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STICKELBERGER ELEMENTS IN FUNCTION FIELDS

David R. Hayes *

Dedicated to the memory of my friend and colleague, George Whaples

§1. Introduction

The group of roots of unity $\mu(F)$ in a global field F is finite. Throughout the paper, W_F denotes the order of $\mu(F)$; and M_F denotes the set of places of F. When F is a function field, $\boldsymbol{D}_F$ denotes the group of divisors of F (in additive notation); and for $x \in F^\times$, $\delta_F(x) \in \boldsymbol{D}_F$ denotes the divisor of x. Except for an example presented briefly in §3 below, we will work entirely over a fixed global function field k as a base. Let $\mathbb{F}_q$ be the exact field of constants of k. If $\mathfrak{d}$ is a divisor of some finite extension field of k, then by "deg $\mathfrak{d}$" we always understand the degree of $\mathfrak{d}$ over $\mathbb{F}_q$. For $\mathfrak{d} = \mathfrak{p}$, a prime divisor, $N_\mathfrak{p} = q^{\deg \mathfrak{p}}$ is the order of the residue class field at $\mathfrak{p}$.

Let K/k be a finite abelian extension with Galois group G_K. Assume given a *non-empty* finite subset T of M_k which contains at least all those places which ramify in K/k. For $\mathfrak{p} \notin T$, the Frobenius automorphism $\sigma_\mathfrak{p} \in G_K$ is well-defined. Let $\hat{G}_K$ be the group of complex valued characters of G_K. For a given $\psi \in \hat{G}_K$, we define the *incomplete L-function of ψ relative to T* as follows:

$$L_T(s, \psi) = \prod_{\mathfrak{p}} \left(1 - \psi(\sigma_\mathfrak{p}) N_\mathfrak{p}^{-s}\right)^{-1} \quad (\mathfrak{p} \in M_k \setminus T) \tag{1.1}$$

where s is a complex value with $\mathrm{Re}(s) > 1$. The Riemann-Roch theorem implies that $L_T(s, \psi)$ is a rational function of q^{-s} which takes a finite value at $s = 0$ (cf. §6 below).

Let "ψ" also denote the linear extension of ψ to the complex group algebra $\mathbb{C}[G_K]$. By character theory, there is a unique element $\theta_{T,K} \in$

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$\mathbb{C}[G_K]$ such that

$$\psi(\theta_{T,K}) = L_T(0, \bar{\psi}) \tag{1.2}$$

for all $\psi \in \hat{G}_K$. We call $\theta_{T,K}$ the *T-incomplete L-function evaluator at* $s = 0$. The reader should note that the definition of $\theta_{T,K}$ is "twisted" by the introduction of the complex conjugate character on the right in (1.2).

DEFINITION 1.1: The element $\omega_{T,K} = W_K \theta_{T,K}$ of $\mathbb{C}[G_K]$ is called the *Stickelberger element of K/k relative to T.*

Deligne ([14], Chapter V) has proved a function field analogue of the conjecture of Brumer-Stark (cf. [16]) for the element $\omega_{T,K}$. This analogue asserts firstly (Brumer) that $\omega_{T,K}$ belongs to the integral group ring $\mathbb{Z}[G_K]$ and annihilates the group C_K of divisor classes of degree zero of K. Given a divisor $\mathfrak{d}$ of K of degree zero, suppose

$$\omega_{T,K} \cdot \mathfrak{d} = \delta_K(\alpha)$$

where $\alpha \in K$. Let λ be some W_K-th root of α so that $K(\lambda)/K$ is a Kummer extension. The analogue asserts secondly (Stark) that $K(\lambda)/k$ is abelian.

Deligne actually proved a more precise result than the analogue of Brumer-Stark. His theorem, which we now state, provides a function field version of the abelian conjectures of Stark [13]

THEOREM 1.1 (*Deligne*): *Let $\mathfrak{P}$ be any prime divisor of K. We have*

(1) $\omega_{T,K} \in \mathbb{Z}[G_K]$.
(2) *If $|T| \geqslant 2$, then there is an element $\alpha_{\mathfrak{P},T} \in K$ such that*

$$\omega_{T,K} \cdot \mathfrak{P} = \delta_K(\alpha_{\mathfrak{P},T}).$$

(3) *If $T = \{\mathfrak{q}\}$, then there is an element $\alpha_{\mathfrak{P},T} \in K$ and an integer $n_{\mathfrak{P}}$ such that*

$$\omega_{T,K} \cdot \mathfrak{P} = \delta_K(\alpha_{\mathfrak{P},T}) + n_{\mathfrak{P}} \cdot (\mathfrak{q})_K$$

where $(\mathfrak{q})_K$ is the simple sum of the places of K which divide $\mathfrak{q}$.
(4) *Let $\lambda_{\mathfrak{P},T}$ be a W_K-th root of the element $\alpha_{\mathfrak{P},T}$ appearing in either* (2) *or* (3). *Then $K(\lambda_{\mathfrak{P},T})/k$ is abelian.*

In the case $W_K = W_k$ (i.e., K/k geometric), Tate (cf. [8]) proved the first assertion of the Brumer-Stark conjecture by using the action of G_K on the Jacobian of K. Deligne's proof of the above stated theorem is based on the same idea, the Jacobian in the non-geometric case being replaced by a *motif*.

Because $\theta_{T,K}$ is defined as an L-function evaluator, one can say that the elements $\alpha_{\mathfrak{P},T}$ of the theorem, or rather their divisors, are generated by analytic processes in $\mathbb{C}$ which are controlled by the arithmetic of k. In this paper we give a proof of Deligne's theorem which is founded on the analytic theory of the elliptic modules of Drinfeld [2]. Let ∞ be the place of k which sits under $\mathfrak{P}$, and let $\mathfrak{m}$ be the conductor of K/k. Using the functorial properties of $\omega_{T,K}$ one can reduce Theorem 1.1 to the case when K is a ray class field split completely over ∞ (see §§2, 3). In this case, we show (§§4–6) that the element $\lambda_{\mathfrak{P},T}$ is an $\mathfrak{m}$-division point of a suitably normalized rank 1 elliptic module relative to ∞. The element $\alpha_{\mathfrak{P},T}$ is then the norm of $\lambda_{\mathfrak{P},T}$ under the natural action of the group of roots of unity of K. This enables us, e.g., to write $\alpha_{\mathfrak{P},T}$ as an infinite product over the lattice $\Gamma = \xi\mathfrak{m}$, where ξ is algebraic over the completion k_∞ of k at ∞.

Perhaps the main interest in these results lies in the comparisons which one can make with number fields. One obvious comparison is with the classical Stickelberger element in cyclotomic fields (cf. [16]). A more illuminating comparison, from the point of view of this paper, can be made with the real subfield of a cyclotomic field. The L-function evaluator at $s = 0$ does not have rational coefficients in this case, but there is nevertheless a very real sense in which the analogues of (2)–(4) of Theorem 1.1 are valid. This comparison is presented briefly in §3 in the hope that it will provide insight for the reader.

The division points of rank 1 elliptic modules over a rational function field k as a base and with ∞ a k-place of degree 1 were studied extensively by Galovich and Rosen in their papers [3], [4] on circular units in "cyclotomic function fields." Their work, which first developed the connection between such division points and the values of incomplete L-functions at $s = 0$, is a basic source of motivation for this paper. In [11] and [12], the results of [9], [3] and [4] were generalized to an arbitrary base field k but with ∞ still a k-place of degree 1. The restriction $\deg \infty = 1$ is removed in §§4–6 below. A conjecture of Goss [6, (2.8)] provided guidance and insight for the author in constructing these successive generalizations.

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§2. Elementary reductions.

A place $\mathfrak{P}$ of K *splits completely over* k if $\mathfrak{P}$ has distinct conjugates under the action of G_K. Let $\mathrm{Sp}(K) \subseteq M_K$ be the set of places which split completely over k. Our proof of Theorem 1.1 involves $\mathfrak{P}$-adic analytic constructions for the $\mathfrak{P} \in \mathrm{Sp}(K)$ "one place at a time." A major aim of this § is to show that Theorem 1.1 follows in general if we prove parts (2), (3) and (4) of the theorem for almost all (i.e., all but finitely many) of the places $\mathfrak{P} \in \mathrm{Sp}(K)$. The techniques required to prove this result are well-established (cf. [15] and [16]). We include a full treatment here in order to fix notation that will be used in §§4 and 5 and for the convenience of the reader.

Let $\mathbb{C}\boldsymbol{D}_K = \mathbb{C} \otimes_{\mathbb{Z}} \boldsymbol{D}_K$, a $\mathbb{C}[G_K]$-module which contains $\boldsymbol{D}_K$ in a natural way. A divisor $\mathfrak{q}$ of any subextension of K/k has a natural image in $\boldsymbol{D}_K$ and hence a natural image in $\mathbb{C}\boldsymbol{D}_K$. We denote these images also by "$\mathfrak{q}$".

For a given strictly positive integer W, let $\mathscr{A}_K(W)$ be the subgroup consisting of those elements $\alpha \in K^\times$ such that $K(\alpha^{1/W})/k$ is *abelian*. This subgroup is well-defined because the abelianess of $K(\lambda)/k$ is independent of the choice of a W-th root λ of α. For any divisor $\mathfrak{d}$ of K, let $BS(K, T, W, \mathfrak{d})$ be the assertion: *There is an element* $\alpha_{\mathfrak{d}} \in \mathscr{A}_K(W)$ *such that*

$$\delta_K(\alpha_{\mathfrak{d}}) = \begin{cases} (W\theta_{T,K})\cdot\mathfrak{d} & \text{if} \quad |T| \geqslant 2 \\ (W\theta_{T,K})\cdot(\mathfrak{d} - f_{\mathfrak{d}}\mathfrak{q}) & \text{if} \quad T = \{\mathfrak{q}\} \end{cases} \tag{2.1}$$

where $f_{\mathfrak{d}} \in \mathbb{Q}^\times$ *is chosen so that* $\deg(\mathfrak{d} - f_{\mathfrak{d}}\mathfrak{q}) = 0$. Equation 2.1 is understood to exist in $\mathbb{C}\boldsymbol{D}_K$.

DEFINITION 2.1: Let $\boldsymbol{D}^*_{K,T}$ be the subgroup of $\boldsymbol{D}_K$ consisting of those divisors $\mathfrak{d}$ such that $BS(K, T, W_K, \mathfrak{d})$ is true.

For $|T| \geqslant 2$ (resp. $T = \{\mathfrak{q}\}$), the truth of $BS(K, T, W, \mathfrak{P})$ for any one of the infinitely many places $\mathfrak{P}$ (resp. $\mathfrak{P} \nmid \mathfrak{q}$) in $\mathrm{Sp}(K)$ implies that $W\theta_{T,K} \in \mathbb{Z}[G]$. Therefore, either of the following two theorems is equivalent to Theorem 1.1:

THEOREM 2.2: *For every* $\mathfrak{P} \in M_K$, $BS(K, T, W_K, \mathfrak{P})$ *is true.*

THEOREM 2.3: *We have* $\boldsymbol{D}^*_{K,T} = \boldsymbol{D}_K$.

We wish to show that Theorem 1.1 is actually a consequence of the following seemingly weaker version of Theorem 2.2.

$\boldsymbol{H}(K/k)$: *For almost all* $\mathfrak{P} \in \mathrm{Sp}(K)$, $BS(K, T, W_K, \mathfrak{P})$ *is true.*

In the remainder of this section, we prove that $\boldsymbol{H}(K/k)$ implies Theorem 2.3. It is necessary first to introduce some notation and make some observations.

Suppose E/k is an abelian overextension of K/k with Galois group G_E. Let S_E be a finite set of k-places containing at least all the places which ramify in E/k; and for each $\mathfrak{p} \in M_k \setminus S_E$, let $\tau_{\mathfrak{p}} \in G_E$ be the Frobenius automorphism associated to $\mathfrak{p}$.

DEFINITION 2.4: Let J_E be the annihilator of $\mu(E)$ in the group ring $\mathbb{Z}[G_E]$.

LEMMA 2.5: *The ideal J_E is the $\mathbb{Z}$-module P generated in $\mathbb{Z}[G_E]$ by the elements $\tau_{\mathfrak{p}} - N_{\mathfrak{p}}$ for $\mathfrak{p} \in M_K \setminus S_E$.*

PROOF: Since every element of J_E is congruent mod P to an integer, it suffices to prove that $W_E \in P$. In fact, W_E is the GCD of the integers $1 - N_{\mathfrak{p}}$ for the $\mathfrak{p} \in M_k \setminus S_E$ with $\tau_{\mathfrak{p}} = 1$ (cf. [1], §2.2). □

The restriction map res: $G_E \to G_K$ takes $\tau_{\mathfrak{p}}$ onto $\sigma_{\mathfrak{p}}$ for all $\mathfrak{p} \in M_k \setminus S_E$. Therefore, we have

COROLLARY 2.6: *The restriction map res: $\mathbb{Z}[G_E] \to \mathbb{Z}[G_K]$ takes J_E onto J_K.*

Put $H = \mathrm{Ker(res)} = \mathrm{Gal}(E/K)$, and let $[H]$ be the sum in $\mathbb{Z}[G_E]$ of the elements in H. Then for all $x \in E^\times$, $x^{[H]} \in K^\times$ is the norm of x. Let $N_{E/K}: \mathbb{C}\boldsymbol{D}_E \to \mathbb{C}\boldsymbol{D}_K$ be the norm map on divisors. Then we have

$$N_{E/K}(\delta_E(x)) = \delta_K(x^{[H]}) \tag{2.2}$$

for all $x \in E^\times$; and since $N_{E/K}$ is a G_E-morphism,

$$N_{E/K}(\eta\mathfrak{d}) = \mathrm{res}(\eta) \cdot N_{E/K}(\mathfrak{d}) \tag{2.3}$$

for all $\eta \in \mathbb{Z}[G_E]$ and $\mathfrak{d} \in \boldsymbol{D}_E$.

In particular, let us consider $E/k = K(\lambda)/k$ where $\lambda^W = \alpha$ with $\alpha \in \mathscr{A}_K(W)$. We note first

Observation 1: $\eta \in J_{K(\lambda)} \Rightarrow \lambda^\eta \in K$.

PROOF: For $\phi \in \mathrm{Gal}(K(\lambda)/K) = H$, we have $\lambda^\phi = \zeta\lambda$ for some $\zeta \in \mu(K(\lambda))$. Since $\eta\phi = \phi\eta$ and since η annihilates $\mu(K(\lambda))$,

$$(\lambda^\eta)^\phi = (\zeta\lambda)^\eta = \lambda^\eta.$$

Thus, $\lambda^\eta \in K$. □

Following Stark ([13], Lemma 6), we use this observation to deduce the remarkable

Observation 2: If $BS(K, T, W, \mathfrak{d})$ is true for *any* strictly positive integer W, then $BS(K, T, W_K, \mathfrak{d})$ is true.

PROOF: Suppose (2.1) holds with $\alpha_{\mathfrak{d}} = \lambda^W$, $\alpha_{\mathfrak{d}} \in \mathscr{A}_K(W)$. Appealing to Corollary 2.6, we choose $\eta \in J_{K(\lambda)}$ so that $W_K = \mathrm{res}(\eta)$, and we set $\alpha_* = \lambda^\eta$. Then $\alpha_* \in K$ by Observation 1, and so

$$\alpha_*^W = \lambda^{W\eta} = \alpha_{\mathfrak{d}}^\eta = \alpha_{\mathfrak{d}}^{W_K}, \tag{2.4}$$

which implies that $\delta_K(\alpha_*) = (W_K/W)\delta_K(\alpha_{\mathfrak{d}})$. Thus, the equality in (2.1) persists if we replace $\alpha_{\mathfrak{d}}$ by α_* and W by W_K. Further, from (2.4) we see that

$$\alpha_*^{1/W_K} = \zeta \alpha_{\mathfrak{d}}^{1/W} = \zeta\lambda$$

for some root of unity ζ. The extension $K(\zeta\lambda)/k \subseteq K(\zeta, \lambda)/k$ is therefore abelian, and so $\alpha_* \in \mathscr{A}_K(W_K)$. □

The following corollary of Observation 2 will be useful in §3.

PROPOSITION 2.7: *Suppose E/k is an abelian extension of K/k which is unramified outside of T. Let $\mathfrak{P}_E \in \mathrm{Sp}(E)$ sit over $\mathfrak{P} \in \mathrm{Sp}(K)$. Then $BS(E, T, W_E, \mathfrak{P}_E) \Rightarrow BS(K, T, W_K, \mathfrak{P})$.*

PROOF: First, we note that

$$\mathrm{res}(\theta_{T,E}) = \theta_{T,K} \tag{2.5}$$

because $\theta_{T,K}$ is uniquely determined as the T-incomplete L-function evaluator at $s = 0$.

Now assume that $BS(E, T, W_E, \mathfrak{P}_E)$ is true, and let $\alpha_* \in \mathscr{A}_E(W_E)$ be chosen so that

$$\delta_E(\alpha_*) = \begin{cases} (W_E\theta_{T,E})\mathfrak{P}_E & \text{if } \quad |T| \geq 2 \\ (W_E\theta_{T,E})(\mathfrak{P}_E - f_{\mathfrak{P}_E}\mathfrak{q}) & \text{if } \quad T = \{\mathfrak{q}\}. \end{cases} \tag{2.6}$$

Since α_* has a W_E-th root in an abelian extension of k, the same is true of α_*^τ for every $\tau \in H$. Therefore, the W_E-th root of $\alpha_*^{[H]}$ is also abelian over k, and so $\alpha_*^{[H]} \in \mathscr{A}_K(W_E)$.

We next apply the divisor norm $N_{E/K}$ to both sides of (2.6). After a short calculation using equations (2.2), (2.3) and (2.5) and noting that $N_{E/K}\mathfrak{P}_E = \mathfrak{P}$ since $\mathfrak{P}_E \in \mathrm{Sp}(E)$, we arrive at (2.1) with $\mathfrak{d} = \mathfrak{P}$ and

$\alpha_{\mathfrak{P}} = \alpha_*^{[H]}$. Thus, $BS(K, T, W_E, \mathfrak{P})$ is valid, and therefore by Observation 2, so is $BS(K, T, W_K, \mathfrak{P})$. □

As noted above, $\boldsymbol{H}(K/k) \Rightarrow \omega_{T,K} \in \mathbb{Z}[G_K]$. We are now in a position to improve this result.

PROPOSITION 2.8: *Hypothesis $\boldsymbol{H}(K/k)$ implies $J_K \theta_{T,K} \subseteq \mathbb{Z}[G_K]$.*

PROOF: Assuming $\boldsymbol{H}(K/k)$ for $|T| \geqslant 2$ (resp. $T = \{\mathfrak{q}\}$) choose $\mathfrak{P} \in \mathrm{Sp}(K)$ (resp. $\mathfrak{P} \in \mathrm{Sp}(K)$, $\mathfrak{P} \nmid \mathfrak{q}$), and let $\lambda^{W_K} = \alpha_{\mathfrak{P}} \in \mathscr{A}_K(W_K)$ satisfy (2.1) with $\mathfrak{d} = \mathfrak{P}$. For $\eta \in J_{K(\lambda)}$, λ^η is a W_K-th root of $\alpha_{\mathfrak{P}}^\eta$ in K by Observation 1. After multiplying (2.1) by η/W_K, we see that $\delta_K(\lambda^\eta) = \eta\theta_{T,K}\mathfrak{P}$ (resp. $\delta_K(\lambda^\eta) = \eta\theta_{T,K}(\mathfrak{P} - f_{\mathfrak{P}}\mathfrak{q})$), which implies $(\operatorname{res}\eta)\theta_{T,K} \in \mathbb{Z}[G_K]$ since $\mathfrak{P} \in \mathrm{Sp}(K)$. We conclude by invoking Corollary 2.6. □

Let $\boldsymbol{P}_K$ be the group of principal divisors of K, and let $\boldsymbol{P}_K^{ab}$ be the subgroup consisting of the divisors of the elements in $\mathscr{A}_K(W_K)$.

LEMMA 2.9: *Hypothesis $\boldsymbol{H}(K/k)$ implies $\omega_{T,K} \cdot \boldsymbol{P}_K \subseteq \boldsymbol{P}_K^{ab}$.*

PROOF: For each $\mathfrak{p} \in M_k \setminus T$, put $\eta_\mathfrak{p} = (\sigma_\mathfrak{p} - N_\mathfrak{p})\theta_{T,K}$. Then $\eta_\mathfrak{p} \in \mathbb{Z}[G_K]$ by Proposition 2.8, and

$$W_K \eta_\mathfrak{p} = (\sigma_\mathfrak{p} - N_\mathfrak{p})\,\omega_{T,K}. \tag{2.7}$$

For $\mathfrak{p}$, $\mathfrak{p}' \in M_K \setminus T$, we have obviously

$$(\sigma_{\mathfrak{p}'} - N_{\mathfrak{p}'})\,\eta_\mathfrak{p} = (\sigma_\mathfrak{p} - N_\mathfrak{p})\,\eta_{\mathfrak{p}'}. \tag{2.8}$$

Now for a given principal divisor $\delta_K(\gamma)$, $\gamma \in K^\times$, let $\alpha = \gamma^{\omega_{T,K}}$ and choose λ over K so that $\lambda^{W_K} = \alpha$. We have to show that $K(\lambda)/k$ is abelian. To that end, imagine K and λ to be embedded in some fixed way in k^{ac}, the algebraic closure of k, and let τ and τ' be k-morphisms of $K(\lambda)$ into k^{ac}. Choose $\mathfrak{p}$ (resp. $\mathfrak{p}'$) so that $\sigma_\mathfrak{p}$ (resp. $\sigma_{\mathfrak{p}'}$) is τ (resp. τ') restricted to K. Then

$$\left(\lambda^{\tau - N_\mathfrak{p}}\right)^{W_K} = \alpha^{\sigma_\mathfrak{p} - N_\mathfrak{p}} = \gamma^{W_K \eta_\mathfrak{p}}$$

by (2.7), and this implies

$$\lambda^{\tau - N_\mathfrak{p}} = \zeta_\tau \gamma^{\eta_\mathfrak{p}} \tag{2.9}$$

for some root of unity $\zeta_\tau \in \mu(K)$. Thus $\lambda^\tau \in K(\lambda)$, and so $K(\lambda)/k$ is a Galois extension. Further, since $\sigma'_\mathfrak{p} - N_{\mathfrak{p}'}$ annihilates $\mu(K)$, when we apply $\tau' - N_{\mathfrak{p}'}$ to both sides of (2.9), we get

$$\lambda^{(\tau' - N_{\mathfrak{p}'})(\tau - N_\mathfrak{p})} = \gamma^{(\sigma_{\mathfrak{p}'} - N_{\mathfrak{p}'})\eta_\mathfrak{p}} = \gamma^{(\sigma_\mathfrak{p} - N_\mathfrak{p})\eta_{\mathfrak{p}'}} = \lambda^{(\tau - N_\mathfrak{p})(\tau' - N_{\mathfrak{p}'})}$$

by (2.8). Thus, $K(\lambda)/k$ is abelian. □

COROLLARY 2.10: *Hypothesis* $\boldsymbol{H}(K/k)$ *implies* $\boldsymbol{P}_K \subseteq \boldsymbol{D}^*_{K,T}$.

PROOF: The corollary follows immediately after one notes that in case $T = \{\mathfrak{q}\}$, $f_{\mathfrak{d}} = 0$ for any principal divisor $\mathfrak{d}$. □

One knows that the canonical images of the places in $\mathrm{Sp}(K)$ outside any finite subset generate the divisor class group $\boldsymbol{D}_K/\boldsymbol{P}_K$. This fact and Corollary 2.10 together suffice to establish our main result:

PROPOSITION 2.11: *Hypothesis* $\boldsymbol{H}(K/k)$ *implies Theorem* 2.3 *and hence also Theorem* 1.1.

Let $I \subseteq \mathrm{Sp}(K)$ have a finite complement. Using class field theory, one can give a short proof that I generates $\boldsymbol{D}_K/\boldsymbol{P}_K$. Since a convenient reference for this proof does not seem to exist, we sketch it here: Let L/K be the maximal abelian unramified extension of K, an extension of infinite degree since it contains all constant field extensions of K. By class field theory, the sequence

$$1 \to \boldsymbol{P}_K \to \boldsymbol{D}_K \xrightarrow{\phi} \mathrm{Gal}(L/K) \to 1,$$

where ϕ is the Artin map, is an exact sequence of topological groups. Let $\boldsymbol{R}$ be the subgroup of $\boldsymbol{D}_K$ generated by I, and let F be the fixed field of $\phi(\boldsymbol{R})$. Since I is nonempty, $\boldsymbol{D}_K/\boldsymbol{P}_K\boldsymbol{R} \cong \mathrm{Gal}(F/K)$ is finite. Now L/k is Galois by construction, and therefore F/k is Galois also. Further, from the definition of $\boldsymbol{R}$, all the places in I split completely in F/K. We see now that the places in M_k which split completely in either K/k or F/k differ by at most a finite set. By the analytic theory (zeta-functions), this forces $F = K$. Therefore, $\boldsymbol{D}_K/\boldsymbol{P}_K\boldsymbol{R}$ is the trivial group.

§3. Hypothesis H(K/k) reformulated

Given a place $\infty \in M_k$, let k_∞ be the completion of k at ∞, and let A_∞ be the ring of functions in k which are holomorphic away from ∞. One knows [2], [10] that the elliptic A_∞-modules of generic characteristic can be constructed by analytic processes over k_∞. In this section, we reduce $\boldsymbol{H}(K/k)$ to a statement $\boldsymbol{S}_\infty$ which is natural to the context of rank 1 elliptic A_∞-modules; and we compare $\boldsymbol{S}_\infty$ to its analogue for the class fields of $\mathbb{Q}$ which are completely split over the archimedean place. This analogue helps to motivate the proof of $\boldsymbol{S}_\infty$ which we give in §§4–6.

Let $\boldsymbol{I}_\infty$ be the group of fractional ideals of A_∞, and let $\boldsymbol{M}_\infty \subseteq \boldsymbol{I}_\infty$ be the monoid of integral ideals. The group $\boldsymbol{I}_\infty$ is naturally isomorphic to the subgroup of $\boldsymbol{D}_k$ consisting of those divisors which are supported away from ∞. From this isomorphism, one can compute the order of $\mathbf{Pic}(A_\infty)$, the ideal class group of A_∞. One finds that $|\mathbf{Pic}(A_\infty)| = hd_\infty$, where h is the class number of k and $d_\infty = \deg \infty$. Let $\mathfrak{e}$ denote the unit ideal of $\boldsymbol{I}_\infty$.

A finite abelian extension field K of k is called a *real class field at* ∞ if there is a k-embedding of K into k_∞ or, equivalently, if ∞ splits completely in K/k. By class field theory, the real class fields at ∞ correspond in the familiar way with the generalized ideal class groups of A_∞. In particular, for every ideal $\mathfrak{m} \in \boldsymbol{M}_\infty$, there is a largest real class field $H_\mathfrak{m}$ of conductor $\mathfrak{m}$. Let $T(\mathfrak{m})$ denote the support of the divisor associated to $\mathfrak{m}$, so that $T(\mathfrak{m})$ is precisely the set of k-places which ramify in $H_\mathfrak{m}/k$. We call $H_\mathfrak{e}$ the *Hilbert Class Field at* ∞ because it is the maximal unramified real class field. Because ∞ splits completely in $H_\mathfrak{e}/k$, $W_\infty = q^{d_\infty} - 1$ is the order of $\mu(H_\mathfrak{m})$ for every $\mathfrak{m} \in \boldsymbol{M}_\infty$.

Let H_∞ be the union of the fields $H_\mathfrak{m}$ for $\mathfrak{m} \in \boldsymbol{M}_\infty$, and let $i_\infty : H_\infty/k \to k_\infty/k$ be an embedding. Then i_∞ determines a unique extension of ∞ to H_∞ and therefore, by restriction, a unique place on each $H_\mathfrak{m}$ above ∞. We use the symbol "∞" to denote all these places.

As a first reformulation of $\boldsymbol{H}(K/k)$ for all K, we state

$\boldsymbol{H}(k)$: For every place $\infty \in M_k$ and every ideal $\mathfrak{m} \neq \mathfrak{e}$ in $\boldsymbol{M}_\infty$, $BS(H_\mathfrak{m}, T(\mathfrak{m}), W_\infty, \infty)$ is true.

PROPOSITION 3.1: *Hypothesis $\boldsymbol{H}(k)$ implies $\boldsymbol{H}(K/k)$ for every finite abelian extension K/k.*

PROOF: Given K and T, let Σ be the finite subset of M_K consisting of those places which restrict to a place of T. Given $\mathfrak{P} \in \mathrm{Sp}(K)\backslash\Sigma$, let ∞ be the k-place sitting under $\mathfrak{P}$, and choose an embedding $K/k \to H_\infty/k$ so that $\mathfrak{P} = \infty$. Next choose a multiple $\mathfrak{m} \in \boldsymbol{M}_\infty$ of the conductor of K/k so that $T(\mathfrak{m}) = T$. Then $K \subseteq H_\mathfrak{m}$ and by Proposition 2.7, $BS(H_\mathfrak{m}, T(\mathfrak{m}), W_\infty, \infty) \Rightarrow BS(K, T, W_K, \mathfrak{P})$. $\square$

For the remainder of the paper, ∞ is a fixed place of k; and H_∞/k is imagined to be embedded in some fixed fashion as a subextension of k_∞/k. Our aim is to prove the truth of $BS(H_\mathfrak{m}, T(\mathfrak{m}), W_\infty, \infty)$ for all $\mathfrak{m} \in \boldsymbol{M}_\infty$, $\mathfrak{m} \neq \mathfrak{e}$.

Let $G_\mathfrak{m}$ be the Galois group of $H_\mathfrak{m}/k$. By class field theory, $G_\mathfrak{e}$ is isomorphic to $\mathbf{Pic}(A_\infty)$; and in general, there is an exact sequence

$$1 \to \mathbb{F}_q^\times \to (A_\infty/\mathfrak{m})^\times \to G_\mathfrak{m} \xrightarrow{\text{res}} G_\mathfrak{e} \to 1 \qquad (3.1)$$

where res is the restriction map. Therefore, if $\Phi(\mathfrak{m})$ is the order of

$(A_\infty/\mathfrak{m})^\times$, then

$$[H_\mathfrak{m}:k]=hd_\infty\Phi(\mathfrak{m})/W_k. \tag{3.2}$$

A place of $H_\mathfrak{m}$ is *infinite* if it is a $G_\mathfrak{m}$-conjugate of ∞; and a divisor $\mathfrak{d}\in \boldsymbol{D}_\mathfrak{m}$, the divisor group of $H_\mathfrak{m}$, is *finite* if $\mathrm{Supp}(\mathfrak{d})$ contains no infinite places. For $x\in H_\mathfrak{m}^\times$, we define an element $l_\mathfrak{m}(x)$ (cf. [4]) of the group ring $\mathbb{Z}[G_\mathfrak{m}]$ by

$$l_\mathfrak{m}(x)=\sum_\sigma v_\infty(x^\sigma)\sigma^{-1}\quad (\sigma\in G_\mathfrak{m}),$$

and we put

$$l_\mathfrak{m}^*(x)=l_\mathfrak{m}(x)\cdot\infty$$

so that $l_\mathfrak{m}^*(x)$ is that part of the divisor of x which is supported by the infinite places of $H_\mathfrak{m}$. We call $l_\mathfrak{m}$ and $l_\mathfrak{m}^*$ *logarithmic maps* because each is a $G_\mathfrak{m}$-morphism from the multiplicative Galois module $H_\mathfrak{m}^\times$ into an additive Galois module.

For brevity, we put $L_\mathfrak{m}(s,\psi)=L_{T(\mathfrak{m})}(s,\psi)$, the incomplete L-function associated to the character $\psi\in\hat{G}_\mathfrak{m}$, and we write

$$\theta_\mathfrak{m}=\theta_{T(\mathfrak{m}),H_\mathfrak{m}}\in\mathbb{C}[G_\mathfrak{m}].$$

DEFINITION 3.2: An element $\alpha\in H_\mathfrak{m}^\times$, $\mathfrak{m}\neq\mathfrak{e}$, is called an *$L_\mathfrak{m}$-function evaluator at $s=0$* if $l_\mathfrak{m}(\alpha)=W_\infty\theta_\mathfrak{m}$.

By Fourier inversion, $\alpha\in H_\mathfrak{m}^\times$ is an $L_\mathfrak{m}$-function evaluator at $s=0$ if and only if

$$L_\mathfrak{m}(0,\psi)=\frac{1}{W_\infty}\sum_\sigma\psi(\sigma)v_\infty(\alpha^\sigma)\quad(\sigma\in G_\mathfrak{m}) \tag{3.3}$$

for all characters $\psi\in G_\mathfrak{m}$.

Let $B_\mathfrak{m}$ be the integral closure of A_∞ in $H_\mathfrak{m}$, and let $\delta_\mathfrak{m}^*$ be the natural Galois isomorphism from the group $\boldsymbol{I}_\mathfrak{m}$ of fractional ideals of $B_\mathfrak{m}$ onto the subgroup of finite divisors of $\boldsymbol{D}_\mathfrak{m}$. For a place $\mathfrak{q}\neq\infty$ of k, let $(\mathfrak{q})_\mathfrak{m}$ denote the product in $\boldsymbol{I}_\mathfrak{m}$ of the prime ideals of $B_\mathfrak{m}$ which sit over $\mathfrak{q}$ viewed as an ideal of A_∞. Since $H_\mathfrak{m}/k$ is Galois,

$$\deg(\mathfrak{q})_\mathfrak{m}=\left([H_\mathfrak{m}:k]/e_\mathfrak{q}\right)\cdot\deg\mathfrak{q} \tag{3.4}$$

where $e_\mathfrak{q}$ is the ramification index of $\mathfrak{q}$ in $H_\mathfrak{m}/k$. For brevity, we write $\mathscr{A}_\mathfrak{m}(W_\infty)=\mathscr{A}_{H_\mathfrak{m}}(W_\infty)$ and $\delta_\mathfrak{m}=\delta_{H_\mathfrak{m}}$.

We can now state

$\boldsymbol{S}_\infty$: For $\mathfrak{m} \in \boldsymbol{M}_\infty$, $\mathfrak{m} \neq \mathfrak{e}$, there is an element $\alpha \in B_\mathfrak{m} \cap \mathscr{A}_\mathfrak{m}(W_\infty)$ which is an $L_\mathfrak{m}$-function evaluator at $s=0$. If $T(\mathfrak{m}) = \{\mathfrak{q}\}$, then α generates the ideal $(\mathfrak{q})_\mathfrak{m}^r$, where $r = W_\infty / W_k$. Otherwise, $\alpha \in B_\mathfrak{m}^\times$.

The following lemmas will be used in proving that $\boldsymbol{S}_\infty$ is equivalent to the truth of BS($H_\mathfrak{m}$, $T(\mathfrak{m})$, W_∞, ∞) for all $\mathfrak{m} \in \boldsymbol{M}_\infty$, $\mathfrak{m} \neq \mathfrak{e}$.

LEMMA 3.3. *When* $T(\mathfrak{m}) = \{\mathfrak{q}\}$,

$$\deg(\mathfrak{q})_\mathfrak{m} = h d_\infty \cdot \deg \mathfrak{q}. \tag{3.5}$$

PROOF: By class field theory, when $T(\mathfrak{m}) = \{\mathfrak{q}\}$ the places over $\mathfrak{q}$ in $H_\mathfrak{m}$ are totally ramified in $H_\mathfrak{m}/H_\mathfrak{e}$. Therefore $[H_\mathfrak{m} : k] = h d_\infty e_\mathfrak{q}$, and so (3.5) follows from (3.4). □

LEMMA 3.4: *Let* $\epsilon_\mathfrak{m} : \mathbb{Z}[G_\mathfrak{m}] \to \mathbb{Z}$ *be the augmentation map. Then*

$$\epsilon_\mathfrak{m}(\theta_\mathfrak{m}) = \begin{cases} 0 & \text{if} \quad |T(\mathfrak{m})| \geqslant 2 \\ -\dfrac{h \deg \mathfrak{q}}{W_k} & \text{if} \quad T(\mathfrak{m}) = \{\mathfrak{q}\}. \end{cases} \tag{3.6}$$

PROOF: Let

$$Z_\mathfrak{m}(s) = \prod_{\mathfrak{p} | \mathfrak{m}} \left(1 - N_\mathfrak{p}^{-s}\right) \cdot Z_k(s)$$

where $Z_k(s)$ is the zeta-function of k. From the definitions,

$$\epsilon_\mathfrak{m}(\theta_\mathfrak{m}) \cdot 1_k = \operatorname{res}(\theta_\mathfrak{m}) = \theta_{T(\mathfrak{m}),k} = Z_\mathfrak{m}(0) \cdot 1_k$$

where res is the restriction down to k. Since

$$Z_k(s) = \frac{P_k(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

where $P_k(u)$ is a polynomial with $P_k(1) = h$, (3.6) follows by a straightforward calculation. □

We can now prove the main result of this §.

PROPOSITION 3.5: *Statement* $\boldsymbol{S}_\infty$ *is equivalent to the truth of* $BS(H_\mathfrak{m}, T(\mathfrak{m}), W_\infty, \infty)$ *for all* $\mathfrak{m} \in \boldsymbol{M}_\infty$, $\mathfrak{m} \neq \mathfrak{e}$.

PROOF: Choose $\mathfrak{m} \in \boldsymbol{M}_\infty$, $\mathfrak{m} \neq \mathfrak{e}$. We consider first the case $|T(\mathfrak{m})| \geqslant 2$. In this case, an element $\alpha_\infty \in H_\mathfrak{m}^\times$ satisfies (2.1) with $K = H_\mathfrak{m}$, $T = T(\mathfrak{m})$,

$W = W_\infty$ and $\mathfrak{d} = \infty$ if and only if α_∞ is an $L_{\mathfrak{m}}$-function evaluator at $s = 0$ such that

$$\delta_{\mathfrak{m}}(\alpha_\infty) = l^*_{\mathfrak{m}}(\alpha_\infty). \tag{3.7}$$

Such an element certainly belongs to $B^\times_{\mathfrak{m}}$.

We consider next the case when $\mathfrak{m}$ is a power of the prime ideal $\mathfrak{q}$. In this case, an element $\alpha_\infty \in H^\times_{\mathfrak{m}}$ satisfies (2.1) with $K = H_{\mathfrak{m}}$, $T = T(\mathfrak{m})$, $W = W_\infty$ and $\mathfrak{d} = \infty$ if and only if α_∞ is an $L_{\mathfrak{m}}$-function evaluator at $s = 0$ with

$$\delta_{\mathfrak{m}}(\alpha_\infty) = \delta^*_{\mathfrak{m}}((\mathfrak{q})^r_{\mathfrak{m}}) + l^*_{\mathfrak{m}}(\alpha_\infty) \tag{3.8}$$

where r is determined by

$$\begin{aligned} r \cdot \deg(\mathfrak{q})_{\mathfrak{m}} &= -\deg l^*_{\mathfrak{m}}(\alpha_\infty) = -d_\infty \epsilon_{\mathfrak{m}}(l_{\mathfrak{m}}(\alpha_\infty)) \\ &= -d_\infty W_\infty \epsilon_{\mathfrak{m}}(\theta_{\mathfrak{m}}) = d_\infty W_\infty h \cdot \deg \mathfrak{q}/W_k, \end{aligned}$$

the last equality following from Lemma 3.4. Invoking Lemma 3.3 to compute $\deg(\mathfrak{q})_{\mathfrak{m}}$ on the left above, we obtain $r = W_\infty/W_k$. □

The proof of $\boldsymbol{S}_\infty$ which we give in the following §§ is motivated in part by the well-known formulas evaluating Dirichlet L-functions at $s = 0$ (cf. [3] and [4]). Given an integer $m > 1$, $m \not\equiv 2 \pmod 4$, let ψ be an even Dirichlet character modulo m, and let

$$L^*_m(s, \psi) = \prod_{p \nmid m} (1 - \psi(p)p^{-s})^{-1}$$

be the L-function associated to ψ. Since ψ is even, $L^*_m(0, \psi) = 0$; but one knows (cf. [13]) that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} L^*_m(s, \psi)\right|_{s=0} = -\tfrac{1}{2} \sum_{\substack{0 < t < m/2 \\ (t,m)=1}} \psi(t) \log|\epsilon_t| \tag{3.9}$$

where $\epsilon_t = (1 - e^{2\pi i t/m})(1 - e^{-2\pi i t/m})$.

Let ∞ denote the archimedean place of $\mathbb{Q}$; and for $x \in \mathbb{R}^\times = Q^\times_\infty$, let $v_\infty(x) = -\log|x|$. Using the "explicit class field theory" of $\mathbb{Q}$, we can interpret (3.9) as a statement about $H_m/\mathbb{Q}$, the maximal abelian extension with conductor m. In fact, H_m is the real subfield of the cyclotomic extension $\mathbb{Q}(\zeta)/\mathbb{Q}$, where ζ is a primitive m-th root of unity. Embedding $\mathbb{Q}(\zeta)$ in $\mathbb{C}$ so that $\zeta = e^{2\pi i/m}$ and viewing ψ as a character of the Galois group $G_m \cong (\mathbb{Z}/m\mathbb{Z})^\times/\{\pm 1\}$ of $H_m/\mathbb{Q}$, we can rewrite (3.9) as

$$\frac{\mathrm{d}}{\mathrm{d}s} L_m^*(s, \psi)\bigg|_{s=0} = \frac{1}{W_\infty} \sum_\sigma \psi(\sigma) v_\infty(\alpha_*^\sigma) \qquad (\sigma \in G_m) \tag{3.10}$$

where $\alpha_* = (1-\zeta)(1-\zeta^{-1})$ and where $W_\infty = W_{H_m} = W_\mathbb{Q} = 2$. Now $L_m^*(s, \psi)$ is the incomplete Artin L-function of ψ except that it is missing a conjectural "Euler factor" $E_\infty(s)$ at ∞. Since ∞ splits completely in $H_m/\mathbb{Q}$, $E_\infty(s)$ should be independent of ψ. We further assume that $E_\infty(s)$ has a simple pole of residue 1 at $s = 0$. If we now define

$$L_m(s, \psi) = E_\infty(s) L_m^*(s, \psi),$$

then $L_m(s, \psi)$ is our incomplete L-function; and we can write

$$L_m(0, \psi) = \frac{1}{W_\infty} \sum_\sigma \psi(\sigma) v_\infty(\alpha_*^\sigma) \qquad (\sigma \in G_m). \tag{3.11}$$

Since α_* is the image of 1-ζ under the norm map from $\mathbb{Q}(\zeta)$ to H_m, $\alpha_* \in H_m$ and therefore deserves to be called "an L_m-function evaluator at $s = 0$".

Let B_m be the ring of integers of H_m. It is a standard fact that $\alpha_* B_m$ is the unique prime ideal of B_m over q if $m = q^e$ is a prime power and that $\alpha_* \in B_m^\times$ otherwise. Since $W_\infty/W_\mathbb{Q} = 1$, this behavior of α_* reflects that asserted for the L_m-function evaluator α in S_∞. If we note further that $\alpha_* = \zeta^{-1}(1-\zeta)^2$, then it is immediate that $H_m(\sqrt{\alpha_*})$ is cyclotomic and hence abelian over $\mathbb{Q}$. Thus the familiar, but special, situation over the base pair $(\mathbb{Q}, \infty)$ provides an exact analogue for S_∞.

Adopting now the philosophy of Hilbert's Twelfth Problem, we may say that α_* has been constructed by analytic processes controlled by the arithmetic of $\mathbb{Q}$. Indeed, noting that $\alpha_* > 0$, we may solve the equations (3.11) for $\log \alpha_*$ in terms of the values of L_m-functions at $s = 0$ and then exponentiate. However, there is a simpler way of computing α_* analytically. Let

$$\lambda_* = 2\mathrm{i} \sin(\pi/m) = \left(\frac{2\pi \mathrm{i}}{m}\right) \prod_{\substack{n \in m\mathbb{Z} \\ n \neq 0}} \left(1 - \frac{1}{n}\right) \tag{3.12}$$

where the infinite product converges conditionally under the natural ordering on the index n. Then since $\alpha_* = -\lambda_*^2$, (3.12) provides an analytic construction of α_* as a single value of the function $4\sin^2(x)$. It is remarkable that the square of the element defined by $2\pi\mathrm{i}/m$ times a simple product over the ideal $m\mathbb{Z}$ is an L_m-function evaluator at $s = 0$ belonging to B_m

If the statement S_∞ is true for function fields, then we can construct $\delta_{\mathfrak{m}}(\alpha)$ but not α itself from the values of the incomplete L-functions at $s=0$. We will show, however, that the W_∞-th power of an element λ defined over k_∞ by an infinite product similar to that in (3.12) is an $L_{\mathfrak{m}}$-function evaluator at $s=0$ meeting the requirements of S_∞.

§4. Normalized elliptic $\mathbf{A}_\infty$-modules.

In this §, we show that the normalization theory introduced in [11], §2 for the case $d_\infty = 1$ can be extended to the general case. The field of constants $\kappa(\infty)$ in the completion k_∞ is isomorphic to the residue class field at ∞ and therefore has degree d_∞ over $\mathbb{F}_q$. Let U_∞ (resp. $U_\infty^{(1)}$) be the group of units (resp. 1-units) at ∞; and let Ω be the completion of the algebraic closure of k_∞.

We recall (see [2] or [10]) that an elliptic A_∞-module over Ω is an $\mathbb{F}_q$-algebra morphism $\rho: A_\infty \to \Omega[\psi]$, where $\Omega[\psi]$ is the twisted polynomial ring relative to the automorphism $w \mapsto w^q$ of Ω. Thus, the elements ρ_x, $x \in A_\infty$, are left polynomials in ψ, where ψ satisfies $\psi w = w^q \psi$ for all $w \in \Omega$. The A_∞-module ρ is said to have rank 1 if $\deg \rho_x = \deg x$ for all $x \in A_\infty \setminus \{0\}$. Let $D: \Omega[\psi] \to \Omega$ be the *differential* which maps each polynomial to its constant term. Throughout the rest of the paper, we understanding the phrase "ρ is an A_∞-module of generic characteristic" to mean that ρ is a rank 1 elliptic A_∞-module over Ω such that the map $x \mapsto D(\rho_x)$ for $x \in A_\infty$ is the inclusion $A_\infty \hookrightarrow \Omega$. We say that such a ρ is *normalized* if the leading coefficient $s_\rho(x)$ of ρ_x belongs to $\kappa(\infty)$ for all $x \in A_\infty \setminus \{0\}$. By [10] Lemma 10.3, each Ω-isomorphism class of A_∞-modules of generic characteristic contains a normalized module.

DEFINITION 4.1: A sign function $\operatorname{sgn}: k_\infty^\times \to \kappa(\infty)^\times$ is a co-section of the inclusion morphism $\kappa(\infty)^\times \hookrightarrow k_\infty^\times$ such that $\operatorname{sgn}(U_\infty^{(1)}) = 1$. In addition, we put $\operatorname{sgn}(0) = 0$. Let σ be an $\mathbb{F}_q$-automorphism of $\kappa(\infty)$. The composite map $\sigma \circ \operatorname{sgn}$ is called a *twisted sign function* or a *twisting of* sgn *by* σ.

LEMMA 4.2: *Let* sgn *and* sgn′ *be sign functions on* k_∞. *Then there is an element* $a \in \kappa(\infty)^\times$ *such that*

$$\operatorname{sgn}(x) = \operatorname{sgn}'(x) \cdot a^{(\deg x)/d_\infty} \tag{4.1}$$

for all $x \in k_\infty$.

PROOF: From the definitions, the quotient $\operatorname{sgn}(x)/\operatorname{sgn}'(x)$ is trivial on U_∞ and therefore factors through $v_\infty: k_\infty^\times \to \mathbb{Z}$. Now, $\deg x = -d_\infty \cdot v_\infty(x)$. □

COROLLARY 4.3: *There are exactly W_∞ sign functions on k_∞.*

Let ρ be a normalized elliptic A_∞-module of generic characteristic. We show now that the map $x \mapsto s_\rho(x)$ is the restriction to A_∞ of a unique twisted sign function on k_∞. Note first that since $\deg \rho_x = \deg x =$ a multiple of d_∞, $s_\rho(x)$ is a multiplicative map on $A_\infty \setminus \{0\}$ into $\kappa(\infty)$ satisfying $s_\rho(a) = a$ for $a \in \mathbb{F}_q^\times$. Therefore, s_ρ has a unique extension to $k^\times$, and

LEMMA 4.4: *We have $s_\rho(U_\infty^{(1)} \cap k^\times) = 1$.*

PROOF: Let $z = x/y \in U_\infty^{(1)}$ with $x, y \in A_\infty$. Then $\deg x = \deg y$ but $\deg(x-y) < \deg y$ because $v_\infty(z-1) = v_\infty((x-y)/y) > 0$. Thus, $\rho_{x-y} = \rho_x - \rho_y$ has degree strictly less than $\deg \rho_y$, and this implies $s_\rho(x) = s_\rho(y)$ and hence $s_\rho(z) = 1$. □

We see by this lemma that s_ρ is continuous on $k^\times$ in the v_∞-topology and therefore has a unique continuous extension to $k_\infty^\times$, also denoted by "s_ρ", which is trivial on $U_\infty^{(1)}$. We can now prove

PROPOSITION 4.5: *The extended map s_ρ on $k_\infty^\times$ is a twisted sign function.*

PROOF: We need only show that s_ρ restricts of an $\mathbb{F}_q$-automorphism of $\kappa(\infty)$. For this, it suffices to prove that $s_\rho(1-a) = 1 - s_\rho(a)$ for all $a \in \kappa(\infty)$. If $a = 1$, this is clear; and so we assume $a \neq 1$. By continuity, we can choose $z = x/y \in U_\infty \setminus U_\infty^{(1)}$ with $x, y \in A_\infty$ so that $s_\rho(z) = s_\rho(a)$ and $s_\rho(1-z) = s_\rho(1-a)$. Then since $z \notin U_\infty^{(1)}$, $\deg(x-y) = \deg x = \deg y$ which implies $s_\rho(x-y) = s_\rho(x) - s_\rho(y)$. Thus, $s_\rho(1-z) = s_\rho((y-x)/y) = 1 - s_\rho(z)$. □

We now choose arbitrarily one of the W_∞ sign functions sgn on k_∞ and incorporate it as part of our base object, which becomes a triple $(k, \infty, \mathrm{sgn})$. An element $z \in k_\infty^\times$ is called *positive* if $\mathrm{sgn}(z) = 1$ and *totally positive* if $\mathrm{sgn}(z^\sigma) = 1$ for every k-isomorphism σ of k_∞. The conditions imposed on the element α by $\boldsymbol{S}_\infty$ uniquely determine the divisor $\delta_{\mathfrak{m}}(\alpha)$. The choice of sign function enables us to specify the element itself by imposing the additional requirement that α be totally positive. As we show in subsequent §§, there is a (necessarily) unique totally positive element $\alpha \in H_{\mathfrak{m}}^\times$ satisfying the conditions of $\boldsymbol{S}_\infty$.

The notion of *monic* elements in function fields has been used by several authors (cf., e.g., Artin's thesis). In the older literature, such elements were called *primary*. The usual way of introducing monic elements is to choose a uniformizer at ∞. This idea has been exploited by Goss [7] in defining characteristic p zeta- and L-functions. Such a procedure leads to a sign function as defined above, and the monic elements are then the elements we have called positive.

We say that a given A_∞-module ρ is sgn-*normalized* if ρ is normalized with s_ρ equal to a twisting of sgn.

PROPOSITION 4.6: *Every elliptic A_∞-module of generic characteristic is Ω-isomorphic to a* sgn-*normalized module ρ.*

PROOF: By [10], Lemma 10.3, there is a normalized ρ' in the Ω-isomorphism class of the given A_∞-module; and by Proposition 4.5 above, $s_{\rho'}$ is a twisting by σ of some sign function sgn′ on k_∞. Choose $a \in \boldsymbol{\kappa}(\infty)$ so that (4.1) holds, choose $w \in \Omega$ so that $w^{W_\infty} = a^\sigma$ and put $\rho = w^{-1}\rho' w$. Then for all $x \in A_\infty \setminus \{0\}$,

$$s_\rho(x) = s_{\rho'}(x) w^{q^{\deg x} - 1} = s_{\rho'}(x) a^{\sigma(q^{\deg x} - 1)/W_\infty}$$

$$= s_{\rho'}(x) a^{\sigma \deg x / d_\infty} = \left[\operatorname{sgn}'(x) a^{\deg x / d_\infty}\right]^\sigma$$

since $a \in \boldsymbol{\kappa}(\infty)$. Thus, s_ρ is a twisting of sgn by σ. □

Now let ρ be a sgn-normalized A_∞-module of generic characteristic, and let $I^*(\rho)$ be the subfield of Ω generated by the coefficients of the polynomials ρ_x, $x \in A_\infty \setminus \{0\}$. By [10], §8, $I^*(\rho)$ contains the Hilbert Class Field H_e; and further, there is an element $w \in \Omega^\times$ such that $\rho' = w\rho w^{-1}$ is defined over H_e. Since $\boldsymbol{\kappa}(\infty) \subseteq H_e$ and since the leading coefficient of ρ'_x belongs to H_e, we see that

$$w^{q^{\deg x} - 1} \in H_e \tag{4.2}$$

for all $x \in A_\infty \setminus \{0\}$. In fact, because d_∞ is the GCD of the integers deg x for $x \in A_\infty \setminus \{0\}$, (4.2) implies that

$$w^{W_\infty} = w^{q^{d_\infty} - 1} \in H_e. \tag{4.3}$$

Thus, $I^*(\rho) \subseteq H_e(w)$ is a Kummer extension of H_e, which implies in particular that $I^*(\rho)/k$ is finite and separable.

For a finite place $\mathfrak{P}$ of H_e, let Norm($\mathfrak{P}$) be its norm down to k viewed as an ideal in M_∞. By the properties of H_e, Norm($\mathfrak{P}$) is a *principal ideal.*

PROPOSITION 4.7: *Let $\mathfrak{P}$ be a finite place of H_e which does not ramify in $H_e(w)/H_e$, and let $\tau_{\mathfrak{P}}$ be the Frobenius automorphism of $H_e(w)$ associated to $\mathfrak{P}$. Let $x_{\mathfrak{P}}$ be one of the generators of* Norm($\mathfrak{P}$). *Then we have*:

(I) $w^{1-\tau_{\mathfrak{P}}} s_\rho(x_{\mathfrak{P}}) \in \mathbb{F}_q^\times$; and

(II) $\tau_{\mathfrak{P}} \rho = s_\rho(x_{\mathfrak{P}})^{-1} \cdot \rho \cdot s_\rho(x_{\mathfrak{P}})$.

PROOF: Let $\mathfrak{P}^*$ be a place of $H_e(w)$ sitting over $\mathfrak{P}$. We know from [10], Lemma 9.4, that the leading coefficient $s_{\rho'}(x_{\mathfrak{P}})$ belongs to $\mathbb{F}_q^\times$ modulo $\mathfrak{P}$, which means that

$$w^{1-N_{\mathfrak{P}}} s_\rho(x_{\mathfrak{P}}) \in \mathbb{F}_q^\times \pmod{\mathfrak{P}}$$

as $N_{\mathfrak{P}} = q^{\deg x_{\mathfrak{P}}}$ by definition of $x_{\mathfrak{P}}$. Now $w^{1-\tau_{\mathfrak{p}}} \in \boldsymbol{\kappa}(\infty)$ by (4.3), and so the congruence

$$w^{1-\tau_{\mathfrak{P}}} s_\rho(x_{\mathfrak{P}}) \in \mathbb{F}_q^\times \pmod{\mathfrak{P}^*}$$

implies (I). For (II), we note from $\rho = w\rho' w^{-1}$ that $w^{\tau_{\mathfrak{P}}-1}$ is an isomorphism from ρ to $\tau_{\mathfrak{P}}\rho$. Since $\mathbb{F}_q^\times \subseteq \mathrm{Aut}(\rho)$, (I) $\Rightarrow$ (II). □

COROLLARY 4.8: *Let* $w_0 = w^{q-1}$. *Then*

(1) $w_0^{\tau_{\mathfrak{P}}} = s_\rho(x_{\mathfrak{P}})^{q-1} w_0$.

(2) $I^*(\rho) = H_e(w_0)$, *and* $[I^*(\rho) : H_e] = r = W_\infty / W_k$.

(3) *A finite place* $\mathfrak{p}$ *of* k *which is unramified in* $I^*(\rho)/k$ *splits completely in* $I^*(\rho)/k$ *if and only if* $\mathfrak{p} = xA_\infty$ *with* $\mathrm{sgn}(x) \in \mathbb{F}_q^\times$.

PROOF: Assertion (1) follows from (I) above, and (1) implies that $[H_e(w_0) : H_e] = r$ since s_ρ is surjective by Proposition 4.5. Now $I^*(\rho) \subseteq H_e(w_0)$ from the definitions, and (II) implies that ρ has r distinct Galois conjugates over H_e since in fact ([10], Corollary 3.9) $\mathrm{Aut}(\rho) = \mathbb{F}_q^\times$. Thus, $I^*(\rho) = H_e(w_0)$. For (3), let $\mathfrak{p} = xA_\infty$ split completely in H_e/k, and let $\mathfrak{P} \in M_{H_e}$ sit over $\mathfrak{p}$. Then we can take $x_{\mathfrak{P}} = x$ since $\mathrm{Norm}(\mathfrak{P}) = \mathfrak{p}$, and so (3) follows from (1). □

Part (3) of this last corollary allows us to identify $I^*(\rho)$ by class field theory. Let J_k be the idèle group of k, and let $U_e^* \subseteq J_k$ be the subgroup consisting of those idèles $\boldsymbol{i}$ such that $\boldsymbol{i}_{\mathfrak{p}}$ is a unit of each finite place $\mathfrak{p}$ of k and such that $\mathrm{sgn}(\boldsymbol{i}_\infty) = 1$. Let π_∞ be a *positive* uniformizer at ∞. Then, as we easily compute, the subgroup

$$J_e^* = k^\times \cdot \pi_\infty^{\mathbf{Z}} \cdot U_e^* \tag{4.4}$$

has index rhd_∞ in J_k and therefore corresponds to an abelian extension E/k of degree rhd_∞. From the definition of J_e^*, E/k is unramified except at ∞ and the ramification index at ∞ is r. We conclude further that the places $\mathfrak{p} \in M_k$ which split completely in E/k are exactly those mentioned in (3) of the corollary. Therefore, the Galois closure of $I^*(\rho)$ over k equals E, and so in fact $I^*(\rho) = E$ since $[E : k] = [I^*(\rho) : k]$. In particular, we see that $I^*(\rho)/k$ is abelian. We see also that $I^*(\rho)$ is independent of the choice of ρ.

DEFINITION 4.9: Let H_e^* be the common field $I^*(\rho)$ for the sgn-normalized A_∞-modules ρ of generic characteristic. We call H_e^* the *normal-*

izing field with respect to sgn (or, for short, the *normalizing field*). Let $G_{\mathfrak{e}}^* \simeq J_k/J_{\mathfrak{e}}^*$ be the Galois group of $H_{\mathfrak{e}}^*/k$.

Since every finite k-place is unramified in $H_{\mathfrak{e}}^*/k$, the Artin automorphism $\tau_{\mathfrak{a}} \in G_{\mathfrak{e}}^*$ is defined for every ideal $\mathfrak{a} \in \boldsymbol{M}_\infty$. We can identify the A_∞-module $\tau_{\mathfrak{a}}\rho$ in terms of the operation $*$ introduced in [10] §3. Let $\rho_{\mathfrak{a}}$ be the isogeny which satisfies $\rho_{\mathfrak{a}} \cdot \rho_x = (\mathfrak{a} * \rho)_x \cdot \rho_{\mathfrak{a}}$ for all $x \in A_\infty$. Then $\deg \rho_{\mathfrak{a}} = \deg \mathfrak{a}$ and so

$$s_{\mathfrak{a} * \rho}(x) = s_\rho(x)^{N_{\mathfrak{a}}} = s_\rho(x)^{\tau_{\mathfrak{a}}} = s_{\tau_{\mathfrak{a}}\rho}(x) \tag{4.5}$$

for $x \in A_\infty$. In particular, we see that $\mathfrak{a} * \rho$ is also sgn-normalized. Now for a prime ideal $\mathfrak{p} \in \boldsymbol{M}_\infty$, the $H_{\mathfrak{e}}$-forms of $\tau_{\mathfrak{p}}\rho$ and $\mathfrak{p} * \rho$ are Ω-isomorphic by [10], Theorem 8.5, which implies that $\tau_{\mathfrak{p}}\rho = a^{-1}(\mathfrak{p} * \rho)a$ for some $a \in \Omega^\times$; and we note from (4.5) that $a \in \boldsymbol{\kappa}(\infty)$. Now let $\mathfrak{P}$ be a place of $H_{\mathfrak{e}}^*$ which divides $\mathfrak{p}$. Since $\tau_{\mathfrak{p}}\rho$ and $\mathfrak{p} * \rho$ have *equal* reductions modulo $\mathfrak{P}$ ([10], Corollary 3.8), a is an automorphism of this reduction and therefore belongs to $\mathbb{F}_q^\times$. We have now shown that $\tau_{\mathfrak{p}}\rho = \mathfrak{p} * \rho$. In order to show $\tau_{\mathfrak{a}}\rho = \mathfrak{a} * \rho$ in general, we proceed by induction on the number of prime ideals dividing $\mathfrak{a}$. Assuming $\tau_{\mathfrak{a}}\rho = \mathfrak{a} * \rho$ for a given $\mathfrak{a}$, we have for any prime ideal $\mathfrak{p}$

$$\tau_{\mathfrak{a}\mathfrak{p}}\rho = \tau_{\mathfrak{p}}(\mathfrak{a} * \rho) = \mathfrak{a} * \tau_{\mathfrak{p}}\rho = \mathfrak{a} * (\mathfrak{p} * \rho) = (\mathfrak{a}\mathfrak{p}) * \rho$$

since $*$ commutes with Galois action.

We have now proved

THEOREM 4.10: *The normalizing field $H_{\mathfrak{e}}^*$ is abelian of degree rhd_∞ over k. The extension $H_{\mathfrak{e}}^*/k$ is unramified except at ∞, and $H_{\mathfrak{e}}^*/H_{\mathfrak{e}}$ is totally ramified at ∞. For a given* sgn*-normalized A_∞-module ρ of generic characteristic, we have $\tau_{\mathfrak{a}}\rho = \mathfrak{a} * \rho$ for every ideal $\mathfrak{a} \in \boldsymbol{M}_\infty$.*

COROLLARY 4.11: *For $x \in A_\infty$, let τ_x be the automorphism assigned to the principal ideal xA_∞ by the Artin map. Then*

$$\tau_x\rho = s_\rho(x)^{-1} \cdot \rho \cdot s_\rho(x). \tag{4.6}$$

PROOF: For $\mathfrak{a} = xA_\infty$, $\mathfrak{a} * \rho$ equals the right hand side of (4.6) by [10], Lemma 3.5. □

In the remainder of this §, ρ is a fixed sgn-normalized A_∞-module of generic characteristic. Let A_∞ act on Ω through ρ, and let Ω_ρ denote the ordinary A_∞-module associated to this action. Consider now the fields $K_{\mathfrak{m}} = H_{\mathfrak{e}}^*(\Lambda_{\mathfrak{m}})$ obtained by adjoining to $H_{\mathfrak{e}}^*$ the submodule $\Lambda_{\mathfrak{m}} \subseteq \Omega_\rho$ of $\mathfrak{m}$-torsion points, $\mathfrak{m} \in \boldsymbol{M}_\infty$, $\mathfrak{m} \neq \mathfrak{e}$. We recall ([10], §2) that $\Lambda_{\mathfrak{m}} \simeq A_\infty/\mathfrak{m}$, which implies that the group of A_∞-automorphisms of $\Lambda_\mathfrak{m}$ is isomorphic to $(A_\infty/\mathfrak{m})^\times$ in a natural way.

The extension $K_\mathfrak{m}/H_\mathfrak{e}^*$ is Galois because $\Lambda_\mathfrak{m}$ is precisely the set of roots of the linear polynomial $\rho_\mathfrak{m}(t) \in H_\mathfrak{e}^*[t]$. Since A_∞ acts on Ω_ρ via polynomials with coefficients in $H_\mathfrak{e}^*$, the A_∞-action on $\Lambda_\mathfrak{m}$ commutes with the Galois action over $H_\mathfrak{e}^*$. Therefore, restriction to $\Lambda_\mathfrak{m}$ provides a natural monomorphism

$$g_\mathfrak{m} : \mathrm{Gal}(K_\mathfrak{m}/H_\mathfrak{e}^*) \to (A_\infty/\mathfrak{m})^\times,$$

which shows that $K_\mathfrak{m}/H_\mathfrak{e}^*$ is abelian. By examining the ramification at the places of $H_\mathfrak{e}^*$ which sit over the primes dividing $\mathfrak{m}$ as in [10], §9, we may show that $[K_\mathfrak{m} : H_\mathfrak{m}^*] = \Phi(\mathfrak{m})$ so that $g_\mathfrak{m}$ is actually an isomorphism.

Let $\mathfrak{P}$ be a place of $H_\mathfrak{e}^*$ which does not ramify in $K_\mathfrak{m}/H_\mathfrak{e}^*$, let $\mathfrak{a}$ be its norm down to k, and let $\sigma_\mathfrak{P} \in \mathrm{Gal}(K_\mathfrak{m}/H_\mathfrak{e}^*)$ be the Frobenius automorphis associated to $\mathfrak{P}$. Since $\tau_\mathfrak{a}$ acts as the identity on $H_\mathfrak{e}^*$ (and $H_\mathfrak{e}$), we deduce from Corollary 4.11 that $\mathfrak{a} = xA_\infty$ is a principal ideal with $\mathrm{sgn}(x) \in \mathbb{F}_q^\times$. Choose $x_\mathfrak{P}$ to be the unique generator of $\mathfrak{a}$ such that $\mathrm{sgn}(x_\mathfrak{P}) = 1$. Then arguing as in the proof of [10], Theorem 9.5, we can show that

$$g_\mathfrak{m}(\sigma_\mathfrak{P}) = \mathrm{can}(x_\mathfrak{P}) \tag{4.7}$$

where can: $A_\infty \to A_\infty/\mathfrak{m}$ is the canonical morphism.

We are now ready to prove that the extension $K_\mathfrak{m}/k$ is abelian and to identify it in the catalogue provided by class field theory. For each finite $\mathfrak{p} \in M_k$, let $\mathfrak{p}^e$ be the highest power of $\mathfrak{p}$ dividing $\mathfrak{m}$; and let $U(\mathfrak{m}) \subseteq J_k$ consist of those idèles $\boldsymbol{i}$ such that $\boldsymbol{i}_\mathfrak{p}$ is a $\mathfrak{p}$-unit satisfy $\boldsymbol{i}_\mathfrak{p} \equiv 1 \pmod{\mathfrak{p}^e}$ for each finite $\mathfrak{p}$. Let $U^*(\mathfrak{m}) \subseteq U(\mathfrak{m})$ consist of the idèles $\boldsymbol{i}$ satisfying the additional condition $\mathrm{sgn}(\boldsymbol{i}_\infty) = 1$. Let π_∞ be a positive uniformer at ∞. Then the subgroup

$$J_\mathfrak{m}^* = k^\times \cdot \pi_\infty^\mathbf{Z} \cdot U^*(\mathfrak{m}) \tag{4.8}$$

has index $\mathrm{rhd}_\infty \Phi(\mathfrak{m})$ in J_k and therefore corresponds to an abelian extension $E_\mathfrak{m}^*/k$ of degree $\mathrm{rhd}_\infty \Phi(\mathfrak{m})$. From the definition (4.8) of $J_\mathfrak{m}^*$, the ramification number in $E_\mathfrak{m}^*/k$ of each finite place $\mathfrak{p}$ is $\Phi(\mathfrak{p}^e)$, and the ramification number at ∞ is W_∞. We conclude also that $\mathfrak{p} \in M_k$ splits completely in $E_\mathfrak{m}^*/k$ if and only if the A_∞-ideal determined by $\mathfrak{p}$ is generated by a *positive* element $x \in A_\infty$ satisfying $x \equiv 1 \pmod{\mathfrak{m}}$. By (4.7), this set of places differs by at most a finite number of places from the set of places which split completely in $K_\mathfrak{m}/k$. Therefore $E_\mathfrak{m}^*$ is the Galois closure of $K_\mathfrak{m}$ over k, and so in fact $K_\mathfrak{m} = E_\mathfrak{m}^*$ as $[K_\mathfrak{m} : k] = [E_\mathfrak{m}^* : k]$. This proves that $K_\mathfrak{m}/k$ is abelian and also that $K_\mathfrak{m}$ is independent of the choice of the sgn-normalized elliptic module ρ.

Let $G_\mathfrak{m}^*$ be the Galois group of $K_\mathfrak{m}/k$. From our observations above, the Artin automorphism $\sigma_{\mathfrak{a}} \in G^*_{\mathfrak{m}}$ is defined for every ideal $\mathfrak{a} \in \boldsymbol{M}_\infty$ which is prime to $\mathfrak{m}$. Let $\rho' = \tau_{\mathfrak{a}}\rho = \mathfrak{a} * \rho$, and let $\Lambda'_{\mathfrak{m}}$ be the set of $\mathfrak{m}$-torsion points of ρ'. Then for fixed $\lambda \in \Lambda_{\mathfrak{m}}$, we have

$$\rho'_x(\lambda^{\sigma_{\mathfrak{a}}}) = [\rho_x(\lambda)]^{\sigma_{\mathfrak{a}}} \tag{4.9}$$

for every $x \in A_\infty$. If $x \in \mathfrak{m}$, (4.9) shows that $\rho'_x(\lambda^{\sigma_{\mathfrak{a}}}) = 0$, which implies that $\sigma_{\mathfrak{a}}$ maps $\Lambda_{\mathfrak{m}}$ into $\Lambda'_{\mathfrak{m}}$. Letting A_∞ act on $\Lambda'_{\mathfrak{m}}$ through ρ', we see further from (4.9) that $\sigma_{\mathfrak{a}}$ is an A_∞-module isomorphism of $\Lambda_{\mathfrak{m}}$ onto $\Lambda'_{\mathfrak{m}}$. Now from the defining equation $\rho_{\mathfrak{a}}\rho_x = \rho'_x\rho_{\mathfrak{a}}$, we see that the map $\lambda \to \rho_{\mathfrak{a}}(\lambda)$ is also an A_∞-module morphism from $\Lambda_{\mathfrak{m}}$ into $\Lambda'_{\mathfrak{m}}$. In fact since $\mathfrak{a}$ is prime to $\mathfrak{m}$, this map is an isomorphism because the roots of $\rho_{\mathfrak{a}}(t)$ are precisely the $\mathfrak{a}$-torsion points for A_∞ acting on Ω through ρ.

THEOREM 4.12: *For every* $\lambda \in \Lambda_{\mathfrak{m}}$,

$$\lambda^{\sigma_{\mathfrak{a}}} = \rho_{\mathfrak{a}}(\lambda). \tag{4.10}$$

PROOF: Consider first the case $\mathfrak{a} = \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal not dividing $\mathfrak{m}$. Let $\mathfrak{P}$ be a place of $K_{\mathfrak{m}}$ lying over $\mathfrak{p}$. Then since the polynomial $\rho_{\mathfrak{p}}(t)/t$ is Eisenstein at any place of $H^*_{\mathfrak{e}}$ laying over $\mathfrak{p}$ ([10], Proposition 7.6) and has degree $N_{\mathfrak{p}}$, we have

$$\lambda^{\sigma_{\mathfrak{p}}} \equiv \lambda^{N_{\mathfrak{p}}} \equiv \rho_{\mathfrak{p}}(\lambda) \pmod{\mathfrak{P}}.$$

Now the elements of $\Lambda'_{\mathfrak{m}}$ are distinct modulo $\mathfrak{P}$ (cf. [10], Lemma 9.3), and so this congruence implies the equality (4.10) for $\mathfrak{a} = \mathfrak{p}$. To prove (4.10) in general, we proceed by induction as in the remarks preceding Theorem 4.10 above. □

Let $C_{\mathfrak{m}}$ be the integral closure of A_∞ in $K_{\mathfrak{m}}$. Since ρ is normalized, the coefficients of each ρ_x, $x \in A_\infty$, are integers away from ∞ ([10], Corollary 7.4). Therefore the torsion points of A_∞ acting on Ω through ρ are also integers away from ∞, and so in particular $\Lambda_{\mathfrak{m}} \subseteq C_{\mathfrak{m}}$.

COROLLARY 4.13: *Let* $\lambda \in \Lambda_{\mathfrak{m}}$, $\lambda \neq 0$. *Then for all* $\sigma \in G^*_{\mathfrak{m}}$, $\lambda^{\sigma - 1} \in C^\times_{\mathfrak{m}}$.

PROOF: Let $\sigma = \sigma_{\mathfrak{a}}$, $\mathfrak{a} \in \boldsymbol{M}_\infty$, $\mathfrak{a}$ prime to $\mathfrak{m}$. Since t divides $\rho_{\mathfrak{a}}(t)$, (4.10) shows that at least $\lambda^{\sigma-1} \in C_{\mathfrak{m}}$. This being true for all $\sigma \in G^*_{\mathfrak{m}}$ and any $\lambda' \neq 0$, $\lambda' \in \Lambda'_{\mathfrak{m}}$, we see that

$$\lambda^{1-\sigma} = \lambda^{\sigma(\sigma^{-1}-1)}$$

also belongs to $C_{\mathfrak{m}}$. □

For $x \in k^\times$, x prime to $\mathfrak{m}$, let $\sigma_x \in G^*_{\mathfrak{m}}$ denote the automorphism assign to the fractional ideal xA_∞ by the Artin map. Then we have

COROLLARY 4.14: *If* $x \in k^{\times}$, $x \equiv 1 \pmod{\mathfrak{m}}$, *then*

$$\lambda^{\sigma_x} = s_\rho(x)^{-1} \cdot \lambda \tag{4.11}$$

for all $\lambda \in \Lambda_{\mathfrak{m}}$.

PROOF: Assume first that $x \in A_\infty$ so that $\mathfrak{a} = xA_\infty$ is an integral ideal. Since $s_\rho(x)$ is the leading coefficient of ρ_x, we have $\rho_{\mathfrak{a}} = s_\rho(x)^{-1}\rho_x$ so that by (4.10)

$$\lambda^{\sigma_x} = s_\rho(x)^{-1} \cdot \rho_x(\lambda).$$

But since $x \equiv 1 \pmod{\mathfrak{m}}$, $\rho_x(\lambda) = \lambda$ and so (4.11) is valid.

In general, let $x = y/z$ with y, $z \in A_\infty$ chosen so that $y \equiv z \equiv 1 \pmod{\mathfrak{m}}$. Then since σ_x is trivial on $H_{\mathfrak{e}} \supseteq \boldsymbol{\kappa}(\infty)$, we have

$$s_\rho(y)^{-1} \cdot \lambda = \lambda^{\sigma_y} = (\lambda^{\sigma_z})^{\sigma_x} = s_\rho(z)^{-1} \cdot \lambda^{\sigma_x}$$

by two applications of (4.11). □

Let $V_{\mathfrak{m}} \subseteq k^{\times}$ consist of those elements x such that $x \equiv 1 \pmod{\mathfrak{m}}$, and let $G_\infty^* \subseteq G_{\mathfrak{m}}^*$ be the image of $V_{\mathfrak{m}}$ under the Artin map. By the Weak Approximation Theorem, sgn takes every possible non-zero value on $V_{\mathfrak{m}}$, and so (4.11) implies that G_∞^* is isomorphic to $\boldsymbol{\kappa}(\infty)^{\times}$. Thus G_∞^* is cyclic of order W_∞.

PROPOSITION 4.15: *The subgroup* G_∞^* *is both the decomposition group and the inertial group at* ∞. *If* $K_{\mathfrak{m},\infty}$ *is the completion of* $K_{\mathfrak{m}}$ *at some place lying over* ∞, *then* $K_{\mathfrak{m},\infty}/k_\infty$ *is a totally ramified Kummer extension of degree* W_∞, *and* $K_{\mathfrak{m},\infty} = k_\infty(\lambda)$, *where* λ *is any non-zero element of* $\Lambda_{\mathfrak{m}}$. *If* $\phi^{(\infty)}$: $k_\infty^{\times} \to G_\infty^*$ *is the norm residue symbol at* ∞, *then*

$$\phi_x^{(\infty)}(\lambda) = s_\rho(x) \cdot \lambda \tag{4.12}$$

for all $x \in k_\infty^{\times}$.

PROOF: Let ϕ: $J_k \to G_{\mathfrak{m}}^*$ be the global reciprocity law. For $x \in V_{\mathfrak{m}}$, let x^* be the idèle which differs from x only in that $x_\infty^* = 1$. By the properties of ϕ, we have $\phi_{x^*} = \sigma_x$ and hence $\phi_x^{(\infty)} = \sigma_x^{-1}$. Thus (4.12) holds for $x \in V_{\mathfrak{m}}$ by the global equation (4.11). Now $V_{\mathfrak{m}}$ is dense in $k_\infty^{\times}$ by weak approximation and so (4.12) is indeed valid for all $x \in k_\infty^{\times}$. It is now clear that G_∞^* is a quotient of the decomposition group at ∞ and that $k_\infty(\lambda)/k_\infty$ is Kummer of degree W_∞. Turning now to the group $J_{\mathfrak{m}}^*$ associated to $K_{\mathfrak{m}}/k$, we observe because $\pi_\infty \in J_{\mathfrak{m}}^*$ that the decomposition and inertial groups at ∞ are indeed equal; and we have already observed that the ramification number at ∞ is W_∞. Thus $K_{\mathfrak{m},\infty} = k_\infty(\lambda)$. □

Let $E_{\mathfrak{m}}$ be the fixed field of G_∞^*. Then ∞ splits completely in $E_{\mathfrak{m}}/k$, and so $E_{\mathfrak{m}}$ is a real class field at ∞. We will soon identify $E_{\mathfrak{m}}$ as the ray class field $H_{\mathfrak{m}}$, but first we note

COROLLARY 4.16: *Let $N_{\mathfrak{m}}^- : K_{\mathfrak{m}} \to E_{\mathfrak{m}}$ be the norm map. Then the subgroup $N_{\mathfrak{m}}^-(K_{\mathfrak{m}}^\times)$ of $E_{\mathfrak{m}}^\times$ consists of totally positive elements.*

PROOF: By definition of $E_{\mathfrak{m}}$, $N_{\mathfrak{m}}^-$ is the restriction to $K_{\mathfrak{m}}$ of the local norm map at ∞. Therefore, $\phi^{(\infty)}$ is trivial on $N_{\mathfrak{m}}^-(K_{\mathfrak{m}}^\times)$ which together with (4.12) shows that this subgroup consists of positive elements in at least one k-embedding of $E_{\mathfrak{m}}$ into k_∞. Since this subgroup is invariant under Galois action over k, we are done. □

Let ϕ: $J_k \to G_{\mathfrak{m}}^*$ be the reciprocity law morphism. In order to identify the abelian extension $E_{\mathfrak{m}}/k$, we will compute its idèle group $\phi^{-1}(G_\infty^*)$. Every element $u \in U(\mathfrak{m})$ can be written $u = u^* u_\infty$ where u_∞ is the idèle whose component at ∞ equals that of u and whose component at any finite place equals 1. Since $u^* \in U^*(\mathfrak{m})$, $\phi(u) = \phi(u_\infty) = \phi_{u_\infty}^{(\infty)} \in G_\infty^*$. Thus, $U(\mathfrak{m}) \subseteq \phi^{-1}(G_\infty^*)$ and so the subgroup

$$J_{\mathfrak{m}} = k^\times \cdot \pi_\infty^{\mathbf{Z}} \cdot U(\mathfrak{m})$$

is also contained in $\phi^{-1}(G_\infty^*)$. But now, as we easily compute, $k^\times \cdot U^*(\mathfrak{m})$ has index W_∞ in $k^\times \cdot U(\mathfrak{m})$, and so $J_{\mathfrak{m}} = \phi^{-1}(G_\infty^*)$.

THEOREM 4.17: *The fixed field of G_∞^* is the ray class field $H_{\mathfrak{m}}$. If λ is a generator of $\Lambda_{\mathfrak{m}}$ as an A_∞-module, then*

$$\alpha = N_{\mathfrak{m}}^-(\lambda) = -\lambda^{W_\infty} \tag{4.13}$$

is a totally positive element of $H_{\mathfrak{m}}$. If $T(\mathfrak{m}) = \{\mathfrak{q}\}$, then α generates the ideal $(\mathfrak{q})_{\mathfrak{m}}^r$ in $B_{\mathfrak{m}}$ where $r = W_\infty / W_k$; otherwise $\alpha \in B_{\mathfrak{m}}^\times$.

PROOF: The computation $J_{\mathfrak{m}} = \phi^{-1}(G_\infty^*)$ above allows us to identify $E_{\mathfrak{m}}$ as $H_{\mathfrak{m}}$, and then Corollary 4.16 shows α to be totally positive.

For the last assertion, let us first consider the case $T(\mathfrak{m}) = \{\mathfrak{q}\}$. Put $\mathfrak{a} = \mathfrak{m}/\mathfrak{q}$, and let $f_{\mathfrak{m}}(t)$ be the quotient polynomial $\rho_{\mathfrak{m}}(t)/\rho_{\mathfrak{q}}(t)$. Then $f_{\mathfrak{m}}(t) \in C_{\mathfrak{e}}[t]$, where $C_{\mathfrak{e}}$ is the integral closure of A_∞ in $H_{\mathfrak{e}}^*$; and the roots of $f_{\mathfrak{m}}(t)$ are precisely the generators of the A_∞-module $\Lambda_{\mathfrak{m}}$.

LEMMA 4.18: *Let $[\mathfrak{q}]_{\mathfrak{e}}$ be the product in $C_{\mathfrak{e}}$ of the prime ideals dividing $\mathfrak{q}$. Then*

$$f_{\mathfrak{m}}(0) \cdot C_{\mathfrak{e}} = [\mathfrak{q}]_{\mathfrak{e}}. \tag{4.14}$$

PROOF: By [10], Proposition 7.6, $f_{\mathfrak{m}}(t)$ is Eisenstein at every prime factor of $[\mathfrak{q}]_{\mathfrak{e}}$. Therefore, we need only show that $f_{\mathfrak{m}}(0)$ is a unit at all the other prime ideals of $C_{\mathfrak{e}}$. To that end, choose $e>0$ so that $\mathfrak{m}^e = xA_\infty$ is principal, and put $\mathfrak{b} = \mathfrak{m}^{e-1}$. Then

$$s_\rho(x)^{-1}\rho_x = (\mathfrak{m} * \rho)_{\mathfrak{b}} \cdot \rho_{\mathfrak{m}}$$

by [10], Theorem 3.10, which shows that $D(\rho_{\mathfrak{m}})$ divides x in $C_{\mathfrak{e}}$. Thus, $f_{\mathfrak{m}}(0)$ also divides x in $C_{\mathfrak{e}}$, and we are done. □

LEMMA 4.19: *Let* $[\mathfrak{q}]_{\mathfrak{m}}$ *be the product in* $C_{\mathfrak{m}}$ *of the prime ideals dividing* $\mathfrak{q}$. *Then*

$$[\mathfrak{q}]_{\mathfrak{m}} = \lambda C_{\mathfrak{m}}. \tag{4.15}$$

PROOF: By Corollary 4.13, the product of the roots of $f_{\mathfrak{m}}(t)$ equals $u\lambda^{\Phi(\mathfrak{m})}$, where $u \in C_{\mathfrak{m}}^\times$. Thus, (4.14) shows that

$$[\mathfrak{q}]_{\mathfrak{e}} \cdot C_{\mathfrak{m}} = \lambda^{\Phi(\mathfrak{m})} C_{\mathfrak{m}}. \tag{4.16}$$

Now because $K_{\mathfrak{m}}/H_{\mathfrak{e}}^*$ is totally ramified at each prime factor of $[\mathfrak{q}]_{\mathfrak{e}}$,

$$[\mathfrak{q}]_{\mathfrak{e}} \cdot C_{\mathfrak{m}} = [\mathfrak{q}]_{\mathfrak{m}}^{\Phi(\mathfrak{m})}$$

which together with (4.16) proves our result. □

Since $\mathfrak{q}$ has ramification number $\Phi(\mathfrak{m})/(q-1)$ in $H_{\mathfrak{m}}/k$ and ramification number $\Phi(\mathfrak{m})$ in $K_{\mathfrak{m}}/k$, each prime factor of $(\mathfrak{q})_{\mathfrak{m}}$ has ramification number $W_k = q-1$ in $K_{\mathfrak{m}}/H_{\mathfrak{m}}$. Thus,

$$(\mathfrak{q})_{\mathfrak{m}}^r \cdot C_{\mathfrak{m}} = [\mathfrak{q}]_{\mathfrak{m}}^{W_\infty} = \lambda^{W_\infty} \cdot C_{\mathfrak{m}} = \alpha C_{\mathfrak{m}}$$

which implies $(\mathfrak{q})_{\mathfrak{m}}^r = \alpha B_{\mathfrak{m}}$, as required.

Finally, we consider the case $|T(\mathfrak{m})| \geqslant 2$. Let $\mathfrak{p}$ and $\mathfrak{q}$ divide $\mathfrak{m}$, $\mathfrak{p} \neq \mathfrak{q}$, and put $\mathfrak{a} = \mathfrak{m}/\mathfrak{p}$ and $\mathfrak{b} = \mathfrak{m}/\mathfrak{q}$. Then $\rho_{\mathfrak{a}}(\lambda)$ and $\rho_{\mathfrak{b}}(\lambda)$ are each respectively generators of $\Lambda_{\mathfrak{p}}$ and $\Lambda_{\mathfrak{q}}$, and so by Lemma 4.19 they generate $[\mathfrak{p}]_{\mathfrak{p}}$ and $[\mathfrak{q}]_{\mathfrak{q}}$ in the subfields $K_{\mathfrak{p}}$ and $K_{\mathfrak{q}}$ of $K_{\mathfrak{m}}$. Since these ideals generate relatively prime ideals in $C_{\mathfrak{m}}$, there are elements a and b in $C_{\mathfrak{m}}$ such that $a\rho_{\mathfrak{a}}(\lambda) + b\rho_{\mathfrak{b}}(\lambda) = 1$. Since λ divides both $\rho_{\mathfrak{a}}(\lambda)$ and $\rho_{\mathfrak{b}}(\lambda)$, we see that $\lambda \in C_{\mathfrak{m}}^\times$; and this certainly implies $\alpha \in B_{\mathfrak{m}}^\times$. This completes the proof of Theorem 4.17. □

§5. The Invariants $\xi(\Gamma)$.

A 1-*lattice* is a rank 1 A_∞-submodule of Ω. If Γ is a 1-lattice, then the infinite product

$$e_\Gamma(z) = z \cdot \prod_\gamma \left(1 - \frac{z}{\gamma}\right) \quad (\gamma \in \Gamma, \gamma \neq 0) \tag{5.1}$$

converges for all $z \in \Omega$ in the v_∞-topology, and the function $e_\Gamma(z)$ which it defines is an $\mathbb{F}_q$-linear endomorphism of Ω with period lattice Γ. One knows further ([2] or §4 of [10]) that A_∞ acts on $e_\Gamma(z)$ by "complex multiplications" through an elliptic A_∞-module ρ^Γ of generic characteristic. This means that

$$e_\Gamma(xz) = \rho_x^\Gamma(e_\Gamma(z)) \tag{5.2}$$

for all $x \in A_\infty$ and all $z \in \Omega$. Further ([10], §5), every elliptic A_∞-module of generic characteristic is ρ^Γ for a uniquely determined 1 lattice Γ.

We say that 1-lattices Γ and Γ' are *isomorphic* if $\Gamma' = w\Gamma$ for some element $w \in \Omega^\times$. Let us call Γ a *special* 1-lattice (relative to the choice of sgn) if ρ^Γ is sgn-normalized. Because $\xi\rho^\Gamma\xi^{-1} = \rho^{\xi\Gamma}$ for all $\xi \in \Omega^\times$, every 1-lattice Γ is isomorphic to a special 1-lattice; and further (cf. Proposition 4.7), the group of constants $\boldsymbol{\kappa}(\infty)^\times$ acts *transitively* on the special 1-lattices belonging to the isomorphism class of Γ. Therefore, if we define the *invariant* of Γ to be an element $\xi(\Gamma) \in \Omega^\times$ such that $\xi(\Gamma) \cdot \Gamma$ is special, then $\xi(\Gamma)$ will be determined up to multiplication by an element of $\boldsymbol{\kappa}(\infty)^\times$. For convenience, we will often ignore the fact that these invariants are not quite uniquely defined when writing equations which involve them.

THEOREM 5.1: *Let $\xi(\Gamma)$ be the invariant of the* 1*-lattice* Γ, *and put* $\rho = \rho^{\xi(\Gamma)\Gamma}$. *Then for any ideal* $\mathfrak{a} \in \boldsymbol{M}_\infty$ *and all* $z \in \Omega$,

$$\rho_{\mathfrak{a}}(\xi(\Gamma)e_\Gamma(z)) = \xi(\mathfrak{a}^{-1}\Gamma)e_{\mathfrak{a}^{-1}\Gamma}(z). \tag{5.3}$$

PROOF: Put $\Gamma_* = \xi(\Gamma)\cdot\Gamma$, a special 1-lattice. By [10], Proposition 5.10, $\Gamma'_* = D(\rho_{\mathfrak{a}})\mathfrak{a}^{-1}\Gamma_*$ is the 1-lattice associated to $\rho' = \mathfrak{a} * \rho$, itself a sgn-normalized module. Therefore, $\xi(\mathfrak{a}^{-1}\Gamma) = D(\rho_{\mathfrak{a}})\xi(\Gamma)$. Now by [10], Equation 5.11,

$$e_{\Gamma'_*}(D(\rho_{\mathfrak{a}})u) = \rho_{\mathfrak{a}}(e_{\Gamma_*}(u))$$

for all $u \in \Omega^\times$. Taking $u = \xi(\Gamma)z$ in this last equation, we arrive easily at (5.3). □

Fix now for the moment a special 1-lattice Γ, and put $\rho = \rho^\Gamma$. For $\mathfrak{m} \in \boldsymbol{M}$, $\mathfrak{m} \neq \mathfrak{e}$, let $\Lambda_{\mathfrak{m}}$ be the module of $\mathfrak{m}$-torsion points for A_∞ acting on Ω through ρ. Whereas in §3 we considered the elements of $\Lambda_{\mathfrak{m}}$ as algebraic objects, we now observe from (5.2) that

$$\Lambda_{\mathfrak{m}} = e_{\Gamma}(\mathfrak{m}^{-1}\Gamma), \tag{5.4}$$

which provides an analytic construction of these elements in the complete field Ω. We see in particular that if we take $\Gamma = \xi(\mathfrak{m})\cdot\mathfrak{m}$, a special 1-lattice isomorphic to $\mathfrak{m}$ itself, then

$$\lambda = \xi(\mathfrak{m})e_{\mathfrak{m}}(1) \tag{5.5}$$

generates $\Lambda_{\mathfrak{m}}$ as an A_∞-module. Thus, the analytic processes imposed by v_∞ allow us to specify a particular element of the field $K_{\mathfrak{m}}$. This element is not quite unique since $\xi(\mathfrak{m})$ is not unique, but the norm $\alpha = N_{\mathfrak{m}}^{-}(\lambda) = -\lambda^{W_\infty} \in H_{\mathfrak{m}}$ is unique (after the choice of sgn). We will show in §6 that this analytically specified element α is an $L_{\mathfrak{m}}$-function evalutor at $s = 0$, thus completing the proof of Theorem 1.1.

For use in §6, we will need the formula for $\xi(\mathfrak{c})$, $\mathfrak{c} \in \boldsymbol{I}_\infty$, provided by the next lemma. We adopt the convention that limits over x mean limits as x runs through a sequence of positive elements of A_∞ such that $N(x) = q^{\deg x}$ becomes infinitely large. The notation $a \in \mathfrak{c} \bmod x$ means that a runs through a complete set of representatives for $\mathfrak{c}$ modulo the subgroup $x\mathfrak{c}$.

LEMMA 5.2: *Let* $\mathfrak{c} \in \boldsymbol{I}_\infty$. *Then*

$$v_\infty(\xi(\mathfrak{c})) = -\lim_{x} \frac{1}{N(x)} \sum_{a \neq 0} v_\infty(e_{\mathfrak{c}}(a/x)) \quad (a \in \mathfrak{c} \bmod x). \tag{5.6}$$

PROOF: Since $\rho = \rho^{\xi(\mathfrak{c})\mathfrak{c}}$ is sgn-normalized, the elements $\xi(\mathfrak{c})e_{\mathfrak{c}}(a/x)$ for $a \in \mathfrak{c} \bmod x$ are the xA_∞ torsion points for A_∞ acting on Ω through ρ and therefore are precisely the roots of the polynomial $\rho_x(t) = t^{N(x)} + \ldots + xt$. Thus

$$x = \prod_{a \neq 0} \xi(\mathfrak{c})e_{\mathfrak{c}}(a/x) = \xi(\mathfrak{c})^{N(x)-1} \cdot \prod_{a \neq 0} e_{\mathfrak{c}}(a/x)$$

which implies (5.6). □

§6. Partial zeta-functions.

In this §, we put $d = d_\infty$ and $A = A_\infty$. For $\mathfrak{a} \in \boldsymbol{I}_\infty$, we write $N(\mathfrak{a}) = q^{\deg \mathfrak{a}}$ so that, in particular, $N(xA) = N(x) = q^{-dv_\infty(x)}$ for $x \in k^\times$. For integral $\mathfrak{a}$, $N(\mathfrak{a}) = |A/\mathfrak{a}|$. For any divisor $\mathfrak{d}$ of k, $L(\mathfrak{d})$ denotes the set of elements $y \in k$ such that $v_{\mathfrak{p}}(y\mathfrak{d}) \geqslant 0$ for all $\mathfrak{p} \in M_k$.

For given $\mathfrak{a} \in \boldsymbol{I}_\infty$, let $D(\mathfrak{a})$ be the least integer greater than or equal to $(\deg \mathfrak{a})/d$, and let $R(\mathfrak{a}) = d\cdot D(\mathfrak{a}) - \deg \mathfrak{a} \geqslant 0$. Since $\deg a \geqslant d\cdot D(\mathfrak{a})$

for all $a \in \mathfrak{a}$, we have

$$\mathfrak{a}\setminus\{0\} = \bigcup_{\nu=0}^{\infty} F_\nu(\mathfrak{a}) \tag{6.1}$$

where

$$F_\nu(\mathfrak{a}) = \{a \in \mathfrak{a} \mid \deg a = d \cdot D(\mathfrak{a}) + d\nu\}$$

for $\nu \geqslant 0$. We call $F_\nu(\mathfrak{a})$ the *ν-th layer* of $\mathfrak{a}$. By the Riemann-Roch Theorem, for δ large

$$T_\delta(\mathfrak{a}) = L(\mathfrak{a}^{-1}\infty^{D(\mathfrak{a})+\delta}) = \{0\} \cup \left(\bigcup_{\nu=0}^{\delta} F_\nu(\mathfrak{a})\right) \tag{6.2}$$

has order

$$|T_\delta(\mathfrak{a})| = q^{R(\mathfrak{a})+\delta d+1-g} \tag{6.3}$$

where g is the genus of k.

PROPOSITION 6.1: *Let $\mathfrak{a} \in \boldsymbol{I}_\infty$ and $t \in k\setminus\mathfrak{a}$ be given. The infinite series*

$$Z_\mathfrak{a}(s, t) = \sum_a \frac{1}{N(a+t)^s} \quad (a \in \mathfrak{a}) \tag{6.4}$$

and

$$V_\mathfrak{a}(s) = \sum_{a \neq 0} \frac{1}{N(a)^s} \quad (a \in \mathfrak{a}) \tag{6.5}$$

converge absolutely for $\mathrm{Re}(s) > 1$. *The functions $Z_\mathfrak{a}(s, t)$ and $V_\mathfrak{a}(s)$ are rational functions of q^{-ds} with no singularity other than a first order pole at $s = 1$. Further, $Z_\mathfrak{a}(0) = 0$ and $V_\mathfrak{a}(0) = -1$.*

PROOF: Choose $N^* \geqslant -v_\infty(t)$ so large that (6.3) holds for all $\delta > N^* - D(\mathfrak{a}) = N$. Then for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
Z_\mathfrak{a}(s, t) - \sum_{\substack{a \\ \deg a \leqslant dN^*}} \frac{1}{N(a+t)^s} &= \sum_{\substack{a \\ \deg a > dN^*}} \frac{1}{N(a)^s} \\
&= q^{-dD(\mathfrak{a})s} \sum_{\nu > N} q^{-d\nu s} |F_\nu(\mathfrak{a})| \\
&= -q^{-d(D(\mathfrak{a})+N+1)s} |T_N(\mathfrak{a})| \\
&\quad + (1 - q^{-ds}) \sum_{\delta > N} q^{-d\delta s} |T_\delta(\mathfrak{a})|
\end{aligned}$$

by partial summation. Invoking (6.3), we see that $Z_{\mathfrak{a}}(s, t)$ is rational in q^{-ds} with denominator $1 - q^{d(1-s)}$. Setting $s = 0$, we find

$$Z_{\mathfrak{a}}(0, t) = \sum_{\substack{a \\ \deg a \leqslant dN^*}} 1 - |T_N(\mathfrak{a})| = 0.$$

The proof for $V_{\mathfrak{a}}(s)$ goes the same way. □

For $\mathfrak{a} \in \boldsymbol{I}_\infty$ and $t \in k \backslash \mathfrak{a}$, we define

$$u_{\mathfrak{a}}(t) = (\log q)^{-1} \cdot \left[\frac{\mathrm{d}}{\mathrm{d}s}(Z_{\mathfrak{a}}(s, t)\right]_{s=0}. \tag{6.6}$$

The functions $u_{\mathfrak{a}} : k \backslash \mathfrak{a} \to \mathbb{R}$ satisfy the *distribution* properties introduced by B. Mazur (cf. [5] and §1 of [11]):

I. For any $x \in k$, $u_{x\mathfrak{a}}(xt) = u_{\mathfrak{a}}(t)$.
II. For all $a \in \mathfrak{a}$, $u_{\mathfrak{a}}(t + a) = u_{\mathfrak{a}}(t)$.
III. Let $\mathfrak{b} \in \boldsymbol{M}_\infty$. Then

$$\sum_a u_{\mathfrak{a}\mathfrak{b}}(t + a) = u_{\mathfrak{a}}(t) \quad (a \in \mathfrak{a} \bmod \mathfrak{b}),$$

where "$a \in \mathfrak{a} \bmod \mathfrak{b}$" means that a runs through a complete set of representatives for the cosets of $\mathfrak{a}\mathfrak{b}$ in $\mathfrak{a}$. The proofs of I-III are straight-forward and are left to the reader.

We turn now to the L-function (1.1) associated to a character $\psi \in \hat{G}_{\mathfrak{m}}$ for a fixed ideal $\mathfrak{m} \in \boldsymbol{M}_\infty$, $\mathfrak{m} \neq \mathfrak{e}$. Let $\boldsymbol{I}(\mathfrak{m}) \subseteq \boldsymbol{I}_\infty$ consist of the fractional ideals prime to $\mathfrak{m}$, and let $\boldsymbol{P}(\mathfrak{m}) \subseteq \boldsymbol{I}(\mathfrak{m})$ consist of the principal ideals which can be generated by an element $x \in k^*$ such that $x \equiv 1 \pmod{\mathfrak{m}}$. Since $H_{\mathfrak{m}}$ is the ray class field which is completely split over ∞, $\boldsymbol{P}(\mathfrak{m})$ is the kernel of the Artin map from $\boldsymbol{I}(\mathfrak{m})$ onto $G_{\mathfrak{m}}$. We may therefore view ψ as defined either on the ideals $\mathfrak{b} \in \boldsymbol{I}(\mathfrak{m})$ or on the "classes" $\mathscr{C} \in \boldsymbol{I}(\mathfrak{m})/\boldsymbol{P}(\mathfrak{m})$. In the sequel, whenever a summation over "$\mathfrak{b}$" (resp. "$\mathscr{C}$") appears, we understand that the summation is over ideals (resp. classes) in $\boldsymbol{I}(\mathfrak{m}) \cap \boldsymbol{M}_\infty$ (resp. $\boldsymbol{I}(\mathfrak{m})/\boldsymbol{P}(\mathfrak{m})$).

Multiplying out the product in (1.1) over the finite places, we find that

$$(1 - q^{-sd}) L_{\mathfrak{m}}(s, \psi) = \sum_{\mathfrak{b}} \frac{\psi(\mathfrak{b})}{N(\mathfrak{b})^s} = \sum_{\mathscr{C}} \psi(\mathscr{C}) \sum_{\mathfrak{b} \in \mathscr{C}} \frac{1}{N(\mathfrak{b})^s}$$

for $\mathrm{Re}(s) > 1$. For a given class $\mathscr{C}$, choose $\mathfrak{c} \in \mathscr{C} \cap \boldsymbol{M}_\infty$. Then $\mathfrak{b} \in \mathscr{C} \cap \boldsymbol{M}_\infty$ if and only if $\mathfrak{b} = x\mathfrak{c}$ for some $x \equiv 1 \pmod{\mathfrak{m}}$, $x \in \mathfrak{c}^{-1}$. Therefore for

$\mathrm{Re}(s) > 1$, we have

$$\zeta_{\mathscr{C}}(s) = \sum_{\mathfrak{b}\in\mathscr{C}} \frac{1}{N(\mathfrak{b})^s} = \sum_{\substack{x\in\mathfrak{c}^{-1}\\ x\equiv 1(\mathrm{mod}\ \mathfrak{m})}} N(x\mathfrak{c})^{-s}$$

$$= N(\mathfrak{c})^{-s} \sum_{b\in\mathfrak{c}^{-1}\mathfrak{m}} N(b+1)^{-s} = N(\mathfrak{c})^{-s} Z_{\mathfrak{c}^{-1}\mathfrak{m}}(s, 1). \tag{6.7}$$

We call $\zeta_{\mathscr{C}}(s)$ the *partial zeta-function* associated to the class $\mathscr{C}$. We see from Proposition 6.1 that $\zeta_{\mathscr{C}}(s)$ is a rational function of q^{-s} with no singularity other than a first order pole at $s = 1$. Therefore, the same is true of

$$(1 - q^{-ds}) L_{\mathfrak{m}}(s, \psi) = \sum_{\mathscr{C}} \psi(\mathscr{C}) \zeta_{\mathscr{C}}(s). \tag{6.8}$$

Since $\zeta_{\mathscr{C}}(0) = 0$, $L_{\mathfrak{m}}(s, \psi)$ is defined at $s = 0$, and we see via l'Hopital from (6.8) that

$$(d \log q) \cdot L_{\mathfrak{m}}(0, \psi) = \sum_{\mathscr{C}} \psi(\mathscr{C}) \zeta'_{\mathscr{C}}(0),$$

which by (6.7) we may rewrite in the form

$$d \cdot L_{\mathfrak{m}}(0, \psi) = \sum_{\mathfrak{c}} \psi(\mathfrak{c}) \cdot u_{\mathfrak{c}^{-1}\mathfrak{m}}(1) \tag{6.9}$$

where $\mathfrak{c} \in \boldsymbol{M}_\infty$ runs through any complete set of representatives for the classes in $\boldsymbol{I}(\mathfrak{m})/\boldsymbol{P}(\mathfrak{m})$. Our aim now is to evaluate the right hand side of (6.9) in terms of the element λ defined by (5.5).

For $\mathfrak{a} \in \boldsymbol{I}_\infty$, $t \in k\backslash\mathfrak{a}$ and $\mathrm{Re}(s) > 1$, we compute from the definition (6.4)

$$(\log q)^{-1} \cdot \left[\frac{\mathrm{d}}{\mathrm{d}s}(Z_{\mathfrak{a}}(s, t))\right] = -\sum_{a\in\mathfrak{a}} \deg(a+t) \cdot q^{-s\deg(a+t)}$$

$$= -\deg t \cdot q^{-s\deg t} - \sum_{\substack{a\in\mathfrak{a}\\ a\neq 0}} \deg(1 + t/a) \cdot q^{-s\deg(a+t)} - H(s)$$

where

$$H(s) = \sum_{\substack{a\in\mathfrak{a}\\ a\neq 0}} \deg(a) \cdot q^{-s\deg(a+t)}.$$

Now $\deg(1 + t/a) = -dv_\infty(1 + t/a) = 0$ for all but finitely many elements $a \in \mathfrak{a}$ and so

$$u_{\mathfrak{a}}(t) = dv_\infty(t) + d \cdot \sum_{\substack{a \in \mathfrak{a} \\ a \neq 0}} v_\infty(1 + t/a) - H(0)$$

$$= dv_\infty(e_{\mathfrak{a}}(t)) - H(0).$$

To evaluate $H(0)$, we note that for $\mathrm{Re}(s) > 1$

$$H(s) = \sum_{\substack{a \neq 0 \\ \deg a \leqslant \deg t}} \deg a \cdot q^{-s \deg(a+t)} + \sum_{\substack{a \\ \deg a > \deg t}} \deg a \cdot q^{-s \deg a}$$

$$= G(s) - (\log q)^{-1} \cdot \frac{\mathrm{d}V_{\mathfrak{a}}}{\mathrm{d}s}$$

where $G(s)$ is a polynomial in q^{-s} such that $G(0) = 0$. Thus, $H(0)$ is independent of t, and our value for $u_{\mathfrak{a}}(t)$ becomes

$$u_{\mathfrak{a}}(t) = dv_\infty(e_{\mathfrak{a}}(t)) + J_{\mathfrak{a}} \tag{6.10}$$

where $J_{\mathfrak{a}}$ is $(\log q)^{-1}$ times $\mathrm{d}V_{\mathfrak{a}}/\mathrm{d}s$ evaluated at $s = 0$. We may use (6.10) to evaluate $J_{\mathfrak{a}}$. For any positive $x \in A$ and $\mathrm{Re}(s) > 1$, we have

$$(1 - N(x)^{-s}) \cdot V_{\mathfrak{a}}(s) = V_{\mathfrak{a}}(s) - V_{x\mathfrak{a}}(s)$$

$$= \sum_{\substack{a \in \mathfrak{a} \bmod x \\ a \neq 0}} Z_{x\mathfrak{a}}(s, a)$$

which implies, by the distribution property I, that

$$\deg x \cdot V_{\mathfrak{a}}(0) = \sum_{\substack{a \in \mathfrak{a} \bmod x \\ a \neq 0}} u_{\mathfrak{a}}(a/x)$$

$$= \sum_{\substack{a \in \mathfrak{a} \bmod x \\ a \neq 0}} dv_\infty(e_{\mathfrak{a}}(a/x)) + (N(x) - 1) \cdot J_{\mathfrak{a}}.$$

Dividing by $N(x)$ above and taking the limit as $N(x)$ gets large, we see from Lemma 5.2 that in fact $J_{\mathfrak{a}} = dv_\infty(\xi(\mathfrak{a}))$. We have proved

THEOREM 6.1: *For any* $\mathfrak{a} \in \boldsymbol{I}_\infty$ *and* $t \in k \backslash \mathfrak{a}$, *we have*

$$u_{\mathfrak{a}}(t) = d \cdot v_\infty(\xi(\mathfrak{a}) e_{\mathfrak{a}}(t)). \tag{6.11}$$

We can now show that the element $\alpha = -\lambda^{W_\infty} \in H_{\mathfrak{m}}$, where λ is defined by (5.5), is an $L_{\mathfrak{m}}$-function evaluator at $s = 0$. Using (6.11) with $t = 1$ to evaluate the quantities $u_{\mathfrak{c}^{-1}\mathfrak{m}}(1)$ appearing on the right hand side of (6.9), we find that

$$L_{\mathfrak{m}}(0, \psi) = \sum_{\mathfrak{c}} \psi(\mathfrak{c}) \cdot v_\infty\big(\xi(\mathfrak{c}^{-1}\mathfrak{m}) e_{\mathfrak{c}^{-1}\mathfrak{m}}(1)\big)$$

$$= \sum_{\mathfrak{c}} \psi(\mathfrak{c}) \cdot v_\infty(\rho_{\mathfrak{c}}(\lambda))$$

by Theorem 5.1 with $\Gamma = \mathfrak{m}$, where $\mathfrak{c} \in \boldsymbol{M}_\infty$ runs through a set of representatives for the classes in $\boldsymbol{I}(\mathfrak{m})/\boldsymbol{P}(\mathfrak{m})$. Appealing to Theorem 4.12, we see that

$$L_{\mathfrak{m}}(0, \psi) = \sum_{\mathfrak{c}} \psi(\mathfrak{c}) v_\infty(\lambda^{\sigma_{\mathfrak{c}}})$$

$$= \frac{1}{W_\infty} \sum_{\mathfrak{c}} \psi(\mathfrak{c}) v_\infty(\alpha^{\sigma_{\mathfrak{c}}})$$

$$= \frac{1}{W_\infty} \sum_{\sigma \in G_{\mathfrak{m}}} \psi(\sigma) v_\infty(\alpha^{\sigma})$$

as required. Putting this evaluation together with the results of §4, we see that the element α does indeed meet the requirement of hypothesis $\boldsymbol{S}_\infty$ of §3.

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