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A CHARACTER APPROACH TO LOOIJENGA'S INVARIANT THEORY FOR GENERALIZED ROOT SYSTEMS

Peter Slodowy

0. Introduction

According to a theorem of Brieskorn [2], cf. [28], the semiuniversal deformation $X \to U$ of a simple singularity of type A_r, D_r, or E_r can be embedded into the adjoint quotient $\chi: G \to T/W$ of the corresponding simply connected complex Lie group G

$$\begin{array}{ccc} X & \hookrightarrow & G \\ \downarrow & & \downarrow \\ U & \hookrightarrow & T/W. \end{array}$$

At other places ([29], [30]), using results of Looijenga and Pinkham ([19], [27], [22]) we have indicated how at least the "simple" part $X^s \to U^s$ of the semi-universal deformation $X \to U$ of a simply elliptic or cusp singularity of degree $\leqslant 5$ can be embedded into a partial adjoint quotient $\mathcal{G} \to \mathcal{T}/W$ for a certain infinite-dimensional group G attached to a Kac-Moody Lie algebra. Such groups contain a Tits system (B, N) such that $T = B \cap N$ is a finite-dimensional torus and $W = N/T$ is an infinite cristallographic Coxeter group ([21], [26], [33]). Following Looijenga [18] one may specify a domain $\mathcal{T} \subset T$ on which W acts properly discontinuously. Thus $\mathcal{T}/W$ inherits the structure of an analytic space, in fact that of a complex manifold. The subset $\mathcal{G} \subset G$ on which one can define a map to $\mathcal{T}/W$ consists of all elements conjugate into B with "T-part" lying in $\mathcal{T}$.

In this paper we shall describe a step towards the goal of extending the partial quotient $\mathcal{G} \to \mathcal{T}/W$ to allow for an embedding of the full semiuniversal deformation $X \to U$. In [19] Looijenga showed how to identify the base space U with a partial compactification $\overline{\mathcal{T}/W}$ of $\mathcal{T}/W$. His construction of $\overline{\mathcal{T}/W}$ parallels procedures in the compactification theory for arithmetic quotients of hermitian symmetric spaces. The space $\overline{\mathcal{T}/W}$ is obtained as a W-quotient $\hat{\mathcal{T}}/W$ of an extension $\hat{\mathcal{T}}$ of $\mathcal{T}$ which can be described in terms of the infinite root system involved, cf., [18]. Our aim here is to show that this partial compactification turns up naturally when one considers the representation theory of the group G. Thus this article may be considered as a supplement to [18].

To be more precise, let us first recall the classical finite-dimensional situation.

Let G be a semisimple simply connected complex Lie group of rank r. Let $\rho_i : G \to GL(V_i)$, $i = 1, \dots r$ denote the fundamental irreducible representations and $\chi_i : G \to \mathbb{C}$, $\chi_i(g) = \text{trace } \rho_i(g)$, the corresponding characters. Then the adjoint quotient of G, i.e., the quotient in the category of algebraic varieties of G by its conjugation action, is given by the map

$$\chi : G \to \mathbb{C}^r$$

$$\chi(g) = (\chi_1(g), \dots, \chi_r(g)).$$

Let $T \subset G$ be a maximal torus, $N \subset G$ its normalizer in G and $W = N/T$ the corresponding Weyl group. Then the restriction of χ to T induces an isomorphism $T/W \xrightarrow{\sim} \mathbb{C}^r$

$$\begin{array}{ccc} T & \xrightarrow{\chi} & \mathbb{C}^r \\ & \searrow \quad \nearrow & \\ & T/W & \end{array}$$

The proof of this fact is the object of the classical exponential invariant theory (cf. [1] or [32]).

Now let G be a group attached to a Kac-Moody algebra $\mathfrak{g}$ with Tits system (B, N) and torus $T = B \cap N$. Let $\pi : N \to N/T = W$ be the projection onto the Weyl group W and let $s = \dim T$. Then there are s fundamental, in general infinite-dimensional, irreducible representations

$$\rho_i : G \to GL(V_i), \quad i = 1, \dots, s.$$

In this paper we only consider the restriction of these representations to N and we specify a domain $\mathcal{N} \subset N$ on which the characters

$$\chi_i(n) = \text{trace } \rho_i(n)$$

can be defined. This domain is the union $\mathcal{N} = \bigcup_w \mathcal{N}(w)$ of its connected components $\mathcal{N}(w) = \mathcal{N} \cap \pi^{-1}(w)$. Here w runs through the set W^p of pure elements of the Weyl group W (cf. 6.2).

We shall show that the map

$$\chi : \mathcal{N} \to \mathbb{C}^s$$

$$\chi(n) = (\chi_1(n), \dots, \chi_s(n))$$

factors through Looijenga's partial compactification

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\chi} & \mathbb{C}^s \\ \tau\downarrow & & \uparrow \\ \hat{\mathcal{T}} & \rightarrow & \hat{\mathcal{T}}/W \end{array}$$

inducing a local isomorphism from $\hat{\mathcal{T}}/W$ near "infinity" to an open subset of $\mathbb{C}^s$. Here $\tau: \mathcal{N} \to \hat{\mathcal{T}}$ is a surjective prolongation of the identity map

$$\mathcal{N} \cap T = \mathcal{T} \to \mathcal{T} \subset \hat{\mathcal{T}}$$

sending each component $\mathcal{N}(w)$, $w \neq 1$, onto a boundary component of $\hat{\mathcal{T}}$.

This result indicates that unlike the case of the simple singularities where the semiuniversal deformation is based at the subregular unipotent conjugacy class of the corresponding Lie group, simply elliptic or cusp singularities are not related to (pro-) unipotent elements in the corresponding groups. Instead, it seems that we now have to look at the elements in the group whose behaviour under the representations resembles that of Weyl group elements of infinite order. We hope that a further analysis of this point will finally allow to complete the full program mentioned before.

1. Kac-Moody algebras

Kac-Moody-Lie algebras are infinite-dimensional generalizations of semisimple Lie algebras. Their study was independently started by Kac and Moody ([8], [24]). In this chapter we shall give a quick description of their construction and some basic properties. Our presentation differs slightly from the usual ones in that we start the construction from a so-called root datum (cf. 1.4.). This seems to be more appropriate for the discussion of associated groups (cf. [33]) and also avoids some unnecessary technical complications (i.e. artificial roots). Moreover, relevant geometric aspects show up only after enlarging the usual Kac-Moody algebras in the way we introduce them (cf. [5], [3], [18]). For discussion of examples (i.e., in the affine case) we refer to [8], [25], [3], [30].

1.1. Cartan matrices

Let r be a natural number, $r \geqslant 1$. An $r \times r$ integer matrix $A = ((A_{ij})) \in M_r(\mathbb{Z})$ is called a (generalized) Cartan matrix if it has the following properties:

(1) $A_{ii} = 2$ for all $i = 1, \ldots, r$,

(2) $A_{ij} \leqslant 0$ for all $i \neq j$, $i, j = 1, \ldots, r$,

(3) $A_{ij} = 0$ if and only if $A_{ji} = 0$ for all $i, j = 1, \ldots, r$.

1.2. Coxeter diagrams

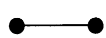
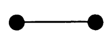
To each Cartan matrix $A \in M_r(\mathbb{Z})$ we associate a Coxeter diagram in the following way:

There are r vertices numbered from 1 to r, and two vertices i and j, $i \neq j$, are connected by an edge if and only if $A_{ij} \neq 0$ ($\Leftrightarrow A_{ji} \neq 0$). In addition, such an edge is valuated by an integer m_{ij} according to the following list:

$A_{ij}A_{ji}$	1	2	3	$\geqslant 4$
m_{ij}	3	4	6	∞

However the valuation 3 will be suppressed, for simplicity.

EXAMPLES:

A	diagram
$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	
$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$	

1.3. Classification of Cartan matrices

A Cartan matrix A is called indecomposable if the corresponding Coxeter diagram is connected. If A is not indecomposable, we may, after possibly rearranging indices, write A as a direct sum

$$A = \begin{pmatrix} \boxed{A_1} & & & \bigcirc \\ & \boxed{A_2} & & \\ \bigcirc & & \ddots & \\ & & & \boxed{A_n} \end{pmatrix}$$

of indecomposable Cartan matrices A_i, $i=1,\dots,n$.

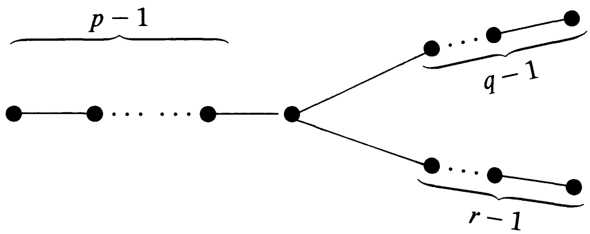
According to Vinberg ([35]) the indecomposable Cartan matrices may be divided into three classes:

(1) the spherical ones, or those of finite type, corresponding to the classical root systems of type A_r, B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2.

(2) the euclidean ones, or those of affine type, which correspond to the affine root systems (cf. [20]). They comprise the Cartan matrices of the extended Dynkin diagrams.

(3) the remaining ones, which are neither spherical nor euclidean, are called of general type.

EXAMPLE: Consider the Coxeter diagram $T_{p,q,r}$, p, q, $r \in N$.



This diagram determines uniquely a corresponding Cartan matrix, which we also denote by $T_{p,q,r}$. Then $T_{p,q,r}$ is of spherical, resp. euclidean, resp. general type if the sum

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$$

is greater than 1, resp. equal to 1, resp. smaller than 1. To any such diagram corresponds an isolated hypersurface singularity in $\mathbb{C}^3$ which is either simple, resp. simply elliptic, resp. a cusp. For other diagrams attached to related singularities cf. [19].

1.4. Root data

Let $A \in M_r(\mathbb{Z})$ be a Cartan matrix. A root datum for A is a triple (X, ∇, Δ) consisting of

- a free $\mathbb{Z}$-module X of finite rank,
- a free indexed subset $\nabla = \{h_1,\dots,h_r\} \subset X$,
- a free indexed subset $\Delta = \{\alpha_1,\dots,\alpha_r\} \subset X^* = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$, such that $\alpha_j(h_i) = A_{ij}$ for i, $j=1,\dots,r$.

The elements $\alpha_i \in \Delta$ (resp. $h_i \in \nabla$) will be called the simple roots (resp. the simple coroots). Let s be the $\mathbb{Z}$-rank of X. We call s the dimension and r the rank of the root datum.

1.5. The construction of Kac-Moody algebras

Let (X, ∇, Δ) be a root datum for a Cartan matrix $A \in M_r(\mathbb{Z})$. We let $\hat{\mathfrak{g}}$ be the complex Lie algebra generated by X and the elements e_i, f_i,

$i = 1, \ldots, r$ subject to the following relations:

$$[h, h'] = 0$$

$$[h, e_i] = \alpha_i(h) e_i$$

$$[h, f_i] = -\alpha_i(h) f_i$$

$$[e_i, f_j] = \delta_{ij}, h_j$$

$$(\text{ad } e_i)^{-A_{ij}+1}(e_j) = 0$$

$$(\text{ad } f_i)^{-A_{ij}+1}(f_j) = 0$$

for all $h, h' \in X$ and $i, j = 1, \ldots, r$.

The elements of X will generate a commutative subalgebra $\hat{\mathfrak{h}} \cong X \otimes_{\mathbf{Z}} \mathbf{C}$ in $\hat{\mathfrak{g}}$, and $\hat{\mathfrak{g}}$ will decompose into a direct sum of finite-dimensional eigenspaces

$$\hat{\mathfrak{g}} = \bigoplus_{\alpha \in \hat{\mathfrak{h}}^*} \hat{\mathfrak{g}}_\alpha$$

Here, for $\alpha \in \hat{\mathfrak{h}}^* = \text{Hom}_{\mathbf{C}}(\hat{\mathfrak{h}}, \mathbf{C})$ we put

$$\hat{\mathfrak{g}}_\alpha = \{ x \in \hat{\mathfrak{g}} \,|\, [h, x] = \alpha(h) x \quad \text{for all} \quad h \in \hat{\mathfrak{h}} \}.$$

Let $\mathfrak{r} \subset \hat{\mathfrak{g}}$ denote that ideal of $\hat{\mathfrak{g}}$ which is maximal among the ideals intersecting trivially $\hat{\mathfrak{h}}$. Then $\mathfrak{g} = \hat{\mathfrak{g}}/\mathfrak{r}$ is the Kac-Moody algebra associated to the root datum (X, ∇, Δ). According to a recent theorem of Gabber and Kac ([4]) the ideal $\mathfrak{r}$ is zero in case the Cartan matrix is "symmetrizable". This condition is fulfilled for all matrices of spherical or euclidean type as well as for those whose Coxeter diagrams are trees.

Let (X, ∇, Δ) and (X', ∇', Δ') be two root data for A of dimensions s and s', $s \leqslant s'$, and let $\mathfrak{g}$ and $\mathfrak{g}'$ be the corresponding Kac-Moody algebras. Then it is easily seen that $\mathfrak{g}'$ is isomorphic to a direct sum of $\mathfrak{g}$ and a commutative Lie algebra $\mathfrak{s}$ of dimension $s' - s$. It is also easily seen that such root data exist (cf. for example [35] §5). If A is indecomposable one has $s \geqslant r + \text{corank}(A)$. It is also possible to construct algebras in the above way by starting from a root datum (X, ∇, Δ) in which ∇ or Δ are not necessarily free (for the $\mathbf{Q}$-classification of such data, cf. [35] §5). The resulting algebras are easily obtained as subquotients of Kac-Moody algebras associated to root data in the sense of 1.4. A glance at the defining relations for a Kac-Moody algebra $\mathfrak{g}$ shows that

(1) the commutator subalgebra $\mathfrak{g}^c = [\mathfrak{g}, \mathfrak{g}]$ is generated by the h_i, e_i, f_i, $i = 1, \ldots, r$,

(2) the quotient $\mathfrak{g}/\mathfrak{g}^c$ is isomorphic to $\mathfrak{d} = \mathfrak{h}/\mathbb{C}\{h_1, \ldots, h_r\}$,

(3) $\mathfrak{g}$ is a semidirect product $\mathfrak{g} \cong \mathfrak{g}^c \rtimes \mathfrak{d}$.

We have $\mathfrak{g} = \mathfrak{g}^c$ if and only if det $A \neq 0$ and dim $X = r$. The algebra $\mathfrak{g}^c$ is the one which usually has been called the Kac-Moody algebra $\mathfrak{g}(A)$ associated to A. When det $A = 0$ this algebra is associated to a "singular" root datum (X, ∇, Δ) in which Δ is not free.

When A is decomposable into a direct sum of Cartan matrices A_i, $i = 1, \ldots, n$, then the commutator subalgebra $\mathfrak{g}(A)^c$ is a direct sum of the commutator subalgebras $\mathfrak{g}(A_i)^c$.

It turns out that $\mathfrak{g}(A)$ is finite-dimensional if and only if A is a direct sum of spherical matrices. In such a case $\mathfrak{g}(A)$ is the corresponding semisimple complex Lie algebra.

1.6. The root decomposition

Let A be a Cartan matrix, (X, ∇, Δ) a root datum for A and $\mathfrak{g}$ the Kac-Moody Lie algebra associated to this datum.

Like in the case of $\hat{\mathfrak{g}}$, the image of X in $\mathfrak{g}$ generates a commutative subalgebra $\mathfrak{h} \simeq X \otimes_{\mathbb{Z}} \mathbb{C}$. With respect to $\mathfrak{h}$ there is a decomposition into finite-dimensional eigenspaces

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

$$\mathfrak{g} = \{x \in \mathfrak{g}[h, x] = \alpha(h)x \qquad \text{for all} \quad h \in \mathfrak{h}\}$$

having the following properties:

(1) $\mathfrak{h} = \mathfrak{g}_0$, i.e. $\mathfrak{h}$ is its own centralizer.

Let $\Sigma = \{\alpha \in \mathfrak{h}^* \setminus \{0\} | \mathfrak{g}_\alpha \neq \{0\}\}$. The elements of Σ are called the *roots* of $\mathfrak{h}$ in $\mathfrak{g}$.

(2) $\Sigma = \Sigma^+ \cup \Sigma^-$, where $\Sigma^- = -(\Sigma^+)$ and $\Sigma^+ = \Sigma \cap \mathbb{N}.\Delta$. In particular $\Sigma \subset X^* \subset \mathfrak{h}^*$.

(3) $\Delta \subset \Sigma$, hence $-\Delta \subset \Sigma$, and $\mathfrak{g}_{\alpha_i}$ (resp. $\mathfrak{g}_{-\alpha_i}$), $\alpha_i \in \Delta$, is spanned by the image of e_i (resp. f_i) in $\mathfrak{g}$.

For simplicity, we shall henceforth denote the images of X, e_i, f_i, in $\mathfrak{g}$ by the same symbols.

Define

$$\mathfrak{u}^{\pm} = \bigoplus_{\alpha \in \Sigma^{\pm}} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}^+$$

We shall call $\mathfrak{h}$ resp. $\mathfrak{b}$ a standard Cartan subalgebra resp. Borel subalgebra.

1.7. The Weyl group

Let W denote the group of automorphisms of X generated by the reflections

$$s_i\colon X \to X, \quad i = 1, \dots, r$$

$$s_i(x) = x - \alpha_i(x) h_i, \quad x \in X.$$

The group W is called the Weyl group of (X, ∇, Δ) or $\mathfrak{g}$. Its contragredient action on X^* is given by analogous formulae

$$s_i(\omega) = \omega - \omega(h_i)\alpha_i, \quad \omega \in X^*.$$

Using for example the geometrical analysis as developed in 4.2, one can prove that $(W, \{s_1, \dots, s_r\})$ is a Coxeter system whose Coxeter diagram is exactly the diagram associated to the Cartan matrix A in 1.3, cf. [10], [26], [35].

The action of W on X induces naturally actions on $X \otimes_{\mathbf{Z}} R$ for any ring R, in particular on $\mathfrak{h}$, as well as on $\mathfrak{h}^*$.

Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the adjoint action of $\mathfrak{g}$, and $\rho_i\colon \mathfrak{sl}_2 \to \mathfrak{gl}(\mathfrak{g})$ the restriction to the $\mathfrak{sl}_2$-subalgebra generated by e_i and f_i. Then ρ_i decomposes into a direct sum of finite-dimensional representations. From this fact one deduces the following:

(1) The Weyl group W permutes the roots.

(2) $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w(\alpha)}$ for all $\alpha \in \Sigma$, $w \in W$.

The set $\Sigma^R = \{w(\alpha) | \alpha \in \Delta, w \in W\}$ is called the set of real or Weyl roots. The complement $\Sigma^I = \Sigma \setminus \Sigma^R$ is called the set of imaginary or complementary roots.

One has $\Sigma^I = \emptyset$ if and only if Σ is finite. For more information on the structure of Σ we refer to [12].

We have $\dim \mathfrak{g}_\alpha = 1$ if $\alpha \in \Sigma^R$. For Lie algebras of general type the dimensions of the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Sigma^I$, are unknown, in general.

Let $\alpha \in \Sigma^R$ be a real root. Then there are $w \in W$ and $\alpha_i \in \Delta$ such that $\alpha = w(\alpha_i)$. Define

$$s_\alpha = w s_i w^{-1}.$$

We call s_α the reflection associated to α.

2. Representations of Kac-Moody algebras

In this chapter we describe the theory of the so-called standard representations of a Kac-Moody algebra $\mathfrak{g}$. When $\mathfrak{g}$ is finite-dimensional these representations coincide with the usual finite-dimensional representations. For the details of the theory we refer to [10], [11], [5].

2.1. Weights

Let $\mathfrak{g}$ be the Kac-Moody algebra attached to a root datum (X, ∇, Δ), and let $\mathfrak{h} = X \otimes \mathbf{C}$ be its standard Cartan subalgebra.

An element $\omega \in \mathfrak{h}^*$ is called a weight if $\omega(h_i) \in \mathbf{Z}$ for all $i = 1, \ldots, r$. A weight ω is called dominant if $\omega(h_i) \in \mathbf{N}$ for all $i = 1, \ldots, r$. Any dominant weight ω_i with the property $\omega_i(h_j) = \delta_{ij}$, $j = 1, \ldots, r$, is called an i-th fundamental dominant weight.

In the context of Lie algebras the integral structure of a root datum is without any importance. Since this will be different when dealing with groups we define: A weight $\omega \in \mathfrak{h}^*$ is admissible for (X, ∇, Δ) if $\omega \in X^* \subset \mathfrak{h}^*$.

We say that the root datum (X, ∇, Δ) is simply connected if one may choose fundamental dominant weights ω_i in X^*, $i = 1, \ldots, r$. In this case we may write

$$X = Q \oplus D$$

where Q is the sublattice generated by the h_i, $i = 1, \ldots, r$, and where D is the common kernel of the functionals $\omega_i \colon X \to \mathbf{Z}$. Moreover, if we put $\mathfrak{d} = D \otimes \mathbf{C}$ we may write $\mathfrak{g}$ as a semidirect product

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \rtimes \mathfrak{d}$$

The set of all weights P then has the following form

$$P = \bigoplus_{i=1}^{r} \mathbf{Z}\omega_i \oplus \mathfrak{d}^*.$$

2.2. Standard representations

Recall the notations (cf. 1.6.)

$$\mathfrak{u} = \mathfrak{u}^+ = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$

$$\mathfrak{u}^- = \bigoplus_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha$$

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}.$$

Let $\mathscr{U}(\mathfrak{g})$, $\mathscr{U}(\mathfrak{b})$, $\mathscr{U}(\mathfrak{u}^-)$ denote the universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{u}^-$ respectively. We then have $\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{u}^-) \cdot \mathscr{U}(\mathfrak{b})$.

Let $\omega \in \mathfrak{h}^*$ be a dominant weight, defining a one-dimensional representation $\mathbf{C}_\omega$ of $\mathfrak{b}$:

$$\mathfrak{b} \to \mathfrak{h} \xrightarrow{\omega} \mathbf{C}.$$

Let $V^{(\omega)} = \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{b})} \mathbb{C}_\omega$ be the induced module of $\mathfrak{g}$. As an $\mathfrak{h}$-space $V^{(\omega)}$ is isomorphic to $\mathscr{U}(\mathfrak{u}^-) \otimes \mathbb{C}_\omega$. By V^ω we denote the quotient of $V^{(\omega)}$ by the maximal $\mathfrak{g}$-submodule which does not contain the line $1 \otimes \mathbb{C}_\omega$. This $\mathfrak{g}$-module V^ω has the following properties:

(1) V^ω is irreducible.

(2) As an $\mathfrak{h}$-module V^ω decomposes into a direct sum of finite-dimensional eigenspaces $V^\omega = \oplus_{\mu \in \mathfrak{h}^*} V^\omega_\mu$, where $V^\omega_\mu = \{ v \in v^\omega | h \cdot v = \mu(h) v \text{ for all } h \in \mathfrak{h} \}$. An element of $\mu \in \mathfrak{h}^*$ is called a *weight* of V^ω if $V^\omega_\mu \neq \{0\}$.

(3) Any weight μ of V^ω has the form

$$\mu = \omega - \sum_{i=1}^{r} c_i \alpha_i, \quad c_i \in \mathbb{N}.$$

(Hence: if ω is admissible for a root datum then μ also is.)

(4) V^ω_ω is the image of $1 \otimes \mathbb{C}_\omega$, and $\dim V^\omega_\omega = 1$.

(5) With respect to the $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ generated by a triple e_i, h_i, f_i the module V^ω decomposes into a direct sum of finite-dimensional $\mathfrak{sl}_2$-modules.

From (5) we get:

(6) The set of weights of V^ω is stable under the action of the Weyl group W. Moreover we have $\dim V^\omega_\mu = \dim V^\omega_{w(\mu)}$ for all $w \in W$.

Properties (3) and (4) mean that V^ω is a highest weight module with highest weight ω. An element $v \in V^\omega_\omega$, $v \neq 0$, is called a highest weight vector. Recall $\mathfrak{u} v = 0$ by construction. The modules V^ω are called the standard modules of $\mathfrak{g}$.

2.3. Formal characters

Let $\mathbb{Z}^P$ denote the set of all functions $P \to \mathbb{Z}$, where P is the group of weights. To each element $\mu \in P$ we associate the function $e(\mu): P \to \mathbb{Z}$ defined by $e(\mu)(\mu') = \delta_{\mu\mu'}$. The Weyl group W operates naturally on P and hence on $\mathbb{Z}^P$.

Let V^ω be the standard module of $\mathfrak{g}$ corresponding to the highest weight ω. Since the weight spaces of V^ω are finite-dimensional the *formal character* of V^ω

$$\chi^\omega := \sum_{\mu \in P} \dim V^\omega_\mu e(\mu)$$

is a well defined element in $\mathbb{Z}^P$. Because of $\dim V^\omega_{w(\mu)} = \dim V^\omega_\mu$, for all $w \in W$, the character χ^ω is a W-invariant element in $\mathbb{Z}^P$.

When the Cartan matrix of $\mathfrak{g}$ is symmetrizable there is an analogue of Weyl's character formula for χ^ω due to Kac [10].

Later we will interpret these formal characters as actual functions, cf. Chapter 5.

3. Groups attached to Kac-Moody algebras

In this chapter we want to attach a group G to a Kac-Moody algebra $\mathfrak{g}$. There are actually many possibilities to do so. We shall restrict ourselves to a group which is minimal with respect to a set of natural properties, and which suffices for the limited purposes of this paper. In [30] we consider a completion of G.

The study of groups G attached to $\mathfrak{g}$ was started by Moody – Teo, Marcuson, and Garland ([26], [21], [7]). Recently Tits gave a uniform treatment ([33], [34]). He defines the groups by an amalgamation process. A different amalgamation procedure has also been proposed by Kac ([9], [11]).

3.1. Definition of the groups

Let $A \in M_r(\mathbb{Z})$ be a Cartan matrix, (X, ∇, Δ) a root datum for A and $\mathfrak{g}$ the corresponding Kac-Moody algebra. The group G we want to associate with $\mathfrak{g}$, more precisely with the root datum (X, ∇, Δ), may be characterized by the following properties:

$$G \text{ is generated by subgroups } T,\ X_i,\ Y_i, \quad i = 1, \ldots, r$$

where

(1) T is the torus $X \otimes_{\mathbf{Z}} \mathbf{C}^* = \mathrm{Hom}_{\mathbf{Z}}(X^*, \mathbf{C}^*)$ with character group X^*,

(2) X_i (resp. Y_i) is the image of an additive one-parameter subgroup x_i (resp. y_i): $\mathbf{C} \hookrightarrow G$ normalized by T such that

$$t x_i(c) t^{-1} = x_i(\alpha_i(t)c)$$

$$t y_i(c) t^{-1} = y_i(-\alpha_i(t)c)$$

for all $t \in T$, $c \in \mathbf{C}$.

Furthermore, we require

(3) For any standard representation

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V^\omega)$$

with admissible highest weight ω there is a homomorphism

$$R : G \to GL(V^\omega)$$

such that for all $i = 1, \ldots, r$, $c \in \mathbf{C}$, $t \in T$, $v \in V$, we have

$$R(x_i(c))v = (\exp(c\rho(e_i)))v$$

$$R(y_i(c))v = (\exp(c\rho(f_i)))v$$

and

$$R(t)v = \mu(t)v$$

if $v \in V_\mu^\omega$.

Note that the exponential series applied to v reduce to polynomials since the action of e_i and f_i is locally nilpotent on V^ω, cf. 2.2.

(4) For any admissible weight ω which is non-zero on each connected component of ∇, the kernel of the representation

$$R: G \to GL(V^\omega)$$

is contained in T. (Here we say that a subset of ∇ is connected if the corresponding set of vertices in the Coxeter diagram is connected).

The existence of such a group G follows from the work of Tits ([33], [34]). One may deduce it also from the work of Marcuson ([21], see [30]). When the Cartan matrix is of finite type we obtain a finite-dimensional reductive group. There is also a more explicit description when the Cartan matrix is affine, due to Garland, cf. [7].

3.2. Subgroups

Let $N \subset G$ be the subgroup generated by T and the elements $n_i(c)$, $c \in \mathbb{C}^*$, $i = 1, \ldots, r$, where

$$n_i(c) = x_i(c)y_i(-c^{-1})x_i(c).$$

Then N contains T as a normal subgroup, and N/T is isomorphic to the Weyl group W introduced in 1.7, cf. [21], [33].

For the limited purpose of this paper we won't use the Borel subgroup $B \subset G$ which together with N forms a Tits system in G. However, for completeness, let us quickly give the definition.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the standard Cartan subalgebra (corresponding to T) and $\Sigma \subset X^* \subset \mathfrak{h}^*$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$. For any real root $\alpha \in \Sigma^R$ we choose a $w \in W$ such that $\alpha = w(\alpha_i)$ for some $\alpha_i \in \Delta$. Let $n \in N$ be an element which projects onto $w \in W$ and define

$$X_\alpha := nX_i n^{-1}.$$

This is an additive one-parameter group normalized by T via the character α.

Let $U \subset G$ be the group generated by all X_α, $\alpha \in \Sigma^+$. Then U is normalized by T and $B := T \ltimes U$.

Let $J \subset \{1, \ldots, r\}$ be a subset and

$$A' = ((A_{ij}))_{i,j \in J}$$

the corresponding submatrix of the Cartan matrix A. Denote $\{h_i | i \in J\}$, resp $\{\alpha_i | i \in J\}$, by ∇', resp. Δ'. Then (X, ∇', Δ') is a root datum for the Cartan matrix A'. Let G' be the group associated to the root datum (X, ∇', Δ'). Exploiting the defining properties of G' and G one establishes a natural isomorphism from G' onto the subgroup of G generated by T and X_i, Y_i, $i \in J$. In particular, if A' is of finite type the last group is a finite-dimensional reductive subgroup of G.

4. Looijenga's theory

In this chapter we will give a survey over Looijenga's work [18]. We first discuss his analysis of the Weyl group action on $\mathfrak{h}$, T and certain subdomains $\mathscr{D}$, $\mathscr{T}$. In the real case such an analysis was also done by Vinberg [35], both following the pattern of Bourbaki's treatment [1]. Finally we describe the construction of the partial compactification of $\mathscr{T}/W$ and its analytic structure.

Starting from section 4.3 we shall assume that the root datum given is a simply connected one. We are interested only in this situation. The results for arbitrary root data may be easily derived from this special case (cf. [28] 4.5).

4.1. Uniformization of the torus

Let G be the group attached to a root datum (X, ∇, Δ). Many of the constructions involving the torus $T = X \otimes_{\mathbb{Z}} \mathbb{C}^*$ can be formulated (sometimes easier) in terms of the Cartan algebra $\mathfrak{h} = \operatorname{Lie} T = X \otimes_{\mathbb{Z}} \mathbb{C}$. In the following we shall consider T as the quotient of $\mathfrak{h}$ by the lattice X, and we fix the exponential map

$$\mathfrak{h} \to T$$

obtained from the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{e} \mathbb{C}^* \to 1$$

$$e(z) = \exp 2\pi i z$$

by tensoring with X.

4.2. The Tits cone

Let (X, ∇, Δ) be a root datum and W the corresponding Weyl group, cf. 1.7. Here we want to study the action of W on the real vector space $V = X \otimes_{\mathbb{Z}} \mathbb{R}$. Let

$$C = \{ v \in V | \alpha(v) < 0 \text{ for all } \alpha \in \Delta \}$$

$$\bar{C} = \{ v \in v | \alpha(v) \leqslant 0 \text{ for all } \alpha \in \Delta \}$$

be the open, resp. closed (anti-) fundamental chamber. The Tits cone I is the union of the W-translates of $\bar{C}$

$$I = W \cdot \bar{C}.$$

Then we have

(1) I is a convex solid cone in V.
(2) $\bar{C}$ is a fundamental chamber for the action of W on I.
(3) Let $v \in I$. Then the stabilizer $W_v = \{ w \in W | w(v) = v \}$ of v is generated by the reflections $s_\alpha \in W$ with $\alpha \in \Sigma^R$, $\alpha(v) = 0$.
(4) W acts properly discontinuously on the interior $\mathring{I}$ of I, in particular for any $v \in \mathring{I}$ there are only finitely many $\alpha \in \Sigma^R$ vanishing on v.
(5) $\mathring{I} = I = V$ if and only if W is finite, i.e. if and only if the Cartan matrix of (X, ∇, Δ) is of finite type.

REMARKS: Of course, similar results hold when we replace $\bar{C}$ by the "true" fundamental chamber $-\bar{C}$. We can also consider fundamental chambers and Tits cones in the dual V^*. Again, there are analogues statements.

4.3. *Looijenga's domain*

From now on our basic root datum (X, ∇, Δ) is assumed to be simply connected. Let $\mathscr{D} \subset \mathfrak{h} = V \otimes_{\mathbf{R}} \mathbf{C} = X \otimes_{\mathbf{Z}} \mathbf{C}$ denote the tube domain

$$V + i\mathring{I} = \{ h \in \mathfrak{h} | \mathrm{Im}(h) \in \mathring{I} \}.$$

Denote the semidirect product $W \ltimes X$ of the Weyl group W with the lattice X by $\tilde{W}$. Then we may extend the complex linear action of W on $\mathscr{D} \subset \mathfrak{h}$ to an action of $\tilde{W}$ by letting X operate on $\mathscr{D}$ via translation in the real direction

$$h \mapsto h + x, \quad h \in \mathscr{D}, \quad x \in X.$$

One can show that this action of $\tilde{W}$ on $\mathscr{D}$ is again properly discontinuous. Moreover, the stabilizers of points in $\mathscr{D}$ are generated by reflections (here we need the simply connectedness). Thus the quotient space $\mathscr{D}/\tilde{W}$ is a complex manifold.

Note that $\mathscr{T} = \mathscr{D}/X$ is a domain in the torus T, and $\mathscr{T}/W = \mathscr{D}/\tilde{W}$.

4.4. *Boundary components*

Let (X, ∇, Δ) be a simply connected root datum as in 4.3. Let ∇' be a subset of ∇, Δ' the corresponding subset of Δ, and

$$\nabla'^* = \{ h_j \in \nabla | \alpha_i(h_j) = 0 \text{ for all } \alpha_i \in \Delta' \}$$

$$\Delta'^* = \{ \alpha_j \in \Delta | \alpha_j(h_i) = 0 \text{ for all } h_i \in \nabla' \}$$

the "orthogonal sets" of ∇' and Δ'.

Let $X(\nabla') = X/\mathbb{Z} \cdot \nabla'$, and denote the projection of ∇'^* into $X(\nabla')$, resp. the injection of Δ'^* into $X(\nabla')^*$ by the same symbols, ∇'^* and Δ'^*. Then (X, ∇', Δ'), $(X, \nabla'^*, \Delta'^*)$, and $(X(\nabla'), \nabla'^*, \Delta'^*)$ are again simply connected root data.

We say that ∇' is a special subset of ∇ if either $\nabla' = \emptyset$ or all connected components of ∇' are of infinite, i.e. non-finite, type.

Let $\mathfrak{h}_{\nabla'}$ be the subspace of $\mathfrak{h}$ generated by a special subset ∇' of ∇. We shall call $\mathfrak{h}_{\nabla'}$ as well as its W-translates special subspaces of $\mathfrak{h}$ of type ∇'. Note $\mathfrak{h}_{\emptyset} = \{0\}$.

For any special subspace $\mathfrak{h}' \subset \mathfrak{h}$ let $\mathscr{D}(\mathfrak{h}')$ denote the image of $\mathscr{D}$ in the quotient $\mathfrak{h}/\mathfrak{h}'$. In particular $\mathscr{D}(\{0\}) = \mathscr{D}$. Define

$$\hat{\mathscr{D}} = \bigcup_{\substack{\mathfrak{h}' \subset \mathfrak{h} \\ \text{special}}} \mathscr{D}(\mathfrak{h}')$$

Then $\tilde{W}$ acts naturally on this union, and we have

$$N(\mathfrak{h}_{\nabla'}) := \{ w \in \tilde{W} | w(\mathscr{D}(\mathfrak{h}_{\nabla'})) = \mathscr{D}(\mathfrak{h}_{\nabla'}) \} = (W_{\Delta'^*} \times W_{\Delta'}) \ltimes X$$

and

$$Z(\mathfrak{h}_{\nabla'}) := \{ w \in N(\mathfrak{h}_{\nabla'}) | w|_{\mathscr{D}(\mathfrak{h}_{\nabla'})} = \mathrm{id} \} = W_{\Delta'} \ltimes (\mathbb{Z} \cdot \nabla')$$

Here W_θ, $\theta \subset \Delta$, denotes the subgroup of W generated by the reflections s_α, $\alpha \in \theta$.

Furthermore the quotient $N(\mathfrak{h}_{\nabla'})/Z(\mathfrak{h}_{\nabla'})$ is isomorphic to the extended Weyl group

$$W_{\Delta'^*} \ltimes X(\nabla')$$

of the root datum $(X(\nabla'), \nabla'^*, \Delta'^*)$ operating in the natural way on $\mathfrak{h}/\mathfrak{h}_{\nabla'} = X(\nabla') \otimes_{\mathbb{Z}} \mathbb{C}$ and properly discontinuously on the subdomain $\mathscr{D}(\mathfrak{h}_{\nabla'}) \subset \mathfrak{h}/\mathfrak{h}_{\nabla'}$.

A topology on $\hat{\mathscr{D}}$ is defined in the following way. Let $x \in \mathscr{D}(\mathfrak{h}')$, $\mathfrak{h}' \subset \mathfrak{h}$ special, be a point of $\hat{\mathscr{D}}$. Then a subset $\mathscr{U}$ of $\hat{\mathscr{D}}$ is called an open neighborhood of x if

(1) $x \in \mathscr{U}$

(2) $\mathscr{U} \cap \mathscr{D}$ is a convex open subset of $\mathscr{D}$, invariant under the stabilizer $\tilde{W}_x$ of x in $\tilde{W}$.

(3) If $\mathfrak{h}''$ is special, $\mathfrak{h}'' \subset \mathfrak{h}'$, then $\mathscr{U} \cap \mathscr{D}(\mathfrak{h}'')$ equals the projection of $\mathscr{U} \cap \mathscr{D}$ under the map $\mathfrak{h} \to \mathfrak{h}/\mathfrak{h}''$.

With respect to this topology $\hat{W}$ acts by continuous transformations. The orbit space $\hat{M} = \hat{\mathscr{D}}/\tilde{W}$ is locally compact, Hausdorff, and admits a countable basis for its topology. In fact $\hat{M}$ is a complex Stein manifold as we will see in the next section.

Note that since each boundary component $\mathscr{D}(\mathfrak{h}')$ of $\hat{\mathscr{D}}$ is stable under the translation group X we obtain a toral analogue of $\hat{\mathscr{D}}$:

$$\hat{\mathscr{T}} = \bigcup_{\substack{\mathfrak{h}' \subset \mathfrak{h} \\ \text{special}}} \mathscr{T}(\mathfrak{h}'), \quad \mathscr{T}(\mathfrak{h}') := \mathscr{D}(\mathfrak{h}')/X.$$

4.5. Analytic functions on $\hat{\mathscr{D}}/\tilde{W}$

Now we shall describe the analytic structure on the orbit space $\hat{M} = \hat{\mathscr{D}}/\tilde{W}$. First we note that $\hat{M}$ is naturally stratified

$$\hat{M} = \bigcup_{\substack{\nabla' \subset \nabla \\ \text{special}}} M(\nabla')$$

where $M(\nabla')$ is the smooth quotient of

$$\bigcup_{\substack{\mathfrak{h}' \subset \mathfrak{h} \\ \text{special of type } \nabla'}} \mathscr{D}(\mathfrak{h}') \quad \text{by } \tilde{W},$$

or equivalently, the quotient of $\mathscr{D}(\mathfrak{h}_{\nabla'})$ by $N(\mathfrak{h}_{\nabla'})/Z(\mathfrak{h}_{\nabla'})$. A continuous function on an open set $\mathscr{U} \subset \hat{M}$ is said to be holomorphic if it induces an analytic function on each stratum $\mathscr{U} \cap M(\nabla')$, $\nabla' \subset \nabla$ special.

With this analytic structure $\hat{M}$ turns out to be a Stein manifold. An essential tool in the proof of this result are the following functions. Let $\omega \in X^*$ be a dominant weight (cf. 2.1). Define the value of the function S_ω: $\hat{\mathscr{D}} \to \mathbf{C}$ at the point $x \in \mathscr{D}(\mathfrak{h}')$ by

$$S_\omega(x) = \sum_{\substack{\mu \in W \cdot \omega \\ \mu|_{\mathfrak{h}'} \equiv 0}} \exp(2\pi i \mu(x)).$$

Then this series converges uniformly and absolutely on compact subsets of $\mathscr{D}(\mathfrak{h}')$ for any special $\mathfrak{h}' \subset \mathfrak{h}$ and defines a $\tilde{W}$-invariant continuous function on $\hat{\mathscr{D}}$. In other words, it defines a holomorphic function on $\hat{M} = \hat{\mathscr{D}}/\tilde{W}$.

5. Holomorphy of characters

In this chapter we shall show that the formal characters of a standard representation can be interpreted as holomorphic functions on a subdomain of the manifold $\hat{M}$. The first results in this direction are due to Lepowsky and Moody [15] who treat the rank two hyperbolic case. It was shown by A. Meurman [23] that already in this case the characters have poles on $M \subset \hat{M}$ outside this subdomain. The domain of convergence we shall obtain here is not an optimal one. Under the assumptions of a symmetrizable Cartan matrix one can exploit the Kac-Weyl character formula for a finer investigation. Such an analysis was recently carried out by Kac and Peterson [14]. They give a precise description of the domain of convergence in that case.

Our approach is inspired by Looijenga's [18].

5.1. A shrinking of the domain $\mathscr{D}$

Let $A \in M_r(\mathbb{Z})$ be a Cartan matrix and (X, ∇, Δ) a simply connected root datum for A. Let $V = X \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{h} = X \otimes_{\mathbb{Z}} \mathbb{C}$. For a positive real number c define

$$C^c = \{ v \in V | \alpha(h) < -c \text{ for all } \alpha \in \Delta \}$$

Let I^c be the convex hull of $W.C^c$, the set of W-translates of C^c. We define

$$\mathscr{D}^c = V + iI^c$$

and for any special subspace $\mathfrak{h}' \subset \mathfrak{h}$ we let $\mathscr{D}^c(\mathfrak{h}')$ denote the image of $\mathscr{D}^c$ in $\mathfrak{h}/\mathfrak{h}'$. Then

$$\hat{\mathscr{D}}^c = \bigcup_{\substack{\mathfrak{h}' \subset \mathfrak{h} \\ \text{special}}} \mathscr{D}^c(\mathfrak{h}')$$

is an open $\tilde{W}$-stable subset of $\hat{\mathscr{D}}$ meeting each boundary component.

5.2. Some estimates

Let $\mathfrak{g}$ be the Lie algebra associated to (X, ∇, Δ) and V^ω a standard representation of $\mathfrak{g}$ with highest weight $\omega \in X^*$. Denote the set of weights of V^ω by P^ω. Then P^ω is a W-stable subset of $\omega - \mathbb{N}.\{\alpha_1, \dots, \alpha_r\}$, cf. 2.2. For each element $\mu = \omega - \sum_{i=1}^r c_i \alpha_i$ in P^ω we define the depth of μ (relative ω) by

$$\operatorname{depth}^\omega(\mu) = \sum_{i=1}^r c_i.$$

LEMMA 1: *Let* μ *be a weight of* V^ω *of depth* n. *Then* $\dim V_\mu^\omega \leqslant r^n$.

PROOF ([15]): By construction of V^ω we have $\dim V_\mu^\omega \leqslant \dim \mathscr{U}(\mathfrak{u}^-)_n$, where $\mathscr{U}(\mathfrak{u}^-)_n$ is the subspace of $\mathscr{U}(\mathfrak{u}^-)$ spanned by the r^n products $f_{i_1} \cdot \ldots \cdot f_{i_n}$, $i_j \in \{1, \ldots, r\}$. Therefore $\dim V_\mu^\omega \leqslant r^n$.

LEMMA 2: *Let* $v \in C^c$ *and* $\mu \in P^\omega$ *such that* $\mu(v) \leqslant n$. *Then* $\mathrm{depth}^\omega(\mu) \leqslant (n - \omega(v))/c$.

PROOF: The weight μ has the form $\mu = \omega - \sum_{i=1}^r c_i \alpha_i$. The condition $\mu(v) \leqslant n$ then reads

$$-\sum c_i \alpha_i(v) \leqslant n - \omega(v)$$

or, since $c < -\alpha_i(v)$,

$$c \cdot \mathrm{depth}^\omega(\mu) \leqslant n - \omega(v).$$

LEMMA 3: *Let* $K \subset I^c$ *be a compact subset and* $\mu \in P^\mu$ *such that* $\min\{\mu(v) | v \in K\} \leqslant n$. *Let* $b := \min\{\omega(v) | v \in K\}$. *Then* $\dim V_\mu^\omega \leqslant r^{(n-b)/c}$.

PROOF: Let $H \subset I^c$ be a convex compact polyhedron containing K such that the extremal points H_0 of H lie in W-translates of C^c. Since a linear function on H attains its minimum at an extreme point $v_0 \in H_0$ we get $\mu(v_0) \leqslant n$. Let $w \in W$ and $v \in C^c$ be such that $v = w(v_0)$. Then

$$w(\mu)(v) = \mu(v_0) \leqslant n$$

and $\mathrm{depth}^\omega(w(\mu)) \leqslant (n - \omega(v0)/c \leqslant (n-b)/c$ by Lemma 2. Hence by Lemma 1

$$\dim V_\mu^\omega = \dim V_{w(\mu)}^\omega \leqslant r^{(n-b)/c}.$$

For any subset S of I^c and any real number n we define $A_{S,\omega}(n)$ to be the number of $\mu \in P^\omega$ such that $\min\{\mu(v) | v \in S\} \leqslant n$.

LEMMA 4: *Let* $K \subset I^c$ *be a compact subset and let* $b = \min\{\omega(v) | v \in K\}$. *Then the number* $A_{K,\omega}(n)$ *is zero for* $n < b$ *and of the order of* n^r *as* $n \to +\infty$.

PROOF: (cf. [18]): As in the proof of Lemma 3 we choose a convex polyhedron $H \subset I^c$, containing K such that the vertices H_0 lie in W-translates of C^c. Then we have

$$A_{K,\omega}(n) \leqslant A_{H,\omega}(n) \leqslant \sum_{v \in H_0} A_{\{v\},\omega}(n)$$

Using the W-stability of P^{ω} and Lemma 2 we get

$$A_{\{v\},\omega}(n) \leqslant \operatorname{card}\{\mu \in P^{\omega} | \operatorname{depth}^{\omega}(\mu) \leqslant (n-b)/c\}.$$

The right hand side is zero for $n < b$ and of the order of n^r as $n \to +\infty$. From this our claim follows.

5.3. Character functions

Let $\chi^{\omega} = \sum_{\mu \in P^{\omega}} \dim V_{\mu}^{\omega} e(\mu)$ be the formal character of the standard representation V^{ω}, cf. 2.3. We want to associate to χ^{ω} a $\tilde{W}$-invariant function, also denoted by χ^{ω},

$$\chi^{\omega}: \hat{\mathscr{D}}^c \to \mathbf{C}, \quad c = (\log r)/2\pi,$$

which induces a holomorphic function on $\hat{\mathscr{D}}^c/\tilde{W}$. Let $x \in \mathscr{D}^c(\mathfrak{h}')$, $\mathfrak{h}' \subset \mathfrak{h}$ special, be a point of $\hat{\mathscr{D}}^c$. Define

$$\chi^{\omega}(x) = \sum_{\substack{\mu \in P^{\omega} \\ \mu|_{\mathfrak{h}'} \equiv 0}} \dim V_{\mu}^{\omega} \exp(2\pi i \mu(x)).$$

PROPOSITION: *For any special subspace $\mathfrak{h}' \subset \mathfrak{h}$ the series χ^{ω} converges uniformly and absolutely on compact subsets of $\mathscr{D}^c(\mathfrak{h}')$. The resulting function on $\hat{\mathscr{D}}^c$ is $\tilde{W}$-invariant and continuous. In particular, χ^{ω} induces a holomorphic function on $\hat{\mathscr{D}}^c/\tilde{W}$.*

PROOF: Since the restriction of χ^{ω} to some boundary component is obtained by omitting terms it is sufficient to prove the first claim for $\mathscr{D}^c(\{0\}) = \mathscr{D}^c$.

Let $K' \subset \mathscr{D}^c$ be a compact subset and let $K \subset I^c$ be the compact image of K' under the projection $\operatorname{Im}: \mathscr{D}^c = V + iI^c \to I^c$, $u + iv \mapsto v$. Note that, because of compactness, K is already contained in some I^a, $a > c$. For any $u + iv \in K'$ we have

$$\sum_{\mu \in P} \dim V_{\mu}^{\omega} |\exp 2\pi i \mu(u + iv)|$$

$$= \sum_{\mu \in P} \dim V_{\mu}^{\omega} \exp(-2\pi \mu(v))$$

$$\leqslant \sum_{n \in \mathbf{Z}} \sum_{\substack{\mu \in P^{\omega} \\ \min\{\mu(v) | v \in K\} \leqslant n}} \dim V_{\mu}^{\omega} \exp(-2\pi n)$$

$$\leqslant \sum_{n \in \mathbf{Z}} A_{K,\omega}(n) r^{(n-b)/a} \exp(-2\pi n)$$

(by Lemma 3 and 4), for some b,

$$\leqslant \text{const.}\left(\sum_{\substack{n\in\mathbf{Z}\\ n\geqslant b}} n^r \exp(-n(2\pi-(\log r)/a))\right)$$

$$\leqslant \text{const.}\int_b^\infty x^r \exp(-x\varepsilon)\mathrm{d}x < \infty$$

where $\varepsilon = 2\pi - (\log r)/a$, and $\varepsilon > 0$ since $a > c = (\log r)/2\pi$.

That χ^ω is a $\tilde{W}$-invariant function reduces to the fact that for a special subset $\nabla' \subset \nabla$ and a dominant weight ν the set $\{\mu \in W.\nu | \mu_{|\nabla'} \equiv 0\}$ forms an orbit under $W_{\Delta'}$, cf. [18] 2.2 and 3.3.

The proof of the continuity of χ^ω is the same as the one given in [18] 3.4 for the function S_ω. The only modifications to be made have already shown up in the convergence arguments above.

6. Factorization of characters

In this chapter we define a domain $\mathscr{N} \subset N$, and we show that the restriction to $\mathscr{N}$ of the character of every standard representation factors over Looijenga's partial compactification $\hat{\mathscr{T}}/W$. The definition of $\mathscr{N}$ and the factorization require some more investigations of the Weyl group action on the dual Tits cone, cf. 6.2, 6.3.

6.1. Algebraic traces

In all this chapter we let G be the group associated to a simply connected root datum (X, ∇, Δ) of dimension s and rank r. Let $\mathfrak{g}$ be the corresponding Kac-Moody algebra.

Let $\omega \in X^*$ be a dominant weight,

$$\rho : \mathfrak{g} \to \mathscr{gl}(V^\omega)$$

the corresponding standard representation and

$$R: G \to GL(V^\omega)$$

the lift of ρ to G, cf. 3.1. The vector space V^ω decomposes into a direct sum $V^\omega = \oplus_{\mu\in P^\omega} V^\omega_\mu$ of finite-dimensional weight spaces.

For each weight $\mu \in P^\omega$ we denote the injection $V^\omega_\mu \hookrightarrow V^\omega$ by i_μ and the obvious projection $V^\omega \to V^\omega_\mu$ by π_μ. Let $R_\mu(n)$ be the composition $\pi_\mu \circ R(n) \circ i_\mu$ for $n \in N$.

We say that $n \in N$ is of algebraic trace class if the sum

$$\mathrm{tr}_\omega(n) = \sum_{\mu\in P^\omega} \mathrm{trace}\, R_\mu(n)$$

is absolutely convergent for all dominant weights $\omega \in X^*$.

Let $n \in N$ be of algebraic trace class and let $m \in N$ be arbitrary with image w in W. Then we have

$$\text{trace } R_{w(\mu)}(n) = \text{trace } R_\mu(mnm^{-1}).$$

Thus mnm^{-1} is also of algebraic trace class and

$$\text{tr}_\omega(n) = \text{tr}_\omega(mnm^{-1}).$$

We call $\text{tr}_\omega(n)$ the algebraic trace of n on V^ω.

Remark: The notion of trace introduced above is a naive one but sufficient for the limited purpose of this paper. If one wants to consider arbitrary elements in G one has to use the analytic trace defined for (pre-) Hilbert spaces. This requires the existence of a positive definite hermitian form on V^ω which should be invariant under the action of a "compact form" of G. Such a form has been constructed by Garland in the affine case (cf. [6], [7]). For symmetrizable Cartan matrices the existence is proved by Kac and Peterson [36].

6.2. *Types of Weyl group elements*

Let $\bar{C}^* = \{\omega \in V^* = X^* \otimes \mathbb{R} \mid \omega(h) \geqslant 0 \text{ for all } h \in \nabla\}$ be the true dual fundamental chamber and $I^* = W.\bar{C}^*$ the corresponding TIts cone. Let $(I^*)^w = \{\omega \in I^* \mid w(\omega) = \omega\}$ be the fixed point set of a Weyl group element $w \in W$. We call

$$\mathfrak{h}_w = \{h \in \mathfrak{h} \mid \omega(h) = 0 \text{ for all } \omega \in (I^*)^w\}$$

the subspace associated to w.

After replacing w by a suitable conjugate in W we may assume that $(I^*)^w$ supports a face of $\bar{C}^*$, i.e. $\bar{C}^* \cap (I^*)^w$ contains an open nonvoid subset of $(I^*)^w$. Then $\mathfrak{h}_w$ is generated as a complex vector space by a subset ∇' of ∇. It follows from 4.2 that w is an element of $W_{\nabla'}$, the subgroup of W generated by the reflections s_h, $h \in \nabla'$, on V^* (equivalently by the reflections s_α, $\alpha \in \Delta'$, on V, where $\Delta' \subset \Delta$ is the subset corresponding to $\nabla' \subset \nabla$). The subset $\nabla' \subset \nabla$ is well defined up to conjugation by W. Abusively we call $\mathfrak{h}_w$ and w of type ∇'.

For any subset $\nabla' \subset \nabla$ there are elements $w \in W$ of type ∇'. Choose some ordering of the elements of ∇'

$$\nabla' = \{h_1, \dots, h_k\}$$

and correspondingly

$$\Delta' = \{\alpha_1, \dots, \alpha_k\}$$

Let $s_i \in W$, $i = 1, \ldots, k$, denote the associated fundamental reflections.

LEMMA: *The product* $w = s_1 \cdot \ldots \cdot s_k$ *is of type* ∇'.

PROOF: We are easily reduced to show

$$(I^*)^w = \{\omega \in I^* | \omega(h) = 0 \text{ for all } h \in \nabla'\}.$$

Let $\omega \in I^*$. Then $w(\omega) = \omega$ implies

$$s_1(\omega) = s_2 \cdot \ldots \cdot s_k(\omega)$$

and

$$s_1(\omega) - \omega = s_2 \cdot \ldots \cdot s_k(\omega) - \omega.$$

The left side is a multiple of α_1, whereas the right side is a linear combination of $\alpha_2, \ldots, \alpha_k$. Since $\Delta' \subset \Delta$ is free both sides must be zero. Inductively we get $s_i(\omega) = \omega$ for all $i = 1, \ldots, k$. Thus $\omega(h_i) = 0$ for $i = 1, \ldots, k$ which proves our assertion.

REMARK: If the Coxeter diagram of ∇' is a forest, then w is a Coxeter element of $W_{\nabla'}$ which up to conjugation in $W_{\nabla'}$ is independent of the ordering of ∇' (cf. [1] V, 6.1). The proof above is an adaptation of Steinberg's ([31] 7.6) to our situation.

Any subset $\nabla' \subset \nabla$ may be decomposed as the disjoint union of connected components. We assemble the components of finite (resp. infinite) type into the finite part ∇'° (resp. the infinite part ∇'^{∞}). Thus ∇' is the disjoint union

$$\nabla' = \nabla'^{\circ} \cup \nabla'^{\infty}.$$

Let $w \in W$ be of type ∇'. According to the decomposition of ∇' into finite and infinite part we have a decomposition of $\mathfrak{h}_{\nabla'}$.

$$\mathfrak{h}_{\nabla'} = \mathfrak{h}_{\nabla'^{\circ}} \oplus \mathfrak{h}_{\nabla'^{\infty}}$$

and, by conjugation, one of $\mathfrak{h}_w$

$$\mathfrak{h}_w = \mathfrak{h}_w^{\circ} \oplus \mathfrak{h}_w^{\infty}$$

We say w, or $\mathfrak{h}_w$, is of

finite, resp. infinite, resp. mixed, resp. pure type

if

$$\nabla'^{\infty} = \emptyset, \quad \text{resp. } \nabla'^{\infty} \neq \emptyset, \quad \text{resp. } \nabla'^{\circ} \neq \emptyset, \quad \text{resp. } \nabla'^{\circ} = \emptyset.$$

Note that the neutral element $1 \in W$ is of pure type. Mixed elements may be infinite or finite.

6.3. Decomposition of Weyl group elements

Let $w \in W$ be an element of type ∇'. According to the decomposition $\nabla' = \nabla'^{\circ} \cup \nabla'^{\infty}$ the Weyl subgroup $W_{\nabla'}$ also decomposes as a product

$$W_{\nabla'} = W_{\nabla'^{\circ}} \times W_{\nabla'^{\infty}}$$

Let $g \in W$ be an element such that $w \in gW_{\nabla'}g^{-1}$. Then we may split w as a product

$$w = w^{\circ} w^{\infty}$$

where $w^{\circ} \in gW_{\nabla'^{\circ}}g^{-1}$ and $w^{\infty} \in gW_{\nabla'^{\infty}}g^{-1}$.

PROPOSITION 1: *The element* w° (*resp.* w^{∞}) *is of type* ∇'° (*resp.* ∇'^{∞}).

PROOF: Without loss of generality we may assume $w \in W_{\nabla'}$.

In case the Cartan matrices for ∇'° and ∇'^{∞} are nonsingular one reduces the situation easily to that of a direct sum of root data where the claims are trivial. For the general case we give a somewhat more complicated argument. It is sufficient to prove

$$\mathfrak{h}_{w^{\circ}} \text{ (resp. } \mathfrak{h}_{w^{\infty}}) \text{ is generated by } \nabla'^{\circ} \text{ (resp. } \nabla'^{\infty})$$

or

$$(I^*)^{w^{\circ}} = \{\omega \in I^* | \omega(h) = 0 \text{ for all } h \in \nabla'^{\circ}\}$$

$$(I^*)^{w^{\infty}} = \{\omega \in I^* | \omega(h) = 0 \text{ for all } h \in \nabla'^{\infty}\}$$

Since $w^{\circ} \in w_{\nabla'^{\circ}}$ and $w^{\infty} \in W_{\nabla'^{\infty}}$ the right-hand sides are contained in the left-hand sides. By conjugating w° in $W_{\nabla'^{\circ}}$ and w^{∞} in $W_{\nabla'^{\infty}}$ we may also assume

$$(I^*)^{w^{\circ}} = \{\omega \in I^* | \omega(h) = 0 \text{ for all } h \in \nabla^{\circ}\}$$

$$(I^*)^{w^{\infty}} = \{\omega \in I^* | \omega(h) = 0 \text{ for all } h \in \nabla^{\infty}\}$$

where ∇° and ∇^{∞} are subsets of ∇'° and ∇'^{∞}, cf. 4.2. Because of

$$(I^*)^{w^{\circ}} \cap (I^*)^{w^{\infty}} \subset (I^*)^{w}$$

or

$$\mathfrak{h}_{w^\circ} + \mathfrak{h}_{w^\infty} \supset \mathfrak{h}_w$$

we get $\nabla^\circ = \nabla'^\circ$ and $\nabla^\infty = \nabla'^\infty$ which had to be shown.

PROPOSITION 2: *An element $w \in W$ is of finite (resp. infinite) type if and only if w is of finite (resp. infinite) order.*

PROOF: We may assume that W is infinite. Otherwise the claim is trivial. We need to prove only one statement. Assume that $w \in W$ is of finite type. Then $w^\circ = w$, $w^\infty = 1$, and w° is contained in a finite subgroup of W. Thus w is of finite order. Conversely assume w is of finite order, say $|w|$. Let $x \in \mathring{I}^*$ be some element in the open Tits cone. Since this cone is convex the element

$$\bar{x} = \frac{1}{|w|} \sum_{l=1}^{|w|} w^l(x)$$

is also contained in $\mathring{I}^*$. Furthermore $\mathring{I}^*$ does not contain 0, therefore $\bar{x}$ is a non-trivial fixed point of w. By 4.2 we get that $\mathfrak{h}_w$ is contained in $\{h \in \mathfrak{h} | \bar{x}(h) = 0\}$ which is of finite type since $\bar{x} \in \mathring{I}^*$. Therefore $\mathfrak{h}_w$, and w, are also of finite type.

6.4. *Normalizer elements*

Let $T \subset G$ be the standard torus of G and $N \subset G$, $N \triangleright T$, the subgroup introduced in 4.2. Then there is a natural projection $N \to N/T = W$. Let $\nabla' \subset \nabla$ be a subset. We say that an element $n \in N$ is type ∇' (resp. of finite, infinite, mixed, or pure type) if its image w in W is of the respective type.

PROPOSITION: *Every element $n \in N$ of mixed type is conjugate under G to an element in N of pure type.*

PROOF: Let n be of type $\nabla' = \nabla'^\circ \cup \nabla'^\infty$, and Δ', Δ'°, Δ'^∞ the corresponding subsets of Δ. Let G°, resp. G^∞, be the subgroup of G corresponding to the root datum $(X, \nabla'^\circ, \Delta'^\circ)$, resp. $(X, \nabla'^\infty, \Delta'^\infty)$. Both groups centralize each other, and their product $G^\circ \cdot G^\infty$ in G is the subgroup G' corresponding to the root datum (X, ∇', Δ'). After replacing n by a conjugate in N we may assume $n \in G'$. Using 6.3 we decompose n as a product $n = n^\circ n^\infty$ with $n^\circ \in N \cap G^\circ$ of type ∇'° and $n^\infty \in N \cap G^\infty$ of type ∇'^∞. Then n° is a semisimple element in the finite-dimensional reductive group G° and conjugate in G° to an element of T. Since G° centralizes G^∞ we may thus conjugate n to an element tn^∞, for some $t \in T$, which is of pure type ∇'^∞.

Remarks: In the proof above we have used the fact that for a finite-dimensional reductive group G every element $n \in N$ is conjugate into T. Let $N(w)$ denote the set of all $n \in N$ mapping onto a given element $w \in W$. Then the conjugates of $N(w)$ in T form a W-stable closed subvariety of T, of the dimension of $T^w = \{t \in T | w(t) = t\}$. However, it is not possible, in general, to define a morphism of algebraic varieties $N(w) \to T$ mapping each element $n \in N(w)$ to a G-conjugate in T. This is possible only after replacing $N(w)$ by a finite cover. The obstructions are already present in the case of GL_2.

We shall denote the pure elements in W, resp. N by W^p, resp. N^p. Let $N(w)$ be the set of elements of N mapping onto $w \in W = N/T$. Since, according to the proposition above mixed elements of N are conjugate to pure elements we shall from now on concentrate on pure elements only. (In the case of symmetrizable Cartan matrices the analytic traces for a mixed element and its pure conjugates will coincide). To simplify investigations further we shall distinguish a class of representatives in N for the elements in $W = N/T$.

Let $w \in W$ be of type ∇', $\nabla' \subset \nabla$, and let $g \in W$ be an element such that $w \in gW_{\nabla'}g^{-1}$. Let G' be the commutator group of the subgroup of G corresponding to the root datum (X, ∇', Δ'). We say that a representative $n(w)$ of w in N is well-chosen if $n(w) \in \tilde{g}G'\tilde{g}^{-1}$, where $\tilde{g} \in N$ represents g.

6.5. The domain $\mathcal{N} \subset N$

Let $T' \subset T$ be a subtorus. We say that T' is special if $\mathfrak{h}' = \text{Lie } T'$ is special in $\mathfrak{h} = \text{Lie } T$. Since any special subspace $\mathfrak{h}' \subset \mathfrak{h}$ is generated over $\mathbb{C}$ by its intersection with X there is always a special subtorus $T' \subset T$ with $\mathfrak{h}' = \text{Lie } T'$. We define

$$\hat{T} = \bigcup_{\substack{T' \subset T \\ \text{special}}} T/T'.$$

Let $c = (\log r)/2\pi$ and

$$\hat{\mathscr{D}}^c = \bigcup_{\substack{\mathfrak{h}' \subset \mathfrak{h} \\ \text{special}}} \mathscr{D}^c(\mathfrak{h}')$$

the extended domain introduced in 5.1, 5.3. Let $\hat{\mathscr{T}}^c$ be the quotient of $\hat{\mathscr{D}}^c$ by X. Then

$$\hat{\mathscr{T}}^c = \bigcup_{\substack{\mathfrak{h}' \subset \mathfrak{h} \\ \text{special}}} \mathscr{T}^c(\mathfrak{h}')$$

where $\mathscr{T}^c(\mathfrak{h}') = \mathscr{D}^c(\mathfrak{h}')/X$ is a domain in the quotient torus T/T', T' being the subtorus of T with Lie $T' = \mathfrak{h}'$. We thus have $\hat{\mathscr{T}}^c \subset \hat{T}$.

Let $w \in W^p$ be a pure element and $T_w \subset T$ the associated special torus (corresponding to $\mathfrak{h}_w$), Let $n(w)$ be a well chosen representative of w in $N(w)$. Then $N(w) = T.n(w)$. We define

$$\tau_w : N(w) = T.n(w) \to T/T_w \subset \hat{T}$$

by

$$\tau_w(tn(w)) = t \bmod T_w.$$

Since all well chosen $n(w)$ differ only by elements in T_w, the definition of τ_w is independent of the choice of $n(w)$. Note that for $1 \in W^p$ the map τ_1 coincides with the identity map $T \to T$.

The union of all τ_w, $w \in W^p$, defines a surjective map

$$\tau : N^p \to \hat{T}$$

We define $\mathscr{N} = \tau^{-1}(\hat{\mathscr{T}}^c)$, and $\mathscr{N}(w) = \mathscr{N} \cap N(w)$ if $w \in W^p$. Then $\mathscr{N}$ (resp. $\mathscr{N}(w)$) is an open subset of N^p (resp. $N(w)$).

6.5. The factorization

Let us first collect some auxiliary results.

Let ∇' be a connected special subset of ∇ and Δ' the corresponding subset of Δ. It follows from [35] and [18] that there is an element

$$x = \sum_{h \in \nabla'} x_h h, \quad x_h \in \mathbb{Z},\ x_h > 0,$$

such that $\alpha(x) = 0$ (resp. $\alpha(x) < 0$) for all $\alpha \in \Delta'$ if ∇' is of affine (resp. of general) type. We then have the following two statements:

LEMMA 1 ([18] 2.2): *For all $\omega \in I^*$ we have $\omega(x) \geqslant 0$.*

LEMMA 2 *Let $\mu \in I^*$ be such that $\mu(h) = 0$ for all $h \in \nabla'$. Then $\mu + \alpha$ is not contained in I^* for all $\alpha \in \Delta'$.*

PROOF: First assume that ∇' is of general type. Let $\alpha \in \Delta'$. Then $(\mu + \alpha)(x) < 0$ since $\mu(x) = 0$ and $\alpha(x) < 0$. Thus $\mu + \alpha \notin I^*$ by Lemma 1.

Now let ∇' be of affine type. Then $(\mu + \alpha)(x) = 0$ for $\alpha \in \Delta'$. It follows from [18] 1.13, 1.17 that all elements $\omega \in I^*$ with $\omega(x) = 0$ are multiples of the fundamental imaginary root δ which has the form $\delta = \sum_{\alpha \in \Delta'} d_\alpha \alpha$, $d_\alpha \in \mathbb{Z}$, $d_\alpha > 0$. Since ∇' has rank at least two we obtain that $\mu + \alpha \notin \mathbb{R}.\delta$ and thus $\mu + \alpha \notin I^*$.

Let $R^{\omega}: G \to GL(V^{\omega})$ be a standard representation as in 6.1. Then the set P^{ω} of weights in V^{ω} is contained in $I^* \cap X$, cf. [5], [11].

LEMMA 3: *Let $w \in W^p$ be a pure element and $n(w) \in N$ a well chosen representative of w. Then $n(w)$ acts as the identity on the weight spaces V^{ω}_{μ}, $\mu \in P^{\omega} \cap (I^*)^w$.*

PROOF: Let ∇' be the type of w. By similar arguments as in the proof of 6.3, Proposition 1, we see that it is sufficient to consider the case of a connected type ∇'. Furthermore, we may assume that $w \in W_{\nabla'}$ and $n(w) \in G'$, where G' is the commutator group of the subgroup of G associated to the root datum (X, ∇', Δ'). We will prove that the whole group G' acts trivially on weight spaces V^{ω}_{μ} with $\mu \in P^{\omega} \cap (I^*)^w$. Let $\mu \in P^{\omega} \cap (I^*)^w$. Since w is of type ∇' we have $\mu(h) = 0$ for all $h \in \nabla'$. Thus Lemma 2 implies that any nontrivial vector $v \in V^{\omega}_{\mu}$ is a highest weight vector for a representation of G' on a subspace of V^{ω} containing v. Let V be a smallest such subspace. Then it follows from [11] that ∇ is a standard module. However, the weight μ is trivial on $T \cap G'$. Therefore $V = \mathbb{C}v$ and G' acts trivially on V. Applying this to all $v \in V^{\omega}_{\mu}$, $v \neq 0$, gives the desired result.

Now we can come to the proof of our main result. Let $\chi^{\omega}: \hat{\mathscr{D}}^c \to \mathbb{C}$ be the $\tilde{W}$-invariant function introduced in 5.3. We may consider χ^{ω} as a function on $\hat{\mathscr{T}}^c$. According to our choice of the exponential map $\mathfrak{h} \to T$ (cf. 4.1) χ^{ω} has the following form:
Let $T' \subset T$ be special, $\mathfrak{h}' = \operatorname{Lie} T'$, and $\bar{t} = t \cdot T'$ an element in $\mathscr{T}^c(\mathfrak{h}') \subset T/T'$. Then

$$\chi^{\omega}(\bar{t}) = \sum_{\substack{\mu \in P^{\omega} \\ \mu|_{T'} \equiv 1}} \mu(t) \dim V^{\omega}_{\mu}.$$

THEOREM: *Each element $n \in \mathscr{N}$ is of algebraic trace class and* $\operatorname{tr}_{\omega}(n) = \chi^{\omega}(\tau(n))$

$$\begin{array}{ccc} \mathscr{N} & \xrightarrow{\operatorname{tr}_{\omega}} & \mathbb{C} \\ & {\scriptstyle\tau}\searrow \quad \nearrow{\scriptstyle\chi^{\omega}} & \\ & \hat{\mathscr{T}}^c & \end{array}$$

PROOF: Let $w \in W^p$ be pure and $n(w) \in N(w)$ a well chosen representative of w. Let T_w be the special torus associated to w. In the notations of 6.1 we denote $\pi_{\mu} R^{\omega}(g) i_{\mu}$ by $R_{\mu}(g)$. Then, for $n = tn(w) \in \mathscr{N}(w)$ we have:

$$\operatorname{tr}_{\omega}(n) = \sum_{\mu \in P^{\omega}} \operatorname{tr}\big(R_{\mu}(n)\big)$$

$$= \sum_{\mu \in P^{\omega} \cap (I^*)^{w}} \operatorname{tr}\big(R_{\mu}(n)\big) \quad \text{since } R^{\omega}(n) V^{\omega}_{\mu} = V^{\omega}_{w(\mu)}$$

$$= \sum_{\mu \in P^{\omega} \cap (I^*)^{w}} \operatorname{tr}\big(R_{\mu}(t)\big) \quad \text{by Lemma 3}$$

$$= \sum_{\substack{\mu \in P^{\omega} \\ \mu|_{T_w} \equiv 1}} \mu(t) \dim V^{\omega}_{\mu}$$

$$= \chi^{\omega}(\tau(n))$$

By the results of 5.3 this gives the absolute convergence of the series $\operatorname{tr}_{\omega}(n)$ for $n \in \mathcal{N}(w)$ and proves what we claimed.

6.6. Fundamental characters

Let $\omega_1, \ldots, \omega_r \in X^*$ be fundamental dominant weights as defined in 2.1 (we assumed (X, ∇, Δ) to be simply connected). Then we may decompose X into a direct sum

$$X = Q \oplus D$$

where Q is the sublattice of X generated by ∇, and where D is the common kernel of the ω_i. We may embed a basis $\{\omega_i\}_{i=r+1,\ldots,s}$ of D^* as dominant weights into X^* by extending the ω_i, $i = r+1, \ldots, s$, trivially on Q. Then $X^* = \oplus_{i=1}^{s} \mathbb{Z}\omega_i$.

Let $R_i : G \to GL(V^{\omega_i})$ denote the corresponding standard representations. For $i = r+1, \ldots, s$ the space V^{ω_i} is one-dimensional and G operates on it by the character ω_i which, in this case, lifts from T to G. For an element $n \in N$ of algebraic trace class we denote $\operatorname{tr}_{\omega_i}(n)$ by $\chi_i(n)$. Let

$$\chi : \mathcal{N} \to \mathbb{C}^s$$

be the map defined by $\chi(n) = (\chi_1(n), \ldots, \chi_s(n))$ and $\bar{\chi} : \hat{\mathcal{T}}^c / W \to \mathbb{C}^s$ the factor of χ given by 6.5, Theorem. Let $\pi : \mathbb{C}^s \to \mathbb{C}^{s-r}$ be the projection onto the last $s-r$ coordinates. Since the situation is well known for (X, ∇, Δ) of finite type, we shall now assume that ∇ is of infinite type and furthermore connected. Then ∇ itself is a special subset of ∇. Let $M^c(\nabla) \subset \hat{M}^c = \hat{\mathcal{T}}^c / W$ be the corresponding stratum, cf. 4.5. Then the restriction of $\pi \circ \bar{\chi}$ to $M^c(\nabla)$ induces an analytic isomorphism from $M^c(\nabla)$ to an open subset $\mathcal{U} \subset \mathbb{C}^{s-r}$.

Now assume that ∇ is of general type. Then it follows from Looijenga's work [18] that $\bar{\chi}$ induces an isomorphism from a neighborhood of $M^c(\nabla)$ onto an open subset of $\mathbb{C}^s$.

In the case that ∇ is affine one has more specific but also less complete information. First one can assume $c = 0$, cf. [14]. Then the set $\mathcal{U}$ has the form $\mathcal{U} = \{u \in (\mathbb{C}^*)^{s-r} \| |\nu(u)| < 1\}$ for some nontrivial character $\nu : (\mathbb{C}^*)^{s-r} \to \mathbb{C}^*$. For $u \in \mathcal{U}$ let $\hat{M}_u = (\pi \circ \bar{\chi})^{-1}(u)$ and $\mathbb{C}^r_u = \pi^{-1}(u)$. Then $\bar{\chi}(\hat{M}_u) \subset \mathbb{C}^r_u$, and $\bar{\chi}$ induces an isomorphism from $\hat{M}_u$ onto $\mathbb{C}^r_u$ for all u in a dense open subset $\mathcal{U}'$ of $\mathcal{U}$. It was recently shown by Peterson that $\mathcal{U}' = \mathcal{U}$ in case ∇ is not of type $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $E_6^{(2)}$, $F_4^{(1)}$ (in the notation of [11], cf. [14]).

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Note added in proof: Since the completion of this paper some progress has been made on the program mentioned in the introduction. Thanks to an idea of Looijenga there is now a set-theoretic extension of the composition $N^p \to \hat{T} \to \hat{T}/W$ to a conjugation-invariant map $G \to \hat{T}/W$. However, an analytic definition of this map over $\hat{\tau}/W$ in terms of characters is still lacking. For details cf. my Habilitationsschrift "Singularitäten, Kac-Moody-Liealgebren, assoziierte Gruppen and Verallgemeinerungen", Universität Bonn, March 1984. This work also supersedes the previously planned elaboration of [30].