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STEVO TODORČEVIĆ

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## REMARKS ON CHAIN CONDITIONS IN PRODUCTS

Stevo Todorčević

### Introduction

We say that a partially ordered set  $\mathfrak{D}$  satisfies the  $\kappa$ -chain condition if every set of pairwise incompatible elements of  $\mathfrak{D}$  has size  $< \kappa$ . A topological space  $X$  is said to satisfy the  $\kappa$ -c.c. if every family of pairwise disjoint open subsets of  $X$  has size  $< \kappa$ . In this note we give several remarks on the well-known problem which asks for which cardinals  $\kappa$  the  $\kappa$ -chain condition is a productive property (see [6; §3]). This problem for the case  $\kappa = \aleph_1$  was first asked by E. Marczewski ([14]), and later by D. Kurepa ([11], [12]) in a more general form. Kurepa [11] showed that the countable chain condition (i.e., the  $\aleph_1$ -c.c.) of a Suslin continuum is not a productive property. He also showed ([13]) that any product of  $\kappa^+$ -c.c. spaces satisfies the  $(2^\kappa)^+$ -c.c. The first examples of non-productive  $\kappa^+$ -c.c. posets, assuming  $2^\kappa = \kappa^+$ , were constructed by F. Galvin and R. Laver ([8]). In [7], W. Fleissner showed that it is consistent with ZFC that  $2^{\aleph_0}$  is large and there exists a c.c.c. space  $X$  such that  $X^2$  does not satisfy the  $2^{\aleph_0}$ -c.c. In this note we show without additional set-theoretical assumptions that for class-many cardinals  $\kappa$  the  $\kappa$ -chain condition is not a productive property. In particular, we show that the  $\text{cf}2^{\aleph_0}$ -c.c. is not productive. This result was announced in [20] where the same result for the  $2^{\aleph_0}$ -c.c. was incorrectly claimed. Our observation uses “entangled” linear orders, and gives a quite general method for constructing non-productive  $\kappa$ -c.c. posets.

### §1. Sierpiński’s construction

In this section we generalize a classical construction of rigid linear suborderings of  $\mathbb{R}$  given by W. Sierpiński [19] from the case  $n = 1$  to the case of any finite  $n$ . Using a simple diagonalization argument Sierpiński ([19]) constructed a one-to-one sequence  $E = \{r_\alpha : \alpha < 2^{\aleph_0}\}$  of real numbers with the following property:

- (1) For any continuous function  $f$  from a  $G_\delta$  subset of  $\mathbb{R}$  into  $\mathbb{R}$  there is an  $\alpha < 2^{\aleph_0}$  such that

$$\forall \beta \geq \alpha \quad f''\{r_\gamma : \gamma < \beta\} \cap E \subseteq \{r_\gamma : \gamma < \beta\}.$$

Using the well-known Lavrentiev Extension Theorem ([10; §35]) he showed that (1) implies the following property of the set  $E$ :

(2) If  $f$  is a homeomorphism between two sets of reals, then

$$\{\alpha < 2^{\aleph_0} : r_\alpha \in \text{dom}(f) \& f(r_\alpha) \neq r_\alpha\} \text{ is not cofinal in } 2^{\aleph_0}.$$

Sierpiński's result was motivated by a problem of M. Fréchet about the number of non-homeomorphic topological spaces. We refer the reader to §§35 and 40 of [10] for further information and generalizations of this result. In this note we shall be interested in the following property of the set  $E$  which easily follows from (2) and which has been quite often used in the theory of uncountable order types (see [4]):

(3) If  $f$  is an one-to-one monotonic function from a set of reals into the reals, then  $\{\alpha < 2^{\aleph_0} : r_\alpha \in \text{dom}(f) \& f(r_\alpha) \neq r_\alpha\}$  is not cofinal in  $2^{\aleph_0}$ .

Since a function may be regarded as a set of ordered pairs, the statement (3) suggests a stronger statement where the pairs are replaced by the  $n$ -tuples from  $E$ . It turns out that in order to get this stronger property one has only replace (1) by a stronger statement which includes any continuous function  $f$  from a  $G_\delta$  subset of  $\mathbb{R}^n$  into  $\mathbb{R}$  for any  $n < \omega$ . Since later we intend to give some applications of this result, let us prove it in the following more general form. But first we need some definitions. Let  $L$  be an infinite linearly ordered set and let  $\{r_\alpha : \alpha < \theta\}$  be a one-to-one enumeration of  $L$ . Then for any  $n < \omega$  and  $x \in L^n$  by  $D(x)$  we denote the set  $\{\alpha < \theta : \exists i < n \ x_i = r_\alpha\}$ . We say that  $A \subseteq L^n$  is cofinal (in  $\theta$ ) iff  $\forall \alpha < \theta \ \exists x \in A \ \alpha < D(x)$ .

**THEOREM 1:** *Assume  $L$  is a linear ordering of size  $2^\lambda$  with a dense subset  $D$  of size  $\leq \lambda$ . Then there is a one-to-one sequence  $E = \{r_\alpha : \alpha < 2^\lambda\}$  of elements of  $L$  such that:*

(4) *For every  $n < \omega$ , for any cofinal set  $A \subseteq E^n$  consisting of one-to-one  $n$ -tuples, and for every  $s \in {}^n 2$ , there exist  $x, y \in A$  such that*

$$\forall i < n \ (x_i < y_i \leftrightarrow s_i = 0).$$

**PROOF:** Let us assume that  $L$  is a dense linear ordering and let  $\mathbb{K}$  be the Dedekind completion of  $L$ . The following fact is an easy generalization of a similar fact for  $\mathbb{R}$  ([10; § 35]).

**LEMMA 1:** *Let  $f$  be a continuous function from a subset of  $\mathbb{K}^n$  into  $\mathbb{K}$  where  $n < \omega$ . Then  $f$  can be continuously extended to a  $G_\lambda$  subset of  $\mathbb{K}^n$ .*

**PROOF:** Let  $A = \text{dom}(f)$ . For  $p \in \mathbb{K}^n$  we define

$$\omega(p) = \bigcap \{ \overline{f''(A \cap I)} : I \text{ is open in } \mathbb{K}^n \& p \in I \}.$$

Thus for each  $p \in \mathbb{K}^n$ ,  $\omega(p)$  is a (possibly empty) compact subset of  $\mathbb{K}^n$ . Let  $A^* = \{p \in \mathbb{K}^n : |\omega(p)| = 1\}$ . Then  $A \subseteq A^*$  and  $f$  extends continuously on  $A^*$ . Using the fact that  $\mathbb{K}$  has a dense subset of size  $\leq \lambda$  one easily shows that  $A^*$  is a  $G_\lambda$  subset of  $\mathbb{K}^n$ .

Since there exist only  $2^\lambda$  continuous functions from  $G_\lambda$  subsets of  $\mathbb{K}^n$  into  $\mathbb{K}$ , ( $n < \omega$ ), we can easily choose a one-to-one sequence  $E = \{r_\alpha : \alpha < 2^\lambda\} \subseteq L$  with the following property:

- (5) For any continuous function  $f$  from a  $G_\lambda$  subset of  $\mathbb{K}^n$  into  $\mathbb{K}$ , ( $n < \omega$ ) there is an  $\alpha < 2^\lambda$  such that

$$\forall \beta \geq \alpha \ f''\{r_\gamma : \gamma < \beta\}^n \cap E \subseteq \{r_\gamma : \gamma < \beta\}.$$

We shall show that this  $E$  satisfies (4). Assume not, and let  $m$  be the minimal  $n < \omega$  for which (4) fails. Clearly  $m > 1$ . Let  $A \subseteq E^m$  be a cofinal set of disjoint one-to-one  $m$ -tuples and let  $s \in {}^m 2$  be such that no  $x, y \in A$  satisfy  $\forall i < m (x_i < y_i \Leftrightarrow s_i = 0)$ . Using a permutation of coordinates of elements of  $A$  and of  $s$  we may assume that

$$(6) \ \forall x \in A \ (x = \langle r_{\alpha_0}, \dots, r_{\alpha_{m-1}} \rangle \Rightarrow \alpha_0 < \dots < \alpha_{m-1})$$

Let  $B = \{x \upharpoonright (m-1) : x \in A\} \subseteq E^{m-1}$ . Define  $f : B \rightarrow \mathbb{K}$  by  $f(z) = r$  iff  $\exists x \in A \ x = z \hat{\ } r$ . By (5), (6) and Lemma 1 it follows directly that  $f$  cannot be continuous on a cofinal subset of  $B$ . So the following lemma gives a contradiction. For  $p \in K^{m-1}$ ,  $\omega(p)$  is defined for our  $f$  and  $B$  as in the proof of Lemma 1.

LEMMA 2: *The set  $B_0 = \{z \in B : |\omega(z)| \geq 2\}$  is not cofinal in  $2^\lambda$ .*

PROOF: Otherwise, by going to a cofinal subset of  $B_0$  and by symmetry, we may assume that there is a  $d \in D$  such that

$$(7) \ \forall z \in B_0 \ \exists r \in \omega(z) \ f(z) < d < r.$$

Assume for definiteness that  $s_{m-1} = 0$ ; the case  $s_{m-1} = 1$  is considered similarly. By the minimality of  $m$  we can find  $u, v \in B_0$  so that  $\forall i < m-1 (u_i < v_i \Leftrightarrow s_i = 0)$ . Let

$$I = \{z \in \mathbb{K}^{m-1} : \forall i < m-1 (u_i < z_i \Leftrightarrow s_i = 0)\}.$$

Then  $I$  is an open subset of  $\mathbb{K}^{m-1}$  which contains  $v$ . Note that by the choice of  $A$  and  $s$  it follows that  $\forall z \in I \cap B, f(z) < f(u) < d$ . Hence  $\omega(v) \subseteq \overline{f''B \cap I} < d$ . But this contradicts (7) and finishes the proof.

### §2. Entangled linear orderings

This section begins with a slight generalization of a notion of Avraham, Rubin and Shelah ([1], [2]) and ends with some applications of the result of §1 which were mentioned in the introduction.

Let  $L$  be an infinite linear ordering, let  $\kappa$  be an infinite cardinal and let  $n < \omega$ . Then by  $(L)^n$  we denote the set of all  $\langle x_0, \dots, x_{n-1} \rangle \in L^n$  such that  $x_0 < \dots < x_{n-1}$ . We say that  $L$  is  $(\kappa, n)$ -entangled iff for every  $A \subseteq (L)^n$  of size  $\kappa$  and for every  $s \in {}^n 2$  there exist  $x, y \in A$  such that  $\forall i < n (x_i < y_i \Leftrightarrow s_i = 0)$ .  $L$  is  $\kappa$ -entangled iff  $L$  is  $(\kappa, n)$ -entangled for every  $n < \omega$ .  $L$  is an entangled linear ordering iff  $L$  is  $\kappa$ -entangled for  $\kappa = \aleph_1$ .

Note that every linear order is  $(\kappa, 0)$ - and  $(\kappa, 1)$ -entangled. Note also that  $L$  is  $(\kappa, 2)$ -entangled iff for every one-to-one monotonic function from a subset of  $L$  into  $L$ , we have that  $|\{r \in \text{dom}(f) : f(r) \neq r\}| < \kappa$ . Clearly if  $L$  is  $(\kappa, 2)$ -entangled then every family of disjoint non-trivial intervals of  $L$  is of size  $< \kappa$ . If  $L$  is  $(\kappa, 3)$ -entangled then  $L$  moreover has a dense subset of size  $< \kappa$ . Thus every  $(\aleph_1, 3)$ -entangled linear ordering is isomorphic with a set of reals. The next easy folklore result shows that uncountable  $\aleph_1$ -entangled sets of reals can exist in certain models of set theory. On the other hand, in the model of Baumgartner [3] there is no uncountable  $(\aleph_1, 2)$ -entangled linear ordering.

**THEOREM 2:** *If  $E$  is any set of Cohen or random reals, then  $E$  is  $\aleph_1$ -entangled.*

To state the result of §1 using the new terminology let  $\text{ded}(\lambda, \theta)$  denote the fact that there is a linear ordering of size  $\theta$  with a dense subset of size  $\leq \lambda$ . Sierpiński [19] showed that  $\text{ded}(\lambda, \lambda^+)$  always holds and that  $\forall \theta < \lambda \ 2^\theta \leq \lambda$  implies  $\text{ded}(\lambda, 2^\lambda)$ . Thus  $\text{ded}(\lambda, 2^\lambda)$  holds if, for example,  $2^\lambda = \lambda^+$  or if  $\lambda$  is a strong limit cardinal. In [16], W. Mitchell constructed a model of ZFC in which  $\text{ded}(\aleph_1, 2^{\aleph_1})$  fails. Theorem 1 can also be written in the following form.

**THEOREM 3:** *Assume  $\text{ded}(\lambda, 2^\lambda)$ . Then for every  $\kappa \leq 2^\lambda$  with  $\text{cf } \kappa = \text{cf } 2^\lambda$ , there is an  $\kappa$ -entangled linear ordering of size  $\kappa$  which moreover has a dense subset of size  $\leq \lambda$ .*

**COROLLARY 4:** *For every  $\kappa \leq 2^{\aleph_0}$  with  $\text{cf } \kappa = \text{cf } 2^{\aleph_0}$  there is an  $\kappa$ -entangled set of reals of size  $\kappa$ .*

**COROLLARY 5:** *(Sierpiński) For every  $\kappa \leq 2^{\aleph_0}$  with  $\text{cf } \kappa = \text{cf } 2^{\aleph_0}$  there is an  $(\kappa, 2)$ -entangled set of reals of size  $\kappa$ .*

The next proposition contains our basic observation which connects entangled linear orders and chain conditions. We shall state later a slightly stronger result of this kind.

**THEOREM 6:** *Let  $\kappa$  be a regular infinite cardinal and let  $L$  be a  $\kappa$ -entangled linear order of size  $\lambda \geq \kappa$ . Then there exist partially ordered sets  $\mathfrak{D}_0$  and  $\mathfrak{D}_1$  such that  $\mathfrak{D}_0$  and  $\mathfrak{D}_1$  satisfy the  $\kappa$ -c.c. but  $\mathfrak{D}_0 \times \mathfrak{D}_1$  does not satisfy the  $\lambda$ -c.c.*

**PROOF:** Let  $\{r_\alpha : \alpha < \lambda\}$  be a one-to-one enumeration of  $L$  and let  $E = \{\langle r_\alpha, r_{\alpha+1} \rangle : \alpha \text{ even} < \lambda\}$ . We consider  $E$  as a subset of  $L \times L$  with the product ordering. Let

$$\mathfrak{D}_0(E) = \{ p \in [E]^{<\aleph_0} : p \text{ is a chain of } E \},$$

$$\mathfrak{D}_1(E) = \{ p \in [E]^{<\aleph_0} : p \text{ is an antichain of } E \},$$

considered as posets under the ordering  $\supseteq$ . Let  $A$  be a subset of  $\mathfrak{D}_0(E)$  or  $\mathfrak{D}_1(E)$  of size  $\kappa$ . By the standard  $\Delta$ -system argument we may assume that the elements of  $A$  are disjoint and of the same cardinality  $n < \omega$ . Since  $L$  is  $\kappa$ -entangled,  $L$  contains a dense subset  $D$  of size  $< \kappa$ . Every member of  $A$  can be separated by a set of  $2n$  intervals with end-points in  $D$ , and by regularity of  $\kappa$  we may assume that the separating set is the same for each  $p \in A$ . Now a simple application of the  $(\kappa, 2n)$ -entangledness of  $L$  gives two compatible members of  $A$ . The product  $\mathfrak{D}_0(E) \times \mathfrak{D}_1(E)$  is not  $\lambda$ -c.c. since  $\{\langle \{e\}, \{e\} \rangle : e \in E\}$  is a pairwise incompatible subset of  $\mathfrak{D}_0(E) \times \mathfrak{D}_1(E)$ . This completes the proof.

Note the following direct way of getting topological spaces which satisfy the conclusion of Theorem 6. Let  $X_0$  be the set of all chains of  $E$  and let  $X_1$  be the set of all antichains of  $E$ . We consider  $X_0$  and  $X_1$  as subspaces of  $\{0, 1\}^E$  under the standard identification via characteristic functions. Then  $X_0$  and  $X_1$  are compact Hausdorff  $\kappa$ -c.c. spaces such that  $X_0 \times X_1$  is not  $\lambda$ -c.c. It is clear that the proof of Theorem 6 also shows that each (finite) power of  $\mathfrak{D}_0(E)$  and of  $\mathfrak{D}_1(E)$  is again a  $\kappa$ -c.c. poset.

By Theorems 3 and 6 it follows directly that  $\text{ded}(\lambda, 2^\lambda)$  implies the existence of two cf  $2^\lambda$ -c.c. posets  $\mathfrak{D}_0$  and  $\mathfrak{D}_1$  such that  $\mathfrak{D}_0 \times \mathfrak{D}_1$  is not cf  $2^\lambda$ -c.c. Thus in particular, the cf  $2^{\aleph_0}$ -c.c. is not productive. We shall later state a more general result of this kind, and in order to do this we need some definitions of Galvin [8].

Let  $\lambda \geq \kappa \geq \aleph_0$  be cardinals and let  $H \subseteq [\lambda]^2$ . Then we let  $\mathfrak{D}(\lambda, H) = \{ F \in [\lambda]^{<\aleph_0} : [F]^2 \subseteq H \}$ ;  $\mathfrak{D}(\lambda, H)$  is partially ordered by  $\supseteq$ . A set  $K \subseteq [\lambda]^2$  is called  $\kappa$ -big if, given any  $n < \omega$  and any  $H_1, \dots, H_n \subseteq [\lambda]^2$ , if  $K \subseteq H_1 \cap \dots \cap H_n$ , and if each of  $H_i$  is the union of less than  $\kappa$

rectangles, then  $\mathfrak{D}(\lambda, H_1) \times \dots \times \mathfrak{D}(\lambda, H_n)$  satisfies the  $\kappa$ -c.c. A set  $K \subseteq [\lambda]^2$  is big iff  $K$  is  $\lambda$ -big. Galvin [8] showed that  $2^\kappa = \kappa^+$  implies the existence of  $\kappa^+$  disjoint big subsets of  $[\kappa^+]^2$  and that MA implies the existence of  $\aleph_0$  disjoint big subsets of  $[2^{\aleph_0}]^2$ . The next proposition shows that entangled linear orderings can also be used in constructing disjoint big subsets of  $[\lambda]^2$ .

**THEOREM 7:** *Let  $\kappa$  be regular and infinite and let  $L$  be a  $\kappa$ -entangled linear order of size  $\lambda \geq \kappa$ . Then there exist  $\aleph_0$  pairwise disjoint  $\kappa$ -big subsets of  $[\lambda]^2$ .*

**PROOF:** Let  $\{r_\alpha : \alpha < \lambda\}$  be a one-to-one enumeration of  $L$ , and let  $\Lambda$  denote the set of all limit ordinals  $< \lambda$ . For  $\alpha, \beta \in \Lambda$  and  $n < \omega$ , we define

$$\{\alpha, \beta\} \in K_n \text{ iff } r_{\alpha+n+1} > r_{\beta+n+1} \ \& \ \forall i \leq n \ r_{\alpha+i} < r_{\beta+i}.$$

It is clear that the proof of Theorem 6 also shows that  $K_n$ , ( $n < \omega$ ) is a family of  $\aleph_0$  disjoint  $\kappa$ -big subsets of  $[\Lambda]^2$ . This finishes the proof.

**COROLLARY 8:** *Assume  $\text{ded}(\lambda, 2^\lambda)$ . Then there exist  $\aleph_0$  pairwise disjoint big subsets of  $[cf\ 2^\lambda]^2$ .*

Note that the proof of Theorem 7 also gives the following proposition where  $\kappa$  is not necessarily a regular cardinal.

**THEOREM 9:** *If there exists a  $\kappa$ -entangled linear ordering of size  $\lambda \geq \kappa$ , then  $\lambda \not\rightarrow [\kappa]_{\aleph_0}^2$ .*

Note that Corollary 4 and Theorem 9 have as an immediate consequence  $\kappa \not\rightarrow [\kappa]_{\aleph_0}^2$  for all  $\kappa \leq 2^{\aleph_0}$  with  $\text{cf } \kappa = \text{cf } 2^{\aleph_0}$ , which is a well-known result of S. Shelah ([9]).

Galvin [8] used families of disjoint big subsets of  $[\lambda]^2$  in constructing several very strong counter-examples to productiveness of the  $\kappa$ -chain condition. Using Theorem 7 and Galvin's ideas we can also get some of his general results. However, here we mention only one instance of such a general result, but let us note that it is possible to prove an analogue of Theorem 4.7 from Galvin [8].

**THEOREM 10:** *Assume  $\text{ded}(\lambda, 2^\lambda)$ . Then for every  $n < \omega$  there is a partially ordered set  $\mathfrak{D}$  such that  $\mathfrak{D}^n$  satisfies the cf  $2^\lambda$ -c.c. but  $\mathfrak{D}^{n+1}$  does not.*

**PROOF:** Let  $L$  be a cf  $2^\lambda$ -entangled linear ordering which exists by Theorem 3. Let  $\{r_\alpha : \alpha < \text{cf } 2^\lambda\}$  be an one-to-one enumeration of  $L$  and let  $\Lambda$  be the set of all limit ordinals  $< \text{cf } 2^\lambda$ . Let  $K_0, \dots, K_n$  be the disjoint

big subsets of  $[\Lambda]^2$  defined in the proof of Theorem 7. Clearly, we may assume that  $n \geq 1$ . For  $i \leq n$ , let  $H_i = \cup\{K_j : j \leq n \& j \neq i\}$ . Let  $\mathfrak{D}_i = \mathfrak{D}(\lambda, H_i)$  for  $i \leq n$ . Since  $K_i$ 's are big sets any product of  $\leq n$  of the posets  $\mathfrak{D}_0, \dots, \mathfrak{D}_n$  satisfies the cf  $2^\lambda$ -c.c., but clearly  $\mathfrak{D}_0 \times \dots \times \mathfrak{D}_n$  is not a cf  $2^\lambda$ -c.c. poset. Hence  $\mathfrak{D} = \mathfrak{D}_0 \oplus \dots \oplus \mathfrak{D}_n$  satisfies the conclusion of the Theorem.

**COROLLARY 11:** *For every  $n < \omega$  there is a partially ordered set  $\mathfrak{D}$  such that  $\mathfrak{D}^n$  satisfies the cf  $2^{\aleph_0}$ -c.c. but  $\mathfrak{D}^{n+1}$  does not.*

**THEOREM 12:** *Let  $\mathfrak{B}_\kappa$  be the standard poset for adding  $\kappa > \aleph_0$  Cohen or random reals. Then  $\mathfrak{B}_\kappa$  forces that for every  $n < \omega$  there is a poset  $\mathfrak{D}$  such that  $\mathfrak{D}^n$  satisfies the c.c.c. but  $\mathfrak{D}^{n+1}$  does not satisfy the  $\kappa$ -c.c.*

**PROOF:** By Theorem 2 and the proof of Theorem 10.

Theorem 12 for the case of Cohen reals was first proved by Fleissner [7]. This theorem also holds for  $\kappa = 1$  (or  $\kappa = \aleph_0$ ) replacing the clause “ $\mathfrak{D}^{n+1}$  does not satisfy the  $\kappa$ -c.c.” by “ $\mathfrak{D}^{n+1}$  does not satisfy the c.c.c.”. This has been proved by J. Roitman [17].

The first construction of an uncountable entangled set of reals under the assumption of CH was (implicitly) given by E. Michael [15]. This fact was first pointed out by E.S. Berney (unpublished) who used it in a construction of an uncountable Boolean algebra with no uncountable antichain, a result which has been also independently proved by R. Bonnet [5]. Michael’s construction is a very nice generalization of the classical construction of concentrated sets of reals from the case  $n = 1$  to the case of any finite  $n$ . His argument uses the Baire category theorem and can also be done with MA instead of CH.

In [3], J. Baumgartner gave, assuming CH, a first generalization of the classical construction ([10; §35]) of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  which is not monotonic on any uncountable subset of  $\mathbb{R}$  from the case  $n = 1$  to the case of any finite  $n$ . Baumgartner’s construction is more flexible than Michael’s, and our §1 owes much to Baumgartner’s construction.

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Department of Mathematics  
University of California  
Berkeley, CA 94720  
USA