restrict further discussion to this case, and set $\delta_1 = \delta$.

First note that if $\nu = 1$ there are three types of behaviour of a point $\omega \in X$ under the action $T$. The point $\omega$ can be fixed, i.e. $T_t\omega = \omega$ for all $t \in \mathbb{R}$, and we denote the set of fixed points by $X_0$. The orbit $t \in \mathbb{R} \mapsto T_t\omega$ can be periodic, i.e., the set of $p > 0$ for which $T_p\omega = \omega$ has a strictly positive greatest lower bound $p(\omega)$. The value $p(\omega)$ is called the period of $\omega$ and the frequency of $\omega$ is defined by $\nu(\omega) = 1/p(\omega)$. Finally the orbit of $\omega$ can be open, i.e., $T_t\omega = \omega$ if and only if $t = 0$, and we then define the frequency of $\omega$ to be zero. Thus we have associated a frequency $\nu(\omega)$ to each $\omega \in X \setminus X_0$.

Next consider a function $g$ over $X \setminus X_0$. Then $g$ is defined to be polynomially bounded if there exists a polynomial $P$, which we may take to be of the form $c(1 + x^k)$ with $c > 0$, such that

$$|g(\omega)| \leqslant P(\nu(\omega))$$

for all $\omega \in \mathrm{X} \setminus \mathrm{X}_0$.

THEOREM 1.2: *Assume $\nu = 1$ and let $H$ be a linear operator from $\mathfrak{A}_\infty$ into $\mathfrak{A}$.*

*A. The following four conditions are equivalent:*

1. $\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f)$, $f \in \mathfrak{A}_\infty$.
2. *If $f \in \mathfrak{A}_\infty$, $\omega \in X$, and $(\tau_t f)(\omega) = 0$ for all $t$ in a neighbourhood of the origin in $\mathbb{R}$ then $(Hf)(\omega) = 0$.*
3. *If $f \in \mathfrak{A}_\infty$ and $(\delta^m f)(\omega) = 0$ for $m = 0, 1, 2, \ldots$ then $(Hf)(\omega) = 0$.*
4. *There exist an $n \geqslant 0$ and a (unique) family of functions $l_0, l_1, \ldots, l_n$ on $X$ with $l_0$ bounded and continuous and $l_1, \ldots, l_n$ equal to zero on the fixed points $X_0$ of $T$ and polynomially bounded and continuous on $X \setminus X_0$ such that*

$$H = \sum_{m=0}^{n} l_m \delta^m \mid_{\mathfrak{A}_\infty}.$$

*Moreover each finite family of functions $l_0, l_1, \ldots, l_n$ with the boundedness and continuity properties specified in Condition 4 determines a linear operator from $\mathfrak{A}_\infty$ into $\mathfrak{A}$ which satisfies these conditions.*

*B. The following two conditions are equivalent:*

1. *a.* $\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f)$, $f \in A_\infty$,
   *b.* $H(\bar{f}f) - \bar{f}(Hf) - (H\bar{f})f \leqslant 0$, $f \in \mathfrak{A}_\infty$.
2. $H = (l_0 + l_1\delta + l_2\delta^2) \mid_{\mathfrak{A}_\infty}$

*where the $l_i$ have the properties of Condition 4 above but $l_0 \geqslant 0$ and $l_2 \leqslant 0$.*

*C. The following two conditions are equivalent:*

1. $H(fg) = (Hf)g + f(Hg)$, $f, g \in \mathfrak{A}_\infty$.
2. *There exists a (unique) function $l_1$ which vanishes on $X_0$ and is polynomially bounded and continuous on $X \setminus X_0$ such that*

$$H = l_1 \delta \mid_{\mathfrak{A}_\infty}.$$

The existence of the continuous function $l_1$ on $X \setminus X_0$ in part C was proved by Batty in [1]. Batty did not determine which functions arise in this way.

The simplest explicit example of a local operator with an unbounded coefficient is given by setting $X = \mathbb{R}^2$, choosing radial co-ordinates $(r, \theta)$, and defining

$$T_t(r, \theta) = \left(r, \theta + \frac{2\pi t}{r}\right), \quad T_t(0, 0) = (0, 0).$$

Thus the orbits of the flow are concentric circles centred at the origin and the orbit of radius $r$ has frequency $1/r$. The origin is a fixed point and the flow is an idealized whirlpool. The generator $\delta$ associated with the flow has the form $(1/r)\partial/\partial\theta$ and the operators $(1/r^n)\delta$ are local operators from $\mathfrak{A}_\infty$ into $\mathfrak{A}$. Locality is easily checked and the fact that the operators are defined on $\mathfrak{A}_\infty$ and map it into $\mathfrak{A}$ follows from the last statement of part A of Theorem 1.2.

The foregoing statements will be proved in Section 3 with the aid of various results on orbits which are derived in Section 2. Generalizations are discussed in Sections 4 and 5. In Section 4 we characterize local operators from $\mathfrak{A}_n$ into $\mathfrak{A}_m$, where

$$\mathfrak{A}_n = \bigcap_{\alpha : |\alpha| \leqslant n} D(\delta^\alpha),$$

as polynomials in the $\delta$, of order $n - m$, whose coefficients satisfy certain regularity properties. In Section 5 we establish Theorem 1.2 for a one-dimensional local flow.

## 2. Extensions from orbits

One natural method of analyzing properties of flows is by restricting to orbits. If one wishes to use this method to analyze operators associated with the flow there are two types of problem. First it is not clear whether the operator has a well defined restriction to each orbit. Second it is difficult to decide whether a given function over the orbit is in the domain of the restriction. To handle this second problem one must be able to show that the function on the orbit has an extension which lies in the domain of the unrestricted operator, and it is useful to be able to construct extensions with good boundedness and support properties etc. The aim of this section is to resolve such problems. The results will then be used in Section 3 to prove Theorems 1.1 and 1.2.

It is necessary to consider the restriction of elements of $\mathfrak{A}_\infty$ to open subsets of the orbits $\Omega_\omega$ of points $\omega$ under $T$. It suffices for most purposes to consider neighbourhoods in $\Omega_\omega$ of $\omega$ of the form

$$I_r = \{T_t\omega ;\ |t| < r\},$$

where $|\cdot|$ is the $l^\infty$ norm on $\mathbb{R}^\nu$, and, by rescaling, one can restrict attention to $I = I_1$.

Let $C_b(I)$ denote the Banach space of bounded continuous functions on $I$ and define $\mathfrak{A}_\infty(I)$ as the subspace of $C_b(I)$ formed by the restrictions $f = F|_I$ to $I$ of those $F \in \mathfrak{A}_\infty$ for which supp$(f)$ is a compact subset of $I$. One can introduce analogues of $\mathfrak{A}_\infty(I)$ for more general subsets of the orbits in an obvious way. If $\mathcal{O} \subseteq \Omega_\omega$ is an open subset of $\Omega_\omega$ then $\mathfrak{A}_\infty(\mathcal{O})$ is the subspace of $C_b(\mathcal{O})$ formed by the restrictions to $\mathcal{O}$ of elements of $\mathfrak{A}_\infty$ whose support is compact in $\mathcal{O}$. But we will only need to consider the $\mathfrak{A}_\infty(I)$. In the case $\nu = 1$, there are three possibilities: (1) $\omega$ is periodic with period $\geqslant 2$ or $\omega$ is aperiodic.

Then $I$ is homeomorphic to $(-1, 1)$. (2) $\omega$ is periodic with period $< 2$. Then $I$ is homeomorphic to a circle. (3) $\omega$ is a fixed point. Then $I = \{\omega\}$.

Now if $F \in \mathfrak{A}_\infty$ and $F(T_t\omega') = 0$ for some $\omega' \in I$ and all small $t$ then $(\delta_i F)(\omega') = 0$ for $i = 1, \dots, \nu$ by definition. Therefore the restrictions of the $\delta_i$ to $\mathfrak{A}_\infty(I)$ are well defined operators from $\mathfrak{A}_\infty(I)$ into $\mathfrak{A}_\infty(I)$, which can be identified with the partial differential operators $D_i = \partial/\partial t_i$. Similarly the monomials $\delta^\alpha$ are well defined in restriction to $\mathfrak{A}_\infty(I)$, and coincide with $D^\alpha$. Therefore $\mathfrak{A}_\infty(I)$ can be identified with a subspace of the infinitely often differentiable functions with compact support on the manifold $I$, which is a quotient of the open unit cube in $\mathbb{R}^\nu$ (in the case $\nu = 1$ we have the three possibilities $I \cong (-1, 1)$, $I \cong \mathbb{T}$ or $I = \{\omega\}$ mentioned above; if $\nu \geq 2$ there are many possibilities. In the special case that $\omega$ is fixed by $T$ then $\mathfrak{A}_\infty(I)$ is isomorphic to $\mathbb{C}$).

Finally for $F \in \mathfrak{A}_\infty$ and $m \geq 0$ we introduce the $C^m$-seminorms

$$\| F \|_m = \sup_{\substack{\omega \in X \\ |\alpha| \leq m}} |(\delta^\alpha F(\omega)|,$$

$$\| F \|_{I,m} = \sup_{\substack{\omega \in I \\ |\alpha| \leq m}} |(\delta^\alpha F)(\omega)|.$$

Note that the latter norms $\| \cdot \|_{I,m}$ are also defined for $f \in \mathfrak{A}_\infty(I)$.

THEOREM 2.1: *Let* $n \in \mathbb{N}$ *and* $\epsilon > 0$, *let* $f \in \mathfrak{A}_\infty(I)$ *and let* $\mathcal{O}$ *be an open subset of* $X$ *containing* $\operatorname{supp}(f) \subseteq I$.

*It follows that there exists an* $F \in \mathfrak{A}_\infty$ *such that*

1. $\| F \|_m \leq (1 + \epsilon) \| f \|_{I,m}$, $m = 0, \dots, n$,
2. $F = f$ on $I$,
3. $\operatorname{supp}(F) \subseteq \mathcal{O}$.

The proof of Theorem 2.1 relies on two lemmas. The first is a general regularization result, which we formulate only for $\nu = 1$. The extension to general $\nu$ is straightforward.

LEMMA 2.2: *Let* $(\mathfrak{A}, \mathbb{R}, \tau)$ *be a* $C^*$*-dynamical system and denote the infinitesimal generator of* $\tau$ *by* $\delta$. *If* $x \in \mathfrak{A}$ *and* $S > 0$ *define* $x_S$ *by*

$$x_S = \frac{1}{2S} \int_{-S}^{S} \mathrm{d}t \; \tau_t(x)$$

*and write* $x = x_{S,0}$ *and* $x_{S,m} = (x_{S,m-1})_S$ *for* $m = 1, 2, \dots$.

*It follows that* $x_{S,n} \in D(\delta^n)$ *and*

$$\| \delta^m(x_{S,n}) \| \leq S^{-m} \| x \|, \quad 0 \leq m \leq n.$$

PROOF: We have that

$$\frac{1}{s}\left(\tau_s(x_S)-x_S\right)=\frac{1}{s}\frac{1}{2S}\left(\int_{-S}^{S}\mathrm{d}t\ \tau_{t+s}(x)-\int_{-S}^{S}\mathrm{d}t\ \tau_t(x)\right)$$

$$=\frac{1}{s}\frac{1}{2S}\left(\int_{S}^{S+s}\mathrm{d}\tau\ \tau_t(x)-\int_{-S}^{-S+s}\mathrm{d}t\ \tau_t(x)\right).$$

Hence by strong continuity of $\tau$, $x_S\in D(\delta)$ and $\delta(x_S)=(1/2S)\,(\tau_S(x)-\tau_{-S}(x))$.

Thus

$$\|\delta(x_S)\|\leqslant\frac{1}{2S}\left(\|\tau_S(x)\|+\|\tau_{-S}(x)\|\right)$$

$$=S^{-1}\|x\|.$$

If $y\in D(\delta)$ then $\delta(y_S)=\delta(y)_S$ because $\delta$ is closed. Thus, by induction $x_{S,n}\in D(\delta^m)$ for $0\leqslant m\leqslant n$ and

$$\delta^m(x_{S,n})=(\delta^m(x_{S,m}))_{S,n-m}.$$

Since $\|y_S\|\leqslant\|y\|$ for all $y\in\mathfrak{A}$, it follows by iteration first that

$$\|\delta^m(x_{S,m})\|\leqslant S^{-m}\|x\|$$

and next that

$$\|\delta^m(x_{S,n})\|=\|\delta^m(x_{S,m})_{S,n-m}\|$$

$$\leqslant\|\delta^m(x_{S,m})\|$$

$$\leqslant S^{-m}\|x\|$$

for $0\leqslant m\leqslant n$.

The next lemma is an existence result for abelian systems.

LEMMA 2.3: *Let* $(\mathfrak{A},\mathbb{R}^\nu,\tau)$ *be an abelian* $C^*$*-dynamical system,* $\omega$ *a point in the spectrum* $X$ *of* $\mathfrak{A}$*,* $n$ *a positive integer, and* $\mathcal{O}$ *an open neighbourhood of the compact set*

$$\{T_t\omega;\ |t|\leqslant 2nS+1\}$$

*where* $S\geqslant 1$.

*It follows that there exists a* $g \in \mathfrak{A}_\infty$ *such that*

$$\operatorname{supp}(g) \subseteq \mathcal{O},$$

$$\| \delta^\alpha g \| \leqslant S^{-|\alpha|}, \qquad |\alpha| \leqslant n,$$
$$(\tau_t g)(\omega) = 1, \qquad |t| \leqslant 1.$$

Proof: We give the proof for $\nu = 1$. The general case is established in a similar manner, but then it is important that $|t|$ is the $l^\infty$-norm of $t$ rather than the Euclidean norm.

Choose an open neighbourhood $\mathcal{O}_0$ of the compact set $C = \{T_t\omega; |t| \leqslant nS + 1\}$ with the property that

$$T_t \mathcal{O}_0 \subseteq \mathcal{O}$$

for $|t| \leqslant nS$. Next choose a continuous function $g_0$ on $X$ with compact support in $\mathcal{O}_0$, with $\| g_0 \| = 1$, and such that $g_0 = 1$ in a neighbourhood of $C$. Finally choose a positive $h \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}(h) \subseteq (-\epsilon, \epsilon)$ where $\epsilon$ is sufficiently small that $T_t(\operatorname{supp}(g_0)) \subseteq \mathcal{O}_0$ for $|t| < \epsilon$, and $\tau_t g_0 = 1$ on $C$ for $|t| < \epsilon$. Normalize $h$ so that

$$\int \mathrm{d}t \, h(t) = 1$$

and then define

$$g_1 = \int \mathrm{d}t \, h(t) \tau_t g_0.$$

It follows that $g_1$ has compact support in $\mathcal{O}_0$, $\| g_1 \| \leqslant 1$, $(\tau_t g_1)(\omega) = 1$ for $|t| \leqslant nS + 1$, and $g_1 \in \mathfrak{A}_\infty$. Note that this last property is a consequence of the regularization with $h$. Next we regularize $g_1$ in the manner of Lemma 2.2 and set $g = (g_1)_{S,n}$.

Since $T_t \mathcal{O}_0 \subseteq \mathcal{O}$ for $|t| \leqslant nS$ and $\operatorname{supp}(g_1) \subseteq \mathcal{O}_0$ one has $\operatorname{supp}(g) \subseteq \mathcal{O}$. Moreover $\| g \| \leqslant 1$ and $(\tau_t g)(\omega) = 1$ for $|t| \leqslant 1$. Finally $\| \delta^m g \| \leqslant S^{-m}$ for $m \leqslant n$ by Lemma 2.2.

Now we return to the proof of Theorem 2.1. Again we consider the case $\nu = 1$. The proof of the general case is very similar.

Proof of Theorem 2.1: First choose $\kappa > 0$ and $S \geqslant 1$ such that

$$(1 + \kappa)(1 + 1/S)^n < 1 + \epsilon.$$

Second choose an open set $\mathcal{O}' \subseteq \mathcal{O}$ such that $\operatorname{supp}(f) \subseteq \mathcal{O}'$ and $\mathcal{O}' \cap \{T_t\omega; |t| \leqslant 2nS + 1\} \subseteq I$. Third choose $g \in \mathfrak{A}_\infty$ with $\operatorname{supp}(g) \subseteq \mathcal{O}'$ and $g = 1$ on $\operatorname{supp}(f)$. This can be arranged by first choosing a continuous $g$ with the last two properties, such that $\operatorname{supp}(g)$ is compact and $g = 1$ in a neighbourhood of supp($f$) in $X$, and then regularizing with a suitably chosen function $h$ as in the proof of Lemma 2.3.

Next let $f_0 \in \mathfrak{A}_\infty$ denote an extension of $f$. Such an extension exists by the definition of $\mathfrak{A}_\infty(I)$. Define $f_1 = f_0 g$. One then has $f_1 = f$ on $I$ and in particular $\| f_1 \|_{I,m} = \| f \|_{I,m}$ for $m = 0, \ldots, n$. Moreover $f_1 \in \mathfrak{A}$ has the property that $(\text{supp}(f_1)) \cap \{T_t\omega;\, |t| \leqslant 2nS + 1\} \subseteq I$, so $f_1 = 0$ on $\{T_t\omega;\, |t| \leqslant 2nS + 1\} \setminus I$. Therefore one may choose an open neighbourhood $\mathcal{O}_1$ of $\{T_t\omega;\, |t| \leqslant 2nS + 1\}$ such that

$$\| f_1 \|_{\mathcal{O}_1, m} = \sup_{\substack{\omega' \in \mathcal{O} \\ k \leqslant m}} |(\delta^k f_1)(\omega')| \leqslant (1 + \kappa) \| f \|_{I,m}$$

$$\text{for} \quad m = 0, \ldots, n.$$

Finally by Lemma 2.3 one can choose an $h \in \mathfrak{A}_\infty$ such that $\text{supp}(h) \subseteq \mathcal{O}_1$, $\| \delta^m h \| \leqslant S^{-m}$ for $0 \leqslant m \leqslant n$, and $h = 1$ on $I$. Define $F = f_1 h = f_0 g h$. It then follows that $F = f_1 = f$ on $I$. But by Leibniz's rule

$$\begin{aligned}
\| F \|_m &= \| F \|_{\mathcal{O}_1, m} \\
&\leqslant \sup_{p \leqslant m} \sum_{k=0}^{p} \binom{p}{k} \| f_1 \|_{\mathcal{O}_1, p-k} \| \delta^k h \| \\
&\leqslant \| f_1 \|_{\mathcal{O}_1, m} \sup_{p \leqslant m} \sum_{k=0}^{p} \binom{p}{k} S^{-k} \\
&\leqslant (1 + \kappa)(1 + 1/S)^m \| f \|_{I,m} \leqslant (1 + \epsilon) \| f \|_{I,m}.
\end{aligned}$$

Moreover $\text{supp}(F) \subseteq \text{supp}(g) \subseteq \mathcal{O}' \subseteq \mathcal{O}$.

Finally we prove the existence of functions $f \in \mathfrak{A}_\infty(I)$ with specified behaviour at $\omega$.

THEOREM 2.4: *Let $(\mathfrak{A}, \mathbb{R}^\nu, \tau)$ be an abelian $C^*$-dynamical system and $\omega$ a point in the spectrum $X$ of $\mathfrak{A}$. Let $\mathcal{O}$ be an open neighbourhood of $\omega$, $M$ and $N$ positive integers and $\gamma \in \mathbb{C}$. Further assume there exists an $\epsilon > 0$ such that $T_{[-\epsilon,\epsilon]^\nu}\omega$ is an injective image of $[-\epsilon, \epsilon]^\nu$, and choose $\epsilon$ sufficiently small that $T_{[-\epsilon,\epsilon]^\nu}\omega$ is contained in $\mathcal{O}$. Finally let $n$ be a $\nu$-tuple of non-negative integers with $|n| \leqslant N$.*

*It follows that there exists $\alpha = \alpha(M, N, \epsilon) > 0$ only depending on $M$, $N$ and $\epsilon$ and an $F \in \mathfrak{A}_\infty$ such that*

$$\text{supp}(F) \subseteq \mathcal{O},$$

$$\| F \|_M < \epsilon,$$

$$(\delta^m F)(\omega) = 0 \quad \text{for} \quad m \neq n, |m| \leqslant N,$$

$$(\delta^n F)(\omega) = \gamma \quad \text{if} \quad |n| > M,$$

*and*

$$(\delta^n F)(\omega) = \alpha(M, N, \epsilon) \quad \text{if} \quad |n| \leqslant M.$$

In fact this result is stronger than necessary for the subsequent discussion. It would suffice to consider the case $|n| \leqslant M$ and omit reference to the statement involving $\gamma$. Nevertheless the more general statement could be of use in similar contexts.

Again we will only give the details in the case $\nu = 1$. An essential ingredient is the following result for $C_{00}^{\infty}(-\epsilon, \epsilon)$, the infinitely often differentiable functions with compact support in the interval $(-\epsilon, \epsilon)$.

LEMMA 2.5: *Let $n$, $M$, $N$ be positive integers with $n \leqslant N$ and let $\epsilon > 0$ and $\gamma \in \mathbb{C}$. There exists an $f \in C_{00}^{\infty}(-\epsilon, \epsilon)$ of the form $g * h$ with $g$, $h \in C_{00}^{\infty}(-\epsilon/2, \epsilon/2)$ such that*

$$\| f \|_M < \epsilon,$$

$$f^{(k)}(0) = 0 \quad \text{for} \quad 0 \leqslant k \leqslant N, \quad k \neq n,$$

$$f^{(n)}(0) = \begin{cases} \alpha(M, N, \epsilon) \neq 0, & \text{if} \quad n \leqslant M, \\ \gamma, & \text{if} \quad n > M. \end{cases}$$

PROOF: Fix $\alpha > 0$, to be specified later, and denote by $g_0$ the polynomial of Nth degree, or less, such that

$$g_0^{(n)}(0) = \begin{cases} \alpha, & \text{if} \quad n \leqslant M, \\ \gamma, & \text{if} \quad n > M, \end{cases}$$

$$g_0^{(k)}(0) = 0 \quad \text{for} \quad 0 \leqslant k \leqslant N, \quad k \neq n.$$

(More precisely, $g_0$ has degree $n$.) In particular,

$$| g_0^{(k)}(0) | < 2\alpha, \quad 0 \leqslant k \leqslant M.$$

Consider the $(M+1)$-dimensional linear space of polynomials $p$ of degree $2M+1$, or less, such that

$$p^{(k)}(0) = 0, \quad 0 \leqslant k \leqslant M.$$

Any polynomial $p \in V$ is determined by the $M+1$ numbers

$$p^{(k)}(\epsilon/4), \quad 0 \leqslant k \leqslant M.$$

Hence since $V$ is finite-dimensional there exists a smallest number $C > 0$ such that

$$\| p \|_{[0,\epsilon/4],M} \leqslant C \| p \|_{\{\epsilon/4\},M}$$

for all $p \in V$, where we have used the notation

$$\| p \|_{F,M} = \sup_{\substack{x \in F \\ 0 \leqslant k \leqslant M}} | p^{(k)}(x) |$$

for $F \subseteq \mathbb{R}$.

Next choose an interval $J = [-\beta, \beta]$ with $0 < \beta < \epsilon/4$ such that

$$\| g_0 \|_{J,M} < 2\alpha.$$

In particular,

$$\| g_0 \|_{\{\pm\beta\},M} < 2\alpha$$

and it follows that the unique polynomials $p_\pm$ of degree $2M + 1$, or less, such that

$$p_\pm^{(k)}(\pm(\beta + \epsilon/4)) = 0, \quad 0 \leqslant k \leqslant M,$$

$$p_\pm^{(k)}(\pm\beta) = g_0^{(k)}(\pm\beta), \quad 0 \leqslant k \leqslant M,$$

satisfy

$$\| p_\pm \|_{\pm[\beta,\beta+\epsilon/4],M} < 2C\alpha.$$

Now define a function $g_1$ on $\mathbb{R}$ by

$$g_1 = g_0 \quad \text{on} \quad [-\beta, \beta],$$

$$g_1 = p_\pm \quad \text{on} \quad \pm[\beta, \beta + \epsilon/4],$$

$$g_1 = 0 \quad \text{on} \quad \pm[\beta + \epsilon/4, \infty).$$

Then $g_1 \in C^M(\mathbb{R})$, and

$$\operatorname{supp}(g_1) \subseteq [-\beta - \epsilon/4, \beta + \epsilon/4] \subseteq (-\epsilon/2, \epsilon/2),$$

$$\| g_1 \|_M < 2 \max(\alpha, C\alpha).$$

Next note that there exists an $h_1 \in C_{00}^\infty(-1, 1)$ such that

$$\int \mathrm{d}t h_1(t) = 1,$$

$$\int \mathrm{d}t t^k h_1(t) = 0, \quad 1 \leqslant k \leqslant N.$$

This follows because if the latter conditions always implied

$$\int \mathrm{d}t h_1(t) = 0$$

then by linear algebra there would exist $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{C}$ such that

$$\int \mathrm{d}t h(t) = \sum_{k=1}^{N} \lambda_k \int \mathrm{d}t t^k h(t), \quad h \in C_0^\infty(-1, 1),$$

and this would lead to the contradiction

$$1 = \sum_{k=1}^{N} \lambda_k t^k, \quad t \in (-1, 1).$$

Now for $\mu > 0$ we define $h_\mu \in C_{00}^\infty(-\mu, \mu)$ by

$$h_\mu(t) = \mu^{-1} h_1(\mu^{-1} t)$$

and observe that

$$\int \mathrm{d}t h_\mu(t) = 1, \int \mathrm{d}t t^k h_\mu(t) = 0, \quad 1 \leqslant k \leqslant N.$$

Moreover

$$\int \mathrm{d}t \,|h_\mu(t)| = \int \mathrm{d}t \,|h_1(t)|.$$

The functions $h_\mu$ also have the property that if $p$ is a polynomial of degree at most $N$ then

$$\int \mathrm{d}t p(t) h_\mu(t-s) = p(s),$$

i.e., for any polynomial $p$ of degree at most $N$

$$p * h_\mu = p.$$

Next fix $\mu = \min\,(\beta/4, \epsilon/4 - \beta)$. Set $g = g_1 * h_\mu$; $h = h_\mu$, and $f = g * h$. Since $\operatorname{supp}(h_\mu) \subseteq (-\mu, \mu)$ with $\mu \leqslant \beta/4$, and $g_1 = g$ in $[-\beta, \beta]$, the function $g$ agrees with the polynomial $g_0$ in a neighbourhood of $[-\beta/2, \beta/2]$, and hence $f = g_1 * h_\mu * h_\mu$ agrees with $g_0$ in a neighbourhood of 0. But

$$\operatorname{supp}(f) \subseteq \operatorname{supp}(g_1) + \operatorname{supp}(h_\mu) + \operatorname{supp}(h_\mu),$$

$$\operatorname{supp}(g_1) \subseteq [-\beta - \epsilon/4, \beta + \epsilon/4],$$

$$\operatorname{supp}(h_\mu) \subseteq (-\mu, \mu),$$

and $\mu \leqslant \epsilon/4 - \beta$. Therefore

$$\operatorname{supp}(f) \subseteq (-\epsilon, \epsilon).$$

Moreover,

$$\| f \|_M \leqslant \| g_1 \|_M \left( \int dt \, | h_\mu(t) | \right)^2$$

$$= \| g_1 \|_M \left( \int dt \, | h_1(t) | \right)^2.$$

Finally if $\alpha$ is specified by

$$\max(\alpha, C\alpha) = \epsilon/2 \left( \int dt \, | h_1(t) | \right)^2$$

then $\| f \|_M < \epsilon$. Note that $\alpha$ is a function of $M$, $N$ and $\epsilon$.

Now we return to the proof of Theorem 2.4 for $\nu = 1$.

PROOF of THEOREM 2.4: First note that $T_{[-\epsilon,\epsilon]}\omega$ is an injective image of $[-\epsilon, \epsilon]$ and is contained in $\mathcal{O}$, by assumption. Moreover there exist, by Lemma 2.5, $g_0$, $h_0 \in C_{00}^\infty(\mathbb{R})$ with support in $(-\epsilon/2, \epsilon/2)$ such that $f_0 = g_0 * h_0$ satisfies

$$f_0^{(n)}(0) = \begin{cases} \alpha(M, N, \epsilon) \neq 0, & \text{if } n \leqslant M \\ \gamma, & \text{if } n > M, \end{cases}$$

$$f_0^{(k)}(0) = 0, \quad 0 \leqslant k \leqslant N, \quad k \neq N,$$

$$\| f_0 \|_M < \epsilon/2.$$

Since $\operatorname{supp}(f_0)$ and $\operatorname{supp}(h_0)$ are contained in $(-\epsilon, \epsilon)$ we may transport $f_0$ and $h_0$ to the orbit $\Omega_\omega$ and consider them as continuous functions on the closure $I^-$ of the interval $I = T_{(-\epsilon,\epsilon)}\omega$. Moreover since $h_0$ is zero outside $[-\epsilon/2, \epsilon/2]$ we can extend it to a function $h_1 \in C_0(X)$ with compact support such that $T_t\,(\operatorname{supp}(h_1)) \subseteq \mathcal{O}$ for $t \in [-\epsilon/2, \epsilon/2]$. Next define $f_1$ by

$$f_1 = \int dt g_0(t) \tau_t h_1.$$

Since $h_1 \in \mathfrak{A}$ and $g_0 \in C_{00}^\infty(\mathbb{R})$ it follows that $f_1 \in \mathfrak{A}_\infty$, and since $T_t(\operatorname{supp}(h_1)) \subseteq \mathcal{O}$ for $t \in [-\epsilon/2, \epsilon/2]$ and $\operatorname{supp}(g_0) \subseteq (-\epsilon/2, \epsilon/2)$ it also follows that $\text{supp}(f_1) \subseteq \mathcal{O}$. Moreover, since $h_1 = h_0$ on $I$, $f_0 = f_1$ on $I$ and hence

$$(\delta^n f_1)(\omega) = \begin{cases} \alpha(M, N, \epsilon) \neq 0, & \text{if} \quad n \leqslant M, \\ \gamma, & \text{if} \quad n > M, \end{cases}$$

$$(\delta^k f_1)(\omega) = 0, \quad 0 \leqslant k \leqslant N, \quad k \neq n,$$

$$\| f_1 \|_{I,M} < \epsilon.$$

Therefore by Theorem 2.1, with $f = \epsilon^{-1} f_1$, $\mathcal{O} = \mathcal{O}$, and $n = M$, there exists $F \in \mathfrak{A}_\infty$ such that $\text{supp}(F) \subseteq \mathcal{O}$, $F = f_1$ on $I$, and $\| F \|_M < \epsilon$. (Note that $\text{supp}(f_1 \mid_I) = \text{supp}(f_0)$ is a compact subset of $I$.)

This completes the proof of Theorem 2.4.

REMARK 2.6: If $\nu = 1$ and $n = 0$ then Theorem 2.4 is true even if $T_{[-\epsilon,\epsilon]}\omega$ is not injective. In this case one can apply Theorem 2.1 as follows. Since $\nu = 1$ the assumption that $T_{[-\epsilon,\epsilon]}\omega$ is not injective means $t \mapsto T_t\omega$ is periodic with period $p(\omega) \leqslant \epsilon$. Now if $f$ denotes the constant function with value $\epsilon/(1+\epsilon)$ on $\Omega_\omega$ then it follows from Theorem 2.1, with $n$ replaced by $M$, that there exists an $F \in \mathfrak{A}_\infty$ such that $\| F \|_M \leqslant (1 + \epsilon) \| f \|_{I,M} = \epsilon$, $F = f$ on $\Omega_\omega$, and $\text{supp}(F) \subseteq \mathcal{O}$. In particular $(\delta^M F)(\omega) = 0$ for $m \geqslant 1$ and $F(\omega) = f(\omega) = \epsilon/(1+\epsilon)$. Thus the conclusion of Theorem 2.4 is valid with $\alpha(M, N, \epsilon) = \epsilon/(1+\epsilon)$. If $T_{[-\epsilon,\epsilon]}\omega$ is injective a similar argument based on Theorem 2.1 shows that $\alpha(M, N, \epsilon)$ can be arranged to have the form $\epsilon/(1+\epsilon)\alpha_M$ where $\alpha_M$ is an increasing sequence with $\alpha_M \geqslant 1$.

## 3. Locality theorems

In this section we prove Theorems 1.1 and 1.2 with the aid of the results of Section 2. We first prove Part A of Theorem 1.2 and hence deduce Part A of Theorem 1.1 for $\nu = 1$. We then comment on the extension of this last result to higher dimensions. Finally we prove Parts B and C of both theorems.

We begin by proving $1 \Rightarrow 4$ in Part A of Theorem 1.2. This is the most difficult of the various implications and it depends upon a series of observations.

OBSERVATION 1: *There is an integer $n$ and a $C_\omega > 0$ such that*

$$|(Hf)(\omega)| \leqslant C_\omega \| f \|_n$$

*for all $f \in \mathfrak{A}_\infty$ and for all $\omega \in X \setminus F$ where $F$ is a finite set.*

PROOF. Assume this is false. Then there are distinct points $\omega_n \in X$ such that $f \in \mathfrak{A}_\infty \mapsto (Hf)(\omega_n)$ is discontinuous with respect to $\|\cdot\|_n$. Let $\omega$ be a limit point of $\omega_n$ in $X \cup \{\infty\}$. Choose $\omega_{n_1} \neq \omega$ and take disjoint open neighbourhoods $\mathcal{O}_{\omega_{n_1}}$ and $\mathcal{O}_\omega^{(1)}$ of $\omega_{n_1}$ and $\omega$. Next choose $\omega_{n_2} \in \mathcal{O}_\omega^{(1)}$ such that $\omega_{n_2} \neq \omega$ and take disjoint open neighbourhoods $\mathcal{O}_{\omega_{n_2}}$, $\mathcal{O}_\omega^{(2)} \subseteq \mathcal{O}_\omega^{(1)}$. Proceeding in this way one obtains an infinite sequence of points $\omega_{n_m}$ with disjoint open neighbourhoods $\mathcal{O}_{\omega_{n_m}}$ such that $f \mapsto (Hf)(\omega_{n_m})$ is discontinuous with respect to $\|\cdot\|_m$.

Next choose functions $g_m \in \mathfrak{A}_\infty$ such that $\operatorname{supp}(g_m) \subseteq \mathcal{O}_{\omega_{n_m}}$ and $g_m = 1$ on an open neighbourhood $\mathcal{U}_{\omega_{n_m}}$ of $\omega_{n_m}$. This is possible by regularization. [It is even possible by functional analysis of the domain of $\delta$ whenever $\delta$ is a closed derivation and $\mathfrak{A}_\infty$ is dense; see, for example, [1], Lemma 2.3.] Then by assumption, there exist $h_m \in \mathfrak{A}_\infty$ such that

$$|(Hh_m)(\omega_{n_m})| \geqslant m$$

and

$$\|h_m\|_m \leqslant 2^{-2m} \|g_m\|_m^{-1}.$$

Now set $f_m = g_m h_m$. One has

$$\|f_m\|_n \leqslant \|f_m\|_m \leqslant 2^{-m}$$

for $m \geqslant n$, where the last estimate follows by Leibniz's rule. Thus the series

$$f = \sum_{m \geqslant 1} f_m$$

converges with respect to the $C^n$-seminorms $\|\cdot\|_n$ for all $n$. It follows that $f \in \mathfrak{A}_\infty$. But from the choice of the $g_m$ one has $f = f_m$ on $\mathcal{O}_{\omega_{n_m}}$ and $f = h_m$ on $\mathcal{U}_{\omega_{n_m}}$. Therefore $(H(f - h_m))(\omega') = 0$ for all $\omega' \in \mathcal{U}_{\omega_{n_m}}$ by Condition 1. But this leads to the inequality

$$|(Hf)(\omega_{n_m})| \geqslant m$$

which contravenes the hypothesis that $Hf$ is bounded.

OBSERVATION 2: *If $\omega \in X \setminus F$, where $F$ is the finite set of Observation* 1, *there exist scalars $l_0(\omega), l_1(\omega), \ldots, l_n(\omega) \in \mathbb{C}$ such that*

$$(Hf)(\omega) = \sum_{m=0}^{n} l_m(\omega)(\delta^m f)(\omega), \quad f \in \mathfrak{A}_\infty.$$

PROOF: Suppose $(\delta^m f)(\omega) = 0$ for $0 \leqslant m \leqslant n$. Then by [1], Proposition 5.2, there exists a sequence $f_p \in \mathfrak{A}_\infty$ such that $f_p = 0$ in an open neighbourhood of $\omega$ and $\| f_p - f \|_n \to 0$. Therefore $(Hf_p)(\omega) = 0$ and

$$\begin{aligned} |(Hf)(\omega)| &= |(H(f-f_p))(\omega)| \\ &\leqslant C_\omega \| f - f_p \|_n \to 0, \end{aligned}$$

i.e., $(Hf)(\omega) = 0$. Thus the joint kernel of $f \in \mathfrak{A}_\infty \mapsto (\delta^m f)(\omega)$, $0 \leqslant m \leqslant n$, is contained in the kernel of $f \in \mathfrak{A}_\infty \mapsto (Hf)(\omega)$ and the desired conclusion follows from elementary linear algebra.

If $\omega \in X \setminus F$ then we have a relation

$$(Hf)(\omega) = \sum_{m=0}^{n} l_m(\omega)(\delta^m f)(\omega)$$

for all $f \in \mathfrak{A}_\infty$ but if $\omega \in X_0$ then $(\delta^m f)(\omega) = 0$ for all $f \in \mathfrak{A}_\infty$ and all $m \geqslant 1$. Hence we may assume $l_m(\omega) = 0$ for $1 \leqslant m \leqslant n$, and with this convention the relation is unique for $\omega \in X_0$. To see this it suffices to take an $f \in \mathfrak{A}_\infty$ with $f(\omega) \neq 0$. But if $\omega \in X \setminus X_0$ then the relation is also unique because in this latter case Theorem 2.4 implies the existence of functions $f_0, f_1, \ldots, f_n \in \mathfrak{A}_\infty$ such that the Jacobian $\det((\delta^i f_j)(\omega))$ is non-zero. Thus we have functions $\omega \in X \setminus F \mapsto (l_0(\omega), l_1(\omega), \ldots, l_n(\omega)) \in \mathbb{C}^{n+1}$.

OBSERVATION 3:
*$F \subseteq X_0$, and the functions $l_0, l_1, \ldots, l_n$ are continuous on $X \setminus X_0$.*

PROOF: If $\omega \in X \setminus X_0$ then by Theorem 2.4 there exists for each $j \leqslant n$ an $f_j \in \mathfrak{A}_\infty$ such that $(\delta^j f_j)(\omega) = 1$ and $(\delta^i f_j)(\omega) = 0$ for $0 \leqslant i < j$. Therefore $\det((\delta^i f_j)(\omega)) = 1$ and, by continuity, $\det((\delta^i f_j)(\omega')) > 0$ for all $\omega'$ in a neighbourhood $\mathcal{O}$ of $\omega$. Thus by solving the equations

$$(*) \qquad (Hf_j)(\omega') = \sum_{i=0}^{n} l_i(\omega')(\delta^i f_j)(\omega')$$

for $\omega' \in (X \setminus F) \cap \mathcal{O}$ one deduces that the $l_m$ are continuous in a deleted neighbourhood of $\omega$ and extend to continuous functions in a neighbourhood of $\omega$. Then the representation $(*)$ is valid in this neighbourhood. Since $\omega$ was arbitrary in $X \setminus X_0$ this demonstrates that the exceptional set $F$ must be contained in the fixed point set $X_0$.

The foregoing argument also establishes that the functions $\omega \in X \setminus F \mapsto l_0(\omega)$ and $\omega \in X \setminus X_0 \mapsto (l_1(\omega), \ldots, l_n(\omega)) \in \mathbb{C}^n$ are continuous.

It remains to show boundedness and continuity of $l_0$ on $X$ and polynomial boundedness of $l_1, \ldots, l_n$ on $X \setminus X_0$.

OBSERVATION 4: *The function* $\omega \in X \setminus F \mapsto l_0(\omega)$ *is bounded.*

PROOF: Suppose the converse is true. Then one can choose a sequence of distinct points $\omega_n \in X \setminus F$ such that

$$l_0(\omega_n) \geqslant n(2^n + 1)\alpha_n,$$

where $\alpha_n \geqslant 1$ is the sequence occurring in Remark 2.6, and one can also choose mutually disjoint neighbourhoods $\mathcal{O}_n \ni \omega_n$ such that $T_{[-2^{-n}, 2^{-n}]}\omega_n \subseteq \mathcal{O}_n$. Consequently by Remark 2.6 there exist functions $F_n \in \mathfrak{A}_\infty$ such that $\operatorname{supp}(F_n) \subseteq \mathcal{O}_n$, $\| F_n \|_n \leqslant 2^{-n}$, $(\delta^m F_n)(\omega_n) = 0$ for $m \geqslant 1$, and

$$F_n(\omega_n) = \begin{cases} 1/(2^n + 1)\alpha_n, & \text{if } \; T_{[-2^{-n}, 2^{-n}]}\omega_n \text{ is injective} \\ 1/(2^n + 1), & \text{if } \; T_{[-2^{-n}, 2^{-n}]}\omega_n \text{ is not injective.} \end{cases}$$

Therefore by the same arguments as used to conclude the proof of Observation 1 it follows that the series

$$F = \sum_{m \geqslant 1} F_m$$

converges with respect to the $C^n$-seminorms to an element $F \in \mathfrak{A}_\infty$ such that

$$\begin{aligned}(HF)(\omega_n) &= (HF_n)(\omega_n) \\ &= l_0(\omega_n) F_n(\omega_n) \geqslant n.\end{aligned}$$

This contravenes the hypothesis that HF is bounded.

OBSERVATION 5: *The functions* $\omega \in X \setminus X_0 \mapsto (l_1(\omega), \ldots, l_n(\omega))$ *are polynomially bounded, i.e. there is a constant* $C > 0$ *and a positive integer* $k$ *such that*

$$| l_m(\omega) | \leqslant C\left(l + \nu(\omega)^k\right)$$

*for all* $\omega \in X \setminus X_0$, *and all* $m = 1, 2, \ldots, n$.

PROOF: Assume conversely that $l_m$ is not polynomially bounded for some $m$, i.e. there exists a sequence $\omega_i \in X \setminus X_0$ such that the sequence $l_m(\omega_i)$ is not polynomially bounded in the frequencies $\nu(\omega_i)$. By passing to subsequences one can distinguish between two cases, one in which all $\nu(\omega_i)$ may be assumed greater than one half, and one in which the $\nu(\omega_i)$ are smaller than or equal to one half.

*Case 1*: $\nu(\omega_i) > 1/2$ for all $i$.

Then the orbit segment $I_i = T_{(-1,1)}\omega_i$ is equal to the whole orbit $\Omega_{\omega_i}$ for each $i$ and this orbit is a circle. The frequency of this orbit is $\nu(\omega_i)$ and we may apply Theorem 2.1 with $f \in \mathfrak{A}_\infty(I_i)$ a scalar multiple of the function $f_i$: $T_t\omega_i \mapsto \exp(2\pi i\nu(\omega_i)t)$. This latter function belongs to $\mathfrak{A}_\infty(I_i)$ because a simple computation shows that

$$f_i = \int \mathrm{d}t g(t)\ \tau_t(f_i)$$

with

$$g(t) = h(t) \exp(-2\pi i\nu(\omega_i)t)$$

where $h \in C_{00}^\infty(\mathbb{R})$ is an arbitrary function with integral one. Necessarily an infinite number of the circles $I_i$ are disjoint, as $l_m$ is bounded on each single $I_i$. Hence by boundedness of the $p(\omega_i) = 1/\nu(\omega_i)$, there exists a subsequence of $I_i$, also denoted by $I_i$, and a sequence of mutually disjoint open subsets $\mathcal{O}_i$ of $X$ with $I_i \subseteq \mathcal{O}_i$. Hence by Theorem 2.1 there exist functions $F_i \in \mathfrak{A}_\infty$ such that

$$\| F_i \|_k \leqslant 2 \| f_i \|_{I_i,k} = 2(2\pi\nu(\omega_i))^k, \quad k = 1, 2, \ldots, i,$$

$$F_i = f_i \quad \text{on} \quad I_i,$$

$$\operatorname{supp}(F_i) \subseteq \mathcal{O}_i.$$

Now, if $\rho_i$ is any sequence in $\mathbb{C}$ which is rapidly decreasing in the sense that

$$\lim_{i\to\infty} \nu(\omega_i)^k \rho_i = 0$$

for $k = 1, 2, \ldots$ then the series

$$F = \sum_{i\geqslant 1} \rho_i F_i$$

converges with respect to the $C^n$-seminorms to an $F \in \mathfrak{A}_\infty$. This follows because for any finite subset $J$ of the set $\{ j,\ j+1, \ldots \}$ one has

$$\begin{aligned} \| \sum_{i\in J} \rho_i F_i \|_j &= \sup_{i\in J} \| \rho_i F_i \|_j \\ &= \sup_{i\in J} | \rho_i | \, \| F_i \|_j \\ &\leqslant \sup_{i\in J} | \rho_i | 2(2\pi\nu(\omega_i))^j \end{aligned}$$

and the last expression converges to zero as $J$ converges to infinity. But then $F \in D(H)$ and by locality

$$(HF)(\omega_l) = (HF_l)(\omega_l)$$
$$= \sum_{m=0}^{n} l_m(\omega_l)(2\pi i\nu(\omega_l))^m \rho_l.$$

Since HF is bounded one concludes that the last expression must be bounded for any sequence $\rho_l$ which is rapidly decreasing in the above sense. Therefore

$$\sum_{m=0}^{n} l_m(\omega_l)(2\pi i\nu(\omega_l))^m$$

must be bounded by a polynomial in the frequencies $\nu(\omega_l)$. (This follows from the general fact that if $(\alpha_l)$ is a sequence of complex numbers such that $(\alpha_l\rho_l)$ is bounded for each $(\nu_l = \nu(\omega_l))$-rapidly decreasing sequence $\rho_l$ in $\mathbb{C}$ then $(\alpha_l)$ is $(\nu_l)$-tempered, i.e. $(\nu_l)$-polynomially bounded. This fact is proved as follows: if $\alpha_l$ increases more rapidly than any polynomial in $\nu_i$ then $\rho_l = |\alpha_i|^{-1/2}$ is $(\nu_l)$-rapidly decreasing but the product sequence $(\rho_i\alpha_l)$ is not bounded.)

Similarly if we replace $\nu(\omega_l)$ by $k\nu(\omega_l)$ in the definition of $f_l$ with $k = 1, 2, \ldots, n+1$ we deduce $(\nu_l)$-polynomial boundedness of

$$\sum_{m=0}^{n} l_m(\omega_l)(2\pi i k\nu(\omega_l))^m.$$

Thus

$$\begin{pmatrix} 1 & 2\pi i\nu_l & \cdots & (2\pi i\nu_l)^n \\ \vdots & \vdots & & \vdots \\ 1 & 2\pi i(n+1)\nu_l & \cdots & (2\pi i(n+1)\nu_l)^n \end{pmatrix} \begin{pmatrix} l_0(\omega_l) \\ \vdots \\ l_n(\omega_l) \end{pmatrix}$$
$$= \Omega \begin{pmatrix} l_0(\omega_i) \\ \vdots \\ l_n(\omega_i) \end{pmatrix}$$

is polynomially bounded in $\nu_l$. But $\Omega$ is a Vandermonde matrix with determinant a non-zero integral multiple of $(2\pi i\nu_i)^{n(n+1)/2}$. As $\nu_l > 1/2$ it follows that det $\Omega$ is bounded away from zero. Hence it follows from the cofactor expression of $\Omega^{-1}$ that

$$\begin{pmatrix} l_0(\omega_i) \\ \vdots \\ l_n(\omega_i) \end{pmatrix} = (\Omega^{-1})\left(\Omega\begin{pmatrix} l_0(\omega_i) \\ \vdots \\ l_n(\omega_i) \end{pmatrix}\right)$$

is polynomially bounded in $\nu_i$, since both quantities in parentheses on the right hand side have this property. But this is inconsistent with the hypothesis.

*Case 2*: $\nu(\omega_i) \leqslant 1/2$ for all $i$.

To reveal an inconsistency in this case it suffices to prove that each sequence $(l_m(\omega_i))$ is uniformly bounded in $i$. Now in this case the map from $(-1, 1)$ to $I_i = T_{(-1,1)}\omega_i$ is injective for all $i$ and we can use Theorem 2.4 for any $0 < \epsilon < 1$.

Assume $|l_m(\omega_i)| \to \infty$ for some $m$ as $i \to \infty$. Now choose an increasing sequence $i_1 < i_2 < i_3 < \dots$ of positive integers such that

$$|l_m(\omega_{i_k})| > k\left(\alpha(k, n, 2^{-k})\right)^{-1}$$

where $\alpha$ is the function occurring in Theorem 2.4. We may now also assume that there is an infinite subsequence of $k \mapsto \omega_{i_k}$ such that the subsets $I_k = T_{[-2^{-k}, 2^{-k}]}\omega_{i_k}$ are all disjoint. (If there is an infinite number of orbits $\Omega_{\omega_i}$ this is clear, if there is a finite number of orbits but the sets $I_k$ do not lie in a bounded part of these finite number of orbits this is again clear because $2^{-k}$ is summable, but finally if all the $I_k$ lie in a bounded part of a finite number of orbits then $i \mapsto l_m(\omega_i)$ is bounded since $l_m$ is continuous by Observation 3 and this is inconsistent with $|l_m(\omega_i)| \to \infty$.) Next for each $\omega_{i_k}$ pick an open neighbourhood $\mathcal{O}_k$ of $I_k$ in such a way that $\mathcal{O}_k \cap \mathcal{O}_{k'} = \emptyset$ if $k \neq k'$. (This is again possible after passing to a subsequence because $l_m$ is continuous on $X \setminus X_0$ and $|l_m(\omega_{i_k})| \to \infty$.) Now replacing $\omega_{i_k}$ by $\omega_k$ and applying Theorem 2.4 with $\nu = 1$, $\omega = \omega_k$, $\mathcal{O} = \mathcal{O}_k$, $M = k$, $N = n$, $n = m$, and $\epsilon = 2^{-k}$ we obtain a sequence $f_k \in \mathfrak{A}_\infty$ such that

$$\operatorname{supp}(f_k) \subseteq \mathcal{O}_k,$$

$$\| f_k \|_k < 2^{-k},$$

$$(\delta^j f_k)(\omega_k) = 0, \quad 0 \leqslant j \leqslant n, \quad j \neq m,$$

$$(\delta^m f_k)(\omega_k) = \alpha(k, n, 2^{-k}).$$

It again follows that the series

$$f = \sum f_k,$$

where the sum is over the last subsequence, converges with respect to the $C^n$-seminorms to an $f \in \mathfrak{A}_\infty$, and

$$(Hf)(\omega_k) = (Hf_k)(\omega_k) = l_m(\omega_k)(\delta^m f_k)(\omega_k).$$

Therefore

$$|(Hf)(\omega_k)| \geqslant k\big(\alpha(k, n, 2^{-k})\big)^{-1}\alpha(k, n, 2^{-k}) = k$$

which is inconsistent with the boundedness of Hf.

This completes the proof of Observation 5.

OBSERVATION 6: *If $f \in \mathfrak{A}_\infty$ and $k$ is a positive integer then the function*

$$\omega \in X \setminus X_0 \mapsto \nu(\omega)^k (\delta f)(\omega)$$

*vanishes at infinity on $X \setminus X_0$.*

PROOF: Let $\omega_i$ be a net of points in $X \setminus X_0$ converging to $\infty$ in $X \setminus X_0$. We must show that $\nu(\omega_i)^k (\delta f)(\omega_i)$ converges to zero. We distinguish two cases. (It suffices to prove that every subnet has a subnet converging to zero; every subnet has a subnet belonging to one of the following two cases.)

*Case 1*: $\nu(\omega_i) \leqslant 1$ for all $i$.

Since $\delta f \in \mathfrak{A}$, and $(\delta f)(\omega) = 0$ for $\omega \in X_0$, $\delta f$ vanishes at $\infty$ as a function on $X \setminus X_0$. As $|\nu(\omega_i)^k| \leqslant 1$ for all $i$, it follows that

$$\lim_{i \to \infty} \nu(\omega_i)^k (\delta f)(\omega_i) = 0.$$

*Case 2*: $\nu(\omega_i) \geqslant 1$ for all $i$.

It is sufficient to consider the case that $f$ is real-valued. The functions

$$t \in \mathbb{R} \mapsto (\delta f)(T_t \omega_i) := g_i(t)$$

are periodic with period $p_i = 1/\nu(\omega_i) \leqslant 1$. Also, as $g_i$ is the derivative of the function $t \mapsto f(T_t \omega_i)$ which is also periodic with period $p_i$, and real-valued, there is a $t_0 \in [0, 1]$ such that $g_i(t_0) = 0$. But then for an arbitrary $t$ we have

$$|g_i(t)| \leqslant |g_i(t_0)| + |t - t_0| \, \|g_i'\| = |t - t_0| \, \|g_i'\|$$

by the mean value theorem, and since $g_i$ is periodic with period $p_i$ we find

$$|g_i(t)| \leqslant \frac{p_i}{2} \|g_i'\| \leqslant p_i \|g_i'\|.$$

(Here $\| g_i' \|$ denote $\sup_{t \in \mathbb{R}} | g_i'(t) |$.) As $f \in \mathfrak{A}_\infty$ we have $g_i \in C^\infty(\mathbb{R})$ and iterating the argument above we obtain

$$\| g_i \| \leqslant p_i^n \| g_i^{(n)} \| = \frac{1}{\nu(\omega_i)^n} \| g_i^{(n)} \|,$$

and hence

$$| \nu(\omega_i)^k (\delta f)(\omega_i) | \leqslant \frac{1}{\nu(\omega_i)^{n-k-1}} \| \delta^n f |_{T_{[0,1]}\omega_i} \|$$

for $n = 1, 2, \ldots$. Now, as $\omega_i$ converges to infinity, so does $T_{[0,1]}\omega_i$, and it follows by choosing $n = k + 1$ that

$$\lim_{i \to \infty} \nu(\omega_i)^k (\delta f)(\omega_i) = 0.$$

This ends the proof of Observation 6.

Observation 7: *If $l_m$: $X \backslash X_0 \to \mathbb{C}$ is a continuous function which is polynomially bounded, then the operator $l_m \delta^m$ defined on $\mathfrak{A}_\infty$ by*

$$(l_m \delta^m f)(\omega) = \begin{cases} l_m(\omega)(\delta^m f)(\omega) & \text{if} \quad \omega \in X \backslash X_0, \\ 0 & \text{if} \quad \omega \in X_0 \end{cases}$$

*maps $\mathfrak{A}_\infty$ into $\mathfrak{A}$, for $m = 1, 2, \ldots$.*

Proof: By hypothesis there is a constant $C > 0$ and a positive integer $k$ such that

$$| l_m(\omega) | \leqslant C\big(1 + \nu(\omega)^k\big)$$

for all $x \in X \backslash X_0$. Thus if $f \in \mathfrak{A}_\infty$ we have

$$| (l_m \delta^m f)(\omega) | \leqslant C\big(1 + \nu(\omega)^k\big) | (\delta^m f)(\omega) |$$

for $\omega \in X \backslash X_0$. Thus $\omega \mapsto l_m(\omega)(\delta^m f)(\omega)$ vanishes at $\infty$ on $X \backslash X_0$ by Observation 6. Thus $l_m \delta^m f$ is continuous on $X$, and vanishes at $\infty$ on $X$, i.e. $l_m \delta^m f \in \mathfrak{A}$, which ends the proof of Observation 7.

The proof of $1 \Rightarrow 4$ in Part A of Theorem 1.2 is now completed by the following observation.

Observation 8: *The function $l_0$ on $X \backslash F$ extends to a bounded, continuous function on $X$, and we have*

$$(Hf)(\omega) = \sum_{m=0}^{n} l_m(\omega)(\delta^m f)(\omega)$$

*for all* $\omega \in X$, *with the convention that* $l_m(\omega) = 0$ *if* $m = 1, \ldots, n$ *and* $\omega \in X_0$.

PROOF: We have already proved this representation for $H$ if $\omega \in X \setminus X_0$. But it follows from Observation 5 and Observation 7 that the functions

$$\omega \in X \mapsto l_m(\omega)(\delta^m f)(\omega)$$

are contained in $\mathfrak{A}$ for any $f \in \mathfrak{A}_\infty$ and $m = 1, 2, \ldots$. Thus

$$\omega \in X \setminus F \mapsto l_0(\omega) f(\omega) = (Hf)(\omega) - \sum_{m=1}^{n} l_m \delta^m f(\omega)$$

extends to a continuous function on $X$ for any $f \in \mathfrak{A}_\infty$. By choosing an $f \in \mathfrak{A}_\infty$ with $f(\omega) \neq 0$ for each $\omega \in F$, we see that $l_0$ extends to a continuous function on $X$, and the representation for $H$ is then valid for all $\omega \in X$. By Observation 4, $l_0$ is bounded.

Since the proof of the implications $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$ is trivial we turn to the last statement of Part A. Thus we consider an operator $H$ of the form

$$H = \sum_{m=0}^{n} l_m \delta^m$$

where $l_0$ is bounded and continuous on $X$ and $l_1, \ldots, l_n$ are polynomially bounded and continuous on $X \setminus X_0$.

First $H$ is defined as an operator on $\mathfrak{A}_n = D(\delta^n)$ as follows:

$$(Hf)(\omega) = \begin{cases} \sum_{m=0}^{n} l_m(\omega)(\delta^m f)(\omega) & \text{if} \quad \omega \in X \setminus X_0, \\ l_0(\omega) f(\omega) & \text{if} \quad \omega \in X_0. \end{cases}$$

Initially it is not clear that $Hf$ is a continuous function but the continuity and boundedness properties of $l_0, \ldots, l_n$ imply that if $f \in \mathfrak{A}_\infty$ then $Hf \in \mathfrak{A}$, by Observation 7 and the trivial fact that $l_0 \mathfrak{A} \subseteq \mathfrak{A}$.

This completes the proof of Part A of Theorem 1.2.

Now consider Part A of Theorem 1.1.

If $\nu = 1$ then the assumption that $T$ is free implies that there are no fixed points or periodic orbits. Thus only orbits of frequency zero occur and Part A of Theorem 1.1 follows from the corresponding statement in Theorem 1.2, which has been established above.

If $\nu > 1$ then all statements of Part A of Theorem 1.1 are straightforward to establish except the implication $1 \Rightarrow 4$. But this implication can be deduced by modification of the reasoning used to prove Observations 1, 2 and 3 and Case 2 in Observation 5.

First by repetition of the proofs of the first three observations one establishes in the multi-dimensional case that

$$(Hf)(\omega) = \sum_{|\alpha| \leqslant n} l_\alpha(\omega)(\delta^\alpha f)(\omega), \quad f \in \mathfrak{A}_\infty$$

for some positive integer $n$ and all $\omega \in X$ where the coefficients $l_\alpha$ are locally bounded and continuous on $X$. These proofs are based on the results of Section 2 and on Proposition 5.2 of [1] and are valid for all $\nu$. Note that since $\mathbb{R}^\nu$ acts freely, $F = \emptyset$, by the argument used in the proof of Observation 3. It remains to prove that the $l_\alpha$ are bounded. But this follows by the argument used in Case 2 of Observation 5. The assumption of free action ensures that the maps $t \in (-1, 1)^\nu \Rightarrow T_t\omega_t \in \Omega_{\omega_t}$ are injective and hence allows application of the multi-dimensional version of Theorem 2.4.

Next we consider the proof of Part B of the Theorems 1.1 and 1.2. In both cases the implication $2 \Rightarrow 1$ is a simple verification and in considering $1 \Rightarrow 2$ one uses the respective Part A to deduce that $H$ is a polynomial in the generators. The new element in the proof is to deduce that this polynomial is of order two by use of the dissipation condition

$$H(\bar{f}f) - \bar{f}(Hf) - (H\bar{f})f \leqslant 0, \quad f \in \mathfrak{A}_\infty.$$

This follows from a number of estimations which we give below which do not use the boundedness and continuity properties of the coefficients $l_\alpha$.

First consider the setting of Theorem 1.2. Thus $\nu = 1$ and

$$(Hf)(\omega) = \sum_{m=0}^{n} l_m(\omega)(\delta^m f)(\omega).$$

Now for any $\omega \in X$ one can find an $f \in \mathfrak{A}_\infty$ with $f = 1$ in an open neighbourhood of $\omega$. Then $(\delta^m f)(\omega) = 0$ and $(\delta^m \bar{f}f)(\omega) = 0$ for $m > 0$. Hence the dissipation inequality

$$(H\bar{f})(\omega)f(\omega) + \bar{f}(\omega)(Hf)(\omega) - (H\bar{f}f)(\omega) \geqslant 0$$

gives

$$l_0(\omega)\,|\,f(\omega)\,|^2 \geqslant 0.$$

Therefore $l_0(\omega) \geqslant 0$.

Next suppose $\omega \in X \setminus X_0$ and choose a real $g \in \mathfrak{A}_\infty$ such that $g(\omega) = 1$ and $(\delta g)(\omega) = 1$. Using the derivation property

$$\delta g^k = k g^{k-1} \delta g$$

one concludes that

$$(\delta^m g^k)(\omega) = k^m + O(k^{m-1})$$

and hence

$$(Hg^k)(\omega) = k^n l_n(\omega) + O(k^{n-1}).$$

Now introduce the notation

$$S(f, g) = (H(\bar{f})g + \bar{f}H(g) - H(\bar{f}g))(\omega).$$

Then

$$S(g^k, g^k) = k^n l_n(\omega)(2 - 2^n) + O(k^{n-1}).$$

Therefore the dissipation inequality $S(g^k, g^k) \geqslant 0$ for large $k$ implies that $l_n(\omega) \leqslant 0$ if $n \geqslant 2$. Next consider the dissipation inequality $S(\lambda g^k + g^l, \lambda g^k + g^l) \geqslant 0$ for $\lambda \in \mathbb{R}$. For this it is necessary that

$$S(g^k, g^k)S(g^l, g^l) \geqslant |S(g^k, g^l)|^2.$$

Setting $l = mk$ then gives, when $n \geqslant 2$ (in which case $l_n(\omega)$ is real),

$$k^{2n} l_n(\omega)^2 \Big((2 - 2^n)^2 m^n - ((m+1)^n - m^n - 1)^2\Big) + O(k^{2n-1}) \geqslant 0.$$

But if $n > 2$ this is impossible for large $m$ unless $l_n(\omega) = 0$. Repeating this argument gives $l_m(\omega) = 0$ for all $m > 2$ and all $\omega \in X \backslash X_0$. But if $m \geqslant 2$ then $l_m(\omega) = 0$ for $\omega \in X_0$ by definition. Finally $l_2(\omega) \leqslant 0$ for all $\omega \in X \backslash X_0$ by the previous argument. Thus $H$ is quadratic in $\delta$ with $l_0 \geqslant 0$ and $l_2 \leqslant 0$.

Now consider the analogous problem in the setting of Theorem 1.1. Thus $\nu$ is arbitrary, $\mathbb{R}^\nu$ acts freely, and $H$ has the form

$$(Hf)(\omega) = \sum_{|\alpha| \leqslant n} l_\alpha(\omega)(\delta^\alpha f)(\omega).$$

Proceeding exactly as before one deduces from the dissipation inequality that $l_0 \geqslant 0$. Now the proof that $l_\alpha = 0$ if $|\alpha| > 2$ is also very similar to the above but one must make a more sophisticated choice of $g$. Fix $x = (x_1, \ldots, x_\nu) \in \mathbb{R}^\nu$.

Choose (as follows) $g = g_x \in \mathfrak{A}_\infty$ such that

$$g(\omega) = 1,$$

$$\delta g(\omega) = (\delta_1 g(\omega), \ldots, \delta_n g(\omega)) = x,$$

$$\delta^{|\beta|} g(\omega) = 0 \qquad \text{if} \quad |\beta| \geqslant 2.$$

(First choose $h_0 \in C_{00}(\mathbb{R}^\nu)$ such that $Dh_0$, the derivative of $h_0$, is equal to $x$ in a neighbourhood $U$ of 0, and $h_0$ is equal to 1 at 0. Choose $\mu > 0$ such that $y \in U$ if $|y_i| \leqslant 2\mu$, $1 \leqslant i \leqslant \nu$. With $h_\mu$ as constructed in the proof of Lemma 2.5 with $N = 1$, define $f \in C_{00}(U)$ by $f(y) = h_\mu(y_1)\dots h_\mu(y_\nu)$. Then for any polynomial $p$ on $\mathbb{R}^\nu$ of degree at most one, $f * p$ agrees with $p$ in a neighbourhood of 0 in $\mathbb{R}^\nu$ (namely, $\{y \in \mathbb{R}^\nu; |y_i| \leqslant \mu\}$), and as $h_0$ is such a polynomial on $U$, $f * h_0$ agrees with $h_0$ on a neighbourhood of 0 in $\mathbb{R}^\nu$. Extend $h_0$ to $h \in \mathfrak{A}$, and set $\int \mathrm{d}t f(t)\tau_t h = g$. Then $g$ is equal to 1 at $\omega$, and $\delta g$ is equal to $x$ in a neighbourhood of $\omega$ in $X$. We use here that $\mathbb{R}^\nu$ acts freely at $\omega$, so that the orbit of $\omega$ may be identified with $\mathbb{R}^\nu$, and in such a way that $\delta$ corresponds to $D$.)

It follows that, for each multiindex $\alpha$ with $|\alpha| \geqslant 1$, and each $k = 1, 2, \dots,$

$$\begin{aligned}\delta^\alpha g^k(\omega) &= k^{|\alpha|}(\delta g(\omega))^\alpha + \mathrm{O}(k^{|\alpha|-1})\\ &= k^{|\alpha|}x^\alpha + \mathrm{O}(k^{|\alpha|-1}),\end{aligned}$$

where $x^\alpha = x_1^{\alpha_1}\dots x_\nu^{\alpha_\nu}$. Consequently,

$$S(g^k, g^k) = k^n(2 - 2^n)\sum_{|\alpha|=n} x^\alpha l_\alpha(\omega) + \mathrm{O}(k^{n-1}).$$

From the dissipation inequality $S(f, f) \geqslant 0$ we deduce first that, if $n \geqslant 2$, then

$$\sum_{|\alpha|=n} x^\alpha l_\alpha(\omega) \leqslant 0.$$

This holding for every $x \in \mathbb{R}^\nu$, it follows that $l_\alpha(\omega)$ is real for each $\alpha$ with $|\alpha| = n$. (It also follows that if $n > 2$ and $n$ is odd then $l_\alpha(\omega) = 0$ for all $\alpha$ with $|\alpha| = n$, but the case that $n$ is odd follows from the case of arbitrary $n > 2$, dealt with below.)

We deduce second, using the Cauchy-Schwarz inequality $S(g^k, g^k)$ $S(g^l, g^l) \geqslant |S(g^k, g^l)|^2$ with $l = mk$, that when $n \geqslant 2$ (in which case $l_\alpha(\omega)$ is real if $|\alpha| = n$),

$$\begin{aligned}&k^{2n}\Big((2-2^n)^2 m^n - \big((m+1)^n - m^n - 1\big)^2\Big)\\ &\quad\times\Big(\sum_{|\alpha|=n} x^\alpha l_\alpha(\omega)\Big)^2 + \mathrm{O}(k^{2n-1}) \geqslant 0.\end{aligned}$$

As before, if $n > 2$ this is impossible for large $m$ unless

$$\sum_{|\alpha|=n} x^{\alpha} l_{\alpha}(\omega) = 0,$$

and as $x \in \mathbb{R}^{\nu}$ is arbitrary this says that $l_{\alpha}(\omega) = 0$, $|\alpha| = n$. Therefore by repetition of the argument $l_{\alpha} = 0$ for all $\alpha$ with $|\alpha| > 2$. Thus $H$ is quadratic in the $\delta_i$, and can be expressed in the form given in Condition B2 of the theorem with $l_{ij} = \bar{l}_{ij} = l_{ji}$. Finally for $\omega \in X$ take $g = \bar{g} \in \mathfrak{A}_{\infty}$ with $g(\omega) = 1$ and note that

$$(\delta_i \delta_j g^k)(\omega) = k^2 (\delta_i g)(\omega)(\delta_j g)(\omega) + \mathrm{O}(k).$$

Therefore

$$S(g^k, g^k) = (2k)^2 \sum_{i,j=1}^{\nu} (-l_{ij}(\omega))(\delta_i g)(\omega)(\delta_j g)(\omega) + \mathrm{O}(k).$$

Consequently the dissipation inequality gives

$$\sum_{i,j=1}^{\nu} (-l_{ij}(\omega))(\delta_i g)(\omega)(\delta_j g)(\omega) \geqslant 0,$$

and this is equivalent to positive-definitioness if, and only if, the maps $\omega\delta_i$ are linearly independent. But this is the case because $\mathbb{R}^{\nu}$ acts freely. Thus $H$ satisfies Condition B2.

Finally we prove Part C of the two theorems, 1.1 and 1.2. Again the implication $2 \Rightarrow 1$ is a simple verification. The proof that $1 \Rightarrow 2$ is based upon the observation that since $H$ is a derivation from $\mathfrak{A}_{\infty}$ into $\mathfrak{A}$ it automatically satisfies the locality conditions $\mathrm{supp}(Hf) \subseteq \mathrm{supp}(f)$, $f \in \mathfrak{A}_{\infty}$. To establish this suppose $f \in \mathfrak{A}_{\infty}$ and $f = 0$ in a neighbourhood $\mathcal{O}$ of $\omega$. Next choose a $g \in \mathfrak{A}_{\infty}$ with the property that $\mathrm{supp}(g) \subseteq \mathcal{O}$ and $g(\omega) = 1$. Therefore $fg = 0$ and $H(fg) = 0$ by linearity. But then

$$\begin{aligned}(Hf)(\omega) &= (Hf)(\omega) g(\omega) \\ &= (Hfg)(\omega) - f(\omega)(Hg)(\omega) = 0\end{aligned}$$

where we have used the derivation property, together with $H(fg) = 0$, and $f = 0$ on $\mathcal{O}$. Then $\mathrm{supp}(Hf) \subseteq \mathrm{supp}(f)$.

It now follows that Condition C1 of the theorems implies Condition B1 for both $\pm H$, this in turn implies Condition B2 for both $\pm H$, and this implies Condition C2.

## 4. Generalizations

There are several interesting extensions of Theorems 1.1 and 1.2 which arise from varying the domain or the range of the operator $H$. First let $M(\mathfrak{A})$ denote the multiplier algebra of $\mathfrak{A}$, i.e. the algebra of all bounded continuous functions on $X$, and assume $H$ maps $\mathfrak{A}_\infty$ into $M(\mathfrak{A})$. It then follows from the proofs in Section 3 that the four conditions of Part A of Theorems 1.1 and 1.2 remain equivalent, and they imply that $H$ maps $\mathfrak{A}_\infty$ into $\mathfrak{A}$. If on the other hand $H$ is only defined on the subspace of elements of $\mathfrak{A}_\infty$ with compact support in $X$ then the four conditions of Theorem 1.1A remain equivalent with the proviso that the coefficients $l_\alpha$ are only locally finitely many, and, furthermore, are otherwise arbitrary continuous functions on $X$. This corresponds to Peetre's original theorem [13] (Theorem 3.33 of [11]).

Second if the range of the local operator $H$ is decreased one can obtain improved smoothness conditions for the $l_\alpha$ and if the domain is enlarged one obtains better boundedness properties. For example, if

$$\mathfrak{A}_n = \bigcap_{|\alpha| \leqslant n} D(\delta^\alpha)$$

then one has the following.

THEOREM 4.1: *Let $H$ be a linear operator from $\mathfrak{A}_n$ into $\mathfrak{A}_m$, and assume $\mathbb{R}^\nu$ acts freely.*

*The following conditions are equivalent.*

1. $\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f)$, $f \in \mathfrak{A}_n$.
2. *If $m > n$ then $H = 0$, and if $m \leqslant n$ then*

$$(Hf)(\omega) = \sum_{|\alpha| \leqslant n-m} l_\alpha(\omega)(\delta^\alpha f)(\omega), \quad f \in \mathfrak{A}_\infty$$

*where $l_\alpha \in D(\delta^\beta)$ for $|\beta| \leqslant m$, and the $\delta^\beta l_\alpha$ are bounded continuous functions.*

*Hence if these conditions are satisfied there exists a $B > 0$ such that*

$$\| Hf \| \leqslant B \| f \|_n, \quad f \in \mathfrak{A}_n.$$

In this statement we have slightly abused notation. The group $\tau$ extends to the bounded continuous functions $M(\mathfrak{A})$, by the definitions $(\tau f)(\omega) = f(T\omega)$, and hence $\delta$ has an extension to $M(\mathfrak{A})$ which one can also denote by $\delta$. This is the convention used in Condition 2.

Note that if $H$ is a closed operator from $\mathfrak{A}_n$ into $\mathfrak{A}$ then $\| Hf \| \leqslant B \| f \|_n$ for some $B > 0$ by the principle of uniform boundedness. In the above theorem this conclusion follows with locality replacing closedness.

Finally, remark that the conclusion $H=0$ if $m<n$ shows that there are no local smoothing operators. There are of course non-local operators which improve smoothness properties.

Next we have the analogue of Theorem 1.2. Note that in the following statement we adopt the convention that $\tau$ is defined on functions $f$ on $X$ which are continuous on $X \backslash X_0$ by $(\tau_t f)(\omega)=f(T_t\omega)$ and we define $\delta$ by $\delta f(\omega)=\lim_{t\to 0}(f(T_t\omega)-f(\omega))/t$ for all $f$ such that this limit exits for all $\omega \in X$.

THEOREM 4.2: *Assume that $\nu=1$ and let $H$ be a linear operator from $\mathfrak{A}_n$ into $\mathfrak{A}_m$, where $n$, $m$ are non-negative integers.*

*The following conditions are equivalent.*

1. $\operatorname{supp}(Hf) \subseteq \operatorname{supp}(f)$, $f \in \mathfrak{A}_n$.
2. *If $n<m$ then $H=0$, and if $n \geqslant m$ then there exists a (unique) family of functions $l_0, l_1, \ldots, l_{n-m}$ on $X$ such that*

 a. *$l_0 \in \bigcap_{p \leqslant m} D(\delta^p)$ and $\delta^p l_0$ is bounded and continuous on $X$ for $0 \leqslant p \leqslant m$,*

 b. *$l_1, \ldots, l_{n-m}$ are zero on $X_0$, $l_p \in \bigcap_{q \leqslant m} D(\delta^q)$, the $\delta^q l_p$ are continuous on $X \backslash X_0$ for $q=1, \ldots, m$, and*

$$\left| \sum_{\substack{p \\ 0 \leqslant p \leqslant n-m \\ r-q \leqslant p \leqslant r}} \binom{q}{r-p} \delta^{q-r+p}(l_p) \right| \leqslant C\left(1+\nu(\omega)^{n-r}\right)$$

*for some $C>0$, all $r=1, \ldots, n-m+q$ and all $q=0,1, \ldots, m$, and*

$$(Hf)(\omega)=\sum_{p=0}^{n-m} l_p(\omega)(\delta^p f)(\omega), \quad f \in \mathfrak{A}_n.$$

*Moreover, if $n-m \geqslant 0$ then each finite family $l_0, l_1, \ldots, l_{n-m}$ of functions with the foregoing boundedness and continuity properties determines a linear operator from $\mathfrak{A}_n$ into $\mathfrak{A}_m$ which satisfies these conditions.*

*If $n=+\infty$, and $m$ is finite or $m=+\infty$, the theorem remains valid if Condition 2 is replaced by*

2′. *There exists a non-negative integer $N$ and a (unique) family of functions $l_0, l_1, \ldots, l_N$ on $X$ such that*

 a. *$l_0 \in \bigcap_{p<m+1} D(\delta^p)$ and $\delta^p l_0$ is bounded and continuous on $X$ for $0 \leqslant p<m+1$,*

 b. *$l_1, \ldots, l_N$ are zero on $X_0$, $l_p \in \bigcap_{q<m+1} D(\delta^q)$, the $\delta^q_{l_p}$ are continuous*

*on* $X \setminus X_0$ *and are polynomially bounded for* $0 \leqslant q < m+1$, *and*

$$(Hf)(\omega) = \sum_{p=0}^{N} l_p(\omega)(\delta^p f)(\omega), \quad f \in \mathfrak{A}_\infty.$$

*Finally if the conditions are satisfied there exists a* $B > 0$ *such that*

$$\| Hf \| \leqslant B \| f \|_M$$

*where* $M = n$ *if* $n$ *is finite, and* $M$ *is a finite integer if* $n = +\infty$.

REMARK: In the special case that $n$ is finite and $m = 0$, the estimate on the coefficient functions $l_p$ in 2.b. simplifies to

$$| l_p(\omega) | \leqslant C\left(1 + \nu(\omega)^{n-p}\right)$$

for $p = 1, \dots, n$. In the case $n = 1$ one obtains that $l_1$ is bounded. If $H$ is a derivation, i.e. if $l_0 = 0$, this boundedness was asserted without proof by Batty in Theorem 5 of [3].

In the case of finite $n$ and general finite $m$, the estimates imply that the leading coefficient $l_{n-m}$ is bounded,

$$l_{n-m} = \mathrm{O}(1),$$

whilst all other coefficients satisfy the estimates

$$\delta^q l_p = \mathrm{O}(1 + \nu^{n-1})$$

for $q = 0, 1, \dots, m$. These estimates can in fact be improved by using the estimate

$$\nu(\omega)^{m-q} | (\delta^q f)(\omega) | \leqslant \sup_{t \in \mathbb{R}} | (\delta^m f)(T_t \omega) |$$

which follows for $m \geqslant q$ from the proof of Observation 6 in Section 3. The improved estimates

$$\delta^q l_p = \mathrm{O}(1 + \nu^{n-m+q-1})$$

are valid for $q = 1, 2, \dots, m$, but not for $q = 0$, and hence it does not seem possible to cast the estimate of Condition 2b in an equivalent simpler form.

The proofs of Theorems 4.1 and 4.2 are variations on the proofs of the corresponding Theorems 1.1A and 1.2A. We will only discuss the changes needed to prove Theorem 4.2.

First remark that since $\mathfrak{A}_\infty \subseteq \mathfrak{A}_n$ and $\mathfrak{A}_n \subseteq \mathfrak{A}$ the operator $H$ is an operator from $\mathfrak{A}_\infty$ into $\mathfrak{A}$. Thus Theorem 1.2A is valid. Consequently Condition 1 implies that

$$H = \sum_{m=0}^{n'} l_m \delta^m |_{\mathfrak{A}_\infty}$$

for some $n'$, where $l_0$ is continuous and bounded, and $l_1, \ldots, l_{n'}$ are continuous on $X \setminus X_0$ and polynomially bounded. Now the details of the general case $H$: $\mathfrak{A}_n \to \mathfrak{A}_m$ can be deduced by combination of the two special cases $H$: $\mathfrak{A}_n \to \mathfrak{A}$ and $H$: $\mathfrak{A}_\infty \to \mathfrak{A}_m$.

*Case 1*: $H$: $\mathfrak{A}_n \to A$

A minor change in the proof of Observation 1 establishes that there are $C_\omega > 0$ such that

$$|(Hf)(\omega)| \leqslant C_\omega \| f \|_n$$

for all $f \in \mathfrak{A}_n$ and $\omega \in X \setminus F$ where $F$ is a finite set, and it follows from Observation 2 that we can take $n' = n$. But the argument of Observation 2 then shows that

$$H = \sum_{m=0}^{n} l_m \delta^m |_{\mathfrak{A}_n}.$$

(Actually the argument only yields directly that $H$ is equal to $\sum_{m=0}^{n} l_m \delta^m$ on $\mathfrak{A}_{2n}$, but equality on $\mathfrak{A}_n$ follows by continuity. Indeed, for each $\omega \in X$ the functionals $f \mapsto Hf(\omega)$ and $f \mapsto \sum_{m=0}^{n} l_m(\omega)(\delta^m f)(\omega)$ are both continuous on $\mathfrak{A}_n$ (by Observation 1), and as a simple regularization argument shows, $\mathfrak{A}_{2n}$ is dense in $\mathfrak{A}_n$ (in the norm of $\mathfrak{A}_n$).)

Next the proof of Observation 5 establishes the polynomial bounds

$$|l_p(\omega)| < C\left(1 + \nu(\omega)^{n-p}\right)$$

as follows. In Case 1 one now considers sequences $\rho_i \in \mathbf{C}$ which are rapidly decreasing in the sense

$$\lim_{i \to \infty} \nu(\omega_i)^k \rho_i = 0$$

for $k = 0, 1, \ldots, n$. Then choosing $f_i$ as before one can repeat the construction of the $F_i$ and

$$\| F_i \|_k \leqslant 2 \| f_i \|_{I_i, k} = 2\left(2\pi\nu(\omega_i)\right)^k$$

for $k = 0, 1, \ldots, n$. Thus

$$F = \sum_{i \geqslant 1} \rho_i F_i$$

converges in the $\|\cdot\|_n$-norm to an $F \in \mathfrak{A}_n$. One concludes in this case that one has bounds of the form

$$\left| \sum_{p=0}^{n} \rho_p(\omega_i)(2\pi i k \nu(\omega_i))^p \right| \leqslant C\big(1 + \nu(\omega_i)^n\big)$$

for $k = 1, \ldots, n+1$. Thus the matrix elements of

$$\begin{pmatrix} 1 & 2\pi i \nu_i & \cdots & (2\pi i \nu_i)^n \\ \vdots & \vdots & & \vdots \\ 1 & 2\pi i(n+1)\nu_i & \cdots & (2\pi i(n+1)\nu_i)^n \end{pmatrix} \begin{pmatrix} l_0(\omega_i) \\ \vdots \\ l_n(\omega_i) \end{pmatrix}$$

$$= \Omega \begin{pmatrix} l_0(\omega_i) \\ \vdots \\ l_n(\omega_i) \end{pmatrix}$$

are bounded by $C'(1 + \nu(\omega_i)^n)$. But the determinant of $\Omega$ is a multiple of $\nu(\omega_i)^{n(n+1)/2}$ and it follows from the cofactor representation of $\Omega^{-1}$ that it has the form

$$\Omega^{-1} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n+1} \\ a_{21}/\nu_i & a_{22}/\nu_i & \cdots & a_{2,n+1}/\nu_i \\ \vdots & \vdots & & \vdots \\ a_{n+1,1}/\nu_i^n & a_{n+1,2}/\nu_i^n & \cdots & a_{n+1,n+1}/\nu_i^n \end{pmatrix}$$

where the $a_{ij}$ are constants, independent of the frequencies. Thus it follows from

$$\begin{pmatrix} l_0(\omega_i) \\ \vdots \\ l_0(\omega_i) \end{pmatrix} = (\Omega^{-1}) \left( \Omega \begin{pmatrix} l_0 \omega_i \\ \vdots \\ l_n(\omega_i) \end{pmatrix} \right)$$

that

$$|l_p(\omega_i)| \leqslant C\big(1 + \nu(\omega_i)^{n-p}\big)$$

for $p = 1, \ldots, n$. (Note that $l_0$ is bounded by Theorem 1.2A.)

Next one shows, as in Observation 6, that if $f \in \mathfrak{A}_n$ then the function

$$\omega \in X \backslash X_0 \mapsto \nu(\omega)^q (\delta^p f)(\omega)$$

vanishes at infinity on $X \backslash X_0$ for $q = 0, 1, \ldots, n-p$.

The remaining details of the proof of the case $H\colon \mathfrak{A}_n \to \mathfrak{A}$ require only minor modification of the earlier proofs.

*Case 2*: $H\colon \mathfrak{A}_\infty \to \mathfrak{A}_m$

Again $H$ has the form

$$H = \sum_{p=0}^{n} l_p \delta^p |_{\mathfrak{A}_\infty}.$$

But by Theorem 2.4 there exist $f_i \in \mathfrak{A}$ such that the equations

$$(Hf_i)(\omega) = \sum_{p=0}^{n} l_p(\omega)(\delta^p f_i)(\omega)$$

are soluble, near $\omega \in X \backslash X_0$, for $l_p(\omega)$ in terms of $(Hf_i)(\omega)$ and $(\delta^p f_i)(\omega)$. Since all the latter functions are in $\mathfrak{A}_\infty$ it follows that $l_p$ is in $D(\delta^m)$ locally, and thus globally.

Next the coefficient functions are polynomially bounded by Theorem 1.2A. But the operators

$$\delta^q H : \mathfrak{A}_\infty \to \mathfrak{A}_{m-q} \subseteq \mathfrak{A}$$

are well defined for $q = 0, 1, \ldots, m$, and as $l_p \in D(\delta^m)$ for all $p$ one deduces by Leibniz's rule that

$$(\delta^q H)(f) = \sum_{p=0}^{n} \sum_{k=0}^{q} \binom{q}{k} \delta^{q-k}(l_n) \delta^{p+k}(f)$$

$$= \sum_{r=0}^{n+q} \left( \sum_{\substack{p \\ 0 \leqslant p \leqslant n \\ r-q \leqslant p \leqslant r}} \binom{q}{r-p} \delta^{q-r+p}(l_p) \right) \delta^r(f).$$

The zeroth order coefficient of this expression is $\delta^q(l_0)$, and hence $\delta^q(l_0)$ is bounded and continuous by Theorem 1.2A, for $q = 0, \ldots, m$. All other coefficients are continuous and polynomially bounded on $X \backslash X_0$, and hence it follows by a recursive argument that $\delta^q(l_p)$ is continuous and polynomially bounded for $q = 0, 1, \ldots, m$.

We now consider the general case $H$: $\mathfrak{A}_n \to \mathfrak{A}_m$. Then, as $H$ maps $\mathfrak{A}_\infty$ into $\mathfrak{A}_m$ it follows from Case 2 that $H$ has the form

$$Hf = \sum_{p=0}^{N} l_p \delta^p f$$

where the coefficient functions $l_p$ are $m$ times continuously differentiable. But as $H$ maps $\mathfrak{A}_n$ into $\mathfrak{A}$ it follows from Case 1 that $N \leqslant n$. But $\delta^q H$ maps $\mathfrak{A}_n$ into $\mathfrak{A}_{m-q} \subseteq \mathfrak{A}$ for $q = 0, 1, \ldots, m$. If $f \in \mathfrak{A}_\infty$ we have in particular

$$\delta H(f) = (\text{Polynomial in } \delta \text{ of degree} \leqslant n)(f) + l_n \delta^{n+1}(f),$$

but as $D(\delta H) = \mathfrak{A}_n$ it follows by uniqueness of this expansion that $l_n = 0$ (if $m \geqslant 1$). Proceeding by induction one finds

$$\delta^q H(f) = (\text{Polynomial in } \delta \text{ of degree} \leqslant n)(f) + l_{n-q+1} \delta^{n+1}(f)$$

and deduces that $l_{n-q+1} = 0$ for $q = 1, \ldots, m$. Hence $N \leqslant n - m$ and $H$ has the form

$$Hf = \sum_{p=0}^{n-m} l_p \delta^p f.$$

The statements concerning the boundedness and continuity properties of the functions $\delta^q l_p$ now follow by applying Case 1 to the operators

$$H,\ \delta H, \ldots, \delta^m H$$

which all map $\mathfrak{A}_n$ into $\mathfrak{A}$, and using the explicit expression for $\delta^q H$ found in Case 2. Conversely, these boundedness and continuity properties of the coefficient functions imply that $H, \delta H, \ldots, \delta^m H$ map $\mathfrak{A}_n$ into $\mathfrak{A}$, i.e. $H$ maps $\mathfrak{A}_n$ into $\mathfrak{A}_m$.

Finally, if $l_0$ is bounded, i.e. $\| l_0 \| < C'$, and

$$| l_p(\omega) | \leqslant C\left(1 + \nu(\omega)^{n-p}\right)$$

one has

$$\begin{aligned} | l_p(\omega)(\delta^p f)(\omega) | &\leqslant C\left(1 + \nu(\omega)^{n-p}\right) |(\delta^p f)(\omega)) | \\ &\leqslant C( \| \delta^p f \| + \| \delta^n f \|) \end{aligned}$$

where the last estimate uses the inequality derived in the proof of Observation 6 in Section 3. Therefore

$$\| Hf \| \leqslant C' \| f \| + \sum_{p=1}^{n} C( \| \delta^p f \| + \| \delta^n f \| )$$

$$\leqslant B \| f \|_n.$$

We remark that, as Example 2.4 of [6] shows, the assumption that $H$ is defined on all of $\mathfrak{A}_\infty$ cannot be relaxed very much. This example describes a flow on $X = [0, 1] \times [0, 1]$ which leaves all the points $(x, 0)$ fixed and a derivation which satisfies Condition 1A of Theorems 1.1 and 1.2 but not Conditions A2 and A3 for the points $\omega = (x, 0)$. (The pair $(\delta, \delta_0)$ in Example 2.4 corresponds to $(H, \delta)$ in the present notation.)

## 5. General groups and local flows

Questions arise concerning actions of more general groups than $\mathbb{R}^\nu$, and also concerning local actions.

It would appear that Theorem 1.1 holds for a Lie group or indeed (via structure theory) for any locally compact group. Some modifications to the proof are needed. For instance, one can use [8] to express a function obtained by the construction of Lemma 2.5 as a finite sum of convolutions with respect to the given group.

On the other hand, while it would be desirable to generalize Theorem 1.2 to the case of an arbitrary locally compact group, one should certainly begin with the case $\mathbb{R}^\nu$, $\nu > 1$, since already in this case the conditions of continuity and polynomial boundedness satisfied by the coefficients of a local operator from $\mathfrak{A}_\infty$ into $\mathfrak{A}$ are more difficult to formulate. The continuity properties were already discussed after the statement of Theorem 1.1. The boundedness properties would be in terms of parameters determining the closed subgroups of $\mathbb{R}^\nu$. (In the case of $\mathbb{R}^1$ these subgroups are determined by one parameter: the frequency.)

We shall now generalize Theorem 1.2 to the case of a local flow of dimension one, rather than a flow. In this case Theorem 1.1 is no longer valid, as the presence of even a single incomplete orbit can permit the coefficients of a local operator to be unbounded. For example, in the case of the local flow generated by $d/dx$ on the positive $x$-axis $(0, +\infty)$, the coefficients, as we shall show, may be any continuous functions $l$ on $(0, +\infty)$ satisfying bounds of the form $|l(x)| \leqslant C(1 + x^{-n})$ for some positive integer $n$. It should be remarked that the presence of an incomplete orbit does not always permit the coefficients to be unbounded; an example where they must be bounded is the flow generated by $d/dx$ on the plane $\mathbb{R}^2$ with the negative $x$-axis $(-\infty, 0]$ removed.

Both of the preceding examples are obtained by restricting a global flow to an open subset. For such a local flow, the proofs of Theorems 2.1 and 2.4 and Remark 2.6 are valid without change. This is also the case for a local flow with the following property: for each compact subset $K$, there exists an open neighbourhood $\mathcal{O}$ of $K$ such that the restriction of the local flow to $\mathcal{O}$ can be obtained as above, i.e. as the restriction to an open subset of a global flow. In fact, any local flow has this property. This can be seen by multiplying the associated derivation by a function in $\mathfrak{A}_1$ of compact support, equal to 1 in an open neighbourhood $\mathcal{O}$ of $K$. (The resulting derivation generates a flow, agreeing on $\mathcal{O}$ with the given local flow.) In other words, Theorems 2.1 and 2.4 together with Remark 2.6 hold for any local flow.

To state Theorem 1.2 for a local flow we must define $\nu(\omega)$ also when the orbit of $\omega$ is incomplete. We define $\nu(\omega)$ to be $1/p(\omega)$ where $p(\omega) \in (0, +\infty]$ is the supremum of the numbers $K > 0$ such that $T_t\omega$ is defined for all $t \in (-K/2, K/2)$. Note that $p(\omega)$ can be defined in this way also when the orbit of $\omega$ is incomplete and not equal to the point $\omega$, by considering only $K$ such that $T_{(-K/2, K/2)}\omega$ is an injective image of the interval $(-K/2, K/2)$. (The requirement of injectivity is automatically fulfilled when the orbit of $\omega$ is incomplete.)

With this definition of $\nu(\omega)$, Theorem 1.2 may now be asserted for local flows with only the following modification: the growth condition on $l_0$ is that it is polynomially bounded on $X \setminus X_0$, and bounded on the set of complete orbits in $X$.

The proof of Theorem 1.2 given in Section 3 is valid in the general case without any essential change except for the proof of Observation 5. Thus, the proof of Observation 4 now shows that $l_0$ is bounded on the set of complete orbits. Observation 5, as before, but now with the new proof given below, shows that each of $l_0, l_1, \ldots, l_n$ is polynomially bounded on $X \setminus X_0$. The proof of Observation 6 still shows that the polynomial growth conditions on $l_1, \ldots, l_n$ are sufficient, and it shows furthermore that for each $f \in \mathfrak{A}_\infty$ and $k = 1, 2, \ldots$ the function

$$\omega \mapsto \nu_0(\omega)^k f(\omega)$$

vanishes at infinity, where

$$\nu_0(\omega) = \begin{cases} 1 \text{ if the orbit of } \omega \text{ is complete,} \\ \nu(\omega) \text{ if the orbit of } \omega \text{ is incomplete.} \end{cases}$$

Hence as in Section 3 it follows that the growth condition for $l_0$ given above is sufficient.

PROOF of OBSERVATION 5 for a one-dimensional local flow: Assume that $l_r(\omega_t)$ is not bounded by a polynomial in $\nu(\omega_t)$.

As before we have two cases. Only Case 1 (i.e. $\nu(\omega_i) > 1/2$ for all $i$) needs a different argument. If the orbit of each $\omega_i$ is periodic we may proceed as before. It remains, after passing to a subsequence, to consider the case that no $\omega_i$ has a periodic orbit. Note that, as $\nu(\omega_i) > 0$, if the orbit of $\omega_i$ is not periodic then it is not complete.

Choose $g \in C_{00}^{\infty}(-1, 1)$ such that $g = 1$ in a neighbourhood of 0. For each $i$, consider the orbit segment $I_i = T_{(-t_i, t_i)}\omega_i$ where $t_i = p(\omega_i)/4$. Writing $g(t/t_i) = g_i(t)$ we have

$$g_i \in C_{00}^{\infty}(-t_i, t_i),$$

$$g_i = 1 \text{ in a neighbourhood of } 0,$$

$$\| g_i \|_k \leqslant t_i^{-k} \| g \|_k = (4\nu(\omega_i))^k \| g \|_k, \quad k = 1, 2, \ldots,$$

where we used $t_i < \frac{1}{2} < 1$ in the last estimate. Denote by $f_i$ the function

$$T_{(-t_i, t_i)}\omega_i \ni T_t\omega_i \mapsto g_i(t) \exp(\mathrm{imt})$$

where $m$ is one of the numbers $1, 2, \ldots, n+1$. By [8], $f_i \in \mathfrak{A}_{\infty}(I_i)$. By Leibniz's rule, as $m \geqslant 1$,

$$\| f_i \|_{I_i, k} \leqslant 2^k m^k \| g_i \|_k$$

$$\leqslant (8m\nu(\omega_i))^k \| g \|_k, \quad k = 1, 2, \ldots.$$

Furthermore

$$f_i^{(k)}(\omega_i) = (im)^k, \quad k = 0, 1, 2, \ldots.$$

Necessarily, for each $j$, infinitely many of the closed orbit segments $I_i^- = T_{[-t_i, t_i]}\omega_i$ must be disjoint from $I_j^-$. (If $I_i^- \cap I_j^- \neq \emptyset$ then $\omega_i \in T_{[-3t_j/2, 3t_j/2]}\omega_j$, and $l_r$ is continuous on this compact set by Observation 3 and therefore bounded there, in contravention of the assumption that $l_r(\omega_i)$ is not polynomially bounded.)

Hence, by the boundedness of the $p(\omega_i) = 1/\nu(\omega_i)$, there exists a subsequence of $I_i$, also denoted by $I_i$, and a sequence of mutually disjoint open subsets $\mathcal{O}_i$ of $X$ with $I_i \subseteq \mathcal{O}_i$.

By Theorem 2.1, applied to $f_i$ and $\mathcal{O}_i$ for each $i$, there exists $F_i \in \mathfrak{A}_{\infty}$ such that

$$\| F_i \|_k \leqslant 2 \| f_i \|_{I_i, k} \leqslant 2(8m\nu(\omega_i))^k \| g \|_k, \quad k = 1, 2, \ldots, i,$$

$$F_i = f_i \text{ on } I_i,$$

$$\mathrm{supp}(F_i) \subseteq \mathcal{O}_i.$$

The proof of Observation 5 in the case that no $\omega_i$ has a complete orbit can now be completed just as in the case that the orbit of every $\omega_i$ is complete (and hence periodic). (The only difference is that the matrices $\Omega$ now are independent of $i$.) We remark that the preceding argument is also valid in the case that the orbits are periodic (dealt with a simpler method in Section 3).

We remark finally that while we have used the theorem of Dixmier and Malliavin [8] in handling a one-dimensional local flow, this could easily be avoided by a suitable approximation (of $f_i$ by $f_i * h_i$ for some $h_i$). On the other hand, the use of this theorem in the case of a non-abelian group seems to be essential. We note that one could also use this theorem (i.e. [8], Théorème 3.1) to prove Lemma 2.5.

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