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STEVO TODORČEVIĆ

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REMARKS ON CELLULARITY IN PRODUCTS

Stevo Todorčević

This is a continuation of our note [30] where it is shown that for many cardinals κ the κ chain condition is not a productive property. Here we consider the closely related problem of whether the cellularity, $c(X)$, of a topological space is a productive property. Under standard translations this problem is equivalent to the problem of whether the κ^+ chain condition is a productive property. Note that [30] does not answer this problem since all cardinals κ for which [30] shows $\kappa cc \times \kappa cc \neq \kappa cc$ are of the form cf 2^λ for some λ, and so all of them might be weakly inaccessible. In this note we show, without additional set-theoretic assumptions, that indeed for many cardinals κ the κ^+ chain condition is not a productive property. So in particular we show the following which solves a problem explicitly stated by Kurepa [16].

THEOREM: *There is a compact Hausdorff space X such that $c(X^2) > c(X)$.*

In the second part of this note we construct for many cardinals $\kappa \leqslant 2^{\aleph_0}$ a *ccc* poset without precaliber κ. In particular, we show the following.

THEOREM: *There is a ccc poset without precaliber $2^{\aleph_0}$.*

This solves a problem explicitly stated in [2, §5] and which has been also asked by many other authors. Actually, our construction gives much more general examples. For instance, we shall construct for each $n \geqslant 2$ a *ccc* poset $\mathscr{P}$ which has property $K_{\theta,n}$ but which does not have property $K_{\theta,n+1}$ for any $\theta \leqslant 2^{\aleph_0}$ such that cf $\theta =$ cf $2^{\aleph_0}$. So in particular, we shall get in ZFC what is in [2] called Gaifman's example ([7]) and its variation due to Argyros ([0]) who has used the Continuum Hypothesis.

All our results are stated in their partial order form but in almost all cases we shall indicate the corresponding spaces which are of independent interest. Our lemmas are more general than needed for our theorems just to indicate possibilities which we do not bother to state explicitly and which might be interesting to an expert reader. The note ends with a remark on the S and L spaces problem.

§1. Cellularity in products

Fix a cardinal κ with $\lambda = \mathrm{cf}\ \kappa$ and a sequence $\langle \kappa_\xi \colon \xi < \lambda \rangle$ of infinite cardinals such that $\sup_\xi \kappa_\xi = \kappa$ and

$$\left| \prod_{\xi < \eta} \kappa_\xi \right| < \kappa_\eta = \mathrm{cf}\ \kappa_\eta \quad \text{for all } \eta < \lambda \tag{1}$$

For distinct $a, b \in \prod_\xi \kappa_\xi$ we define $\rho(a, b) = \min\{\xi \colon a(\xi) \neq b(\xi)\}$. From now on when writing $\rho(a, b)$ we always assume $a \neq b$. When we say that $\mathscr{F}$ is a filter on λ we also assume that $\mathscr{F}$ is nontrivial and that $\mathscr{F}$ contains all cobounded subsets of λ. By $\mathscr{F}_\lambda$ we denote the filter of all cobounded subsets of λ. For a filter $\mathscr{F}$ on λ and $a, b \in \prod_\xi \kappa_\xi$ we define

$$\begin{aligned} a =_{\mathscr{F}} b \quad &\text{iff} \quad \{\xi < \lambda \colon a(\xi) = b(\xi)\} \in \mathscr{F}, \\ a \leqslant_{\mathscr{F}} b \quad &\text{iff} \quad \{\xi < \lambda \colon a(\xi) \leqslant b(\xi)\} \in \mathscr{F}, \\ a <_{\mathscr{F}} b \quad &\text{iff} \quad \{\xi < \lambda \colon a(\xi) < b(\xi)\} \in \mathscr{F}. \end{aligned}$$

If $\mathscr{F} = \mathscr{F}_\lambda$, then we use $=^*$, $\leqslant^*$ and $<^*$ to denote $=_{\mathscr{F}}$, $\leqslant_{\mathscr{F}}$ and $<_{\mathscr{F}}$, respectively. By $\mathscr{F}^+$ we denote the set of all subsets I of λ such that $I \cap J \neq \emptyset$ for all $J \in \mathscr{F}$. For $I \in \mathscr{F}^+$ and $A \subseteq \prod_\xi \kappa_\xi$ we say that A is $\leqslant_{\mathscr{F}}$-unbounded in $\prod_\xi \kappa_\xi$ on I if there is no $b \in \prod_\xi \kappa_\xi$ such that for all $a \in A$ for some $J \in \mathscr{F}$, $a(\xi) \leqslant b(\xi)$ for all $\xi \in I \cap J$. Similarly we define $<_{\mathscr{F}}$-unbounded on I. We do not mention I if $I = \lambda$.

For $A \subseteq \prod_\xi \kappa_\xi$ and $I \subseteq \lambda$ we define

$$\mathscr{P}_I(A) = \{ p \subseteq A \colon p \text{ is finite and } \rho(a, b) \in I \text{ for all } a, b \in p \}$$

and consider $\mathscr{P}_I(A)$ as a poset under the ordering $\supseteq$. Let $X_I(A)$ be the set of all $B \subseteq A$ such that $\rho(a, b) \in I$ for all $a, b \in B$ considered as a compact subspace of $\{0, 1\}^A$ under standard identification via characteristic functions.

LEMMA 1: *Let $\mathscr{F}$ be a filter on λ and let θ be a regular cardinal. Let $I \in \mathscr{F}^+$ and let $A = \{a_\alpha \colon \alpha < \theta\}$ be $\leqslant_{\mathscr{F}}$-increasing and $\leqslant_{\mathscr{F}}$-unbounded on I. Then $\mathscr{P}_I(A)$ is a θcc poset.*

PROOF: Fix a sequence $\langle p_\sigma \colon \sigma < \theta \rangle$ of elements of $\mathscr{P}_I(A)$. By a standard Δ-system argument we may assume p_σ's are disjoint and all of size $n \geqslant 1$. We may also assume that there is $\xi_0 < \lambda$ such that for all σ, $\rho(a, b) < \xi_0$ for all $a, b \in p_\sigma$. For $\sigma < \theta$ and $\xi < \lambda$ set

$$D_\sigma = \{\alpha < \theta \colon a_\alpha \in p_\sigma\} \text{ and } b_\sigma(\xi) = \min\{a_\alpha(\xi) \colon \alpha \in D_\sigma\}.$$

Then $\{b_\sigma \colon \sigma < \theta\}$ is $\leqslant_{\mathscr{F}}$-unbounded on I, so we can find $\eta \in I \setminus \xi_0$ such that $\{b_\sigma(\eta)\colon \sigma < \theta\}$ is unbounded in κ_η. By (1) we can find $C \subseteq \theta$ of order type κ_η and $\{t_1, \ldots, t_n\} \subseteq \prod_{\xi<\eta}\kappa_\xi$ such that:

$$a_\alpha(\eta) < a_\beta(\eta) \text{ for all } \sigma < \tau \text{ in } C \text{ and } \alpha \in D_\sigma,\ \beta \in D_\tau, \tag{2}$$

$$\text{if } \sigma \in C, \text{ if } 1 \leqslant i \leqslant n, \text{ and if } \alpha \text{ is the } i^{\text{th}} \text{ member of } D_\sigma,$$
$$\text{then } a_\alpha \restriction \eta = t_i. \tag{3}$$

It is now clear that for all σ, $\tau \in C$ and all a, $b \in p_\sigma \cup p_\tau$, $\rho(a, b) \in I$. This shows that p_σ is compatible with p_τ for all σ, $\tau \in C$ and finishes the proof of Lemma 1.

Clearly, the proof of Lemma 1 also gives the following.

LEMMA 2: *Let $\mathscr{F}$, θ, I, and A be as in Lemma* 1. *Then for any finite sequence $I_0, \ldots, I_n$ of subsets of λ containing I, $\mathscr{P}_{I_0}(A) \times \ldots \times \mathscr{P}_{I_n}(A)$ is a θcc poset.*

Note that if $|A| = \theta$ and if $I \cap J = \emptyset$ then $\mathscr{P}_I(A) \times \mathscr{P}_J(A)$ is not θcc since $\{\langle\{a\}, \{a\}\rangle\colon a \in A\}$ is a pairwise incompatible set. The following lemma says a little more than this.

LEMMA 3: *If A has size θ of cofinality $> \kappa$ and if I and J are almost disjoint, then $\mathscr{P}_I(A) \times \mathscr{P}_J(A)$ is not θcc.*

Note that many cardinals θ satisfy the hypothesis of Lemma 1 for the filter $\mathscr{F}_\lambda$, so we have already shown $\theta cc \times \theta cc \neq \theta cc$ for many cardinals θ. Actually, assuming $\lambda > \omega$ and changing $\langle \kappa_\xi\colon \xi < \lambda\rangle$ using the $\leqslant^*$-minimal function which bounds a given κ^+-sequence of elements of $\prod_\xi \kappa_\xi$ it is easy to show that κ^+ can satisfy the hypotheses of Lemmas 1 and 3. In [24], Shelah proved that this can be done even if $\lambda = \omega$ assuming κ is sufficiently closed under exponentiation. In order to fit our purpose here, we need the following formulation of his Lemma 11 (D) from [24].

LEMMA 4: *Assume $\rho^\lambda < \kappa$ for all $\rho < \kappa$. Let $\theta > \kappa$ be regular and let $\{a_\alpha\colon \alpha < \theta\}$ be an $<^*$-increasing sequence from $\prod_\xi \kappa_\xi$. Then there exist another $\langle \kappa'_\xi\colon \xi < \lambda\rangle$ satisfying* (1) *and $<^*$-increasing $\{a'_\alpha\colon \alpha < \theta\}$ in $\prod_\xi \kappa'_\xi$ which is $<^*$-cofinal*[1] *in $\prod_\xi \kappa'_\xi$.*

PROOF: Inductively on $\sigma < \bar\sigma$ define $\{b_\sigma\colon \sigma < \bar\sigma\} \subseteq \prod_\xi(\kappa_\xi + 1)$ such that:

$$b_\tau \leqslant^* b_\sigma \text{ and } b_\tau \text{ is not} =^* \text{ to } b_\sigma \text{ for all } \sigma < \tau < \bar\sigma, \tag{4}$$

$$a_\alpha <^* b_\sigma \text{ for all } \alpha < \theta \text{ and } \sigma < \bar\sigma. \tag{5}$$

[1] I.e., for all $b \in \prod_\xi \kappa'_\xi$ there is an $\alpha < \theta$ such that $b <^* a'_\alpha$.

The ordinal $\bar{\sigma}$ is determined as the first place where the induction stops. By $(2^\lambda)^+ \to (\lambda^+)^2_\lambda$ it follows easily that $\bar{\sigma} < (2^\lambda)^+$. We claim that $\bar{\sigma}$ is a successor ordinal. Suppose not, and for $\xi < \lambda$ define $B_\xi = \{b_\sigma(\xi)\colon \sigma < \bar{\sigma}\}$. Let $B = \prod_\xi B_\xi$. Then $|B| \leqslant 2^\lambda$, so there is $\gamma < \theta$ such that if $b \in B$ and $a_\gamma <^* b$, then $a_\alpha <^* b$ for all $\gamma \leqslant \alpha < \theta$. Define $b \in \prod_\xi(\kappa_\xi + 1)$ by $b(\xi) = \min\{\beta\colon \beta \in B_\xi \text{ and } \beta > a_\gamma(\xi)\}$. From (4) and (5) it follows directly that $a_\alpha <^* b \leqslant^* b_\sigma$ for all $\alpha < \theta$ and $\sigma < \bar{\sigma}$. Since $\bar{\sigma}$ is a limit ordinal for no $\sigma < \bar{\sigma}$, $b =^* b_\sigma$ holds. But this contradicts the definition of $\bar{\sigma}$.

Set $\bar{b} = b_\sigma$ where $\sigma = \bar{\sigma} - 1$, and for $\xi < \lambda$ define $\bar{\kappa}_\xi = \operatorname{cf} \bar{b}(\xi)$. For $\xi < \lambda$, pick a set C_ξ of order type $\bar{\kappa}_\xi$ cofinal in $\bar{b}(\xi)$. Now for $\alpha < \theta$ define $\bar{a}_\alpha \in \prod_\xi \bar{\kappa}_\xi$ by

$$\bar{a}_\alpha(\xi) = \begin{cases} tp\big(C_\xi \cap a_\alpha(\xi)\big) & \text{if } a_\alpha(\xi) < \bar{b}(\xi) \\ 0 & \text{otherwise.} \end{cases}$$

CLAIM 1: *$\{\bar{a}_\alpha\colon \alpha < \theta\}$ is $\leqslant^*$-unbounded in $\prod_\xi \bar{\kappa}_\xi$ on any $I \in [\lambda]^\lambda$.*

PROOF: Assume that for some $I \in [\lambda]^\lambda$ and $c \in \prod_\xi \bar{\kappa}_\xi$, $\{\bar{a}_\alpha\colon \alpha < \theta\}$ is $\leqslant^*$-bounded by c on I. Since $\theta > \lambda$ we can find $E \in [\theta]^\theta$ and $\xi_0 < \lambda$ such that for all $\alpha \in E$ and $\xi \in I \backslash \xi_0$, $\bar{a}_\alpha(\xi) \leqslant c(\xi)$. We may also assume that for all $\alpha \in E$ and $\xi \in \lambda \backslash \xi_0$, $a_\alpha(\xi) < \bar{b}(\xi)$. Thus for all $\alpha \in E$ and $\xi \in \lambda \backslash \xi_0$, $\bar{a}_\alpha(\xi) = tp(C_\xi \cap a_\alpha(\xi))$. Define $d \in \prod_\xi(\kappa_\xi + 1)$ by $d(\xi) = \bar{b}(\xi)$ if $\xi \notin I \backslash \xi_0$ and $d(\xi) =$ the unique $\delta \in C_\xi$ such that $c(\xi) = tp(C_\xi \cap \delta)$ if $\xi \in I \backslash \xi_0$. Then it is easily seen that d satisfies (4) and (5) in place of $b_{\bar{\sigma}}$ contradicting the fact that $b_{\bar{\sigma}}$ could not have been defined.

CLAIM 2: *If $\alpha < \theta$ then for some $\gamma < \theta$, $\bar{a}_\alpha <^* \bar{a}_\beta$ for all $\gamma \leqslant \beta < \theta$.*

PROOF: First of all note that $\bar{a}_\alpha \leqslant^* \bar{a}_\beta$ for $\alpha < \beta$. Hence if Claim 2 fails for some α, then since $\theta > 2^\lambda$ there are $I \in [\lambda]^\lambda$ and $F \in [\theta]^\theta$ such that $\bar{a}_\alpha(\xi) = \bar{a}_\beta(\xi)$ for all $\beta \in F$ and $\xi \in I$. This contradicts Claim 1.

Hence by going to a cofinal subset of θ we may assume that $\bar{a}_\alpha <^* \bar{a}_\beta$ for all $\alpha < \beta < \theta$. Since $\rho^\lambda < \kappa$ for all $\rho < \kappa$ there is an increasing sequence $\{\zeta_\xi\colon \xi < \lambda\} \subseteq \lambda$ such that if we put $\kappa'_\xi = \bar{\kappa}_{\zeta_\xi}$ for $\xi < \lambda$ then $\langle \kappa'_\xi\colon \xi < \lambda\rangle$ satisfies (1). For $\alpha < \theta$, define $a'_\alpha \in \prod_\xi \kappa'_\xi$ by $a'_\alpha(\xi) = \bar{a}_\alpha(\zeta_\xi)$. By Claim 1 it follows directly that $\langle \kappa'_\xi\colon \xi < \lambda\rangle$ and $\langle a'_\alpha\colon \alpha < \theta\rangle$ satisfy the conclusion of Lemma 4.

THEOREM 1: *Assume $\rho^{\operatorname{cf}\kappa} < \kappa$ for all $\rho < \kappa$. Then for each finite $n \geqslant 1$ there is a poset $\mathscr{P}$ such that $\mathscr{P}^n$ satisfies the κ^+cc but $\mathscr{P}^{n+1}$ does not.*

PROOF: Note that if $\langle \kappa_\xi\colon \xi < \lambda\rangle$ satisfies (1), then $\prod_\xi \kappa_\xi$ contains an $<^*$-increasing κ^+ sequence $A = \{a_\alpha\colon \alpha < \kappa^+\}$. So by Lemma 4 we may assume that, in fact, A is $\leqslant^*$-unbounded on any $I \in [\lambda]^\lambda$. Fix $I_0, \ldots, I_n$ in $[\lambda]^\lambda$ such that $\bigcap_{i \leqslant n} I_i = \emptyset$ but $|\bigcap_{i \in N} I_i| = \lambda$ for all $N \subseteq n+1$ of size

n. Define $\mathscr{P}=\mathscr{P}_{I_0}(A)\oplus\ldots\oplus\mathscr{P}_{I_n}(A)$. By Lemma 2, $\mathscr{P}^n$ is a κ^+cc poset, and by the argument of Lemma 3, $\mathscr{P}_{I_0}(A)\times\ldots\times\mathscr{P}_{I_n}(A)$ is not κ^+cc. So $\mathscr{P}^{n+1}$ is not κ^+cc.

COROLLARY 1. *If* $\kappa=(2^{\aleph_0})^{+\omega}$, *then for each finite* $n\geqslant 1$ *there is a compact Hausdorff space* X *such that* $c(X^n)=\kappa$ *but* $c(X^{n+1})>\kappa$.

In concluding this section, let us remark that Theorem 1 does not use the whole power of Lemma 1. As an illustration of another use of Lemma 1 we note that the following can be proved using the fact that there are more ultrafilters on ω than cardinals $\leqslant 2^{\aleph_0}$: If $2^{\aleph_0}$ is a singular cardinal, then $\{\kappa<2^{\aleph_0}:\ \kappa cc\times\kappa cc\neq\kappa cc\}$ is unbounded in $2^{\aleph_0}$.

§2. Negative partition relations

Negative partition relations are closely connected to the problem of κcc in products. In this section we give information which explains this connection.

For a family $\mathscr{G}$ of sets "$c(\mathscr{G})$" and "$\mathscr{G}$ satisfies the θcc" are defined similarly to the corresponding notions for posets or spaces. If $\mathscr{G}$ and $\mathscr{H}$ are families of sets then $\mathscr{G}\otimes\mathscr{H}$ denotes the family of all sets of the form $G\times H$ where $G\in\mathscr{G}$ and $H\in\mathscr{H}$.

PROPOSITION 1. $\kappa\nrightarrow(\theta_\xi)^2_{\xi<\lambda}$ *iff for each* $\xi<\lambda$ *there is a* $\theta_\xi cc$ *family* $\mathscr{G}_\xi$ *such that* $\otimes_{\xi<\lambda}\mathscr{G}_\xi$ *is not* κcc.

The direct implication of this result has been essentially proved by Sierpiński [27] using a variant of the following construction (see also [18] and [12; p. 286]). Let $\prec$ be a fixed well-ordering of $\mathbb{R}$ and define disjoint $[\mathbb{R}]^2=S_0\cup S_1$ by

$$\{r,s\}\in S_0\quad\text{iff } r<s \text{ and } r\prec s.$$

For $i<2$ let X_i be the set of all $A\subseteq\mathbb{R}$ such that $[A]^2\subseteq S_i$ and consider X_i as a compact subspace of $\{0,1\}^{\mathbb{R}}$. For $i<2$ and $r\in\mathbb{R}$ let $G_i(r)$ be the set of all $A\in X_i$ which contain r. Then $\mathscr{G}_i=\{G_i(r):\ r\in\mathbb{R}\}$ is a family of open subsets of X_i such that $c(\mathscr{G}_i)=\aleph_0$. However, the product $\mathscr{G}_0\otimes\mathscr{G}_1$ is not $2^{\aleph_0}cc$ as the family $\{G_0(r)\times G_1(r):\ r\in\mathbb{R}\}$ shows. It can be proved that for all $i<2$, X_i is not $2^{\aleph_0}cc$. Actually, for no $2^{\aleph_0}cc$ space Y there is 1-1 h: $\mathscr{G}_i\to\tau(Y)$ such that $G_0\cap G_1=\emptyset$ implies $h(G_0)\cap h(G_1)=\emptyset$ for $G_0,G_1\in\mathscr{G}_i$.

The converse implication of Proposition 1 has essentially been proved by Kurepa [17] who used $(2^\kappa)^+\rightarrow((2^\kappa)^+,\kappa^+)^2$ (due independently to Erdös ([4][5]) and himself ([14][15])) in getting the first known upper bound for the cellularity in products.

Almost all known proofs of $\kappa cc \times \kappa cc \neq \kappa cc$ give as byproducts strong negative partition relations. This is also true for the construction of this paper as the next result shows. However, let us note that constructing a partition for $\kappa cc \times \kappa cc \neq \kappa cc$ is a much more delicate task than just constructing a counterexample to an ordinary partition relation.

THEOREM 2: *If $\rho^{\mathrm{cf}\kappa} < \kappa$ for all $\rho < \kappa$ then $\kappa^+ \nrightarrow [\kappa^+]^2_{\mathrm{cf}\kappa}$.*

PROOF: Let $\langle \kappa_\xi \colon \xi < \lambda \rangle$ and $\{ a_\alpha \colon \alpha < \kappa^+ \}$ be as in the proof of Theorem 1. Fix disjoint $\{ I_\xi \colon \xi < \lambda \} \subseteq [\lambda]^\lambda$ covering λ. For $\xi < \lambda$ define $K_\xi \subseteq [\kappa^+]^2$ by

$$\{ \alpha, \beta \} \in K_\xi \quad \text{iff } \rho(a_\alpha, a_\beta) \in I_\xi .$$

Then clearly $[\kappa^+]^2 = \cup_{\xi < \lambda} K_\xi$ is a disjoint partition. We claim that $[\kappa^+]^2 = \cup_\xi K_\xi$ is a witness of $\kappa^+ \nrightarrow [\kappa^+]^2_\lambda$. Namely, if $\xi < \lambda$ and if $D \in [\kappa^+]^{\kappa^+}$, then we must have $[D]^2 \cap K_\xi \neq \emptyset$ since $\mathscr{P}_{I_\xi}(A)$ is a $\kappa^+ cc$ poset.

The following is a slight improvement over a similar result of Galvin [8] which we give without proof.

THEOREM 3: *Assume $n \geqslant 1$ is finite and there is a poset $\mathscr{P}$ such that $\mathscr{P}^n$ is κcc but $\mathscr{P}^{n+1}$ is not κcc. Then $\kappa \nrightarrow [\kappa]^2_{2n+2}$.*

The next result generalizes at the same time known results of Fleissner [6] and Roitman [20] about *ccc* in products.

THEOREM 4: *If we add one Cohen or random real, then in the extension there is a ccc poset $\mathscr{P}$ such that $\mathscr{P}^2$ is not $2^{\aleph_0} cc$.*

PROOF: Let $\{ r_\alpha \colon \alpha < 2^{\aleph_0} \}$ be a ground model 1-1 enumeration of ω^ω. Let $c \colon \omega \to 2$ be a Cohen or random real. Define $[2^{\aleph_0}]^2 = K_0 \cup K_1$ by $\{ \alpha, \beta \} \in K_i$ iff $c(\rho(r_\alpha, r_\beta)) = i$. For $i < 2$ let $\mathscr{P}_i$ be the set of all finite $p \subseteq 2^{\aleph_0}$ such that $[p]^2 \subseteq K_i$. We claim that each $\mathscr{P}_i$ is a *ccc* poset. This is proved by a standard forcing argument in the ground model. The random real case uses the following lemma, which we state in a more general form than needed.

LEMMA 5: *Assume $A \subseteq \mathbb{R}$ is uncountable and $G \subseteq [A]^2$ has uncountable chromatic number. Then for any sequence $\langle B_a \colon a \in A \rangle$ of Borel sets of positive measure there is a sequence $\{ a_n \colon n < \omega \}$ from A converging to $a \in A$ such that $B_a \cap \bigcap_{n < \omega} B_{a_n}$ has positive measure and $\{ a, a_n \} \in G$ for all n.*

Note that the proof of Theorem 4 uses the same kind of partition as the proof of Theorem 2, and that $2^{\aleph_0} \not\to [\aleph_1]^2_{\aleph_0}$ can be added to the conclusion of Theorem 4. This also follows from the following more general result.

THEOREM 5: *If we add one Cohen or random real, then in the extension any ground model $G \subseteq [2^{\aleph_0}]^2$ with no uncountable independent set can be split into two disjoint subsets with the same property.*

PROOF: Let $\{r_\alpha\colon \alpha < 2^{\aleph_0}\}$ and $c\colon \omega \to 2$ be as in the proof of Theorem 4. Define $G = G_0 \cup G_1$ by $\{\alpha, \beta\} \in G_i$ iff $\{\alpha, \beta\} \in G$ and $c(\rho(r_\alpha, r_\beta)) = i$. The proof that this works uses a standard forcing argument in the ground model which in the random real case needs Lemma 5.

Hence if we add $> \aleph_1$ Cohen or random reals then any $G \subseteq [\omega_1]^2$ with no uncountable independent set can be split into two disjoint subsets with the same property. It is an open problem whether this can be done in ZFC alone.

§3. Precalibers of *ccc* posets

A poset $\mathscr{P}$ has precaliber θ if every θ-sequence of elements of $\mathscr{P}$ contains a θ-subsequence with the finite compatibility property in $\mathscr{P}$. We say that $\mathscr{P}$ has property $K_{\theta,n}$ $(n \in \omega\setminus 2)$ if in every θ-sequence from $\mathscr{P}$ there is a θ-subsequence with the n compatibility property. $\mathscr{P}$ is κ-n-linked if there is a decomposition $\mathscr{P} = \cup_{\xi<\kappa}\mathscr{P}_\xi$ where each $\mathscr{P}_\xi$ has the n compatibility property in $\mathscr{P}$. Clearly if $\mathscr{P}$ is κ-n-linked then $\mathscr{P}$ has property $K_{\theta,n}$ for all θ with cf $\theta > \kappa$. For $\bar{p} \in \mathscr{P}^n$ by cal $\bar{p}$ we denote maximal k for which there is an $N \subseteq n$ of size k such that $\{p_i\colon i \in N\}$ is extendable in $\mathscr{P}$. If $\mathscr{S} \subseteq \mathscr{P}$, then the intersection number, $i(\mathscr{S})$, of $\mathscr{S}$ in $\mathscr{P}$ is the infimum of all numbers of the form cal $\bar{p}/n$ where $n \in \omega\setminus 1$ and $\bar{p} \in \mathscr{S}^n$. A classical result of Kelley [11] says that a BA $\mathscr{B}$ has a finitely additive strictly positive measure iff $\mathscr{B}\setminus\{0\}$ is the union of countably many sets each with positive intersection number.

Fix now an infinite cardinal κ with $\lambda = \mathrm{cf}\ \kappa$ and a sequence $\langle\kappa_\xi\colon \xi < \lambda\rangle$ such that $\sup_\xi \kappa_\xi = \kappa$ and

$$\text{if } \lambda < \kappa \text{ then } \langle\kappa_\xi\colon \xi < \kappa\rangle \text{ satisfies (1) and if } \lambda = \kappa \text{ then } \kappa \text{ is strongly inaccessible and } \kappa_\xi = \kappa \text{ for all } \xi < \lambda. \tag{6}$$

For $A \subseteq \prod_\xi \kappa_\xi$ and $f\colon \lambda \to \omega\setminus 2$ define

$$\mathscr{P}_f(A) = \{p \in [A]^{<\omega}\colon \text{if for some } q \subseteq p \text{ and } \xi < \lambda,\ \rho(a, b) = \xi \text{ for all } a, b \in q, \text{ then } |q| \leqslant f(\xi)\}.$$

Let $X_f(A)$ be the set of all $B \subseteq A$ such that if for some $C \subseteq B$ and $\xi < \lambda$, $\rho(a, b) = \xi$ for all $a, b \in C$, then $|C| \leqslant f(\xi)$, and consider $X_f(A)$ as a compact subspace of $\{0, 1\}^A$. Let $\mathscr{B}_f(A)$ be the clopen algebra of $X_f(A)$.

LEMMA 6: *If $f(\xi) \geqslant n$ for almost all ξ, then $\mathscr{P}_f(A)$ and $\mathscr{B}_f(A)$ are both κ-n-linked.*

PROOF: Let $\xi_0 < \lambda$ be such that $f(\xi) \geqslant n$ for all $\xi \geqslant \xi_0$. For each p in $\mathscr{P}_f(A)$ we fix $\xi_p \geqslant \xi_0$ and $t_p \subseteq \prod_{\xi < \xi_p} \kappa_\xi$ such that $p \restriction \xi_p = t_p$ and $\rho(a, b) < \xi_p$ for all $a, b \in p$. Suppose $p_1, \ldots, p_n$ are such that for some ξ and t, $\xi_{p_i} = \xi$ and $t_{p_i} = t$ for all $i \in \{1, \ldots, n\}$. Then it is easily seen that $p_1 \cup \ldots \cup p_n$ is a member of $\mathscr{P}_f(A)$. Since there are only κ such pairs ξ and t the result follows directly.

LEMMA 7: *Assume that for some filter $\mathscr{F}$ on λ and $cf\ \theta > \kappa$, A contains an $\leqslant_{\mathscr{F}}$-increasing $\leqslant_{\mathscr{F}}$-unbounded θ-sequence in $\prod_\xi \kappa_\xi$. Then $\mathscr{P}_f(A)$ and $\mathscr{B}_f(A)$ do not have precaliber θ, and if, moreover, $f(\xi) \leqslant n$ for almost all ξ, then $\mathscr{P}_f(A)$ and $\mathscr{B}_f(A)$ do not have property $K_{\theta, n+1}$.*

PROOF: We prove only the second conclusion since the proof of the first conclusion is similar. Let $\{a_\alpha\colon \alpha < \theta\} \subseteq A$ be $\leqslant_{\mathscr{F}}$-increasing and $\leqslant_{\mathscr{F}}$-unbounded in $\prod_\xi \kappa_\xi$. Let $C \in [\theta]^\theta$. We shall show that $\{\{a_\alpha\}\colon \alpha \in C\}$ has no $n+1$ compatibility property. We may assume that for some $\xi_0 < \lambda$, $f(\xi) \leqslant n$ for all $\xi \geqslant \xi_0$ and $a_\alpha \restriction \xi_0 = a_\beta \restriction \xi_0$ for all $\alpha, \beta \in C$. Since $\{a_\alpha\colon \alpha \in C\}$ is $\leqslant_{\mathscr{F}}$-unbounded there must be $\xi \geqslant \xi_0$ such that $\{a_\alpha(\xi)\colon \alpha \in C\}$ is unbounded in κ_ξ. Let η be a minimal such ξ. By (6) we can find $D \subseteq C$ of order type κ_η and $t \in \prod_{\xi < \eta} \kappa_\xi$ such that for all $\alpha < \beta$ in D, $a_\alpha(\eta) < a_\beta(\eta)$ and $a_\alpha \restriction \eta = t$. Hence if $N \subseteq D$ has size $n+1$, then $\{\{a_\alpha\}\colon \alpha \in N\}$ cannot be extended to a condition in $\mathscr{P}_f(A)$.

LEMMA 8: *If A is $\leqslant_{\mathscr{F}}$-unbounded in $\prod_\xi \kappa_\xi$ for some filter $\mathscr{F}$ on λ then $\mathscr{P}_f(A)$ and $\mathscr{B}_f(A)$ are not κ unions of sets with positive intersection numbers.*

PROOF: It suffices to show that if A is $\leqslant_{\mathscr{F}}$-unbounded in $\prod_\xi \kappa_\xi$, then $\mathscr{S} = \{\{a\}\colon a \in A\}$ has intersection number in $\mathscr{P}_f(A)$ equal to 0. Working as in Lemma 7, we can find $\eta < \kappa$, $t \in \prod_{\xi < \eta} \kappa_\xi$ and $D \in [A]^{\kappa_\eta}$ such that $\rho(a, b) = \eta$ and $a \restriction \eta = t$ for all $a, b \in D$. Then for any 1-1 $\bar{p} \in \{\{a\}\colon a \in D\}^n$ we have cal $\bar{p} \leqslant f(\eta)$. Hence for all n, $i(\mathscr{S}) \leqslant f(\eta)/n$.

Note that Lemmas 6–8 are much less restrictive than the lemmas of §1, however, much as in §1 we shall mention only the simplest applications. For information concerning the cardinal θ needed in Lemma 7, we refer the reader to [25; Ch. XIII].

THEOREM 6: *Assume $\rho^{\mathrm{cf}\,\kappa} < \kappa$ for all $\rho < \kappa$. Then for each finite $n \geqslant 2$ there is a κ-n-linked BA $\mathscr{B}_n$ which does not have property $K_{\kappa^+, n+1}$ and which is not κ union of sets with positive intersection numbers.*

PROOF: Let $\langle \kappa'_\xi : \xi < \mathrm{cf}\,\kappa \rangle$ and $A = \{a'_\alpha : \alpha < \kappa^+\}$ be as in the conclusion of Lemma 4. Let f be the constant function $\xi \mapsto n$. Then by Lemmas 6–8, $\mathscr{B}_n = \mathscr{B}_f(A)$ satisfies the theorem.

COROLLARY 2: *If* $\kappa = (2^{\aleph_0})^{+\omega}$, *then for each finite* $n \geqslant 2$ *there is a poset* $\mathscr{P}_n$ *which has property* $K_{\kappa^+, n}$ *but which does not have property* $K_{\kappa^+, n+1}$.

COROLLARY 3: *If* $\kappa = (2^{\aleph_0})^{+\omega}$, *then there is a BA* $\mathscr{B}$ *which is* κ*-n-linked for each* n, *but which does not have strictly positive finitely additive measure nor precaliber* κ^+.

If $\kappa = \omega$ and if b is the minimal cardinal of an $<^*$-unbounded subset of ω^ω, then the following results are proved by letting f equal $i \mapsto n$ and $i \mapsto i+2$, respectively.

THEOREM 7: *For each* $n \geqslant 2$ *there is a* σ*-n-linked BA* $\mathscr{B}_n$ *without strictly positive finitely additive measure and without property* $K_{b, n+1}$.

THEOREM 8: *There is a BA* $\mathscr{B}$ *which is* σ*-n-linked for each* n, *which does not have strictly positive finitely additive measure nor precaliber* b.

The poset $\mathscr{P}_f(A)$ is a member of a much larger class of posets which, for example, includes Gaifman's algebra [7], almost disjoint forcing [10], uniformization forcing [25,Ch.II], poset for forcing a subtree of a given tree [29], …, etc. It is interesting to note that Gaifman [7] used the Baire category argument in proving properties of his algebra, and that a similar Baire category argument has been used more recently by several other authors ([0],[1],[28]). Let us now present two new constructions of *ccc* posets without certain precalibers, just indicating the easy generalizations to higher cardinals.

The amoeba algebra of Solovay ([19]) is the Boolean algebra generated by the following poset $\mathscr{Q}$. The elements of $\mathscr{Q}$ are open subsets of [0, 1] of measure $< \frac{1}{2}$; the ordering of $\mathscr{Q}$ is $\supseteq$. Clearly $\mathscr{Q}$ is σ-n-linked for each $n \geqslant 2$. Let κ_0 be the minimal κ for which there is a κ-sequence of measure 0 subsets of [0, 1] whose union is not of measure 0.

THEOREM 9: *The amoeba algebra does not have precaliber* κ_0.

PROOF: Let $\langle A_\xi : \xi < \kappa_0 \rangle$ be an increasing sequence of measure 0 subsets of [0, 1] whose union has exterior measure equal to 1. For each $\xi < \kappa_0$ fix an open set U_ξ of measure $< \frac{1}{2}$ containing A_ξ. Then it is easily seen that $\{U_\xi : \xi < \kappa_0\}$ witnesses that $\mathscr{Q}$ does not have precaliber κ_0.

Let Y_0 be the set of all subsets C of κ_0 such that $\cup_{\xi \in F} U_\xi$ has measure $< \frac{1}{2}$ for all finite $F \subseteq C$ considered as a compact subspace of

$\{0, 1\}^{\kappa_0}$. Then a simple forcing argument ([9; p. 568]) shows that Y_0 does not have strictly positive measure.

Let $\mathbb{R} = \omega^\omega$ and fix f: $\omega \to \omega \setminus 1$ and 1-1 H: $\mathbb{R} \to \mathbb{R}$. Define $\mathscr{R}_f(H)$ to be the set of all finite $p \subseteq \mathbb{R}$ such that

$$\begin{aligned}&\text{if for some } a \in p,\ q \subseteq p \setminus \{a\} \text{ and } n < \omega,\\ &\quad \rho(a, b) \neq \rho(a, c) \text{ but } \rho(H(a), H(b)) = \rho(H(a), H(c)) = n\\ &\quad \text{for all distinct } b, c \in q,\\ &\quad \text{then } |q| \leqslant f(n).\end{aligned} \tag{7}$$

Let $Z_f(H)$ be the set of all $A \subseteq \mathbb{R}$ satisfying (7) in place of p considered as a compact subspace of $\{0, 1\}^{\mathbb{R}}$. Let $\mathscr{C}_f(H)$ be the clopen algebra of $Z_f(H)$.

LEMMA 9: *If $f(i) \geqslant n - 1$ for almost all i, then $\mathscr{R}_f(H)$ and $\mathscr{C}_f(H)$ are both σ-n-linked.*

PROOF: For each $p \in \mathscr{R}_f(H)$ fix $k_p < \omega$, s_p, $t_p \subseteq \omega^{k_p}$ and h_p: $s_p \to t_p$ such that: $p \restriction k_p = s_p$, $H''p \restriction k_p = t_p$, $h_p(a \restriction k_p) = H(a) \restriction k_p$ for all $a \in p$, and $\rho(a, b)$, $\rho(H(a), H(b)) < k_p$ for all a, $b \in p$. Suppose $p_1, \dots, p_n$ are such that for some k, s, t and h, we have $k_{p_i} = k$, $s_{p_i} = s$, $t_{p_i} = t$, and $h_{p_i} = h$ for all i. Then it is easily seen that $p_1 \cup \ldots \cup p_n$ satisfies (7) provided that $f(i) \geqslant n - 1$ for all $i \geqslant k$. Since there are only $\aleph_0$ such quadruples $\langle k, s, t, h\rangle$ the result follows.

LEMMA 10: *Assume $f(i) \leqslant n - 1$ for all i and that for some $A \subseteq \mathbb{R}$, $\{\{a\}$: $a \in A\}$ satisfies the $n + 1$ compatibility condition in $\mathscr{R}_f(H)$. Then $H \restriction A$ is continuous.*

PROOF: Suppose that for some $a \in A$ and $\{a_i$: $i < \omega\} \subseteq A$, $\lim a_i = a$ but either $\lim H(a_i)$ does not exist or it is not equal to $H(a)$. Then we may assume that for some l and all i, $\rho(H(a), H(a_i)) = l$ and $\rho(a, a_i) < \rho(a, a_{i+1})$. Hence a and $q = \{a_0, \dots, a_{n-1}\}$ do not satisfy (7) and so $\{a\}, \{a_0\}, \dots, \{a_{n-1}\}$ are not compatible in $\mathscr{R}_f(H)$.

Clearly the proof of Lemma 10 also shows the following.

LEMMA 11: *If $\{\{a\}$: $a \in A\}$ is centered in $\mathscr{R}_f(H)$, then $H \restriction A$ is continuous.*

LEMMA 12: *If $\{\{a\}$: $a \in A\}$ has positive intersection number in $\mathscr{R}_f(H)$, then $H \restriction A$ is continuous.*

PROOF: See the proofs of Lemmas 8 and 10.

Fix now an 1-1 enumeration $\{a_\alpha$: $\alpha < 2^{\aleph_0}\}$ of $\mathbb{R}$ and an 1-1 H: $\mathbb{R} \to \mathbb{R}$ which is not continuous on any $A \subseteq \mathbb{R}$ with $\{\alpha$: $a_\alpha \in A\}$ cofinal in $2^{\aleph_0}$ ([13; §35]). Then the following results follow from Lemmas 9-12 by letting f be $i \mapsto i+1$ and $i \mapsto n-1$, respectively.

THEOREM 10: *There is a BA $\mathscr{C}$ which is σ-n-linked for each n which does not have strictly positive finitely additive measure nor precaliber θ for all $\theta \leqslant 2^{\aleph_0}$ with* cf $\theta =$ cf $2^{\aleph_0}$.

THEOREM 11: *For each $n \geqslant 2$ there is a σ-n-linked BA $\mathscr{C}_n$ which does not have strictly positive finitely additive measure nor property $K_{\theta, n+1}$ for all $\theta \leqslant 2^{\aleph_0}$ with* cf $2^{\aleph_0}$.

COROLLARY 4: *For each $n \geqslant 2$ there is a ccc poset $\mathscr{P}$ which has property $K_{2^{\aleph_0}, n}$ but which does not have property $K_{2^{\aleph_0}, n+1}$.*

We have already noted that Theorems 10 and 11 have been proved first by Gaifman [7] and Argyros [0], respectively, using the Continuum Hypothesis (see also [22] and [2]). It is clear that in all these results we can replace $\aleph_0$ by any cardinal κ with the property ded(κ, 2^{κ}) (see [30]). We conclude this section with one more application of the same ideas.

THEOREM 12: *There is a compact* 0-*dimensional Hausdorff space X which has a discrete subspace of any size* $< 2^{\aleph_0}$ *but which does not have discrete subspace of size* $2^{\aleph_0}$.

PROOF: Using a classical diagonalization argument ([13; §35]) fix two disjoint dense sets E_0 and E_1 in [0, 1] of size $2^{\aleph_0}$ such that:

There is a strictly increasing mapping from E_0 into E_1 of any size $< 2^{\aleph_0}$. (8)

There is no 1-1 monotonic function from a subset of E_0 of size $2^{\aleph_0}$ into E_1. (9)

For $i < 2$, let K_i be the linearly ordered set obtained from [0,1] by replacing any $r \in E_i$, by two points $r^0 < r^1$. We clain that $X = K_0 \times K_1$ satisfies the theorem. By (8) it follows directly that X has discrete subspace of any size $< 2^{\aleph_0}$. So let us show that X has no discrete subspace of size $2^{\aleph_0}$. Suppose $D \subseteq X$ is a discrete subspace. Since no K_i has an uncountable discrete subset, we may assume that D is a 1-1 function from a subset of K_0 into K_1. Furthermore, we may assume that for some ϵ_0, $\epsilon_1 \in \{0, 1\}$ any member of D has the form $\langle r^{\epsilon_0}, s^{\epsilon_1} \rangle$ for some $r \in E_0$ and $s \in E_1$. We may also assume that there are rationals q_0 and q_1 such that the first coordinates of separating neighborhoods of D contain q_0 and the second containing q_1. Now it follows directly that D induces an increasing or decreasing function from E_0 into E_1 depending on whether $\epsilon_0 = \epsilon_1$ or $\epsilon_0 \neq \epsilon_1$. Then by (9), $|D| < 2^{\aleph_0}$.

§4. Productive *ccc* posets

A poset $\mathscr{P}$ is productively θcc if $\mathscr{P}\times\mathscr{Q}$ is θcc for any θcc poset $\mathscr{Q}$. A classical result ([2; §2]) says that if $\mathscr{P}$ has property K_θ $(\equiv K_{\theta,2})$ then $\mathscr{P}$ is productively θcc. In this section we show, in ZFC, that the convers of this result fails for many cardinals θ. The countable case of our example gives a ZFC analogue of the well-known CH-example of Kunen [32][3][2; §7]. The section ends with an application of the same methods to the S and L spaces problem or General Topology. For example, we shall construct a ZFC counterexample to the generalized S and L spaces problem.

We start with a lemma which is also useful in many other situations, for example, in considering nowhere-dense set-mappings on sets of reals.

LEMMA 13: *Let X be a topological space with a basis of size κ and let $\prec$ be a well-ordering of X. Suppose F maps X into the family of all closed subsets of X such that $F(x)\subseteq\{y\in X\colon y\preccurlyeq x\}$ for all $x\in X$. Let*

$$\mathscr{S}_X=\{s\in[X]^{<\omega}\colon x\notin F(y) \text{ for all } x\neq y \text{ in } s\}.$$

Then $\mathscr{S}_X$ is a productively κ^+cc poset.

PROOF: Fix a sequence $\langle s_\alpha\colon \alpha<\kappa\rangle$ of elements of $\mathscr{S}_X$. As usual, we may assume s_α's are disjoint and all of the same size n. For $\alpha<\kappa^+$, let $\{s_\alpha^1,\dots,s_\alpha^n\}$ be the $\prec$-increasing enumeration of s_α. We shall identify s_α with $\langle s_\alpha^1,\dots,s_\alpha^n\rangle\in X^n$. Since X^n has a basis of size κ, we may assume that for some open sets $U_1,\dots,U_n$ of X, we have that for all $\alpha<\kappa^+$ and $1\leqslant i\neq j\leqslant n$, $s_\alpha^i\in U_i$ and $F(s_\alpha^j)\cap U_i=\emptyset$. Furthermore we may assume $s_\alpha^i\prec s_\beta^i$ for $\alpha<\beta<\kappa^+$ and $1\leqslant i\leqslant n$. Fix a $\gamma<\kappa^+$ such that $\{s_\alpha\colon \alpha<\gamma\}$ is dense in $\{s_\alpha\colon \alpha<\kappa^+\}\subseteq X^n$. Choose $\delta>\gamma$. By the property of F, $V_i=U_i\setminus F(s_\gamma^i)$ is an open set containing s_δ^i for all $1\leqslant i\leqslant n$. Since $\{s_\alpha\colon \alpha<\gamma\}$ is dense in $\{s_\alpha\colon \alpha<\kappa^+\}$ there is a $\beta<\gamma$ such that $s_\beta^i\in V_i$ for all $1\leqslant i\leqslant n$. Then it follows directly that $s_\beta\cup s_\gamma$ is a member of $\mathscr{S}_X$.

Let $\mathscr{Q}$ be a κ^+cc poset. The hypothesis of Lemma 13 clearly remains satisfied in the forcing extension of $\mathscr{Q}$. So $\mathscr{S}_X$ is κ^+cc in the forcing extension by $\mathscr{Q}$, and this means that $\mathscr{Q}\times\mathscr{S}_X$ is a κ^+cc poset ([9; p. 236]). This completes the proof.

A similar proof shows that Lemma 13 also holds under the dual assumption: for all $x\in X$, $F(x)\subseteq\{y\in X\colon x\preccurlyeq y\}$.

From now on we fix a cardinal $\kappa\geqslant\omega$ of cofinality λ such that $\kappa^\lambda=\kappa$ and a sequence $\langle\kappa_\xi\colon \xi<\lambda\rangle$ of regular cardinals with supremum κ such that

$$\text{if } \lambda<\kappa, \text{ then } \sup_{\xi<\eta}\kappa_\xi<\kappa_\eta \text{ for all } \eta<\lambda \text{ and} \tag{11}$$

$$\text{if } \lambda=\kappa, \text{ then } \kappa_\xi=\kappa \text{ for all } \xi<\lambda. \tag{12}$$

For a, $b \in \prod_\xi \kappa_\xi$, we define

$$a \leqslant b \quad \text{iff space } a(\xi) \leqslant b(\xi) \text{ for all } \xi < \lambda.$$

Clearly $\leqslant$ is a partial ordering on $\prod_\xi \kappa_\xi$. For $A \subseteq \prod_\xi \kappa_\xi$, we define

$$\mathscr{S}_A = \{ s \in [A]^{<\omega} \colon s \text{ is an } \leqslant\text{-antichain} \}.$$

The following well-known consequence of $(2^\lambda)^+ \to ((2^\lambda)^+, (\lambda^+)_\lambda)^2$ has been already used earlier in this note.

LEMMA 14: *Assume* $\{a_\alpha \colon \alpha < \theta\} \subseteq \prod_\xi \kappa_\xi$ *where* $\theta = (2^\lambda)^+$. *Then there exists a* $B \in [\theta]^\theta$ *such that* $a_\alpha \leqslant a_\beta$ *for all* $\alpha < \beta$ *in* B.

LEMMA 15: *Suppose* $A = \{a_\alpha \colon \alpha < \theta\}$ *is an* $<_{\mathscr{F}}$*-increasing sequence in* $\prod_\xi \kappa_\xi$ *for some filter* $\mathscr{F}$ *on* λ. *Then* $\mathscr{S}_A$ *is a productively* κ^+*cc poset.*

PROOF: For $t \in \bigcup_{\eta < \lambda} \prod_{\xi < \eta} \kappa_\xi$ we put $B_t = \{a \in \prod_\xi \kappa_\xi \colon a \upharpoonright l(t) = t\}$. Then B_t's form a basis of size κ for the usual Baire topology σ on $\prod_\xi \kappa_\xi$. For $a \in A$, put $F(a) = \{b \in A \colon b \leqslant a\}$. Then $F(a_\alpha) \subseteq \{a_\beta \colon \beta \leqslant \alpha\}$ and $F(a_\alpha)$ is closed in A, σ. So the result follows from Lemma 13.

THEOREM 13: *Let* $\lambda = \operatorname{cf} \kappa$ *and suppose* $2^\lambda \leqslant \kappa$ *and* $\kappa^\lambda = \kappa$. *Then there is a productively* κ^+*cc poset that does not have property* K_{κ^+}.

PROOF: $\prod_\xi \kappa_\xi$ contains an $<^*$-increasing sequence A of length κ^+. $\mathscr{S}_A$ does not have property K_{κ^+} by Lemma 14.

Note that any strong limit singular cardinal satisfies the hypothesis of Theorem 13 and that $\kappa = (2^{\aleph_0})^{+\omega}$ is a minimal such cardinal. Before considering the case $\kappa = \omega$, let us mention the following result which is proved by a simple diagonalization argument.

THEOREM 14: *Assume* $2^{\underset{\smile}{\kappa}} = \kappa$ *and* $2^\kappa = \kappa^+$. *Then* κ^κ *contains an* $<^*$*-increasing* κ^+*-sequence with no* $\leqslant$*-antichain of size* κ^+. *Hence there is a productively* κ^+*cc poset that does not have property* K_{κ^+}.

LEMMA 16: *Suppose* $\{a_\alpha \colon \alpha < \theta\} \subseteq \omega^\omega$ *is* $<^*$*-increasing and* $<^*$*-unbounded in* ω^ω *and that each* a_α *is an increasing function. Then there exist* $\alpha < \beta < \theta$ *such that* $a_\alpha \leqslant a_\beta$.

PROOF: Fix $\gamma < \theta$ such that $\{a_\alpha \colon \alpha < \gamma\}$ is dense in $\{a_\alpha \colon \alpha < \theta\}$. Fix also cofinal $D \subseteq \theta$ and $n_0 < \omega$ such that $a_\gamma(n) < a_\delta(n)$ for all $\delta \in D$ and $n \geqslant n_0$. We also assume that for some $t_0 \in \omega^{n_0}$, $t_0 \subset a_\delta$ for all $\delta \in D$. Let n_1 be the minimal $n < \omega$ such that $\{a_\delta(n) \colon \delta \in D\}$ is unbounded in ω. So, there is a $t_1 \in \omega^{n_1}$ and increasing $\{\beta_i \colon i < \omega\} \subseteq D$ such that $t_1 \subset a_{\beta_i}$ and $a_{\beta_i}(n_1) < a_{\beta_{i+1}}(n_1)$ for all $i < \omega$. Pick $\alpha < \gamma$ such that $t_1 \subset a_\alpha$. Let $n_2 \geqslant n_1$ be such that $a_\alpha(n) < a_\gamma(n)$ for all $n \geqslant n_2$. Let $i < \omega$ be such that $a_{\beta_i}(n_1) > a_\gamma(n_2)$. Then $a_\alpha(n) \leqslant a_{\beta_i}(n)$ for all $n < \omega$.

THEOREM 15: *There is a productively ccc poset that does not have property* K_b.

A space X is said to be right-separated if there is a well-ordering of X in which every initial segment is open in X; X is left-separated if there is a well-ordering of X in which every initial segment is closed. Define $hd(X) = \sup\{|Y|: Y \subset X \text{ is left separated}\}$ and $hL(X) = \sup\{|Y|: Y \subseteq X \text{ is right separated}\}$. The generalized S and L spaces problem asks whether $hd(X) \leqslant hL(X)$ or $hL(X) \leqslant hd(X)$ hold in the class of all regular spaces (see [21][23]) because Sierpiński [26] showed that neither of these two inequalities hold in the class of all Hausdorff spaces. Using the above methods we shall show now that neither of these two inequalities hold even in the class of all 0-dimensional T_2 spaces. This solves Problem 5 of [23]. We refer the reader to the two most recent survey papers of this area, [21] and [23], for the extensive mathematical, historical, and bibliographical information concerning the S and L problem. $hd(X)$ and $hL(Y)$ for our counterexamples X and Y are not equal to $\aleph_0$, and it is known ([31]) that at least in the case of first inequality such a counterexample cannot be constructed in ZFC.

Fix now $\kappa \geqslant \omega$ of cofinality λ such that $\kappa^{\lambda} = \kappa$ and a non-decreasing sequence $\langle \kappa_\xi : \xi < \lambda \rangle$ with supremum κ. For $a \in \prod_\xi \kappa_\xi$, define

$$[a, \cdot) = \{b \in \textstyle\prod_\xi \kappa_\xi : a \leqslant b\} \text{ and } (\cdot, a] = \{b \in \textstyle\prod_\xi \kappa_\xi : b \leqslant a\}.$$

For $a \in \prod_\xi \kappa_\xi$ and $\eta < \lambda$, define

$$B_\eta^+(a) = B_{a \restriction \eta} \cap [a, \cdot) \quad \text{and} \quad B_\eta^-(a) = B_{a \restriction \eta} \cap (\cdot, a].$$

It is easily checked that $\{B_\eta^+(a)\}$ and $\{B_\eta^-(a)\}$ form bases for λ-additive 0-dimensional topologies τ^+ and τ^- on $\prod_\xi \kappa_\xi$, respectively, which extend the usual Baire topology σ on $\prod_\xi \kappa_\xi$. Let ${}^+\prod_\xi \kappa_\xi$ denote $\prod_\xi \kappa_\xi$, τ^+ and let ${}^-\prod_\xi \kappa_\xi$ denote $\prod_\xi \kappa_\xi$, τ^-.

LEMMA 17: *If D is a discrete subspace of* ${}^+\prod_\xi \kappa_\xi$ *or of* ${}^-\prod_\xi \kappa_\xi$, *then D is the union of* κ $\leqslant$*-antichains.*

LEMMA 18: $hL({}^+\prod_\xi \kappa_\xi)$, $hd({}^-\prod_\xi \kappa_\xi) \leqslant \kappa \cdot 2^\lambda$.

PROOF: This follows from Lemma 14. Clearly, the same conclusion holds for any (finite) power of ${}^+\prod_\xi \kappa_\xi$ and ${}^-\prod_\xi \kappa_\xi$.

LEMMA 19: *Suppose* $A = \{a_\alpha : \alpha < \theta\} \subseteq \prod_\xi \kappa_\xi$ *is* $<_{\mathscr{F}}$*-increasing for some filter* $\mathscr{F}$ *on* λ. *Then A is left-separated in* ${}^+\prod_\xi \kappa_\xi$ *and right-separated in* ${}^-\prod_\xi \kappa_\xi$.

THEOREM 16: *Suppose* $2^\lambda \leqslant \kappa$ *and* $\kappa^\lambda = \kappa$ *where* $\lambda = \operatorname{cf} \kappa$. *Then* $hL({}^+\prod_\xi \kappa_\xi) < hd({}^+\prod_\xi \kappa_\xi)$ *and* $hd({}^-\prod_\xi \kappa_\xi) < hL({}^-\prod_\xi \kappa_\xi)$.

PROOF: $\prod_\xi \kappa_\xi$ contains an $<^*$-increasing sequence of length κ^+.

Clearly any strong limit singular cardinal κ satisfies the hypothesis of Theorem 16. In this case $hd({}^+\prod_\xi \kappa_\xi)$ and $hL({}^-\prod_\xi \kappa_\xi)$ are very closely related to the value of $\kappa^{\mathrm{cf}\,\kappa} = 2^\kappa$. In fact, using Lemmas 18 and 19 and some recent results on the singular cardinals problem (see [25; XIII §5]) it can be shown that any strong limit singular cardinal κ of cofinality $\lambda > \omega$ satisfies the following:

$$hL({}^+\kappa^\lambda) = \kappa \quad \text{and} \quad hd({}^+\kappa^\lambda) = 2^\kappa \tag{13}$$ [2]

$$hd({}^-\kappa^\lambda) = \kappa \quad \text{and} \quad hL({}^-\kappa^\lambda) = 2^\kappa. \tag{14}$$

The following is a typical case of Theorem 16 and it gives two first countable 0-dimensional counterexamples to the generalized S and L spaces problem.

COROLLARY 5: $hL({}^+\prod_{n<\omega}(2^{\aleph_0})^{+n}) < hd({}^+\prod_{n<\omega}(2^{\aleph_0})^{+n})$ *and* $hd({}^-\prod_{n<\omega}(2^{\aleph_0})^{+n}) < hL({}^-\prod_{n<\omega}(2^{\aleph_0})^{+n})$.

If we are willing to assume $\kappa^{\underline{\kappa}} = \kappa$ and $2^\kappa = \kappa^+$, then Theorem 14 gives κ-additive 0-dimensional counterexamples to the generalized S and L problem at level κ. If $\kappa = \omega$ then Lemma 16 gives the following

THEOREM 17: ${}^+\omega^\omega$ *contains a left-separated subspace of type b with no discrete subspace of size b, and* ${}^-\omega^\omega$ *contains a right-separated subspace of type b with no discrete subspace of size b.*

This result gives a ZFC analogue of the well-known CH-construction of van Douwen and Kunen ([3]) of S and L subspaces of $\mathscr{P}(\omega)$ with the Vietoris topology.

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Miller Institute for Basic Research in Science
University of California at Berkeley
Berkeley, CA 94720
USA