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ON THE DISCRIMINANT OF THE ARTIN COMPONENT

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Introduction

By a normal surface singularity (V, p) we understand the germ of a normal complex surface V at the singular point p. Let π: $M \to V$ be the minimal resolution. Laufer, [12], has shown that there exists a 1-convex flat map ω: $\mathfrak{M} \to R$ over a complex manifold R of dimension m $= \dim_{\mathbb{C}} H^1$ $(M; \Theta_M)$ which represents the semi-universal deformation of the germ of M at the exceptional set E, see also [7]. If (V, p) is rational, i.e. $R^1\pi_* \mathcal{O}_M = 0$, then ω simultaneously blows down to a deformation of (V, p). This procedure yields a holomorphic map germ Φ: $(R, 0) \to (S, 0)$ where $(S, 0)$ denotes the base space of the semi-universal deformation ϑ: $(\mathscr{V}, p) \to$ (S, 0) of the given rational singularity. Results of Artin, [1], say that the blowing down map Φ is finite and that the germ of the image $S_a := \Phi(R)$ is an irreducible component of the deformation space $(S, 0)$ which is also called the Artin component. The aim of this paper is to study the base change given by Φ, the discriminant $\Delta_a := \Delta \cap S_a$ of the Artin component, and the singularities of the fibers corresponding to generic points of Δ_a.

Basic examples are provided by the rational double points (RDP's) which arise as singularities of quotients of $\mathbb{C}^2$ by actions of finite subgroups of $SL_2(\mathbb{C})$. Let Γ be the weighted dual graph associated to the minimal resolution of such a singularity. Then it is well known that such Γ correspond uniquely to the Dynkin diagrams which classify those simple Lie algebras having root systems with only roots of equal length. It is the work of Brieskorn, [2], which makes this connection more precise. In particular it turns out that the map germ Φ: $(R, 0) \to (S, 0)$ can be represented by a Galois covering whose group of automorphisms is the Weyl group of the corresponding Lie algebras. Further the discriminant $\Delta \subset S$ of the semi-universal deformation δ is an irreducible hypersurface such that the fiber over a generic point of Δ has an ordinary double point as its only singularity. Our main result, Theorem 2, generalizes these results to arbitrary rational singularities. In this general case, the group of automorphisms which represents the map germ Φ: $(R, 0) \to (S_a, 0)$ is isomorphic to a direct product of Weyl groups which correspond to the maximal RDP-configurations on the minimal resolution of the given rational singularity.

A crucial point is to verify smoothability of the exceptional divisors which "carry the cohomology of M", see Theorem 1. The author's main result in [8] plays a significant role in the proof of it. As an application of our method it is shown that the smoothings of a given divisor occur on irreducible components of the same dimension for which we can give an explicit formula. Assuming the smoothability, this has been proved by Wahl, [19], in the formal category. Here we carry out a different proof using the concept of weak lifting introduced by Laufer, [11].

Under the smoothability condition, the essential parts of Theorem 2 have been also already stated in [18], [19] but Wahl's approach only works well for the deformation theory taking place on the category of artin (resp. complete) local $\mathbb{C}$-algebras. The reason is that it is not known yet if there exists an analytic semi-universal deformation of 1-convex spaces with isolated singularities. This has been only proved in the smooth case, [12]. So we cannot use Wahl's arguments in a straightforward way.
Our proof is based on an explicit description of the action of the direct product of Weyl groups on the deformation space R of M, see §3.

NOTATIONS AND CONVENTIONS: We write $h^i(X;\ \mathscr{F}:=\dim_{\mathbb{C}} H^i(X;\ \mathscr{F})$, $\chi(\mathscr{F}):=\Sigma(-1)^i h^i(X;\ \mathscr{F})$, $0 \leqslant i \leqslant \dim X$, and use the standard symbols $\mathcal{O}$, Θ, Ω^k in order to denote the sheaf of germs of holomorphic functions, the tangent sheaf and the sheaf of differential k-forms. There will be no systematic distinction between germs and spaces representing them whenever there is no serious likelihood of confusion.

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§1. Smoothing of cycles

1.1 Let π: $M \to V$ be the minimal resolution of a normal surface singularity (V, p) with exceptional set E. In this paper we additionally assume that the minimal resolution is good, i.e. the irreducible exceptional components $E_1, \dots, E_r$ are smooth and have normal crossings. A cycle D on M is a divisor on M which is given by an integral linear combination of the irreducible exceptional components. Suppose D is a positive cycle. Then the corresponding curve (supp(D), $\mathcal{O}_D$) will be also denoted by D. By the *topological type* of D we understand the weighted dual graph associated to the embedding of supp(D), the support of D, in M together with the multiplicities of the irreducible components of D.

1.2 Let γ: $\mathfrak{M} \to Q$ be a flat map which represents a deformation of the germ (M, E) over the germ of a complex space Q at a distinguished point 0. We may always assume that γ is a 1-convex map and that each fiber $\mathfrak{M}_q$ is a strictly pseudoconvex manifold with a well defined exceptional set E_q, [15]. By $\mathscr{E}$ we denote the union of the exceptional sets E_q, $q \in Q$, provided with the reduced complex structure.

1.3 One says that a positive cycle D on M *lifts* to the germ $(Q, 0)$ if there exists a complex subspace $\mathscr{D}$ of $\mathfrak{M}$ such that, after possibly shrinking of Q, the restriction $\lambda: = \gamma|\mathscr{D}: \mathscr{D} \to Q$ is a deformation of $D = \mathscr{D}_0$ which is also called a *lifting* of D over $(Q, 0)$. Equivalently, a lifting of D is given by a relative Cartier divisor $\mathscr{D}$ on $\mathfrak{M}$ whose intersection with $M = \mathfrak{M}_0$ gives D. This concept yields a contravariant functor $\mathscr{L}_D(-)$ from the category of germs of complex spaces to the category of sets which is defined by

$$\mathscr{L}_D((Q, 0)): = \text{set of equivalence classes of}$$

$$(M, E) \text{ over } (Q, 0) \text{ together with a lifting of } D.$$

Let Def((M, E), $-$) denote the deformation functor of (M, E), and let $\mathbb{C}_1$ denote the 0-dimensional germ $(0, \mathbb{C}\langle t\rangle/(t^2))$. Then it turns out that, via the well-known identification of Def((M, E), $\mathbb{C}_1$) with $H^1(M; \Theta_M)$, we have a natural isomorphism

$$\mathscr{L}_D(\mathbb{C}_1) \overset{\approx}{\to} \operatorname{Ker}(\gamma^*: H^1(\Theta_M) \to H^1(\mathscr{O}_D(D))$$

where γ^* is induced by the homomorphism γ: $\Theta_M \to \mathscr{O}_D(D)$ of sheaves which can be locally described as follows. If θ is a vector field near a point $x \in M$ and $f(z)$ is a local defining equation for D near x, then $\gamma(\theta) = \theta(f)$, compare [19], [11], [7].

Furthermore it can be shown without any difficulties that there exists a semi-universal formal lifting, i.e. $\mathscr{L}_D(-)$ has a hull in the sense of Schlessinger on the category of 0-dimensional complex spaces. Unfortunately it is not known whether there exists a lifting which is semi-universal with respect to germs of complex spaces of arbitrary dimension. To avoid this unpleasant difficulty at least in the case of reduced parameter spaces, Laufer, [11], has introduced a weaker notion of a lifting.

1.4 Suppose that Q is reduced. Then a positive cycle D *weakly lifts* to the germ $(Q, 0)$ if for each $q \in Q$, q near 0, there is a (necessarily unique) cycle $D_q > 0$ on $\mathfrak{M}_q$ such that D and D_q are homologous in $\mathfrak{M}$. Note that the family of cycles $\{D_q\}$, $q \in Q$, also called a *weak lifting* of D, does not define in general a Cartier divisor on $\mathfrak{M}$. But his is true if Q is smooth. So a positive cycle lifts to a smooth space if and only if it weakly lifts. By semi-continuity $\chi(\mathscr{D}_q) = \chi(D)$ for a lifting of D. Hence, using a resolution of given parameter space Q, it can be readily seen that $\mathrm{X}(D_q)=\mathrm{X}(D)$ and $D_q\cdot D_q=D\cdot D$ for a weak lifting $\{D_q\}$ of D over $(Q, 0)$, too. Now the point is that to each deformation $\gamma\colon \mathfrak{M}\to Q$ of (M, E) over a reduced space Q there exists a maximal reduced subspace $(Q_D, 0)\subset(Q, 0)$ to which a given positive cycle D weakly lifts, [11; Proposition 2.7].

1.5 Throughout this paper, $\omega\colon \mathfrak{M}\to R$ denotes a flat 1-convex map with smooth parameter space R which represents a versal deformation of the minimal resolution germ (M, E) of the singularity (V, p), i.e. the Kodaira-Spencer map $T_0R\to H^1(M;\ \Theta_M)$ is surjective. If this map is also injective, then ω represents the semi-universal deformation of (M, E), see [12] or [7; §7].

Standard arguments in deformation theory show that the obstruction space $\mathrm{ob}(\mathscr{L}_D)$ for the functor $\mathscr{L}_D(-)$ is given by

$$\mathrm{ob}(\mathscr{L}_D)=\mathrm{coker}\ \gamma^*=H^1\big(D;\ \mathscr{T}_D^1\big),$$

where $\mathscr{T}_D^1$ is the sheaf of germs of infinitesimal deformations of D. If there are no obstructions, then the notions of lifting and weak lifting coincide. More precisely, if $\mathrm{supp}(D)$ is connected and $h^1(\mathscr{T}_D^1)=0$, then

(i) R_D is smooth,
(ii) fibers of $\lambda:\mathscr{D}\to R_D$ are generically smooth,
(iii) $\mathrm{codim}\ R_D=h^1(\mathscr{O}_D(D))$,

see [13; Theorem 3.5], [7; 11.8]. Clearly, $h^1(\mathscr{T}_D^1)=0$ if D is reduced.

1.6 From now on assume $h^0(\mathscr{O}_D)=1$, e.g. take D to be the fundamental cycle. Note that the vanishing of $H^1(\mathscr{T}_D^1)$ implies that $h^0(\mathscr{O}_D)=1$ if $\mathrm{supp}(D)$ is connected, [7]. Then straightforward computations show that

$$h^1\big(\mathscr{T}_D^1\big)=h^1\big(\mathscr{O}_{D-D_{\mathrm{red}}}(D)\big),$$

where $D_{\mathrm{red}}=\Sigma E_i$, $E_i\subset\mathrm{supp}(D)$. Thus it is easy to find examples of D (even in case of rational singularities of multiplicity $\geqslant 4$) for which $H^1(\mathscr{T}_D^1)$ does not vanish. Now a major problem is to find useful weaker conditions that guarantee that D is smoothable over R_D.

1.7 Proposition: *Suppose D admits a decomposition $D=\Sigma k_i\cdot D_i$, $k_i\geqslant 0$, $1\leqslant i\leqslant s$, such that $h^1(\mathscr{T}_{D_i}^1)=0$ for $1\leqslant i\leqslant s$. Then R contains an irreducible component of codimension*

$$\leqslant\sum_{1\leqslant i\leqslant s}h^1\big(\mathscr{O}_{D_i}(D_i)\big).$$

Using 1.5, the proof is clear since $R_D\supset\bigcap_{1\leqslant i\leqslant s}R_{D_i}$ by hypothesis.

1.8 Without loss of generality assume $D_{\text{red}} = E$. Then

$$R_D \supseteq \bigcap_{1 \leqslant i \leqslant r} R_{E_i} =: \Sigma$$

where the R_{E_i}'s are smooth subspaces of R of codimension $= h^1(\mathcal{O}_E(E_i))$ which transversally intersect in a smooth subspace Σ, see 1.5 and [11]. If ω represents the semi-universal deformation, then $\dim \Sigma = h^1(\Theta_M(\log E))$, [17]. But note that Σ is the moduli space for the functor of equitopological deformations of M introduced by Laufer, [10] see also [7; 11.14.3]. Hence ω induces a locally trivial deformation of every positive cycle Y over Σ. Thus D cannot be smoothable over Σ.

1.9 DEFINITION: Assume that $D_{\text{red}} = E$. Then a decomposition $D = \Sigma k_i \cdot D_i$, $1 \leqslant i \leqslant n$, $k_i \geqslant 0$, is called a *good decomposition* of D if

$$\sum_{1 \leqslant i \leqslant n} \operatorname{codim} R_{D_i} < \operatorname{codim} \Sigma .$$

1.10 Let R'_D be an irreducible component of R_D at 0. By [11; 2.1–2.2], there exists a nowhere dense analytic subset S of R'_D such that ω induces an equitopological deformation over $R'_D - S$. Hence the topological type of weak lifting of D is uniquely determined over $R'_D - S$. Let $t \in R'_D - S$, t near 0, then we call t a generic point of R'_D and a positive cycle D_t on $\mathfrak{M}_t$ which is homologous to D in $\mathfrak{M}$ a *generic weak lifting* of D. We may always assume that t is a smooth point of R'_D.

1.11 LEMMA:
a) *Assume that $D_{\text{red}} = E$. Then, for every irreducible component R'_D of R_D at* 0, *the support of a generic weak lifting D_t of D is the full exceptional set of $\mathfrak{M}_t$.*
b) *Suppose the support of D is not smooth. If D admits a good decomposition, then, for every irreducible component R'_D of R_D of maximal dimension, a generic weak lifting D_t of D is not topologically equivalent to D. In particular the pairs (M, E) and $(\mathfrak{M}_t, E_t)$ are not of the same topological type.*

PROOF. Let t be a generic point of R'_D. Denote by ω_t the deformation of $\mathfrak{M}_t$ over a sufficiently small representative of the germ (R, t) induced by ω. By the openness of versality, [12], ω_t represents a versal deformation of the germ of $\mathfrak{M}_t$ along E_t. It is obvious that a sufficiently small representative of the germ (R'_D, t) is the maximal reduced subspace of R to which D_t weakly lifts with respect to ω_t. Now suppose that $E_t \neq \operatorname{supp}(D_t)$. Then it would follow from Theorem 3.6 in [13] that the deformation over the germ (R'_D, t) is not equitopologial; a contradiction to the generic choice of t, see 1.10. To prove part b), assume that R'_D is of maximal dimension. Then codim $R'_D <$ codim Σ. So we can verify our assertion by similar arguments as before.

THEOREM 1: *Suppose (V, p) is a rational singularity. let $\omega: \mathfrak{M} \to R$ be a flat 1-convex map which represents the semi-universal deformation of the minimal resolution germ (M, E) of (V, p). If D is a positive cycle on M with $\chi(D) = 1$, then we have:*

(*i*) *If R'_D is an irreducible component of R_D, then, for generic $t \in R'_D$, the cycle D_t is smooth and is the (full) exceptional set of $\mathfrak{M}_t$, $t \neq 0$.*

(*ii*) dim $R'_D = h^1(\Theta_M) + 1 + D \cdot D$

(*iii*) *R_D is smooth if D is almost reduced, i.e. D is reduced at the non-2 curves.*

REMARK: It is most likely to expect that the R_D's are irreducible but we cannot yet prove it.

1.12 COROLLARY: *Assumptions as in Theorem 1. Then D is the only positive cycle which weakly lifts to R'_D and satisfies $\chi(D) = 1$.*

PROOF: Take Y to be an arbitrary positive cycle with $\chi(Y) = 1$ which weakly lifts to R'_D. Then Y weakly lifts to a cycle Y_t on $\mathfrak{M}_t$ where t is a generic point on R'_D as in (i). Since $\chi(Y_t) = 1$ and D_t is the full exceptional set of $\mathfrak{M}_t$, we observe that $Y_t = D_t$. Hence Y and D are homologous in M. But this is only true if $D = Y$ because the intersection form on M is negative definite.

1.13 PROOF OF THEOREM 1: The last statement follows immediately from 1.5 and 1.6, see also [11]. So it remains to prove (i) and (ii). First let us assume that D admits a good decomposition. Then we can continue as follows.

We may assume that D is supported on the full exceptional set E. Recall that $\chi(D)$ equals 1 if and only if D appears as part of a computation sequence for the fundamental cycle Z on M. Hence $h^0(\mathcal{O}_D) = 1$ and there are only finitely many positive cycles Y on M which satisfy $Y \leqslant D$ and $\chi(Y) = 1$. We do induction on this number N to verify the first statement.

If $N \leqslant 6$, then D is automatically reduced and we are done, see 1.5. For $N > 6$ consider an irreducible component of R_D at 0, say R'_D, and let D_t, $t \in R'_D$, be a generic weak lifting of D. By 1.11 the support of D_t is the full exceptional set of $\mathfrak{M}_t$.

$$D_t \text{ is smooth.} \tag{1.13.1}$$

Recall that a sufficiently small representative of the germ (R'_D, t) is the maximal reduced subspace to which D_t weakly lifts with respect to the versal deformation ω_t of $\mathfrak{M}_t$ induced by ω, compare the proof of 1.11. By definition of a generic point, ω_t determines an equitopological deformation over (R'_D, t). Denote by N_t the number of positive cycles $F \leqslant D_t$ on $\mathfrak{M}_t$ which satisfy $\chi(F) = 1$. We observe that $N_t = 1$ if and only if D_t is smooth. If $1 < N_t < N$, then, by induction, it follows from 1.11b) that ω_t is not equitopological over (R'_D, t) yielding a contradiction. If $N_t = N$, then it can be readily seen that there exists a curve C on R'_D passing through 0 such that all points $s \in C$, $s \neq 0$, are generic points of R'_D and such that every cycle $Y \leqslant D$ with $\chi(Y) = 1$ weakly lifts to C. Therefore ω is an equitopological deformation over C. This implies that D_t and D are of the same topological type. Now it is not difficult to see that a good decomposition of D induces one of D_t. Thus it follows again from 1.11b) that ω_t is not equitopological over (R'_D, t) yielding a contradiction. This finishes the proof of (1.13.1) and hence of (i).

To do the second part, recall that the Kodaira-Spencer map $\rho_t : T_t R \to H^1(\Theta_{\mathfrak{M}_t})$ associated to ω_t is surjective. Thus we know from 1.5 that D_t lifts to a smooth subgerm (R_{D_t}, t) of (R, t) of dimension

$$d = \dim\left(\gamma_t^* \circ \rho_t : T_t R \to H^1\left(\Theta_{\mathfrak{M}_t}\right) \to H^1\left(\mathcal{O}_{D_t}(D_t)\right)\right)$$

$$= h^1(\Theta_M) - h^1\left(\Theta_{\mathfrak{M}_t}\right) + \dim \operatorname{Ker} \gamma_t^*,$$

see 1.3. But $\dim \operatorname{Ker} \gamma_t^* = h^1(\Theta_{\mathfrak{M}_t}) + 1 + D_t \cdot D_t$, compare 1.5. Putting together, we obtain formula (ii) since $D_t \cdot D_t = D \cdot D$ and $(R_{D_t}, t) = (R'_D, t)$.

It remains to check that every positive cycle D with $\chi(D) = 1$ admits a good decomposition. We again do induction on N. We may assume that $N > 6$ and that D is not reduced. Recall that $\chi(D) = 1$ if and only if D is part of a computation sequence for the fundamental cycle $Z = \Sigma z_i \cdot E_i$, $1 \leqslant i \leqslant r$., see [11;3.1]. Thus there is an irreducible component of D, say E_k, such that $\chi(D - E_k) = 1$ and $E_k \cdot (D - E_k) = 1$. We claim that $D = D - E_k + E_k$ is a good decomposition. By induction and application of previous arguments,

$$\begin{aligned} \operatorname{codim} R_{D-E_k} + \operatorname{codim} R_{E_k} &= -1 - (D - E_k) \cdot (D - E_k) \\ &\quad -1 - E_k \cdot E_k \\ &= -D \cdot D. \end{aligned}$$

Now suppose that $-D \cdot D \geqslant \operatorname{codim} \Sigma = \Sigma(e_i - 1)$, $1 \leqslant i \leqslant r$, where $e_i = -E_i \cdot E_i$. Then it would follow that $-Z \cdot Z \geqslant \operatorname{codim} \Sigma$ since $D \cdot D \geqslant Z \cdot Z$ as it can be readily seen. But this is impossible because

$$-Z \cdot Z < \operatorname{codim} \Sigma. \tag{1.13.2}$$

To prove this inequality, we first show:

Z admits a good decomposition $Z = d_1 Z_1 + \dots + d_n Z_n$ such that

$$h^1\left(\mathcal{T}^1_{Z_i}\right) = 0 \quad \text{and} \quad \sum_{1 \leqslant i \leqslant n} h^1\left(\mathcal{O}_{Z_i}(Z_i)\right) < \operatorname{codim} \Sigma. \tag{1.13.3}$$

By the Main Lemma in [8], there exists a reduced, non-irreducible cycle $L \leqslant Z$ with $\chi(L) = 1$ which additionally satisfies following property: There is a positive integer k such that $k \cdot L \leqslant Z$ and $z_i = k$ for every irreducible component E_i satisfying $E_i \cdot L < 0$. We set $Z_1 := L$ and denote by $Z_2, \dots, Z_n$ the irreducible components E_i of E which do not appear in L or satisfy $E_i \cdot L = 0$ and $z_i > k$. Then we can write $Z = \Sigma d_i Z_i$, $1 \leqslant i \leqslant n$, with $d_i \geqslant 1$ and $d_1 = k$. We assert that this decomposition of Z is a good one. Since $h^1(\mathcal{T}^1_{Z_i}) = 0$ for $1 \leqslant i \leqslant n$, we have $\operatorname{codim} R_{Z_i} = h^1(\mathcal{O}_{Z_i}(Z_i))$, $1 \leqslant i \leqslant n$, see 1.5. By Riemann-Roch we get $h^1(\mathcal{O}_{Z_i}(Z_i)) = -1 - Z_i \cdot Z_i$ for $1 \leqslant i \leqslant n$. Since any irreducible component E_i with $E_i \cdot Z_1 = 0$ does not give a contribution to $Z_1 \cdot Z_1$, it is clear that

$$\sum_{1 \leqslant i \leqslant n} h^1\left(\mathcal{O}_{Z_i}(Z_i)\right) < \sum_{1 \leqslant i \leqslant r} (e_i - 1) = \operatorname{codim} \Sigma.$$

Now we can prove the inequality in (1.13.2). By 1.7 there exists an irreducible component of R_Z at 0, say R'_Z, such that $\operatorname{codim} R'_Z \leqslant \Sigma h^1(\mathcal{O}_{Z_i}(Z_i))$, $1 \leqslant i \leqslant n$. Let t be a generic point of R'_Z, and let Z_t be a generic weak lifting of Z. Recall that Z_t is the fundamental cycle on $\mathfrak{M}_t$, see the proof of Proposition 2.3 in [8]. By Theorem 3 in [8] Z_t is smoothable. Therefore Z_t is smooth for otherwise ω_t does not induce an equitopological deformation over the maximal reduced subgerm of (R, t) to which Z_t weakly lifts, compare our arguments in the proof of Lemma 1.11. So we can conclude as above that $\operatorname{codim} R'_Z = -1 - Z \cdot Z$. Gathering together, we get

$$-1 - Z \cdot Z \leqslant \Sigma(-1 - Z_i \cdot Z_i), \quad 1 \leqslant i \leqslant n.$$

It remains to show that this inequality is strict. First we observe that the family $\{Z_1, \dots, Z_n\}$ satisfies the hypothesis of Corollary 3.9 in [9]. Hence there exists a 1-convex deformation $\psi : \mathcal{N} \to B$ of M over a smooth curve B to which Z lifts. Furthermore, if $\lambda : \mathcal{Z} \to B$ denotes the induced deformation of Z, it turns out that $A_s := \operatorname{supp}(\mathcal{Z}_s)$, $s \in B$, is a connected component of the exceptional set of $\mathcal{N}_s$ and that $\mathcal{Z}_s$ is the fundamental cycle on A_s, compare [9]. It is clear that A_s blows down to a rational singularity. Let $A_s = A_{s,1} \cup \dots \cup A_{s,n_s}$ be the decomposition into irreducible components. Then, for $s \neq 0$, $n_s = n$ and $A_{s,i} \cdot A_{s,i} = Z_i \cdot Z_i$, $1 \leqslant i \leqslant n$. Applying (1.13.3) to $\mathcal{Z}_s$, $s \neq 0$, yields the strictness of the inequality above for $\mathcal{Z}_s \cdot \mathcal{Z}_s = Z \cdot Z$. This completes the proof of Theorem 1.

§2. The main result

2.1 Let $\varphi: \mathfrak{M} \to Q$ be a 1-convex map which represents a deformation of the minimal resolution germ (M, E) of a rational singularity (V, p). Consider the unique relative Stein-factorization

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\tau} & \mathscr{X} \\ \varphi \searrow & & \swarrow \tilde{\varphi} \\ & Q & \end{array}$$

i.e. $\mathscr{X}$ is a normal Stein space and τ is a proper, surjective holomorphic map such that $\tau_* \mathcal{O}_{\mathfrak{M}} = \mathcal{O}_{\mathscr{X}}$ and τ is biholomorphic on $\mathfrak{M} - \mathscr{E}$. Since (V, p) is rational, it is known, [15], that $\tilde{\varphi}$ is flat and $\tau \mid \mathfrak{M}_t : \mathfrak{M}_t \to \mathscr{X}_t$ is the Stein factorization of $\mathfrak{M}_t$, $t \in Q$. Hence $(\mathscr{X}_0, x)$, $x := \tau(E)$, is isomorphic to (V, p) and $\tilde{\varphi}: \mathscr{X} \to Q$ defines a deformation of (V, p). We say $\tilde{\varphi}$ arises by *simultaneously blowing down* of φ. Conversely, consider a deformation $\delta: \mathscr{Y} \to Q$ of the rational singularity (V, p) such that δ is isomorphic over Q to the relative Stein factorization of a deformation $\varphi: \mathfrak{M} \to Q$ of M. Then we call the diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\tau} & \mathscr{Y} \\ \varphi \searrow & & \swarrow \delta \\ & Q & \end{array}$$

a *simultaneous resolution of* δ.

2.2 One can generalize the construction above as follows. Let $A \subset E$ be an exceptional subset (not necessarily connected). Then φ induces a 1-convex deformation $f: \mathscr{N} \to Q$ of a strictly pseudoconvex neighborhood N of A. Blowing down of A yields a 1-convex normal space M^* which has singularities corresponding to the connected components of A. Let E^* be the exceptional set of M^*. Then it is a rather easy excercise to show that the relative Stein factorization with respect to f extends (after possibly shrinking of $\mathfrak{M}$) to a commutative diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\tau^*} & \mathfrak{M}^* \\ \varphi \searrow & & \swarrow \varphi^* \\ & Q & \end{array}$$

where τ^* is a proper holomorphic map and φ^* is a 1-convex deformation of M^* inducing the deformation $\tilde{f}$ of the singularities of M^*. We say φ^* arises by *partially blowing down of* φ *relative* A.

2.3 Let $\vartheta: \mathscr{V} \to S$ denote the semi-universal deformation of (V, p). Replace φ by the semi-universal deformation $\omega: \mathfrak{M} \to R$ of (M, E). Then ω blows down simultaneously to a deformation $\tilde{\omega}: \mathfrak{V} \to R$ of (V, p). Hence there exists a map-germ $\Phi: (R, 0) \to (S, 0)$ uniquely determined up to first order which yields the cartesian diagram

$$\begin{array}{ccccc} \mathfrak{M} & \xrightarrow{\tau} & \mathfrak{V} & \to & \mathscr{V} \\ & \searrow & \downarrow \tilde{\omega} & & \downarrow \vartheta \\ & \varphi & R & \xrightarrow[\Phi]{} & S \end{array}$$

By a result of Artin, [1], one knows that Φ is a finite map and that the image $S_a := \Phi(R)$ is an irreducible component of S, the so called Artin-component. Via the identification $T_0 R \xrightarrow{\approx} H^1(M; \Theta_M)$, the kernel of the tangent map $T_0\Phi: T_0 R \to T_0 S$ can be identified with the local cohomology group $H^1_E(M; \Theta_M)$. It turns out that

$$h^1_E(\Theta_M) = \#\{-2 \text{ curves on } M\}.$$

Hence Φ is a local embedding if M does not contain any -2 curve. For more details compare [16]. Following result is the analytic version of Proposition 6.2 in [19].

2.4 PROPOSITION: *Let $\Delta \subset S$ be the discriminant of the semi-universal deformation ϑ of a rational singularity (V, p), and let $\Delta_a := \Delta \cap S_a$ be the discriminant of the Artin-component. Then*

$$\Delta_a = \cup \Delta_D, \quad D \in \Lambda_+,$$

where $\Delta_D := \Phi(R_D)$ and Λ_+ is the set of positive cycles D on the minimal resolution M with $\chi(D) = 1$.

PROOF: Let $R^* := \cup R_D$, $D \in \Lambda_+$. Suppose there is a $t \in R - R^*$, t near 0, such that the exceptional set E_t of $\mathfrak{M}_t$ is non empty. Let C be an irreducible component of E_t. Then C must appear in some irreducible component $\mathscr{E}'$ of $\mathscr{E} \subset \mathfrak{M}$. Let $Q := \omega(\mathscr{E}')$. It follows from [11; Theorem 2.1] that $\mathscr{E}'$ defines a weak lifting of a cycle $Y \in \Lambda_+$ over Q. But this gives a contradiction. Hence the fibers of ω are Stein manifolds over $R - R^*$, and we are done.

2.5 The matrix $-(E_i \cdot E_j)$, $1 \leqslant i, j \leqslant r$, defines an inner product $\langle\ ,\ \rangle$ on $H := H_2(M; \mathbb{R})$. Consider the finite sets

$$\Lambda'_+ = \{D \in \Lambda_+ \mid D \cdot D = -2\} \quad \text{and} \quad \Lambda = \Lambda_+ \cup \Lambda_-$$

$$\text{with } \Lambda_- = \{-D \mid D \in \Lambda_+\}.$$

It is easy to see that each element of Λ'_+ is supported on an exceptional subset of E which blows down to a RDP. Associating to each element of Λ its fundamental class, we may consider Λ as a subset of H. Each element D of $\Lambda' := \Lambda'_+ \cup \Lambda'_-$ defines a reflection, say s_D, at the hyperplane H_D orthogonal to D which is given by

$$s_D(x) = x - \langle x, D \rangle \cdot D, \quad x \in H. \tag{2.5.1}$$

NOTATIONS:

(i) By W we denote the subgroup of $GL(H)$ which is generated by the reflections s_D, $D \in \Lambda'$. It is easy to check that s_D sends $\Lambda_+ - \{D\}$ into itself and D to $-D$.

(ii) Let A denote the union of all -2 curves on M, and let $A_1, \ldots, A_n$ be the connected components of A. We call A_i a RDP-configuration and A the maximal RDP-configuration on M.

2.6 LEMMA:

(i) If (V, p) is a RDP, then $\Lambda = \Lambda'$ and Λ is a root system of H with Λ_+ as the set of positive roots. The group W is the Weylgroup of this root system, and the associated Dynkindiagram is given by the weighted graph Γ associated to the minimal resolution M.

(ii) Suppose (V, p) is rational. By W_i, $1 \leqslant i \leqslant n$, denote the Weylgroups corresponding to the RDP-configurations A_i on M. Let H_i, $1 \leqslant i \leqslant n$, be the subspaces of H generated by the irreducible components of A_i. By restriction one obtains a faithful representation $W \to GL(\bigoplus_{1 \leqslant i \leqslant n} H_i)$. Furthermore the induced action is equivariantly isomorphic to the action of $\prod_{1 \leqslant i \leqslant n} W_i$ on $\bigoplus_{1 \leqslant i \leqslant n} H_i$.

(iii) $s(Y_1) \cdot s(Y_2) = Y_1 \cdot Y_2$ for $s \in W$ and $Y_1, Y_2 \in \Lambda_+$.

PROOF: The first part is an easy exercise, see also [19; Lemma 6.6]. The statements in (ii) follow immediately from (i) and (2.5.1). Therefore, to prove (iii), it suffices to check the identity for reflections s_D where D is carried on a -2 curve. But this can be easily done.

THEOREM 2: *Let $\pi : M \to V$ be the minimal resolution of a rational surface singularity (V, p).*

(i) There is an analytic W-action on $(R, 0)$ such that

$$s(R_D) = R_{s(D)} \quad \text{for } s \in W \text{ and } D \in \Lambda_+.$$

(*ii*) $\Phi : (R, 0) \to (S_a, 0)$ *is a Galois covering and its group of automorphism is* W. *Further*, S_a *is smooth.*

(*iii*) $\Delta_a = \cup \Delta_D$ *where* $D \in \Lambda_+$ *runs through a fundamental set of the* W*-action on* Λ.

(*iv*) Δ_D *is an irreducible component if* R_D *is irreducible.*

(*v*) $\dim \Delta'_D = \dim S_a + D \cdot D + 1$ *for each irreducible component* Δ'_D *of* Δ_D.

(*vi*) *Over a generic point of* Δ'_D, *the fiber of the semi-universal deformation* $\vartheta : \mathscr{V} \to S$ *has a cone singularity of degree* $d = -D \cdot D$ *as its only singularity.*

§3. Proof of Theorem 2

The last two statements of the theorem are easy corollaries of Theorem 1. The crucial point is to define an action of W on R and to show that it is the "right" action. Note that we cannot argue as in [18] since it is not known if there exists a semi-universal deformation of strictly pseudoconvex spaces with isolated singularities. We retain the notations of previous sections.

3.1 Let M_A denote a strictly pseudoconvex neighborhood of the maximal RDP-configuration A on M. Then M_A is a disjoint union of strictly pseudoconvex neighborhoods, say M_i, of the RDP-configurations A_i, $1 \leqslant i \leqslant n$. The semi-universal deformation $\omega : \mathfrak{M} \to R$ induces deformations of (M_A, A), respectively (M_i, A_i), which we denote by $\omega_A : \mathfrak{M}_A \to R$, and $\omega_i : \mathfrak{M}_i \to R$ respectively.

3.2 Proposition: *Let* $\omega : \mathfrak{M} \to R$ *be a sufficiently small representative of the semi-universal deformation of* (M, E). *Then there is a product decomposition* $R = R_0 \times R_1 \times \ldots \times R_n$ *which satisfies following properties:*

(*i*) $R_0 \times \{0\} = \cap R_{A_{ij}}$ *where the* A_{ij}*'s run through the set of irreducible components of* A_i, $1 \leqslant i \leqslant n$.

(*ii*) *The restriction of* ω *to* $R_1 \times R_2 \times \ldots \times R_n$ *is a representative of a semi-universal deformation of* (M_A, A), *say* $\eta : \mathscr{N} \to R_1 \times \ldots \times R_n$. *Further there is an isomorphism* $h : \mathfrak{M}_A \to R_0 \times \mathscr{N}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{M}_A & \xrightarrow{h} & R_0 \times \mathscr{N} \\ \omega_A \searrow & & \swarrow \mathrm{id} \times \eta \\ & R = R_0 \times \ldots R_n & \end{array}$$

(*iii*) *The restriction of* η *to* R_i, $1 \leqslant i \leqslant n$, *is a representative of a semi-universal deformation of* (M_i, A_i), *say* $\eta_i : \mathscr{N}_i \to R_i$. *Further there is* *a commutative diagram*

$$\begin{array}{ccc} \mathfrak{M}_i & \xrightarrow{h_i} & \mathrm{R}_0 \times \ldots \times R_{i-1} \times \mathcal{N}_i \times R_{i+1} \times \ldots \times R_n \\ & \omega_i \searrow \quad \swarrow \mathrm{id} \times \eta_i \times \mathrm{id} & \\ & \mathrm{R} = \mathrm{R}_0 \times \ldots \times R_n & \end{array}$$

where h_i is an isomorphism.

PROOF: The obstruction map γ_A for lifting all -2 curves sits in an exact sequence

$$0 \to H^1(\Theta_M(\log A)) \to H^1(\Theta_M) \overset{\gamma_A}{\to} \bigoplus_{i,j} H^1\big(\mathcal{O}_{A_{ij}}(A_{ij})\big) \to 0 \quad (3.2.1)$$

where $\gamma_A = \oplus \gamma_{ij}^*$, $1 \leqslant i \leqslant r_i$, $1 \leqslant i \leqslant n$, see 1.3. For each RDP-configuration A_i we can consider the exact sequence analogous to (3.2.1):

$$0 \to H^1\big(\Theta_{M_i}(\log A_i)\big) \to H^1\big(\Theta_{M_i}\big) \to \bigoplus_{1 \leqslant j \leqslant \tau_i} H^1\big(\mathcal{O}_{A_{ij}}(A_{ij})\big) \to 0 \quad (3.2.2)$$

Note that $H^1(\Theta_{M_i}(\log A_i)) = 0$ since rational double points are taut. Now recall Laufer's construction of the semi-universal deformation ω, [12]. Take a Stein cover $\mathcal{U} = \{U_s\}$, $1 \leqslant s \leqslant l$, of M such that $\overline{U}_r \cap \overline{U}_s \cap \overline{U}_t = \emptyset$ for $r \neq s \neq t$. Let $\{\theta_{qs}^{(1)}\}, \ldots, \{\theta_{qs}^{(m)}\}$ be a set of cocycles in $Z^1(\mathcal{U}; \Theta_M)$ which represent a basis of $H^1(\Theta_M)$. Take R to be a small polydic in $\mathbb{C}^m$. Then $\mathfrak{M}$ will be obtained by patching together the sets $U_s \times R$, $1 \leqslant s \leqslant l$. The transition functions are of type

$$(x, t) \mapsto \big(h_{qs}(x, t), t\big)$$

where $h_{qs}(x, t)$ is defined by integration along $t_1 \cdot \theta_{qs}^{(1)} + \ldots + t_m \cdot \theta_{qs}^{(m)}$ for time 1. Now the point is to choose the set of cocycles above in a suitable way. First we arrange it that the cover $\mathcal{U}$ satisfies following requirements:

(a) To each singular point y of E there exists a unique neighborhood $U_s \in \mathcal{U}$ with $y \in U_s$.

(b) $\mathcal{U}_i := \{U_{l_{i-1}+1}, \ldots, U_{l_i}\}$, $l_0 := 0$, is a cover of M_i, $1 \leqslant i \leqslant n$, and $\mathcal{U}^* := \{U_s \mid s > l_n$ is a cover of a strictly pseudoconvex neighborhood M^* of E^* where E^* contains precisely the irreducible components E_i of E with $E_i \cdot E_i \neq -2$.

Let $d_0 < d_1 < \ldots < d_n$ be an increasing sequence of integers such that

$$d_i - d_{i-1} = h^1\big(\Theta_{M_i}\big), \quad 1 \leqslant i \leqslant n.$$

Then we may choose cocycles $\{\theta_{qs}^{(j)}\}$, $d_{\iota-1} < j \leqslant d_\iota$, which represent a basis of $H^1(\Theta_{M_\iota})$ and vanish on $U_q \cap U_s$ if $U_q \cap U_s \not\subset M_\iota$. It follows from the exact sequences in (3.2.1) and (3.2.2) that the corresponding cohomology classes in $H^1(\Theta_M)$ are linearly independent and do not sit in the kernel of the obstruction map γ_A. Finally let $\{\theta_{qs}^{(1)}\}, \ldots, \{\theta_{qs}^{(d_0)}\}$ by cocycles which define a basis of $H^1(\Theta_M(\log A))$. Since $H^1(\Theta_{M_\iota}(\log A)) = 0$, we can arrange it that these cocycles vanish on $U_q \cap U_s$ if $(U_q \cap U_s) \cap M^* = \emptyset$. It is now clear how to complete the proof of Proposition 3.2.

3.3 Definition: Let $g:(R, 0) \to (R, 0)$ be an analytic automorphism. Then $g^*\tilde{\omega}: \mathfrak{V} \times_R R \to R$ is the relative Stein-factorization of the pull back $g^*\omega: \mathfrak{M} \times_R R \to R$ of ω via g. We call g a SR-*automorphism* (SR := simultaneous resolution) if $\tilde{\omega}: \mathfrak{V} \to R$ and $g^*\tilde{\omega}: \mathfrak{V} \times_R R \to R$ are representatives of isomorphic deformations of (V, p) over $(R, 0)$. By $\mathscr{G}$ we denote the group of SR-automorphisms.

3.4 Suppose g is a SR-automorphism of R. Then we have a cartesian diagram

$$\begin{array}{ccc} \mathfrak{V} \times_R R & \rightarrow & \mathscr{V}_a \\ {\scriptstyle g^*\bar{\omega}} \downarrow & & \downarrow {\scriptstyle \vartheta_a} \\ R & \underset{\Phi}{\rightarrow} & S_a \end{array}$$

In [1; Theorem 1], Artin has shown that the functor Res is representable, see also [7; Satz 9.16] for an analytic version which is weaker but sufficient for our purposes. From this it follows that the diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\quad g \quad} & R \\ & \Phi \searrow \quad \swarrow \Phi & \\ & S & \end{array}$$

So Φ factorizes via the quotient $R/\mathscr{G}$. Note, since Res is representable, a SR-automorphism is already uniquely determined by its first-order map.

3.5 Elementary transformations: Let C be a -2 curve on M. Then C lifts to a smooth hypersurface $R_C \subset R$, see Theorem 1, and the lifting $\lambda: \mathscr{C} \to R_C$ defines a trivial deformation of C. As in 3.2, ω induces a deformation $\varphi: \mathscr{X} \to R$ of a strictly pseudoconvex neighborhood X of C. Same arguments as in the proof of Proposition 3.2 show that there is a smooth 1-dimensional subgerm $(B, 0)$ of $(R, 0)$ such that $R = B \times R_C$ and that following holds: The restriction of φ to $B := B \times \{0\}$, say $\varphi_C: \mathscr{X}_C \to B$ represents the semi-universal deformation of (X, C). Further, we have an isomorphism

$$\begin{array}{ccc} \mathscr{X}_C \times R_C & \xrightarrow{f} & \mathscr{X} \\ \varphi_C \times \mathrm{id} \searrow & & \swarrow \varphi \\ & B \times R_C & \end{array}$$

over $R = B \times R_C$. Clearly, f induces a trivialization of $\lambda : \mathscr{C} \to R_C$. Since C does not lift to B, the normal bundle of C in $\mathscr{X}_C$ may be identifies with $\mathscr{O}_C(-1) \oplus \mathscr{O}_C(-1)$. Let $\sigma_C : \mathfrak{M}_C \to \mathscr{X}_C$ be the monoidal transformation of $\mathscr{X}_C$ with center at C. The inverse image of C is a rational ruled surface $\Sigma_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. The proper transform is its diagonal. By [5], see also [6], Σ_0 can be blown down to $\mathbb{P}^1 \cong C$ in two different ways. One gives nothing but $\varphi_C : \mathscr{X}_C \to B$. Let $\varphi_C^* : \mathscr{X}_C^* \to B$ be the deformation obtained from the other blowing down which again represents a semi-universal deformation of (X, C). Thus there is an automorphism $\tau : (B, 0) \to (B, 0)$ of order two inducing φ_C^* from φ_C. It is a straightforward excercise to check that τ is actually a SR-automorphism. Now consider the monoidal transformation $\sigma : \mathfrak{N} \to \mathfrak{M}$ of $\mathfrak{M}$ with center at $\mathscr{C}$. Using the trivialization f, it is obvious that the inverse image of $\mathscr{C}$, call it $\mathscr{C}'$, has a neighborhood U which is isomorphic to $\mathfrak{N}_C \times R_C$. Further, the corresponding isomorphism is compatible with the mappings $\omega \cdot \sigma \mid U$ and $(\varphi_C \times \mathrm{id}) * (\sigma_C \times \mathrm{id}) : \mathfrak{N}_C \times R_C \to B \times R_C$. thus we may identify $\mathscr{C}'$ with $\Sigma_0 \times R_C$, and it makes sense to say that we blow down $\mathscr{C}'$ in two different directions. By construction, this procedure yields a SR-automorphism $\tau_C : (R, 0) \to (R, 0)$ of order two which is necessarily given by $\tau_C = \tau \times \mathrm{id}$ and is called the *elementary transformation* of $(R, 0)$ defined by C.

3.6 COROLLARY: *We retain the notations of* 3.5. *Let* A_{ij} *be an irreducible component of* A_i, *and let* τ_{ij} *be the induced elementary transformation of* $(R, 0)$. *Then the restriction* $\tau_{ij} \mid R_i$ *defines an elementary transformation of* $(R_i, 0)$ *with respect to the semi-universal deformation* $\omega_i : \mathfrak{M}_i \to R_i$.

We omit the easy proof.

3.7 PROPOSITION: *Let* W^* *be the group of SR-automorphisms of* $(R, 0)$ *generated by the elementary transformations defined by all -2 curves on* M. *Then* W^* *is isomorphic to* W, *and the induced action of* W *on* $(R, 0)$ *is faithful and compatible with that one on* Λ, *i.e.*

$$s(R_D) = R_{s(D)} \quad \text{for } s \in W \text{ and } D \in \Lambda_+$$

Here we use the convention $R_D =: R_{-D}$ *for* $D \in \Lambda_+$.

PROOF: Let g be a SR-automorphism of $(R, 0)$. By 3.4 and Proposition 2.4, g induces an automorphism of $T := \cup R_D$, $D \in \Lambda_+$. Suppose $(T', 0)$ is an irreducible component of $(T, 0)$. Then Corollary 1.12 says that there exists a unique cycle $Y \in \Lambda_+$ such that $T' \subset R_Y$. Hence

$$g(R_D) = R_Y \quad \text{for a unique } Y \in \Lambda_+.$$

It follows from the definition of the elementary transformation τ_{ij} that the deformations $\omega : \mathfrak{M} \to R$ and $\tau_{ij}^* \omega : \mathfrak{M} \times_R R \to R$ are isomorphic over the complement $R - R_{A_{ij}}$. Applying the arguments given in [3; Remark 7.8], the corresponding isomorphism induces a reflection

$$H \to H \quad \text{given by } x \mapsto x - \langle x, A_{ij} \rangle \cdot A_{ij}. \tag{3.7.1}$$

We observe that (3.7.1) defines a representation

$$\lambda : W^* \to GL(H)$$

$$\text{with } w(R_D) = R_{\overline{D}},\ \overline{D} = \lambda(w)(D), \quad \text{for } w \in W^*. \tag{3.7.2}$$

Thus, because of Lemma 2.6, it still remains to show that

$$\lambda \text{ is a faithful representation.} \tag{3.7.3}$$

The point is to compare it with the representation $\rho : W^* \to GL_{\mathbf{C}}(H^1(\Theta_M))$ which is given by the linearization of the action of W^* on $(R, 0)$. By 3.4 and the fact that the functor Res is "representable", it follows that

$$\rho \text{ is a faithful representation.}$$

The obstruction map γ_{ij} to lifting the -2 curve A_{ij} yields a direct sum decomposition

$$H^1(\Theta_M) \cong H^1\big(\Theta_M(\log A_{ij})\big) \oplus H^1_{A_{ij}}(\Theta_M).$$

Recall that $\mathscr{L}_{A_{ij}}(\mathbf{C}_1) = H^1(\Theta_M(\log A_{ij}))$. So $\rho(\tau_{ij})$ is a reflection on $H^1(\Theta_M)$ which is -1 on the line $H^1_{A_{ij}}(\theta_M)$ and $+1$ on $H^1(\Theta_M(\log A_{ij}))$. On the other hand, it follows from [4; 1.10] and the proof of Proposition 3.2 that there is a direct sum decomposition

$$H^1(\Theta_M) \cong H^1(\Theta_M(\log A)) \oplus H^1_{A_1}(\Theta_M) \oplus \dots \oplus H^1_{A_n}(\Theta_M). \tag{3.7.4}$$

Further Proposition 3.2 and Corollary 3.6 imply that $\rho(\tau_{ij})$ is $+1$ on each of above direct sum components which is not equal to $H^1_{A_i}(\Theta_M)$ and that $\rho(\tau_{ij}) \mid H^1_{A_i}(\Theta_M)$ is a reflection which is -1 on $H^1_{A_{ij}}(\Theta_M) \subset H^1_{A_i}(\Theta_M)$

$\cong H^1(\Theta_{M_i})$ and $+1$ on $H^1(\Theta_{M_i}(\log A_{ij}))$. The latter may be identified with $H^1(\Theta_M(\log A_{ij})) \cap H^1_{A_i}(\Theta_M)$. Hence the direct sum components of $H^1(\Theta_M)$ in (3.7.4) are $\rho(W^*)$ invariant subspaces. Let ρ_i, $0 \leqslant i \leqslant n$, denote the restriction of ρ to $H^1(\Theta_M(\log A))$, $i=0$, respectively to $H^1_{A_i}(\Theta_M)$. Then we have $\rho = \rho_0 \oplus \rho_1 \oplus \ldots \oplus \rho_n$. Let W_i^*, $1 \leqslant i \leqslant n$, be the subgroup of W generated by the elementary transformations τ_{ij}, $1 \leqslant j \leqslant n_i$. Then $\rho_i(W_k^*) = 1$ for $k \neq i$, and $\rho_i : W_i^* \to GL_{\mathbb{C}}(H^1_{A_i}(\Theta_M))$, $1 \leqslant i \leqslant n$, is the linearization of the action of W_i^* on $(R_i, 0)$. Hence there is a natural isomorphism

$$W^* \cong \prod_{1 \leqslant i \leqslant n} W_i^* \tag{3.7.5}$$

Thus, to prove (3.7.3), it suffices to show that $\lambda : W_i^* \to GL(H)$ is a faithful representation. Since W_i^* acts on $(R_i, 0)$, this is true if we knew that the induced representation $\lambda_i : W_i^* \to GL(H_i)$ is faithful.

Now we recall the commutative diagram, see [19; (6.21.1)]:

$$\begin{array}{ccccc} H^1(\Theta_{M_i}) & \xrightarrow{\cong} & H^1(\Omega_{M_i}) & \xrightarrow{\cong} & H^2(M_i; \mathbb{C}) \\ \gamma_{A_i} \downarrow & & \downarrow \cong & & \downarrow \cong \\ \bigoplus_j H^1(\mathcal{O}_{A_{ij}}(A_{ij})) & \xrightarrow{\cong} & \bigoplus_j H^1(\Omega_{A_{ij}}) & \xrightarrow{\cong} & \oplus H^2(A_{ij}; \mathbb{C}) \end{array}$$

where γ_{A_i} is the obstruction map to lifting all irreducible components of A_i. Let β_i denote the isomorphism $H^1(\Theta_{M_i}) \to H^2(M_i; \mathbb{C})$, and let $\lambda_i^{\mathbb{C}}$ denote the complexification of λ_i. Then straightforward computations show that β_i induces an equivalence between the representations ρ_i and $\lambda_i^{\mathbb{C}}$. Since ρ_i is faithful, we are done.

3.8 PROPOSITION: *The blowing down map $\Phi : R \to S$ factorizes via the quotient R/W and the induced map $\Phi_W : R/W \to S$ is a local embedding at 0. Further R/W is smooth and $\mathcal{G} \cong W$.*

PROOF: Let N be the normal strictly pseudoconvex space which one obtains from M by blowing down the maximal RDP-configuration A. Since $H^1(\Theta_N) < \infty$, it follows that there exists a formal deformation $\bar{\xi} : \bar{\mathcal{N}} \to \bar{T}$ of N which is semi-universal for the functor Def$((N, Y), -)$ on the category of artin $\mathbb{C}$-algebras. Here Y is the exceptional set of N. General obstruction theory shows that $\bar{T}$ is smooth. In a natural way, the deformations $\omega : \mathfrak{M} \to R$ and $\vartheta : \mathcal{V} \to S$ define formal deformations, say $\bar{\omega} : \bar{\mathfrak{M}} \to \bar{R}$ and $\bar{\vartheta} : \bar{\mathcal{V}} \to \bar{S}$, which are hulls for the corresponding deformation functors on the category of artin $\mathbb{C}$-algebras. Let $\bar{\Phi} : \bar{R} \to \bar{S}$ be the induced formal blowing down morphism. Then $\overline{\Phi}$ factorizes via $\overline{T}$, i.e. there exists a commutative diagram

$$\begin{array}{ccc} \overline{R} & \xrightarrow{\overline{\Phi}} & \overline{S} \\ & \overline{\eta}\searrow \quad \swarrow \overline{\epsilon} & \\ & \overline{T} & \end{array}$$

such that $\overline{\eta}^*\overline{\xi}: \mathcal{N} \times_{\overline{R}} \overline{R} \to \overline{R}$ is isomorphic to the formal deformation of N which one obtains by partially blowing down of ω relative A, see 2.2, and such that there is a cartesian diagram, see [18]:

$$\begin{array}{ccc} \overline{\mathcal{N}} & \to & \overline{\mathcal{V}}_a \\ \overline{\xi}\downarrow & & \downarrow \overline{\vartheta}_a \\ \overline{T} & \xrightarrow[\overline{\epsilon}]{} & \overline{S}_a \end{array}$$

Now Lipman's result in [14] implies that $\overline{\epsilon}$ is an isomorphism. Further it can be easily checked that τ_{ij} acts on $\overline{R}$ such that $\overline{\eta} \circ \tau_{ij} = \overline{\eta}$. Hence $\overline{\eta}$ factorizes via the quotient $\overline{R}/W$. Since the action of $\overline{W}$ is faithful, it follows from [18; Thm. 1.3] and Prop. 3.7 that $\overline{T}$ and $\overline{R}/W$ may be identified. Putting altogether it follows that R/W is smooth and that Φ_W necessarily defines an isomorphism between $(R/W, 0)$ and $(S_a, 0)$.

To finish the proof of Theorem 2, it still remains to check statement (iii). But this is an immediate consequence of Corollary 1.12.

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