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ON SPHERICAL SPACE FORMS WITH META-CYCLIC FUNDAMENTAL GROUP WHICH ARE ISOSPECTRAL BUT NOT EQUIVARIANT COBORDANT

Peter B. Gilkey *

Abstract

Ikeda constructed examples of irreducible spherical space forms with meta-cyclic fundamental groups which were isospectral but not isometric. We use the eta invariant of Atiyah-Patodi-Singer to show these examples are not equivariantly cobordant. We show two such examples which are strongly π_1 isospectral are in fact isometric.

0. Introduction

Let M be a compact Riemannian manifold of dimension m without boundary and let Δ denote the scalar Laplacian. Let $\operatorname{spec}(\Delta)_M$ denote the spectrum of the Laplacian where each eigenvalue is repeated according to its multiplicity. Two manifolds M_1 and M_2 are said to be isospectral if $\operatorname{spec}(\Delta)_{M_1}$ and $\operatorname{spec}(\Delta)_{M_2}$ are the same. The question of to what extent the global geometry of M is reflected by $\operatorname{spec}(\Delta)_M$ is a very old one. It was phrased by Kac in the form "can you hear the shape of a drum." We refer to Millman's survey article [11] for further historical information.

It is clear that if there exists an isometry between two manifolds, then they are isospectral. That the converse need not hold was shown by Milnor [12] who gave examples of isospectral tori which were not isometric. In 1978, Vigneras [15] gave examples of isospectral manifolds of constant negative curvature which were not isometric. If $m \geqslant 3$, then these manifolds have different fundamental groups so are not homotopic. The fundamental groups are all infinite and the calculations involved some fairly deep results in quaternion algebras.

In 1983 Ikeda [10] constructed examples of spherical space forms (compact manifolds without boundary with constant positive sectional curvature) which were isospectral but not isometric. As DeRham [2] had shown that diffeomorphic spherical space forms are isometric, Ikeda's examples were not diffeomorphic. Unlike Vigneras examples, Ikeda's

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examples involved finite fundamental groups and were rather easily studied.

Let G be a finite group and M a compact smooth manifold without boundary. We assume all manifolds oriented henceforth. All maps are assumed to be orientation preserving unless otherwise specified. A G-structure on M is a principal G-bundle over M. Given two manifolds M_i with G-structures, we say they are G-cobordant if there exists a smooth compact (oriented) manifold N so $dN = M_1 - M_2$ and so the given G structure on the boundary can be extended to all of N. We will also need the notion of SPIN_c cobordism. We refer to [7] for the definition of a SPIN_c structure; two manifolds M_i with fixed G-structures and SPIN_c structures are said to be G-SPIN_c-cobordant if the manifold N can be chosen to be both a G-cobordism and a SPIN_c-cobordism (i.e. we can extend the SPIN_c structure from the boundary of N to all of N).

Of particular interest is the case in which G is a homomorphic image of the fundamental group. We suppose given a fixed homomorphism $\epsilon: \pi_1(M) \to G$. This representation defines a principal G-bundle over M and defines a natural G-structure. Changing ϵ will of course in general change the bundle. If ϵ is an isomorphism the resulting notion is called π_1-cobordism (and π_1-SPIN_c cobordism respectively). We emphasize that it depends upon the choice of the marking ϵ (or equivalently upon the choice of a fixed isomorphism between $\pi_1(M_1)$ and $\pi_1(M_2)$). Changing the marking in general changes the cobordism type.

Let G be a finite group and let $\tau: G \to U(l)$ be a unitary representation. We say τ is fixed point free if $\det(1 - \tau(g)) \neq 0$ for $g \neq 1$. Such a τ is faithful and the existence of such a representation places severe restrictions on the group G. In particular, all the Sylow subgroups corresponding to odd primes must be cyclic while the Sylow subgroup corresponding to the prime 2 is either cyclic or generalized quaternionic. The groups admitting fixed point free representations have all been classified by Wolf [16]. The easiest to work with are the type I groups: these are the groups with all Sylow subgroups cyclic which admit fixed point free representations. (There are also types II-VI in Wolf's classification).

Let τ be fixed point free. $\tau(G)$ acts without fixed points on the unit sphere S^{2l-1} in C^l and we let $M(\tau) = S^{2l-1}/\tau(G)$. We shall always assume $l > 1$ so τ induces an isomorphism (or marking) between G and $\pi_1(M(\tau))$. M is said to be irreducible if τ is irreducible. If G is Abelian, then G is cyclic and the resulting $M(\tau)$ are the lensspaces. We will be primarily interested in the non-Abelian case; τ irreducible and $l > 1$ implies G is non-Abelian.

$M(\tau)$ inherits a natural orientation and Riemannian metric from the sphere S^{2l-1}. It also inherits a natural Cauchy-Riemann structure and SPIN_c structure. The metric has constant positive sectional curvature.

Such a manifold is called a spherical space form. All odd dimensional compact manifolds without boundary with metrics of constant positive sectional curvature arise in this way; the only even dimensional spherical space forms are the spheres S^{2l} and the projective spaces RP^{2l}.

We shall let the word "natural" for a differential operator mean natural in the category of oriented Riemannian manifolds; we refer to Epstein and Stredder [4,14] for further details. The Laplacian acting on p-forms and the tangential operator of the signature complex (if m is odd) are both natural in this setting. We say that two manifolds are strongly isospectral if $\operatorname{spec}(P)_{M_1} = \operatorname{spec}(P)_{M_2}$ for all elliptic self-adjoint natural operators P.

The notion of strong isospectrality is not enough to distinguish between spherical space forms of the same dimension as we shall see. Let $\epsilon : \pi_1(M) \to G$ be a marking of the fundamental group where G is finite. Let ρ be a representation of G (necessarily unitary) and let P be a self-adjoint natural elliptic partial-differential operator. Let P_ρ denote the operator P with coefficients in the locally flat (unitary) bundle defined by ρ; this is self-adjoint and elliptic as well. We say that M_1 and M_2 are strongly π_1 isospectral if $\operatorname{spec}(P_\rho)_{M_1} = \operatorname{spec}(P_\rho)_{M_2}$ for all such (P, ρ); this depends of course upon the marking chosen. (There is a similar notion of strong $\mathrm{SPIN_c}$ and strong $\mathrm{SPIN_c}$ π_1 isospectrality of course).

We can now summarize the main results of this paper:

THEOREM 0.1: *Let $\tau : G \to U(d)$ be an irreducible fixed point free representation of a type I group where $d > 1$. Let $M(\tau) = S^{2d-1}/\tau(G)$ with the natural orientation, marking, and $SPIN_c$ structure.*

(*a*) *If $\tilde{\tau}$ is any other irreducible fixed point free representation of G, then $d = \tilde{d}$. The manifolds $M(\tau)$ and $M(\tilde{\tau})$ are strongly isospectral.*

(*b*) *Let d be odd and assume either of the following two conditions*:

(*b*1) *$M(\tau)$ and $M(\tilde{\tau})$ are π_1 cobordant*

(*b*2) *$M(\tau)$ and $M(\tilde{\tau})$ are strongly π_1 isospectral*

Then $M(\tau)$ and $M(\tilde{\tau})$ are isometric.

(*c*) *Let d be either odd or even and assume either of the following two conditions*:

(*c*1) *$M(\tau)$ and $M(\tilde{\tau})$ are π_1-$SPIN_c$ cobordant*

(*c*2) *$M(\tau)$ and $M(\tilde{\tau})$ are strongly π_1-$SPIN_c$ isospectral*

Then $M(\tau)$ and $M(\tilde{\tau})$ are isometric.

REMARK: In fact one does not need the full strength of π_1 to make these arguments go; one can work with an Abelian homomorphic image of π_1 as we shall see in Theorem 1.5. Furthermore, one can draw some weak conclusions regarding $M(\tau)$ and $M(\tilde{\tau})$ even if d is even just in the oriented category. To get the full strength however, one must work in the $\mathrm{SPIN_c}$ category; the prime 2 is always exceptional in this subject and ordinary oriented cobordism does not seem to be enough to fully detect the 2-torsion involved.

This suggests that the appropriate spectral question to study in the non-simply connected case is not just the spectrum of the Laplacian but also the spectrum of certain auxilary operators with coefficients in representations of the fundamental group; in light of this theorem, it will not suffice just to study the spectrum of the Laplacian on p-forms to distinguish Ikeda's examples.

We shall see in section one that there are lots of representations τ giving rise to non-isometric manifolds $M(\tau)$ in general, and we will be able to use Theorem 0.1(b) to show these manifolds are all inequivalent in the category of oriented Riemannian manifolds. This proof will be independent of DeRham's theorem and construct isospectral and inequivalent manifolds if d is odd. If d is even, the manifolds are in fact inequivalent, but this will not follow from Theorem 0.1 as we had to impose a SPIN_c structure and there are many.

Although our motivation was based on the isospectral problem, this paper is really a paper in topology. We are constructing and computing combinatorial invariants to distinguish between manifolds that have the same homotopy, homology, and K-theory groups. We will use the eta invariant of Atiyah-Patodi-Singer [1]. This is a particularly nice invariant since it is both a spectral and cobordism invariant so that the spectral geometry is being used to reflect the topology.

This paper is divided into four sections. In the first section, we shall discuss briefly the material from Wolf's book [16] which we shall need regarding the fundamental groups of spherical space forms. We will also review the argument showing Ikeda's examples are all strongly isospectral and prove 0.1(a).

In the second section, we will review the material concerning the eta invariant which we shall need. We will complete the proof of Theorem 0.1 in the case that $d = p^a$ is a prime power. This case is particularly simple and the reader who is only interested in finding isospectral manifolds which are not diffeomorphic need only consult the first two sections.

In the third section we will complete the proof of Theorem 0.1 by considering dimensions divisible by different primes. In the final section, we will draw some further consequences of these calculations.

It is a pleasure to acknowledge helpful correspondence with both Ikeda and Millman regarding this subject.

1. Geometry of spherical space forms

Let $\tau: G \to U(l)$ be a fixed point free representation of a finite group. We say that G is type I if all the Sylow subgroups of G are cyclic. This is necessarily the case if I is odd, but there are other groups which can occur if l is even. The type 1 groups are all meta-cyclic and the following classification theorem can be found in Wolf [16].

THEOREM 1.1 (WOLF): *Classification of type I non-Abelian groups.*

(*a*) *Let* (m, n, d, r) *be a* 4*-tuple of integers each of which is at least* 2 *so that*:

(*i*)m *is coprime to* $n \cdot r \cdot (r-1)$.

(*ii*)d *is the order of* r *in the group of units in the ring* $Z_m = Z/mZ$.

(*iii*)$n = dn_1$ *and each prime which divides* d *also divides the integer* n_1.

Let $G = G(m, n, d, r)$ *be the group on two elements* $\{A, B\}$ *such that* $A^m = B^n = 1$ *and* $BAB^{-1} = A^r$. G *is a non-Abelian meta-cyclic semi-direct product of* Z_m *with* Z_n. $|G| = mn$ *and* B^d *generates the center of* G.

(*b*) *Let* H *be the subgroup generated by* A *and* B^d. *This is a normal cyclic subgroup and* $G/H \simeq Z_d$. *Define a linear representation* $\rho_{u,v}$ *on* H *by setting* $\rho_{u,v}(A) = \mathrm{e}^{2\pi i u/m} = \alpha$ *and* $\rho_{u,v}(B^d) = \beta = \mathrm{e}^{2\pi i v/n_1}$. *Let* $\pi_{u,v} = \rho_{u,v}^G$ *be the induced representation of* G *in* $U(d)$. *We can find a basis* e_j *for* C^d *so relative to this basis,* $\pi_{u,v}$ *is represented by the matrices*: $\pi_{u,v}(A) = \operatorname{diag}(\alpha, \alpha^r, \dots, \alpha^{r^{d-1}})$ *and* $\pi_{u,v}(B) = \gamma\sigma$ *where* $\gamma^d = \beta$ *and where* σ *is the cyclic permutation matrix* $\sigma(e_1) = e_d$, $\sigma(e_j) = e_{j-1}$ *otherwise.* $\pi_{u,v}$ *is fixed point free and irreducible if* $(u, m) = (v, n) = 1$.

(*c*) *Let* $\tau : G \to U(l)$ *be fixed point free. We can factor* $l = dt$ *and decompose (up to unitary equivalence)* $\tau = \oplus_j \pi_{u_j, v_j}$ *where* $(u_j, m) = (v_j, n) = 1$ *for* $1 \leqslant j \leqslant t$.

(*d*) *In the isomorphism* $G = G(m, n, d, r)$, *the indices* (m, n, d) *are unique. We can replace* r *by* r^c *for any* c *coprime to* d *and obtain an isomorphic group.*

(*e*) *If* $\tau : \tilde{G} \to U(l)$ *is fixed point free where* $\tilde{G}$ *is an arbitrary non-Abelian finite group, then* $\tilde{G} = G(m, n, d, r)$ *for some* (m, n, d, r) *if* l *is odd. Groups with other structure can occur if* l *is even.*

REMARK: Such groups exist with irreducible representations in any dimension d. Let $n = d^2$ and apply Dedekind's theorem to find m prime $m \equiv 1(d)$. Let ϕ be the Euler function so the group of units of Z_m is a cyclic group of order $\phi(m) = m - 1$ divisible by d. We can therefore find r so (m, n, d, r) is admissible. We also note that once the fundamental group G has been fixed, then the dimension of the irreducible fixed free representations is constant which was one of the assertions of Theorem 0.1(a).

We have chosen a slightly different matrix representation from that which appears in Wolf. Let $\{e_i\}$ denote the standard basis for C^d. Wolf represents the $\pi_{u,v}$ by:

$$\pi_{u,v}(A)e_j = \alpha^{r^{j-1}} \cdot e_j \qquad 1 \leqslant j \leqslant d \text{ for } \alpha = \mathrm{e}^{2\pi i u/m}$$

$$\pi_{u,v}(B)e_j = e_{j-1} \qquad 1 < j \leqslant d$$

$$\pi_{u,v}(B)e_1 = \beta e_d \qquad \text{for } \beta = \mathrm{e}^{2\pi i v/n_1}$$

We let $\gamma = e^{2\pi i v/n}$ and let $\bar{e}_j = \gamma^j e_j$ be a new basis. Then:

$$\pi_{u,v}(A)\bar{e}_j = \alpha^{r^{j-1}} \cdot \bar{e}_j \qquad 1 \leqslant j \leqslant d$$

$$\pi_{u,v}(B)\bar{e}_j = \pi_{u,v}(B)\gamma^j e_j = \gamma \cdot \gamma^{j-1} e_{j-1} = \gamma \cdot \bar{e}_{j-1} \qquad 1 < j \leqslant d$$

$$\pi_{u,v}(B)\bar{e}_1 = \pi_{u,v}(B)\gamma e_1 = \gamma\beta e_d = \gamma \cdot \gamma^d e_d = \gamma \cdot \bar{e}_d$$

which shows that these two representations are equivalent. We will use this trick again in proving Lemma 1.4.

Henceforth we shall reserve the notation $\pi_{u,v}$ for the fixed point free representations of such a G; i.e. we shall suppose $(u, m) = (v, n_1) = 1$ in this notation. It is clear from the first description that it only depends upon the residue class of v modulo n_1 even though the second matrix representation makes it appear to depend on the residue class of v modulo n. Let $\{\epsilon_1, \dots, \epsilon_d\}$ denote the d^{th} roots of unity and $\gamma = e^{2\pi i v/n}$ satisfy $\gamma^d = \beta$. Then the eigenvalues of $\pi_{u,v}(B)$ are $\{\epsilon_1\gamma, \dots, \epsilon_d\gamma\}$ and $\pi_{u,v}(B^d) = \beta$ is diagonal.

The type I groups have a large number of automorphisms:

THEOREM 1.2: *Adopt the notation of Theorem* 1.1 *and let* $G = G(m, n, d, r)$.

(*a*) $\pi_{u,v}$ *is unitarily equivalent to* $\pi_{\bar{u},\bar{v}}$ *if and only if* $u \equiv \bar{u}r^c(m)$ *for some* c *and* $v \equiv \bar{v}(n_1)$.

(*b*) *Let* $(s, m) = 1$ *and let* t *and* j *be arbitrary. There exists an automorphism of* G $\psi = \psi(s, t, j)$ *so* $\psi(A) = A^s$ *and* $\psi(B) = A^t B^{(1+dj)}$. $\pi_{u,v}\psi = \pi_{su,(1+dj)v}$.

(*c*) *If* Ψ *is any automorphism of* G, *then* $\Psi = \psi(s, t, j)$ *for some* (s, t, j).

PROOF: We refer to Wolf [16] for details; our notation differs slightly from his.

Let $d = l$ and let $M(u, v) = M(\pi_{u,v})$ be an irreducible spherical space form. We use the orientation induced from the natural orientation on S^{2d-1} and let the G structure be given by using $\pi_{u,v}$ to identify G with $\pi_1(M(u, v))$. If we replace $\pi_{u,v}$ by $\pi_{u,v}\psi$ then the underlying manifold is unchanged and thus $M(u, v)$ is diffeomorphic to $M(su, (dj + 1)v)$ although they have different G-structures. Let $\bar{d} = \gcd(d, n_1)$ be the greatest common divisor. Since v is only defined modulo n_1 and since we can always replace v by $(dj + 1)v$ and u by su without changing the diffeomorphism structure, we see that the oriented diffeomorphism type of $M(u, v)$ only depends upon the congruence class of v modulo $\bar{d}$. Thus there are at most $\phi(\bar{d})$ distinct such manifolds in the category of oriented smooth manifolds corresponding to a given group G. If d is odd, then the map $z \to \bar{z}$ reverses the orientation of S^{2d-1} so that $M(u, v) = -M(-u, -v)$ and there are at most $\phi(\bar{d})/2$ distinct such manifolds in the oriented category.

We must now investigate the structure of these groups more closely:

LEMMA 1.3: *Let $G = G(m, n, d, r)$ and adopt the notation of Theorem 1.1. Enumerate the elements of G in the form $\{A^j B^k\}$ for $0 \leqslant j < m$, $0 \leqslant k < n$.*

(a) If $(k, d) = 1$, then $A^j B^k$ and B^k are in the same conjugacy class

(b) Let $X = \operatorname{diag}(x_1, \dots, x_\nu)$ be a diagonal unitary matrix. Let Y be a permutation matrix on C^ν of order ν acting transitively on the standard basis. Let $x = \det(X)$ and $y^\nu = x$. Then XY is conjugate to yY in $U(\nu)$.

(c) $\pi_{u,v}(A^j B^k)$ and $\pi_{1,1}(A^{ju} B^{kv})$ are conjugate in $U(d)$.

PROOF: The defining relation $BA = A^r B$ implies $A^{-s} B^k A^s = A^{s(r^k - 1)} B^k$. Suppose $(k, d) = 1$. If we can show $(r^k - 1, m) = 1$ then we can solve the relation $s(r^k - 1) \equiv j(m)$ which will prove (a). Let p be prime and let $p \mid m$. Suppose $r^k \equiv 1(p)$. As $r^d \equiv 1(p)$ and $(k, d) = 1$ we have $r \equiv 1(p)$. This contradicts the assumption of Theorem 1.1 that $(r - 1, m) = 1$ and proves (a).

(b) is a technical fact we shall use in the proof of (c). Any two such permutation matrices are conjugate. By simply renumbering the basis, we may assume the basis for C^ν to be chosen so $Xe_j = x_j e_j$ and so that $Ye_1 = e_\nu$ and $Ye_j = e_{j-1}$ otherwise. Define a new basis $\bar{e}_j = y^j x_j \dots x_\nu e_j$ and calculate:

$$XY\bar{e}_1 = yx_1 \dots x_\nu XYe_1 = yxx_\nu e_\nu = yy^\nu x_\nu e_\nu = y\bar{e}_\nu$$

$$XY\bar{e}_j = y^j x_j \dots x_\nu XYe_j = y^j x_{j-1} x_j \dots x_\nu e_{j-1} = y\bar{e}_{j-1} \quad \text{for } j > 1$$

which proves (b) as this is clearly conjugate to yY.

From the definition of Theorem 1.1 we see $X = \pi_{u,v}(A^j) = \pi_{1,1}(A^{uj})$. If $\gamma_0 = e^{2\pi i/n}$ then $Y = \pi_{u,v}(B^k) = \gamma_0^{kv}\sigma^k$ and $y_1 = \pi_{1,1}(B^{vk}) = \gamma_0^{kv}\sigma^{kv}$. We must show XY and XY_1 are conjugate in $U(d)$. Let $\mu = \gcd(k, d)$ and $\nu = d/\mu$. If $\mu > 1$, then σ^k is not transitive on the standard basis for C^d. Instead, σ^k breaks up into μ cycles each of which is of length ν. Decompose $C^d = V_1 \oplus \dots \oplus V_\mu$ where $\dim(V_i) = \nu$ and where on each V_i σ^k represents a transitive permutation matrix on the basis. As v is assumed to be coprime to n, σ^{kv} is also a transitive permutation matrix on each V_i. On each subspace, we apply (b) to see that XY is conjugate to $\gamma_0^{kv} \det(X \text{ on } V_i)^{1/\nu}\sigma^k$ and XY_1 is conjugate to $\gamma_0^{kv} \det(X \text{ on } V_i)^{1/\nu}\sigma^{kv}$. Two transitive permutation matrices are conjugate so XY and XY_1 are conjugate on each V_i and hence on all of C^d. This completes the proof.

The correspondence $A^j B^k \to A^{uj} B^{vk}$ is *not* a group homomorphism for $v \not\equiv 1 \bmod d$. Nevertheless, this correspondence is at the heart of many of our calculations. We can use it to prove the following mild generalization of Theorem 0.1(a):

THEOREM 1.4: *Let $G = G(m, n, d, r)$ and adopt the notation of Theorem 1.1. Let $\tau = \pi_{u_1,v_1} \oplus \ldots \oplus \pi_{u_\nu,v_\nu}$ be a fixed point free representation of G in $U(\nu d)$. Let c be coprime to mn and define $\tau(c) = \pi_{cu_1,cv_1} \oplus \ldots \oplus \pi_{cu_\nu,cv_\nu}$. Let P be a self adjoint elliptic differential operator which is natural in the category of oriented Riemannian manifold. Then* $\operatorname{spec}(P)_{M(\tau(c))}$ *is independent of c so these manifolds are all strongly isospectral.*

PROOF: Let P denote the operator on $S^{2d\nu-1}$ and $P(c)$ the corresponding operator on the quotient $M(\tau(c))$. Let $\lambda \in R$ and let $E(\lambda)$ and $E(\lambda,c)$ denote the eigenspace of the operator P and $P(c)$ respectively. We must show $\dim(E(\lambda, c))$ is independent of c for all λ.

The unitary group $U(d\nu)$ acts on $S^{2d\nu-1}$ by orientation preserving isometries. The hypothesis of naturality permits us to extend this action to an action we shall denote by $e(\lambda)$ and $E(\lambda)$. Again, the naturality implies that the eigenspace $E(\lambda, c)$ is just the subspace of $E(\lambda)$ invariant under $e(\lambda)\tau(c)(G)$. We apply the orthogonality relations to compute:

$$\dim E(\lambda, c) = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(e(\lambda)\tau(c)(g)).$$

We parametrize the elements of G in the form A^jB^k. By Lemma 1.4, $\tau(c)(A^jB^k)$ is conjugate in $U(d\nu)$ to $\tau(A^{cj}B^{ck})$. Consequently $\operatorname{Tr}(e(\lambda)\tau(c)(A^jB^k)) = \operatorname{Tr}(e(\lambda)\tau(A^{cj}B^{ck}))$ so that:

$$\dim E(\lambda, c) = \frac{1}{m \cdot n} \sum_{0 \leqslant j < m,\, 0 \leqslant k < n} \operatorname{Tr}\big(e(\lambda)\tau(A^{cj}B^{ck})\big).$$

As we sum over all (j, k) we have (cj, ck) also ranges over all appropriate indices and thus $\dim E(\lambda, c) = \dim E(\lambda, 1)$ which completes the proof.

We now take $\nu = u_1 = v_1 = 1$. We apply the Chineese remainder theorem to find $c \equiv u \bmod m$ and $c \equiv v \bmod n$. Then $M(1, 1)$ is strongly isospectral to $M(u, v)$. This implies any two $M(u, v)$ and $M(\bar{u}, \bar{v})$ are isospectral which completes the proof of Theorem 0.1(a).

The subgroup generated by A is a normal subgroup and the natural map $A^jB^k \to k \bmod n$ is a group homomorphism. If $e \mid n$, this defines a Z_e structure on M. We use this to define Z_e cobordism and strong Z_e isospectrality. We can now state a mild generalization of Theorem 0.1($b\&c$):

THEOREM 1.5: *Let $G = G(m, n, d, r)$ and adopt the notation of Theorem 1.1. Let $M(u, v) = S^{2d-1}/\pi_{u,v}(G)$. Decompose $n_1 = d_1 q$ where $(q, d) = 1$ and where each prime dividing d_1 divides d and conversly. Let $\mathscr{D}$ be the least common multiple of d and d_1. Let $M(u, v)$ have the $Z_{\mathscr{D}}$ structure defined above.*

(*a*) *Let d be odd and assume either of the two following conditions*:
 (*a*1) *$M(u, v)$ and $M(\bar{u}, \bar{v})$ are $Z_{\mathscr{D}}$ cobordant*
 (*a*2) *$M(u, v)$ and $M(\bar{u}, \bar{v})$ are strongly $Z_{\mathscr{D}}$ isospectral*
 then $v \equiv \bar{v} \bmod d_1$ and $M(u, v)$ and $M(\bar{u}, \bar{v})$ are isometric.

(*b*) *Let d be either odd or even and assume either of the following two conditions*:
 (*b*1) *$M(u, v)$ and $M(\bar{u}, \bar{v})$ are $Z_{\mathscr{D}}$-$SPIN_c$ cobordant*
 (*b*2) *$M(u, v)$ and $M(\bar{u}, \bar{v})$ are strongly $Z_{\mathscr{D}}$-$SPIN_c$ isospectral*
 then $v \equiv \bar{v} \bmod d_1$ and $M(u, v)$ and $M(\bar{u}, \bar{v})$ are isometric.

PROOF: We shall complete the proof of Theorem 1.5 in sections 2 and 3.

All possible changes of marking are parametrized by replacing a given $\pi_{u,v}$ by $\pi_{u,v}\psi(s, t, j) = \pi_{su,v(dj+1)}$. This doesn't change the congruence class of v modulo $\bar{d} = \gcd(d, d_1) = \gcd(d, n_1)$. Suppose d is odd and there exists an oriented diffeomorphism from $M(u, v)$ to $M(\bar{u}, \bar{v})$. By changing the choice of marking we can assume that this is a π_1 diffeomorphism. This gives rise immediately to a π_1 cobordism and hence v is congruent to $\bar{v} \bmod \bar{d}$. Thus $M(u, v)$ and $M(\bar{u}, \bar{v})$ must be isometric. This gives another proof of DeRham's result in this context. We combine this with the previous isospectral result:

THEOREM 1.6: *Let $G = G(m, n, d, r)$ and adopt the notation of Theorem 1.1. Let d be odd and set $\bar{d} = (d, n/d)$. The manifolds $M(u, v) = S^{2d-1}/\pi_{u,v}(G)$ are all strongly isospectral. The following conditions are equivalent*:
 (*i*) $v \equiv \bar{v} \bmod \bar{d}$,
 (*ii*) *There exists a π_1 cobordism between $M(u, v)$ and $M(\bar{u}, \bar{v})$ after a suitable change of the marking of the fundamental group,*
 (*iii*) *$M(u, v)$ and $M(\bar{u}, \bar{v})$ are strongly π_1 isospectral after a suitable change of the marking of the fundamental group.*

In the category of oriented manifolds, there are $\phi(\bar{d})$ inequivalent manifolds among the $M(u, v)$ for a fixed G. In the category of unoriented manifolds, there are $\phi(\bar{d})/2$ inequivalent manifolds among the $M(u, v)$ for a fixed G.

REMARK: If $d \geqslant 5$ is odd, we can always take $n = d^2$ so $d = \bar{d} = d_1$. Since $\phi(d) \geqslant 4$, this construction gives at least two distinct isospectral manifolds which are not diffeomorphic in dimension $m \equiv 1(4)$ for $m \geqslant 9$.

The case $d = d_1$ is a particularly nice one.

LEMMA 1.7: *Let $G = G(m, n, d, r)$ and adopt the notation of Theorem 1.1. Let $d = \bar{d} = d_1$ and let $\epsilon: G \to Z_d$ be defined by $\epsilon(A^j B^k) = k \bmod d$. If M is a manifold so that there exists an isomorphism $\pi_1(M) \simeq G$, then the corresponding Z_d structure on M is independent of the particular isomorphism chosen. In this context, Z_d cobordism becomes an oriented diffeomorphism invariant of the manifold M.*

PROOF: If ψ is an automorphism, then $\psi(A^j B^k) = A^{js} B^{k(1+dx)} A^{tk}$ for some (s, t, x) so that $\epsilon\psi(A^j B^k) = k(1 + dx) \bmod d$ which completes the proof.

We conclude this section with some facts concerning the representations of the cyclic group $Z_n = Z/nZ$ (and we will use both notations as convenient).

LEMMA 1.8: *Let $\rho_{s,n}(j) = e^{2\pi i js/n}$. The $\{\rho_s\}_{0 \leqslant s < n}$ parametrize the irreducible representations of the cyclic group Z_n. If $n = \bar{n}c$ there is a natural surjection $Z_n \to Z_{\bar{n}}$ and under this surjection $\rho_{s,\bar{n}}(j) = \rho_{sc,n}(j)$. Finally, if $n = p^a$ is a prime power, then there is a representation $\delta = \delta_n$ of Z_n such that $\mathrm{Tr}(\delta(0)) = p^{a-1}$ and $\mathrm{Tr}(\delta(j)) = 0$ if p divides j and $0 < j < n$.*

PROOF: This is immediate except for the construction of δ. If $a = 1$, we take $\delta = 1$. Otherwise define:

$$\delta = \sum_{0 \leqslant j < p^{a-1}} \rho_{j,n}.$$

Let $q = p^{a-1}$ and factor $(x^q - 1) = (x - 1)(x^{q-1} + \ldots + 1)$. Let $\lambda = e^{2\pi i s/n}$. Suppose that p divides s but that $s \not\equiv 0(n)$. Then $\lambda \neq 1$ but $\lambda^q = 1$ so λ satisfies the equation $\lambda^{q-1} + \ldots + 1 = 0$ which is precisely the assertion that $\mathrm{Tr}(\delta(s)) = 0$. This completes the proof as $\mathrm{Tr}(\delta(0)) = p^{a-1}$.

We shall often omit the subscript "n" and simply discuss $\rho_s = \rho_{s,n}$ when n is understood. If τ is a representation, we let $\mathrm{Tr}(\tau)$ denote the corresponding class function. Let $R(G)$ be the group representation ring over Z generated by the irreducible unitary representations of G and let $\mathrm{ch}(G)$ denote the ring of virtual characters. Let $R_0(G)$ and $\mathrm{ch}_0(G)$ be the ideals corresponding to representations of virtual dimension 0. The map $\tau \to \mathrm{Tr}(\tau)$ is a ring bijection and we will often omit the words "Tr" when speaking of the character defined by a representation when no confusion is likely to result.

2. The eta invariant

The eta invariant defined by the tangential operator of the signature and SPIN_c complexes will provide the tool we shall need to distinguish between the manifolds defined in the first section. We review briefly the properties of this invariant and refer to [1,5] for the details.

Let M be a compact Riemannian manifold of odd dimension $2k-1$ without boundary and let P be a first order self-adjoint elliptic differential operator on M. Let $\{\lambda_\nu\} = \operatorname{spec}(P)$. There will be an infinite number of both positive and negative eigenvalues in general. We define

$$\eta(s, P) = \sum_\nu \operatorname{sign}(\lambda_\nu)\,|\lambda_\nu|^{-s}$$

as a signed generalization of the Riemann zeta function. The sum converges absolutely if $\operatorname{Re}(s) \gg 0$ and has a meromorphic extension to C with isolated simple poles on the real axis. The value at $s=0$ is regular and we define

$$\eta(P) = \tfrac{1}{2}\{\eta(0, P) + \dim N(P)\} \in R \bmod Z.$$

as a measure of the spectral asymmetry of P.

We reduce mod Z to ensure that η will be continuous under deformations. Let $P(a)$ be a smooth 1-parameter family of such operators. As eigenvalues cross the origin under the spectral flow defined by the parameter a, $\eta(0, P(a))$ has twice integer jumps. Dividing by $1/2$ and reducing mod Z defines a smooth invariant of the parameter a. In $R \bmod Z$, $\eta(-P) = -\eta(P)$.

Let $G = \pi_1(M)$ be the fundamental group and let ρ be a unitary representation of G. Let P_ρ denote P with coefficients in the locally flat bundle defined by ρ. Define $\eta(\rho, P) = \eta(P_\rho)$. η is additive with respect to direct sums so $\eta(\rho_1 \oplus \rho_2, P) = \eta(\rho_1, P) + \eta(\rho_2, P)$ and therefore $\eta(*, P)$ extends to a Z-linear map of the group representation ring $R(G) \to R \bmod Z$. $R_0(G)$ is the augmentation ideal of all virtual representations of virtual dimension 0. We define $\operatorname{ind}(\rho, P)$ to be the restriction of η to $R_0(G)$.

LEMMA 2.1:

(*a*) *Let $P(a)$ be a smooth* 1*-parameter family of such operators and let $\rho \in R_0(G)$. Then* $\operatorname{ind}(\rho, P(a))$ *is independent of the parameter a. If G is a finite group, then* $\operatorname{ind}(\rho, P) \in Q \bmod Z$ *is a torsion invariant.*

(*b*) *Let $M = dN$ be the boundary of a compact manifold N. Suppose there is an elliptic first order complex $Q\colon C^\infty(V_1) \to C^\infty(V_2)$ over N. Near M, we use the symbol of the inward unit normal to identify V_1 with V_2 and express Q in the form $Q = \partial/\partial n + Q_{\mathrm{T}}$ where Q_{T} is a tangential elliptic first order operator on M. Q_{T} is called the tangential part of the complex. Suppose Q_{T} is self-adjoint and that the virtual bundle V_ρ extends as a flat virtual bundle V_ρ over N. Then* $\operatorname{ind}(\rho, Q_{\mathrm{T}}) = 0$.

REMARK: (a) shows that $\operatorname{ind}(\rho, P)$ is an invariant which only depends on the homotopy class of the leading symbol of P within the given class. (b) shows that this index can be interpreted in equivariant cobordism. In [5] we used this invariant to calculate the K-theory of spherical space forms; in [6] we used the invarint to detect PIN_c cobordism so this is an extremely useful invariant. We refer to [6] for the proof.

We suppose that M is oriented and odd dimensional. Let $N = Mx[0, \infty)$ with the inherited orientation and product metric. Let $P = Q_T$ be the tangential operator of the signature complex. Modulo signs, $P = \pm d * \pm * d$ on $C^\infty(\Lambda(T^*M))$. We refer to [1,5,7] for further details. P is elliptic and self-adjoint. We can connect any two Riemannian metrics on M by a smooth 1-parameter family of metrics. This creates a smooth 1-parameter family of such operators and by Lemma 2.1(a), $\mathrm{ind}(\rho, P)$ is constant under such perturbations. It defines an invariant in the category of smooth oriented manifolds. If we reverse the orientation we replace P by $-P$ and change the sign. We let $\mathrm{ind}(\rho, \mathrm{sign}, M) = \mathrm{ind}(\rho, P)$. If M has a SPIN_c structure, we may define $\mathrm{ind}(\rho, \mathrm{SPIN}_c, M)$ in a similar fashion by using the tangential operator of the SPIN_c complex.

Let G be a finite group, not necessarily $\pi_1(M)$, and let $\rho \in R_0(G)$. Let M have a G-structure. Either by considering directly the locally flat bundle defined by ρ or by pulling back ρ to $R_0(\pi_1(M))$, we can define $\mathrm{ind}(\rho, *, M)$.

LEMMA 2.2: *Let M be a compact oriented manifold without boundary of odd dimension and let M have a G-structure where G is some finite group. Let $\rho \in R_0(G)$. Then:*

(a)$\mathrm{ind}(\rho, \mathrm{sign}, M) \in \mathrm{Q} \bmod Z$. *This is both a G-cobordism and a G-spectral invariant.*

(b) $\mathrm{ind}(\rho, SPIN_c, M) \in Q \bmod Z$. *This is both a G-$SPIN_c$ cobordism and a G-$SPIN_c$-spectral invariant.*

REMARK: It is not in fact necessary to assume G is a finite group. G can be an arbitrary discrete infinite group and ρ an arbitrary finite dimensional representation. The resulting invariants in this case are in $C \bmod Z$.

This is a combinatorial invariant which can be computed quite explicitly for spherical space forms. Let $A \in U(l)$ have eigenvalues $\{\lambda_\nu\}$. We may also regard $A \in \mathrm{SO}(2l)$ as being defined by rotation angles $\{\theta_\nu\}$. Assume A is fixed point free- i.e. $\det(A-1) \neq 0$ or equivalently $0 < \theta_\nu \leqslant \pi$. Let

$$\mathrm{defect}(A; \mathrm{sign}) = \prod_\nu \{(\lambda_\nu + 1)/(\lambda_\nu - I)\} = \prod_\nu \{\mathrm{i}\cot(\theta_\nu/2)\}$$

$$\mathrm{defect}(A, \mathrm{SPIN}_c) = \prod_\nu \{\lambda_\nu/(\lambda_\nu - 1)\} = \det(A)/\det(A - I)$$

be the terms appearing in the Lefschetz fixed point formula for isolated fixed points for the signature or Dolbeault (SPIN_c) complex. The work of [1,3,13] leads immediately to the following combinatorial formula.

LEMMA 2.3: *Let G be a finite group (not necessarily of type I) and let $\tau : G \to U(l)$ be a fixed point free representation. Let $M(\tau) = S^{2l-1}/\tau(G)$. We give $M(\tau)$ the natural structures and marking. Let $\rho \in R_0(G)$, then:*

$$(a)\ \operatorname{ind}(\rho, \text{sign}, M(\tau)) = \frac{1}{|G|} \sum_{g \in G,\, g \neq I} \times \operatorname{Tr}(\rho(g))\, \mathit{defect}(\tau(g); \text{sign})$$

$$(b)\ \operatorname{ind}(\rho, \mathit{SPIN}_c, M(\tau)) = \frac{1}{|G|} \sum_{g \in G,\, g \neq I} \times \operatorname{Tr}(\rho(g))\, \mathit{defect}(\tau(g); \mathit{SPIN}_c).$$

REMARK: This is a generalized Dedekind or cotangent/cosecant sum which can easily be evaluated numerically on a computer for $|G|$ not too large.

The somewhat surprising fact is that this invariant is polynomial if the group G in question is cyclic. Let $\vec{x} = (x_1, \dots)$ be a collection of indeterminates. Let $L_j(\vec{x})$ and $\mathrm{Td}_j(\vec{x})$ denote the Hirzebruch and Todd polynomials [7,8]. We list the first few values for future reference:

$$\begin{aligned} &L_0(\vec{x}) = 1 && \mathrm{Td}_0(\vec{x}) = 1 \\ &L_1(\vec{x}) = 0 && \mathrm{Td}_1(\vec{x}) = \tfrac{1}{2} \textstyle\sum x_j \\ &L_2(\vec{x}) = \tfrac{1}{3} \textstyle\sum x_j^2 && \mathrm{Td}_2(\vec{x}) = \tfrac{1}{12} \left\{ \displaystyle\sum_{j<k} x_j x_k + \Big(\sum_j x_j \Big)^2 \right\}. \end{aligned}$$

(We have reindexed the L-polynomials for convenience). We introduce an auxilary parameter s which formally corresponds to the Chern class of a line bundle and define:

$$L_l(s; \vec{x}) = \sum_{j+k=l} \frac{(2s)^j}{j!} L_k(\vec{x})$$

$$\mathrm{Td}_l(s; \vec{x}) = \sum_{j+k=l} \frac{s^j}{j!} \mathrm{Td}_k(\vec{x}).$$

At least formally speaking, these are the integrands of the Hirzebruch signature theorem and of the Riemann-Roch formula. (For a spherical space form with the cannonical $SPIN_c$ structure, the tangential operator of the $SPIN_c$ complex and the tangential operator of the Dolbeault complex agree). These are rational polynomials; we let $\mu(l)$ be the least common denominator of these two polynomials.

Let Z_n be the cyclic group of order n and let $\vec{q}=(q_1,\dots,q_l)$ be a collection of integers coprime to n. Let

$$\tau=\rho_{q_1,n}\oplus\dots\oplus\rho_{q_l,n}\colon Z_n\to U(l).$$

This is a fixed point free representation of Z_n and up to unitary equivalence any fixed point free representation of Z_n has this form. Let $L(n;\,\vec{q})=S^{2l-1}/\tau(Z_n)$ be the resulting lensspace. Then one has the following combinatorial formula:

LEMMA 2.4: *Let $M=L(n;\,\vec{q})$ be a lensspace of dimension $2l-1$. Choose e so that $e\cdot Q_1\dots Q_e\equiv 1 \bmod n\cdot\mu(l)$. Let $\rho=\rho_{s,n}-\rho_{t,n}\in R_0(Z_n)$. In $Q \bmod Z$ we have the identities*:

$$\operatorname{ind}(\rho,\,\text{sign},\,L(n;\,\vec{q}))\equiv\frac{-e}{n}\{L_l(s;\,n,\,\vec{q})-L_l(t;\,n,\,\vec{q})\}$$

$$\operatorname{ind}(\rho,\,SPIN_c,\,L(n;\,\vec{q}))\equiv\frac{-e}{n}\{\mathrm{Td}_l(s;\,n,\,\vec{q})-\mathrm{Td}_l(t;\,n,\,\vec{q})\}.$$

PROOF: We refer to [5] for the proof. This is a simple residue calculation based on the results of Hirzebruch-Zagier [9].

It is possible using the Brauer induction formula to reduce the sum of Lemma 2.3 over an arbitrary fixed point free group to sums over cyclic subgroups so in practice Lemma 2.4 is all that one ever needs to evaluate these formulas combinatorially.

We conclude section 2 by proving:

LEMMA 2.5: *Let $G=G(m,\,n,\,d,\,r)$ and adopt the notation of Theorem 1.1. Let $d=p^x$ be a prime power and decompose $n_1=p^yq$ where $(q,\,p)=1$. Let $M(u,\,v)=S^{2d-1}/\pi_{u,v}(G)$. If d is odd, let $*$ = either sign or $SPIN_c$; if d is even we let $*=SPIN_c$. Let $\Omega(p,\,u,\,v)=\operatorname{ind}(\rho,\,*,\,M(u,\,v))\in Q \bmod Z$. Suppose*

$$\Omega(\rho,\,u,\,v)=\Omega(p,\,\bar{u},\,\bar{v})\ \textit{for all}\ \rho\in R_0(Z_{p^y}).$$

Then $v\equiv\bar{v} \bmod p^y$ and $M(u,\,v)$ and $M(\bar{u},\,\bar{v})$ are isometric.

REMARK: By Lemma 2.2, Ω is both a cobordism and a π_1-spectral invariant in the appropriate category. This lemma completes the proof of Theorems 0.1 and 1.5 for prime power dimensions; since $R_0(Z_{p^y})\subseteq R_0(Z_{\mathscr{D}})\subseteq R_0(G)$.

Let $k \in Z$ and let $\nu(k) = \nu_p(k)$ be the power of p dividing k; $\nu(0) = \infty$. We must show that $\nu(v - \bar{v}) \geqslant \nu(n_1) = y$. We suppose the contrary. We will get a contradiction using a bootstrapping argument. Let $a = \nu(v - \bar{v})$ and $b = y - a$. Let $\Sigma'_{j,k}$ denote the sum over the non-identity elements of G; $0 \leqslant j \leqslant m$, $0 \leqslant k \leqslant n$, $(j, k) \neq (0, 0)$. Let w and $\bar{w}$ solve the congruences $wv \equiv 1(n)$, $wu \equiv 1(m)$, $\bar{w}\bar{v} \equiv 1(n)$, and $\bar{w}\bar{u} \equiv 1(m)$. It is clear $\nu(w - \bar{w}) = \nu(v - \bar{v}) = a$. We calculate using the appropriate defect that:

$$\Omega(\rho, u, v) = \frac{1}{m \cdot n} \sum_{j,k}' \operatorname{Tr}(\rho(k)) \operatorname{defect}\big(\pi_{u,v}(A^j B^k), *\big)$$

$$= \frac{1}{m \cdot n} \sum_{j,k}' \operatorname{Tr}(\rho(k)) \operatorname{defect}\big(\pi_{1,1}(A^{uj} B^{vk}), *\big)$$

$$= \frac{1}{m \cdot n} \sum_{j,k}' \operatorname{Tr}(\rho(kw)) \operatorname{defect}\big(\pi_{1,1}(A^j B^k), *\big).$$

We go from the first to the second line by applying Lemma 1.3 and the fact that the defect is a class function; we go from the second to the third line by making the change of variables $j \to jw$ and $k \to kw$. From this equation, it is clear that $\Omega(\rho, u, v) = \Omega(\rho, v)$ since only the congruence class of w modulo p^y plays a role in the third line. The hypothesis of the lemma ensures the identity:

$$0 = \frac{1}{m \cdot n} \sum_{j,k}' \{\operatorname{Tr}(\rho(kw)) - \operatorname{Tr}(\rho(k\bar{w}))\} \cdot \operatorname{defect}\big(\pi_{1,1}(A^j B^k), *\big),$$

for all $\rho \in R_0(Z_{p^y})$. (Where this is regarded as in Q mod Z).

We now select a suitable subcollection of representations:

SUBLEMMA 2.6: *We adopt the notation of Lemma* 2.5. *Let* $a = \nu(v - \bar{v}) = \nu(w - \bar{w})$ *and let* $b = y - a$. *Assume* $b \geqslant 1$. *If* $b = 1$, *let* $\delta = 1$. *Otherwise let* δ *be given by Lemma* 1.8 *so*:

$$\operatorname{Tr}(\delta(j)) = p^{b-1} \quad \text{if } \nu(j) \geqslant b \quad \text{and}$$

$$\operatorname{Tr}(\delta(j)) = 0 \quad \text{for } 1 \leqslant \nu(j) < b.$$

Let $\rho \in \delta R_0(Z_{p^y})$, *then* $\operatorname{Tr}(\rho(wk)) - \operatorname{Tr}(\rho(\bar{w}k)) = 0$ *if* $p \mid k$.

PROOF: Although $\delta \in R(Z_{p^b})$ by construction, we can regard $\delta \in R(Z_{p^y})$ since $p^b \mid p^y$. We check cases. Let $p \mid k$ and suppose first $0 < \nu(k) < b$. Then $\nu(wk) = \nu(\bar{w}k) = \nu(k)$ so $\operatorname{Tr}(\delta(wk)) = \operatorname{Tr}(\delta(\bar{w}k)) = 0$ and the conclusion follows. Otherwise suppose $b \leqslant \nu(k)$. Then $y = a + b \leqslant \nu(k) + \nu(w - \bar{w}) = \nu(kw - k\bar{w})$ so $kw \equiv k\bar{w} \bmod p^y$. The desired equality now holds because ρ is a representation of Z/p^yZ. This completes the proof; the small powers of p are controlled by δ while the large powers are controlled by the induction assumption.

For such a representation, we use the Frobenius-Weilandt method. We may sum over $(k, p) = 1$ since the coefficient is zero if $p \mid k$. By Lemma 1.3, we have A^jB^k and B^k are conjugate in G so defect$(\pi_{1,1}(A^jB^k))$ = defect$(\pi_{1,1}(B^k))$ is independent of the index j. We group the m equal terms for $0 \leqslant j < m$ together to conclude that we have the identity:

$$0 = \frac{1}{n} \sum_{0<k<n,\,(k,\,p)=1} \{\mathrm{Tr}(\rho(kw)) - \mathrm{Tr}(\rho(k\bar{w}))\} \cdot \mathrm{defect}\big(\pi_{1,1}(B^k),\, *\big)$$

for all $\rho \in \delta R_0(Z_{p^\nu})$.

The defect for such a matrix telescopes:

SUBLEMMA 2.7: *We adopt the notation of Lemma* 2.5. *Let* $(k, p) = 1$ *and let* $\beta = e^{2\pi i/n_1}$. *Then defect* $(\pi_{1,1}(B^k), *) = \mathit{defect}(\beta^k, *)$.

PROOF: Let $\{\epsilon_\mu\}_{1 \leqslant \mu \leqslant d}$ parametrize the d^{th} roots of unity. As $(k, p) = 1$ we have $(k, d) = 1$ so $\{\epsilon_\mu^k\}$ is just a re-ordering of $\{\epsilon_\mu\}$. Let $\gamma^d = \beta$. By Theorem 1.1 the eigenvalues of $\pi_{1,1}(B^k)$ are $\{\epsilon_\mu^k\gamma^k\} = \{\epsilon_\mu\gamma^k\}$. Suppose first d is odd. When we take the product over the roots of unity, we compute:

$$\begin{aligned}\mathrm{defect}\big(\pi_{1,1}(B^k), \mathrm{sign}\big) &= \prod_\mu{}' \big\{\big(\epsilon_\mu\gamma^k + 1\big)/\big(\epsilon_\mu\gamma^k - 1\big)\big\}\\ &= \big(\gamma^{dk} + 1\big)/\big(\gamma^{dk} - 1\big) = \big(\beta^k + 1\big)/\big(\beta^k - 1\big)\\ \mathrm{defect}\big(\pi_{1,1}(B^k), \mathrm{SPIN_c}\big) &= \prod_\mu{}' \big\{\epsilon_\mu\gamma^k/\big(\epsilon_\mu\gamma^k - 1\big)\big\}\\ &= \gamma^{dk}/\big(\gamma^{dk} - 1\big) = \beta^k/\big(\beta^k - 1\big).\end{aligned}$$

If d is even, the corresponding formulas become:

$$\begin{aligned}\mathrm{defect}\big(\pi_{1,1}(B^k), \mathrm{SPIN_c}\big) &= \prod_\mu{}' \big\{\epsilon_\mu\gamma^k/\big(\epsilon_\mu\gamma^k - 1\big)\big\}\\ &= -\gamma^{dk}/\big(-\gamma^{dk} + 1\big) = \beta^k/\big(\beta^k - 1\big).\end{aligned}$$

This completes the proof. (The identity fails if $*$ = sign and $p = 2$, of course).

We apply Sublemma 2.7 to derive the identity:

$$0=\frac{1}{n}\sum_{0<k<n,\,(k,\,p)=1}\{\mathrm{Tr}(\rho(wk))-\mathrm{Tr}(\rho(\bar{w}k))\}\cdot\mathrm{defect}(\beta^k,\,*).$$

Only the residue class of k mod n_1 is relevant; we group the d equal terms together to obtain:

$$0=\frac{1}{n_1}\sum_{0<k<n_1}\{\mathrm{Tr}(\rho(wk))-\mathrm{Tr}(\rho(\bar{w}k))\}\cdot\mathrm{defect}(\beta^k,\,*).$$

We drop the condition $(k,\,p)=1$ since the denominator in the defect is well defined for all k in the range while the Trace term vanishes if $(k,\,p)\neq 1$.

Let $f(x,\,\mathrm{sign})=x+1$ and $f(x,\,\mathrm{SPIN_c})=x$ so that defect $(x,\,*)=f(x,\,*)/(x-1)$. β^q is a primitive $(p^y)^{\mathrm{th}}$ root of unity. We set $\rho=\delta\hat{\rho}$ where $\mathrm{Tr}(\hat{\rho}(\mathrm{j}))=\beta^{\mathrm{jq}}-1$ and $\hat{\rho}\in R_0(Z_{p^y})$. Then we obtain

$$0=\frac{1}{n_1}\sum_k{}'\{\mathrm{Tr}(\delta(wk))(\beta^{qwk}-1)-\mathrm{Tr}(\delta(\bar{w}k))(\beta^{q\bar{w}k}-1)\}$$

$$\times f(\beta^k,\,*)/(\beta^k-1).$$

We use the identity $(\xi^j-1)/(\xi-1)=\xi^{j-1}+\xi^{j-2}+\ldots+1$ for $\xi=\beta^k$ to divide by (β^k-1) and obtain

$$0=\frac{1}{n_1}\sum_k{}'\{\mathrm{Tr}(\delta(wk))(\beta^{qw(k-1)}+\ldots+1)-\mathrm{Tr}(\delta(\bar{w}k))$$

$$\times(\beta^{qw(k-1)}+\ldots+1)\}f(\beta^k,\,*)$$

This is now well defined at the omitted value $k=0$. If we sum over the entire group, we get an integer by the orthogonality relations. We are working in Q mod Z and conclude therefore

$$0=\frac{1}{n_1}\{\mathrm{Tr}(\delta(0))(qw)-\mathrm{Tr}(\delta(0))(q\bar{w})\}\cdot f(0,\,*)$$

$$=\frac{1}{n_1}\cdot p^{b-1}(qw-q\bar{w})f(0,\,*)=\frac{1}{p^y}(w-\bar{w})p^{b-1}f(0,\,*)$$

from which it follows that $f(0,\,*)(w-\bar{w})\equiv 0 \bmod p^{y+1-b}=p^{a+1}$. If

$* = \mathrm{SPIN}_c$, $f(0, *) = 1$ so we have $w \equiv \bar{w} \bmod p^{a+1}$ and hence $v \equiv \bar{v} \bmod p^{a+1}$. This has bootstrapped a congruence mod p^a into one mod p^{a+1} which will complete the proof. If $* = $ sign, $f(0, *) = 2$. As d is necessarily odd in this case, $2(w - \bar{w}) \equiv 0 \bmod p^{a+1}$ implies $(w - \bar{w}) \equiv 0 \bmod p^{a+1}$.

We remark that defect$(\pi_{1,1}(B^k)$; sign$) = 1$ if d is even and $(k, p) = 1$. Thus there is no denominator to divide and using the signature complex gives no information in this instance.

3. Groups $G(m, n, d, r)$ with d divisible by different primes

In section 2, we proved Theorems 0.1 and 1.5 if $d = p^x$ is a prime power. We shall restrict to suitably chosen subgroups to discuss the general case. Although the original representation is irreducible, its restriction need not be. Thus we must consider non-irreducible representations, although of a very special sort. We generalize Lemma 2.5 as follows:

LEMMA 3.1: *Let $d = p^x$ be a prime power and let $G = G(m, n, d, r)$ with the notation of Theorem* 1.1. *Let $(c, mn) = 1$ and let $(t, p) = 1$. Let $\tau = \oplus \pi_{u_i, v_i}$ and $\tau(c) = \oplus \pi_{cu_i, cv_i}$ be fixed point free representations of G into $U(dt)$. If pt is odd, let $*$ be either* sign *or $SPIN_c$. If pt is even, let $* = SPIN_c$. Let $\Omega(\rho, c) = \mathrm{ind}(\rho, *, S^{2dt-1}/\tau(c)(G))$. Decompose $n_1 = p^y q$ for $(p, q) = 1$ and assume*

i) $v_i = v_j$ for $1 \leqslant i \leqslant j \leqslant t \bmod p^y$

ii) $\Omega(\rho, c) = \Omega(\rho, \bar{c})$ for all $\rho \in R_0(z_{p^y})$.

Then $c \equiv \bar{c} \bmod p^y$.

REMARK: c is a scaling constant. Since we are working modulo scaling constants, we can assume without loss of generality that $v_1 \equiv 1 \bmod p^y$ for $1 \leqslant i \leqslant t$ in the proof.

PROOF: We generalize the argument given to prove Lemma 2.5. Instead of the defects telescoping to define a Dedekind sum for the circle, they will telescope to define a Dedekind sum for a $2t - 1$ dimensional lensspace. This Dedekind sum will be evaluated using Lemma 2.4 to derive the desired congruences.

Let $a = \nu(c - \bar{c})$ be the power of p dividing $c - \bar{c}$ and let $b = y - a$. We must show that $y \leqslant a$. Suppose on the contrary that $0 \leqslant a < y$ so that $1 \leqslant b$. Let $wc \equiv 1 \bmod mn$ and $\bar{w}\bar{c} \equiv 1 \bmod mn$. Apply sublemma 2.6 and let $\rho \in \delta R_0(Z_{p^y})$. Then $\mathrm{Tr}(\rho(wk)) - \mathrm{Tr}(\rho(\bar{w}k)) = 0$ if $p \mid k$. Let $\beta = e^{2\pi i/n_1}$ and $\vec{v} = (v_1, \ldots, v_t)$. Let $\hat{\tau}(\mathrm{k}) = \mathrm{diag}(\beta^{kv_1}, \ldots, \beta^{kv_t})$ define the lensspace $L(n_1; \vec{v})$. The same argument given in the proof of Lemma 2.5

shows that in Q mod Z we have the identity:

$$
\begin{aligned}
0 &= \Omega(\rho, c) - \Omega(\rho, \bar{c}) \\
&= \frac{1}{m \cdot n} \sum_{j,k}{}' \{\mathrm{Tr}(\rho(kw)) - \mathrm{Tr}(\rho(k\bar{w}))\} \cdot \mathrm{defect}(\tau(A^j B^k), *) \\
&= \frac{1}{n} \sum_{0<k<n,\,(k,\,p)=1} \{\mathrm{Tr}(\rho(kw)) - \mathrm{Tr}(\rho(k\bar{w}))\} \\
&\quad \cdot \mathrm{defect}(\tau(B^k), *) \\
&= \frac{1}{n_1} \sum_{0<k<n_1} \{\mathrm{Tr}(\rho(kw)) - \mathrm{Tr}(\rho(k\bar{w}))\} \cdot \mathrm{defect}(\hat{\tau}(k), *).
\end{aligned}
$$

If $t = 1$, we may apply Lemma 2.5 directly to complete the proof so we assume henceforth $t \geqslant 2$ (if pt is odd, $t \geqslant 3$). In the case $t = 1$, we analysed the relevant Dedekind sum using elementary methods to derive the desired congruence relating w and $\bar{w}$ (or equivalently c and $\bar{c}$). In this more general setting, we must use Lemma 2.4. The algebra involved becomes a bit more cumbersome and we must first introduce some additional notation. If $\rho \in R_0(Z_{n_1})$, define:

$$
F(\rho, *) = \mathrm{ind}(\rho, L(n_1; \vec{v}), *) = \frac{1}{n_1} \sum_{0<k<n_1}
$$
$$
\times \mathrm{Tr}(\rho(k))\mathrm{defect}(\hat{\tau}(k), *).
$$

We define w_* acting on $R_0(Z_{n_1})$ by $w_*(\rho)(k) = \rho(wk)$. If G is any function on $R_0(Z_{n_1})$, we define:

$$
G_\delta(\rho) = G(\delta\rho) \quad \text{and} \quad w_* G(\rho) = G(w_* \rho).
$$

w_* is a ring homomorphism and $w_*(G_\delta) = (w_* G)_{w_*\delta}$. The results above may be summarized in the form:

SUBLEMMA 3.2: *We adopt the notation above. Then* $w_* F_\delta(\rho, *) - \bar{w}_* F_\delta(\rho, *) = 0$ *in* Q mod Z *for all* $\rho \in R_0(Z_{p^\nu})$.

We apply Lemma 2.4 to obtain some information regarding the general form of the invariant $F(\cdot, \cdot)$. There exists e coprime to n_1 so that if $F(s; *) = F(\rho_{s,n_1} - \rho_{0,n_1}, *)$, then:

$$F(s, \text{sign}) = -\frac{e}{n_1}\left\{\frac{(2s)^t}{t!} + \frac{(2s)^{t-2}}{(t-2)!}\tfrac{1}{3}\left(n_1^2 + v_1^2 + \ldots + v_t^2\right) + \ldots\right\}$$

$$F(s, \text{SPIN}_c) = -\frac{e}{n_1}\left\{\frac{s^t}{t!} + \frac{s^{t-1}}{(t-1)!}\tfrac{1}{2}(v_1 + \ldots + v_t) + \ldots\right\}.$$

In these two expressions, we omit the constant term since $F(0, *) = 0$. These polynomials are very complicated and only the leading terms are tractible from a combinatorial point of view. We decompose

$$F(s, *) = F_t(s, *) + F_{t-1}(s, *) + \ldots + F_1(s, *)$$

where the F_j are homogeneous of degree j in s. We wil use the two leading terms to prove Lemma 3.1. We note that $w_*(F_j) = w^j F_j$.

Let $\vec{s} = (s_1, \ldots, s_j)$ be a collection of integer variables. Define

$$F^j(\vec{s}; *) = F\left\{\prod_{i=1}^{j}(\rho_{s_i,n_i} - \rho_{0,n_1}), *\right\}.$$

This is a symmetric function. The F^j are defined recursively from F using a difference operator. If G is a function of j-variables, define:

$$\Delta G(s_0, s_1, \ldots, s_j) = G(s_0 + s_1, s_2, \ldots, s_j) + G(0, s_2, \ldots, s_j)$$
$$- G(s_0, s_2, \ldots, s_j) - G(s_1, s_2, \ldots, s_j).$$

It is immediate that $F^j = \Delta F^{j-1}$. We define $F_k^j = \Delta^{j-1}(F_k)$ and let $F_k^1 = F_k$. As Δ is a linear operator. $F^j = F_t^j + \cdots + F_1^j$. The upper index refers to the number of variables and the lower index to the degree of homogeneity. As $F(0) = 0$, the constant term is always missing.

We compute these functions recursively using the binomial theorem. Let $\alpha = (a_1, \ldots, a_j)$ be a collection of non-negative integers. We say $\alpha > 0$ if all the a_i's are positive. Define the length $l(\alpha) = j$ and define:

$$\alpha! = \prod_i (a_i)!, \quad \vec{s}^{\alpha} = \prod_i (s_i)^{a_i}, \quad |\alpha| = \sum_i a_i.$$

SUBLEMMA 3.3: *Let k be a positive integer and let $G^1(s)=s^k/k!$. Let $G^j=\Delta G^{j-1}$. Then $G^j=0$ for $j>k$. If $j\leqslant k$, then $G^j(s)=\sum_{|\alpha|=k\,\alpha>0,\,l(\alpha)=j}\vec{s}^{\alpha}/\alpha!$. In particular*:

$$G^k(\vec{s})=s_1\dots s_k$$

$$G^{k-1}(\vec{s})=s_1\dots s_{k-1}\left\{\tfrac{1}{2}\sum_i s_i\right\}$$

$$G^{k-2}(\vec{s})=s_1\dots s_{k-2}\left\{\tfrac{1}{6}\sum_i s_i^2+\tfrac{1}{4}\sum_{i<l}s_is_l\right\}.$$

PROOF: We proceed by induction. If $j=1$, this is just the defining relation. Let $\vec{s}'=(s_3,\dots,s_j)$ and $\alpha=(a,\beta)$. Then:

$$G^j(s_1,\dots,s_j)=\sum\left\{(s_1+s_2)^a(\vec{s}')^{\beta}-s_1^a(\vec{s}')^{\beta}-s_2^a(\vec{s}')^{\beta}\right\}/a!\beta!$$

where the sum is over $a>0$, $\beta>0$, $a+|\beta|=k$. We use the binomial theorem to decompose $\{(s_1+s_2)^a-s_1^a-s_2^a\}/a!=\sum_{a_1+a_2=a,a_1>0,a_2>0}s_1^{a_1}s_2^{a_2}/a_1!a_2!$. When this is substituted into this equation, the desired formula results which completes the proof.

We use this formalism to begin to apply the formula of sublemma 3.2. To avoid complicated subscripts, we temporarily change notation and define

$$\rho(j,n;k)=e^{2\pi ijk/n}$$

so $\{\rho(j,n;\cdot)\}$ parametrize the irreducible representations of Z_n for $0\leqslant j<n$. We have defined

$$\delta=\sum_{0\leqslant i<p^{b-1}}\rho(i,p^b;\cdot)=\sum_{0\leqslant i<p^{b-1}}\rho(iqp^a,n_1;\cdot)$$

when this is expressed as a representation of Z_{n_1}. Therefore:

$$\begin{aligned}F_{\delta}(s,*)&=F\big(\delta\oplus(\rho_{s,n_1}-\rho_{0,n_1};*)\big)\\&=\sum_{0\leqslant i<p^{b-1}}\{F(s+iqp^a;*)-F(iqp^a;*)\}\\&=\sum_{0\leqslant i<p^{b-1}}\\&\quad\times\{F(s+iqp^a;*)-F(iqp^a;*)-F(s;*)+F(s;*)\}\\&=p^{b-1}F(s;*)+\sum_{0<i<p^{b-1}}F^2(iqp^a,s,*).\end{aligned}$$

We apply Δ^{t-1} to eliminate all but the leading term. Let $F = F_t + F_{t-1} + \ldots + F_1$. By sublemma 3.3, $F^t = F_t^t + F_{t-1}^t + \ldots + F_1^t = F_t^t$. Therefore:

$$F_\delta^t(\vec{s};\ *) = p^{b-1}F^t(\vec{s};\ *) + \sum_{0 \leqslant i < p^{b-1}} F^{t+1}(iqp^{a+1}, \vec{s};\ *)$$

$$= p^{b-1}F_t^t(\vec{s};\ *).$$

By sublemma 3.3, this is just a multiple of $s_1 \ldots s_t$. We take $\vec{s} = \vec{q}$ to obtain a representation of Z_p, and not just of Z_{n_1}. By sublemma 3.2, in $Q \bmod Z$

$$0 = w_* F_\delta(\vec{q};\ *) - \bar{w}_* F_\delta(\vec{q};\ *) = (w^t - \bar{w}^t) p^{b-1} q^t F_t^t(\vec{1};\ *)$$

as F_t^t is homogeneous. We compute:

$$F_t^t(\vec{1};\ \text{sign}) = -e \cdot 2^t/n_1$$

$$F_t^t(\vec{1};\ \text{SPIN}_c) = -e/n_1$$

in $Q \bmod Z$. As $n_1 = p^{a+b}q$ and $(q,\ p) = (e,\ p) = 1$, we conclude that:

$$(w^t - \bar{w}^t) p^{b-1} \equiv 0 \bmod p^{a+b} \qquad \text{if} \quad * = \text{SPIN}_c$$

$$2^t (w^t - \bar{w}^t) p^{b-1} \equiv 0 \bmod p^{a+b} \qquad \text{if} \quad * = \text{sign}.$$

If $* = $ sign, then p is necessarily odd. This proves:

SUBLEMMA 3.4: *With the notation defined above*, $w^t \equiv \bar{w}^t \bmod p^{a+1}$.

This congruence arises from considering the leading term of the polynomial defined in Lemma 2.3. We will use the next non-zero term in this polynomial to establish the congruences

$$w^{t-1} \equiv \bar{w}^{t-1} \bmod p^{a+1} \qquad \text{if} \quad * = \text{SPIN}_c$$

$$w^{t-2} \equiv \bar{w}^{t-2} \bmod p^{a+1} \qquad \text{if} \quad * = \text{sign}.$$

We conclude $w = \bar{w} \bmod p^{a+1}$ if $* = \text{SPIN}_c$. If $* = $ sign, then t is odd so $(t,\ t-2) = 1$ and again $w \equiv \bar{w} \bmod p^{a+1}$. This completes the proof of Lemma 3.1 and gives the needed bootstrapping argument. If $p = 2$ and $* = $ sign, then the telescoping fails and we obtain no congruence at all. If p is odd and t is even, then we obtain the congruence $w^2 \equiv \bar{w}^2 \bmod p^{a+1}$ instead of the needed congruence.

The argument using the next lower term in the polynomial is somewhat complicated. Let $G(s)=s^t/t!$. Then sublemma 3.3 implies:

$$2\cdot G^{t-1}(\vec{1})=t \text{ and } 12\cdot G^{t-2}(\vec{1})=2t+3t(t-1)/2.$$

We take $\vec{s}=\vec{q}=(q,\dots,q)$. We introduce the factors of "2" and "12" to avoid denominators. We first study the leading term $F_{t,\delta}^{t-1}(\vec{q};\ \mathrm{SPIN_c})$ and $F_{t,\delta}^{t-2}(\vec{q};\ \mathrm{sign})$.

$$\begin{aligned}2\cdot F_{t,\delta}^{t-1}(\vec{q};\ \mathrm{SPIN_c}) &= 2p^{b-1}F_t^{t-1}(\vec{q};\ \mathrm{SPIN_c})\\ &\quad +2\sum_{0\leqslant i<p^{b-1}} F_t^{t}(ip^{q}q,\ \vec{q};\ \mathrm{SPIN_c})\\ &= -\frac{e}{n_1}\left(tp^{b-1}q^t+2\sum_{0\leqslant i<p^{b-1}} ip^aq^t\right)\\ &= -\frac{e}{n_1}p^{b-1}q^t\left(t+p^a\left(p^{b-1}-1\right)\right)\\ &= -\frac{e}{n_1}p^{b-1}q^t.\end{aligned}$$

$$\begin{aligned}12\cdot F_{t,\delta}^{t-2}(\vec{q};\ \mathrm{sign}) &= 12p^{b-1}F_t^{t-2}(\vec{q};\ \mathrm{sign})\\ &\quad +12\sum_{0\leqslant i<p^{b-1}} F_t^{t-1}(ip^aq,\ \vec{q};\ \mathrm{sign})\\ &= -\frac{e}{n_1}\left(p^{b-1}2^tq^t(2t+3t(t-1)/2)\right)\\ &\quad -\frac{e}{n_1}\left(2^tq^t\left(6\sum_{0\leqslant i<p^{b-1}}\left(i^2p^{2a}+(t-2)ip^a\right)\right)\right.\\ &= -\frac{e}{n_1}\left(p^{b-1}2^tq^t(2t+3t(t-1)/2)\right)\\ &\quad -\frac{e}{n_1}\left(p^{b-1}2^tq^t\left\{p^{2a}\left(p^{b-1}-1\right)\left(2p^{b-1}-1\right)\right.\right.\\ &\quad \left.+(t-2)p^a3\left(p^{b-1}-1\right)\right\}\\ &= -\frac{e''}{n_1}p^{b-1}q^t.\end{aligned}$$

In this expression, e' and e'' are integers. From the congruence $w^t \equiv \bar{w}^t \bmod p^{a+1}$ we conclude:

SUBLEMMA 3.5: *We adopt the notation above. Then*:

(*a*) $2 \cdot (w_* F_{t,\delta}^{t-1} - \bar{w}_* F_{t,\delta}^{t-1})(\vec{q};\ SPIN_c) = (w^t - \bar{w}^t) 2 F_{t,\delta}^{t-1}(\vec{q};\ SPIN_c) \in Z$

(*b*) $12 \cdot (w_* F_{t,\delta}^{t-2} - \bar{w}_* F_{t,\delta}^{t-2})(\vec{q};\ \text{sign}) = (w^t - \bar{w}^t) 12 F_{t,\delta}^{t-2}(\vec{q};\ \text{sign}) \in Z$.

REMARK: We are working in Q mod Z. It seems as though, therefore, we are throwing away information by multiplying by 2 and by 12. In fact, no additional information is obtained by neglecting these coefficients and the analysis of the total formula is simply more complicated. Sublemma 3.5 fails if these coefficients are omitted in general.

We can now finally complete the proof of Lemma 3.1. In Q mod Z, we can ignore integer terms. We apply sublemma 3.2:

$$\begin{aligned}
0 &= 2\left(w_* F_\delta^{t-1} - w_* F_\delta^{t-1}\right)(\vec{q};\ \text{SPIN}_c)\\
&= 2\left(w_* F_{t,\delta}^{t-1} - \bar{w}_* F_{t,\delta}^{t-1}\right)(\vec{q};\ \text{SPIN}_c)\\
&\quad + 2\left(w_* F_{t-1,\delta}^{t-1} - \bar{w}_* F_{t-1,\delta}^{t-1}\right)(\vec{q};\ \text{SPIN}_c) = 0 + (w^{t-1} - \bar{w}^{t-1})\\
&\quad \times \left(2p^{b-1} F_{t-1}^{t-1}(\vec{q};\ \text{SPIN}_c) + 2 \sum_{0 \leqslant i < p^{b-1}} F_{t-1}^{t}(ip^q q,\ \vec{q};\ \text{SPIN}_c)\right)\\
&= 0 + (w^{t-1} - \bar{w}^{t-1})\left(-\frac{eq^{t-1}}{n_1}\right)(n_1 + v_1 + \dots + v_t)(p^{b-1}) + 0.
\end{aligned}$$

This leads to the congruence

$$(w^{t-1} - \bar{w}^{t-1})(v_1 + \dots + v_t) \equiv 0 \bmod p^{a+1}.$$

We *now* finally use the hypothesis that each $v_t \equiv 1 \bmod p^y$ so $v_1 + \dots + v_t \equiv t \bmod p^{a+1}$. Again, we use the hypothesis that $(t, p) = 1$ so finally we see $w^{t-1} \equiv \bar{w}^{t-1} \bmod p^{a+1}$ which completes the proof of Lemma 3.1 if $* = \text{SPIN}_c$.

Similarly, we calculate:

$$\begin{aligned}
0 &= 12\left(w_* F_\delta^{t-2} - \bar{w}_* F_\delta^{t-2}\right)(\vec{q};\ \text{sign})\\
&= 12\left(w_* F_{t,\delta}^{t-2} - \bar{w}_* F_{t,\delta}^{t-2}\right)(\vec{q};\ \text{sign})\\
&\quad + 12\left(w_* F_{t-2,\delta}^{t-2} - \bar{w}_* F_{t-2,\delta}^{t-2}\right)(\vec{q};\ \text{sign}) = 0 + (w^{t-2} - \bar{w}^{t-2})\\
&\quad \times \left(12 p^{b-1} F_{t-2}^{t-2}(\vec{q};\ \text{sign}) + 12 \sum_{0 \leqslant i < p^{b-1}} F_{t-2}^{t-1}(ip^q q,\ \vec{q};\ \text{sign})\right)
\end{aligned}$$

$$= 0 + \left(w^{t-2} - \overline{w}^{t-2}\right)\left(4 \cdot - \frac{e}{n_1} \cdot 2^{t-2} q^{t-2} p^{b-1}\left(n_1^2 + v_1^2 + \ldots + v_t^2\right)\right)$$

$$+0.$$

This leads to the congruence $(w^{t-2} - \overline{w}^{t-2})(v_1^2 + \ldots + v_t^2) \equiv 0 \bmod p^{a+1}$ since p is odd. Since $v_1^2 + \ldots + v_t^2$ is coprime to p, we conclude $w^{t-2} \equiv \overline{w}^{t-2} \bmod p^{a+1}$ which completes the proof of the lemma.

Since $\operatorname{ind}(\rho, *, M(u, v))$ is both a spectral and a cobordism invariant, we shall complete the proof of Theorems 0.1 and 1.5 by proving:

LEMMA 3.6: *Let $G = G(m, n, d, r)$ with the notation of Theorem 1.1. Let $M(u, v) = S^{2d-1}/\pi_{u,v}(G)$. Decompose $n = dn_1$ and $n_1 = d_1\bar{q}$ where $(\vec{q}, d) = 1$ and where every prime dividing d_1 divides d and conversly. Let $\mathscr{D}$ be the least common multiple of d and d_1. If d is odd, let $*$ =* sign *or $SPIN_c$; if d is even, let $* = SPIN_c$. Assume that*

$$\operatorname{ind}(\rho, *, M(u, v)) = \operatorname{ind}(\rho, *, M(\bar{u}, \bar{v}))$$

$$\text{for all} \quad \rho \in R_0(Z_{\mathscr{D}}).$$

Then $v \equiv \bar{v}$ mod *d_1 and there is an isometry between $M(u, v)$ and $M(\bar{u}, \bar{v})$.*

PROOF: The existence of the desired isometry will follow from the congruence. We establish this congruence one prime at a time. Let p be a prime dividing d. Decompose $d = p^x q$ and $n_1 = p^y q_1$ where $(p, q) = (p, q_1) = 1$. Let G_p be the subgroup of G generated by $\{A, B^q\}$. Let κ be the right regular representation of Z_q so $\operatorname{Tr}(\kappa(k)) = 0$ if $q \nmid k$ and $\operatorname{Tr}(\kappa(k)) = q$ if $q \mid k$. Let $\rho \in R_0(Z_{p^y})$, then $\kappa\rho \in R_0(Z_{\mathscr{D}})$ is admissible. We use Frobenius reciprocity.

$$\operatorname{ind}(\kappa\rho, *, M(u, v)) = \frac{1}{|G|} \sum_{j,k}{}' \operatorname{Tr}(\kappa\rho(k)) \operatorname{defect}\left(\pi_{u,v}(A^j B^k), *\right)$$

$$= \frac{q}{|G|} \sum_{j, q \mid k}{}' \operatorname{Tr}(\rho(k)) \operatorname{defect}\left(\pi_{u,v}(A^j B^k), *\right)$$

$$= \operatorname{ind}\left(\rho \,|_{G_p}, *, M\left(\pi_{u,v} \,|_{G_p}\right)\right)$$

can be calculated from information regarding the subgroup G_p. We therefore have:

$$0 = \operatorname{ind}\left(\rho \,|_{G_p}, *, M\left(\pi_{u,v} \,|_{G_p}\right)\right) - \operatorname{ind}\left(\rho \,|_{G_p}, *, M\left(\pi_{\bar{u},\bar{v}} \,|_{G_p}\right)\right)$$

for all $\rho \in R_0(Z_{p^y})$.

We will verify that the hypothesis of Lemma 3.1 hold; this will lead to the congruence $v \equiv \bar{v} \bmod p^y$. The congruence for all p dividing d will complete the proof of Lemma 3.6.

We use the structure theorem for Type I groups. All the Sylow subgroups of G are cyclic so the same is true of G_p. Since $B^qAB^{-q} \neq A$ as B^q isn't central in G, we conclude G_p is non-Abelian. Since $\pi_{u,v}|_{G_p}$ is a fixed point free representation, we conclude G_p is a type I group so $G_p = G(m_p, n_p, d_p, r_p)$. Furthermore, d_p divides the degree of $\pi_{u,v}$ so $d_p \mid p^xq$. Although G_p is generated by $\{A, B^q\}$, these need not be the cannonical generators.

For a type I group, all the Sylow subgroups are cyclic. Thus any two subgroups of the same prime power order must be conjugate. Let $\bar{p}$ be prime and let $J_{\bar{p}}$ be a Sylow $\bar{p}$-subgroup. If $\bar{p} \mid m$, factor $m = \bar{p}^a\overline{m}$. Then $A^{\overline{m}}$ can be taken to be a generator of $J_{\bar{p}}$. As A generates a normal cyclic subgroup, $J_{\bar{p}}$ is a normal subgroup of G. If $Z(G)$ is the center, then $J_{\bar{p}} \cap Z(G) = \{1\}$. Next suppose $\bar{p} \mid n$ but $(\bar{p}, d) = 1$. If we factor $n = \bar{p}^a\bar{n}$, then $B^{\bar{n}}$ can be taken as a generator of $J_{\bar{p}}$. Since $d \mid \bar{n}$, we have $J_{\bar{p}} \subseteq Z(G)$ in this case. Thus in either event, if $\bar{p} \nmid d$, we have a unique normal cyclic Sylow $\bar{p}$-subgroup.

If $\bar{p} \mid d$, then $d = \bar{p}^a\bar{d}$ and $n_1 = \bar{p}^b\bar{n}_1$. $B^{\bar{d}\bar{n}_1}$ can be taken to be a generator of $J_{\bar{p}}$. The Sylow subgroup is not unique in this instance. It is immediate that: $|J_{\bar{p}} \cap Z(G)| = \bar{p}^b$ and $|J_{\bar{p}} : J_{\bar{p}} \cap Z(G)| = \bar{p}^a$. Consequently we can recover d from the Sylow subgroup structure:

$$d = \prod_{\text{primes } \bar{p} \text{ so } J_{\bar{p}} \cap Z(G) \neq \{1\}} |J_{\bar{p}} : J_{\bar{p}} \cap Z(G)|.$$

We apply this to calculate d_p for the group G_p. Suppose $\bar{p} \mid q$. Then if $n_1 = \bar{p}^b\bar{n}_1$, we have $B^{d\bar{n}_1}$ generates $J_{\bar{p}}$. This is central in G and hence central in G_p. Thus $J_{\bar{p}}$ is contained in $Z(G_p)$ so $\bar{p} \nmid d_p$. As d_p divides $d = p^xq$, we conclude d_p is a power of the prime p being studied. B^{QQ_1} generates J_p in G. This is contained in G_p and is therefore the Sylow subgroup there as well. B^{QQ_1k} commutes with A if and only if $p^x \mid k$ and thus $J_p \cap Z(G) = J_p \cap Z(G_p)$. Therefore $d_p = p^x$, so this hypothesis of Lemma 3.1 is satisfied by the group G_p.

By the Chineese remainder theorem, we choose $(c, |G|) = 1$ so that $c \equiv u(m)$ and $c \equiv v(n)$. Similarly we choose $\bar{c} \equiv \bar{u}(m)$ and $\bar{c} \equiv \bar{v}(n)$. $\pi_{u,v} = \pi_{c,c}$ and $\pi_{\bar{u},\bar{v}} = \pi_{\bar{c},\bar{c}}$. We wish to show $c \equiv \bar{c} \bmod p^y$. Let $\pi^p_{u_i,v_i}$ denote the irreducible representations of G_p which are fixed point free. We may decompose

$$\pi_{1,1}|_{G_p} = \tau = \oplus \pi^p_{u_i,v_i}$$

for some suitably chosen collection of indices $(u_1, v_1, \ldots, u_q, v_q)$. Let $\gamma = e^{2\pi i/mn}$ and let $Q(\gamma)$ be the cyclotonic number field. The $\pi_{u,v}$ can be viewed as matrices with values in $Q(\gamma)$. Let ϵ_c be the element of the Galois group of this field so $\epsilon_c(\gamma)=\gamma^c$ for $(c, mn)=1$. Then we compute:

$$\pi_{u,v}|G_p=\pi_{c,c}|G_p=\epsilon_c(\pi_{1,1})|G_p=\oplus\epsilon_c\left(\pi^p_{u_i,v_i}\right)$$
$$=\oplus\pi^p_{cu_i,cv_i}=\tau(c)$$

in the notation of Lemma 3.1. Consequently, we have the identity:

$$\operatorname{ind}(\rho, M(\tau(c)), *)=\operatorname{ind}(\rho, M(\tau(\bar{c})), *)$$

for all $\rho\in R_0(Z_{p^y})$ for the group G_p. The exact embedding of $R_0(Z_{p^y})$ into $R_0(G_p)$ of course depends upon the choice of the cannonical generators. This set, however, is independent of the generators chosen and thus this condition is generator independent. If we can verify that $v_i\equiv v_j$ for all $1\leqslant i, j\leqslant q$, then all the hypothesis of Lemma 3.1 will be satisfied and we can conclude $c\equiv\bar{c}$ or equivalently $v\equiv\bar{v}$ mod p^y as desired.

Let $C=B^{qq_1}$ generate the Sylow p-subgroup J_p of both G and G_p; $D=C^{p^x}$ generates $J_p\cap$ the center. Then $\pi_{1,1}(D)=e^{2\pi i/p^y}\cdot I$ is a constant multiple of the identity. Similarly $\pi^p_{1,1}(D)=e^{2\pi i\mu/p^y}$ is a constant multiple of the identity where $(\mu, p)=1$ and where μ reflects the marking chosen for G_p. Therefore $\oplus\pi^p_{u_i,v_i}(D)=\oplus e^{2\pi i\mu v_i/p^y}$. As this must be a multiple of the identity, we have $v_i\equiv v_j$ mod p^y for all (i, j) which completes the proof.

4. Other results

We proved Theorems 0.1 and 1.5 by studying the primes which divided the dimension d. In this section, we will study the other primes dividing the order of the group. We shall need the following analogue of Lemma 3.1 for lensspaces.

LEMMA 4.1: *Let $L(n;\vec{q})$ be a lensspace of dimension $2t-1$. If n is odd, we let* $*=$ sign *or $SPIN_c$; if n is even, let* $*=SPIN_c$. *Let* $(c, n)=(\bar{c}, n)=1$ *and assume* $\operatorname{ind}(\rho, *, L(n; c\vec{q}))=\operatorname{ind}(\rho, *, L(n;\bar{c}\vec{q}))$ *for all* $\rho\in R_0(Z_n)$. *Then* $c^t\equiv\bar{c}^t$ mod n. *Suppose we assume additionally that* $\vec{q}=(q_1,\dots,q_1)$ *is diagonal and that* $(t, n)=1$. *Then we can conclude that* $c\equiv\bar{c}$ mod n.

PROOF: We use the same argument as that used to prove Lemma 3.1. Let $wc\equiv 1$ mod n and $\overline{wc}\equiv 1$ mod n. Let $F(\rho)=\operatorname{ind}(\rho, *, L(n;\vec{q}))$ in Q mod Z, then the hypothesis of the lemma implies that

$$(w_*F-\bar{w}_*F)(\rho)=0 \qquad \text{for all} \quad \rho\in R_0(Z_n).$$

As in the proof of Lemma 3.1, we define $F(s) = F(\rho_{s,n} - \rho_{0,n})$ and $F(\vec{s})$ by difference operators. We compute the leading term; let $\vec{s} = (1, \dots, 1)$ t-times, then:

$$(w_* F - \bar{w}_* F)(1, \dots, 1) = (w^t - \bar{w}^t) F_t'(\vec{1})$$

since the lower order terms all vanish. We apply Lemma 2.4 and the methods of the third section to evaluate

$$F_t'(\vec{1}) = -\frac{e}{n} \quad \text{if } * = \mathrm{SPIN_c}$$

and

$$F_t'(\vec{1}) = -\frac{e2^t}{n} \quad \text{if } * = \text{sign}.$$

This gives the desired identity $w^t \equiv \bar{w}^t \bmod n$ and proves the first part of the lemma. The proof of the second part of the lemma uses the next lower term in the expansion for F; the assumptions of the theorem imply that $(q_1 + \dots + q_t)$ and $(q_1^2 + \dots + q_t^2)$ are coprime to the index n. The remainder of the argument is exactly the same as that given in the proof of Lemma 3.1 and is therefore omitted. This lemma is simpler in that we do not need to introduce the representation δ and we do not need to consider any telescoping.

We use this lemma to establish:

LEMMA 4.2: *Let $G = G(m, n, d, r)$ with the notation of Theorem* 1.1. *Let $M(u, v) = S^{2d-1}/\pi_{u,v}(G)$. If $|G|$ is odd, let $*$ = sign or $SPIN_c$; if $|G|$ is even, let $* = SPIN_c$. Suppose that* $\mathrm{ind}(\rho, *, M(u, v)) = \mathrm{ind}(\rho, *, M(\bar{u}, \bar{v}))$ *for all $\rho \in R_0(G)$. Then $v \equiv \bar{v}$* mod *n_1 and $u^d \equiv \bar{u}^d$* mod *m.*

PROOF: We have already established the portion of the congruence which relates to the primes dividing d. Let $p \mid n$ be prime and suppose $(p, d) = 1$. We decompose $n_1 = p^y q$ where q is coprime to p. We must show $v \equiv \bar{v} \bmod p^y$. Let $C = B^{dq}$ be an element of order p^y. C generates a cyclic subgroup J_p of G which is the Sylow p-subgroup of G. For $\rho \in R_0(J_p)$, let ρ^G be the induced virtual character on G; this vanishes on the identity. As defect is a class function, Frobenius reciprocity implies that:

$$\begin{aligned}\mathrm{ind}(\rho^G, *, M(u, v)) &= \frac{1}{|G|} \sum_{g \in G}{}' \mathrm{Tr}(\rho^G(g))\mathrm{defect}(\pi_{u,v}(g), *) \\ &= \frac{1}{|J_p|} \sum_{g \in J}{}' \mathrm{Tr}(\rho(g))\mathrm{defect}\big(\pi_{u,v}|_{J_p}(g), *\big) \\ &= \mathrm{ind}\big(\rho, *, M(\pi_{u,v}|_{J_p})\big).\end{aligned}$$

By definition, we have $\pi_{u,v}(C) = e^{2\pi i v q d / n_o q d}$ is diagonal. Therefore $M(\pi_{u,v} | _{J_p}) = L(p^y;\ v, \dots, v)$ is a lensspace corresponding to a diagonal representation. We have assumed that $(p, d) = 1$ so we can apply Lemma 4.1 to conclude $v \equiv \bar{v} \bmod p^y$ as desired.

We apply a similar argument to the primes dividing m. Since the corresponding lensspace will not be diagonal, we can only conclude $u^d \equiv \bar{u}^d$ modulo m. This completes the proof.

The case in which m is prime is particularly tractible. Since d is the order of r in the group of units of Z_m, we have r satisfies the equation:

$$\prod_{i=1}^{d} (x - r^i) \equiv x^d - 1 \bmod m.$$

The congruence $u^d \equiv \bar{u}^d$ given by Lemma 4.2 yields the identity:

$$\prod_{i=1}^{d} (u - r^i \bar{u}) \equiv 0 \bmod m.$$

Since m is prime, this implies $u \equiv r^i \bar{u}$ for some i. This implies $\pi_{u,v}$ and $\pi_{\bar{u},\bar{v}}$ are unitarily equivalent. Conversly, if these two representations are unitarily equivalent, then $M(u, v) = M(\bar{u}, \bar{v})$ where the identification is an isometry preserving all the structures involved. This proves:

THEOREM 4.3: *Let $G = G(m, n, d, r)$ with the notation of Theorem 1.1. Let $M(u, v) = S^{2d-1}/\pi_{u,v}(G)$ and assume that m is prime. If $|G|$ is odd, let $* =$ oriented or $SPIN_c$; if $|G|$ is even, let $* = SPIN_c$. The following conditions are equivalent:*

i) $\pi_{u,v}$ and $\pi_{\bar{u},\bar{v}}$ are unitarily equivalent representations
ii) $M(u, v)$ and $M(\bar{u}, \bar{v})$ are G-$*$-cobordant
iii) $M(u, v)$ and $M(\bar{u}, \bar{v})$ are strongly $*$-π_1-isospectral.

There is one other special case which is worth discussing. Let $n = d^2 q$ where $(q, d) = 1$. In this case, $\mathscr{D} = d = d_1$ and the natural Z_d bundle over $M(u, v)$ is independent of the marking chosen for the fundamental group. The invariants ind(ρ, $*$, $M(u, v)$) for $\rho \in R_0(Z_d)$ are then diffeomorphism invariants of the manifold. Then the results we have proved already show:

THEOREM 4.4: *Let $G = G(m, n, d, r)$ with the notation of Theorem 1.1. Assume d is odd and that $n = d^2 q$ for $(q, d) = 1$. Let $M(u, v) = S^{2d-1}/\pi_{u,v}(G)$. We give the natural Z_d structure to $M(u, v)$. The following conditions are equivalent:*

(i) There exists an isometry between $M(u, v)$ and $M(\bar{u}, \bar{v})$.
(ii) $M(u, v)$ and $M(\bar{u}, \bar{v})$ are Z_d cobordant
(iii) $M(u, v)$ and $M(\bar{u}, \bar{v})$ are strongly Z_d isospectral.

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