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FAMILIES OF QUINTIC SURFACES AND CURVES *

Edmond E. Griffin II

Introduction

Horikawa has given a complete description of the family of all numerical quintics, that is, the smooth surfaces of general type with $p_g = 4$, $q = 0$, and $c_1^2 = 5$. This family has two 40 dimensional components, I and II. I is the component parametrizing quintic surfaces in P^3 with at worst rational double point singularities. The other component, II, parametrizes numerical quintics, S, on which the linear system $|K_S|$ has one simple base point P. These components meet transversely in a 39 dimensional sub-locus of II called II_b. Horikawa shows, by analytic methods, that if $S \in \text{II}_b$ then

$$h^1(S, \Theta_S) = 41$$

and that there is one obstruction to deforming S whose leading term is of the form $xy = 0$. In particular, there are small deformations of any such S that are smooth quintic surfaces in $\mathbb{P}^3$.

At the Montreal summer conference on Algebraic Geometry in August 1980, Miles Reid posed an open problem [Reid]. The problem is to give an explicit algebraic family, $\mathscr{S}$, such that,

(1) S_t is a smooth quintic surface in $\mathbb{P}^3$, for $t \neq 0$ and

(2) S_0 is a smooth member of II_b.

Again, Horikawa shows that the canonical map,

$$\varphi|K_{S_t}|: S_t \to \mathbb{P}^3$$

is an embedding for $t \neq 0$, while

$$\varphi|K_{S_0}|: S_0 \to \mathbb{P}^3$$

is a 2 to 1 map onto a quadric cone. Thus the "canonical model" of the family, $\mathscr{S}$, in $\mathbb{P}^3$ is a family of smooth quintics degenerating to a doubled cone pulse a plane.

* This work forms a major part of the author's doctoral thesis written at Harvard University in 1982 under the advice of Phillip Griffiths. I wish to thank Phil again for all his help and patience.

Note that given such an explicit family, the generic hyper-plane section of S_t would be a smooth plane quintic curve to $t \neq 0$. The surface in II are characterized as ramified double covers of rational surfaces, as can be seen from the fact that the generic element, C, in $|K_{S_0}|$ is a smooth hyper-elliptic curve. By the adjunction formula,

$$2g(C) - 2 = 2K_{S_0}^2 = 10$$

or

$$g(C) = 6.$$

Thus the generic hyper-plane section of $\mathscr{S}$ is a family of smooth plane quintics for $t \neq 0$, such that C_0 is a smooth hyper-elliptic curve. Hence the firest step in our program is to study hyper-elliptic curves of genus 6 equipped with semi-canonical divisors. Once one sees how to deform such a curve to a smooth plane quintic it will be fairly straight-forward to do the same for the surface case.

This observation points to the following very suggestive idea: It is clear that knowledge of families of surfaces yields knowledge of families of curves. As will be seen, the methods used here work with equal ease in either setting. Perhaps there is some deep connection between the study of pairs, $L \to C$, of curves with special line bundles (e.g. giving an embedding of C in $\mathbb{P}^2$) and the study of surfaces of general type. Of course, this connection will not be made explicit here, but it seems that this example of the quintics may, in part, point the way to examining it.

The author wishes to give special thanks to Miles Reid whose suggestions and comments were very helpful. Especially nice was the idea for a purely algebraic proof of Theorem II.

1. Preliminaries

Denote by $R(X, D)$ the ring,

$$R(X, D) = \bigoplus_{n \geqslant 0} H^0(X, \mathcal{O}(nD)),$$

where X is an algebraic variety, D is a divisor on X, and the multiplication is the usual "cup" product. In the cases at hand this will be a graded ring (generally *not* generated as an algebra by its degree one part!) which is finitely generated over $H^0(X, \mathcal{O}) \simeq C$. Let

$$R_m(X, D) = H^0(X, \mathcal{O}(mD))$$

be the m-th degree part of $R(X, D)$. In all the computations to follow D

will be a divisor such that for some $m \in \mathrm{II}$, $mD \in |K_X|$, the canonical linear system on X.

By $\mathbb{P}(e_1^{n_1}, e_2^{n_2}, \dots, e_k^{n_k})$ we will denote the "weighted projective space," [Mori].

$$\mathrm{Proj}\left(C[x_{11}, x_{12}, x_{13}, \dots, x_{1n_1}, x_{21}, x_{22}, \dots, x_{2n_2}, \dots, x_{kl}, \dots, x_{kn_k}]\right)$$

where the grading is given by weight$(x_{ij}) = e_i$.

Let C be a smooth algebraic curve of genus g. *C possesses a g_d^r* if there is a line bundle, $L \to C$, such that:

(1) $\deg_C L = d$ and

(2) $\dim H^0(C, L) = L^0(C, L) \geqslant r + 1$.

Any *$r + 1$ dimensional* linear sub-system of $|L|$ will be called a g_d^r on C.

2. Curves

Consider a family, $\mathscr{L} \to C \to \Delta$ such that $L_t \to C_t$ satisfies,

(1) C_t is a smooth, genus 6 curve for all t,

(2) $\deg_{C_t} L_t = 5$ for all t,

(3) $h^0(C_t, L_t) = 3$ for $t \neq 0$, and

(4) L_t is very ample on C_t for $t \neq 0$.

By the Upper Semi-Continuity Theorem [Hartshorne],

$$h^0(C_0, L_0) \geqslant 3$$

and by Clifford's Theorem, equality must hold. So, C_0 comes equipped with a

$$g_5^2 = |L_0| = |D_0|$$

where D_0 is some effective divisor of degree 5 on C_0. Now it is simple to see that either $|D_0|$ has no base point, in which case the morphism

$$\varphi_{|D_0|}: C_0 \to \mathbb{P}^2$$

is an embedding, or $|D_0|$ has one base point P. In this case $D_0 - P$ is a g_4^2 on C_0, and hence, by Clifford's Theorem, C_0 is hyper-elliptic and,

$$|D_0| = |4Q| + P$$

where Q is a Weierstrass point on C_0. Thus, either C_0 has a very ample g_5^2 or C_0 is hyper-elliptic.

More can be said about $D_0 = 4Q + P$ in the latter case. The adjunction formula yields,

$$\mathcal{O}(K_{C_t}) = \mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2} + C_t)|_{C_t} = \mathcal{O}_{\mathbb{P}^2}(-3H + 5H)|_{C_t} = \mathcal{O}_{\mathbb{P}^2}(2)|_{C_t}.$$

So,

$$K_{C_t} \simeq L_t^2, \quad t \neq 0.$$

Now, consider the family, $K_{C_t} \otimes L_t^{-2} =$ (by def.) $M_t \to C_t$. For $t \neq 0$ the above remark shows that

$$M_t \simeq \mathcal{O}_{C_t}$$

and for $t = 0$,

$$\deg_{C_0} M_0 = 0.$$

Since $h^0(C_t, M_t) = 1$, one has

$$h^0(C_0, M_0) \geqslant 1,$$

which, of course, implies that $M_0 \simeq \mathcal{O}_{C_0}$ or,

$$K_{C_0} \simeq L_0^2.$$

Thus,

$$2D_0 \equiv 8Q + 2P \equiv K_{C_0} \equiv 10Q \qquad \text{or,}$$

$$2P \equiv 2Q,$$

so that

$$2P \in |2Q| = \text{the} \quad g_2^1 \quad \text{on} \quad C_0.$$

This implies that P must be a Weierstrass point on C! Therefore.

Proposition 1: *If C is a smooth, genus 6, hyperelliptic curve, in a family of smooth plane quintics then its*

$$g_5^2 = |L_0| = |D_0| = 2g_2^1 + P = |5P|$$

where the base point P is a Weierstrass point.

Up until this point no indication has been given as to whether or not such a pair, $L_0 \to C_0$, actually occurs as the "limit" of smooth plane quintics. The main theorem of this section rectifies this situation.

Theorem I: *Given a smooth hyper-elliptic curve, C, of genus 6, together with a Weierstrass point $P \in C$, set*

$$L = \mathcal{O}_C(2g_2^1 + P) = \mathcal{O}_C(5P).$$

Then there exists a flat family, $\mathscr{L} \to \mathscr{C} \to \Delta$, *such that*

(1) $L_0 \to C_0$ *is isomorphic to* $L \to C$

and

(2) $L_t \to C_t$ *is a smooth plane quintic curve together with the line bundle that embeds it in* $\mathbb{P}^2$.

(See [Chang] for a recently published, independent proof.)

REMARK: The theorem and Proposition 1 combine to show that the closure of the locus of plane quintics in the moduli space of genus 6 curves, $\mathscr{M}_6$, consists exactly of the quintics themselves and the locus of hyper-elliptic curves. Also note that the scheme $\mathscr{W}^2_{5,6}$[ACGH] (which parametrizes pairs $L \to C$ with $g(C) = 6$, $\deg_C L = 5$, and $h^0(C, L) \geqslant 3$) has two components, W_1 and W_2, each of dimension 12. The first, W_1, parametrizes the smooth quintics with their unique g^2_5. The second, W_2,
parametrizes the smooth quintics with their unique g^2_5. The second, W_2,
parametrizes the pairs, $L \to C$, where C is a hyper-elliptic curve and $|L| = 2g^1_2 + P$, P an arbitrary base point on C. The theorem says that W_1 meets W_2 precisely in the co-dimension 1 sub-locus of W_2 parametrizing pairs, $L \to C$, as above, with P a Weierstrass point.

For the proof of the theorem we will need two standard results.

FACT 1: Let S be a graded ring over k and let T be another graded ring such that

$$T(n) = S(nd)$$

for all $n \geqslant 0$ and some fixed d. Then

$$\mathrm{Proj}(T) \simeq \mathrm{Proj}(S).$$

[Hartshorne Ex. II.5.13]

FACT 2: Let C be a smooth curve of genus $\geqslant 2$. Then $R(C, K_C)$ is finitely generated (not necessarily in degree one!) and

$$C \simeq \mathrm{Proj}(R(C, K_C)).$$

PROOF OF THEOREM I: Begin by computing $R = R(C, P)$. Recall

$$R(C, P) = \bigoplus_{n \geqslant 0} H^0(C, \mathcal{O}_C(np)).$$

In degree

$$\begin{array}{lll} n=0 & h^0(C, \mathcal{O}_C)=1 & \text{generator} \quad 1 \\ n=1 & h^0(C, \mathcal{O}_C(P))=1 & \text{generator} \quad u \\ z=2 & h^0(C, \mathcal{O}_C(2P))=2 & \text{generators} \quad u^2, v. \end{array}$$

The sections u^2, v of $\mathcal{O}_C(2P)$ form a basis of the $g_2^1=|2P|$ on C, since P is a Weierstrass point.

It is clear that there can be no polynomial relation of the form $p(u, v)=0$ in R. On the other hand

$$\begin{aligned} &\text{the number of monomials of degree } n \text{ in } u \text{ and } v \\ &=\left[\frac{n}{2}\right]+1 \\ &=h^0(C, \mathcal{O}(nP)) \end{aligned}$$

for $0 \leqslant n \leqslant 12$. The last equality is from the Riemann-Roch Theorem. Thus, up to degree 12, the ring is generated by u and v.

In degree 13 we note that $|13P|$ is very ample and thus there must be a new generator! There is exactly one since $\#\{u^iv^j | i+2j=13\}=7$ and

$$n=13 \qquad h^0(C, \mathcal{O}(13P))= \quad \text{new generator} \quad w.$$

It is easy to see that w is "odd" with respect to the natural involution i: $C \to C$. That is, while

$$\begin{aligned} &i^*u=u \quad \text{and} \quad i^*v=v \\ &i^*w=-w \end{aligned}$$

In $R(C, P)$. Set $R^+=\{x \in R | i^*x=x\}$ and $R^-=R-R^+$. Then R^+ is spanned by the monomials in u and v and R^+ by monomials in which w appears to an odd power. Thus $w^2 \in R^+$ and must be equal to a homogeneous polynomial of degree 26 in u and v. This can be seen by computing dimensions as well. For all n, $\#\{u^iv^i | i+2j=n\}=[\frac{n}{2}]+1$ and for $n \geqslant 11$ $h^0(C, \mathcal{O}(NP))=n-5$. So in degree

$$\begin{array}{ll} n=26 & h^0(26P)=21 \\ & \#\{u^iv^j | i+2j=26\}=14 \\ & \#\{wu^iv^j | i+2j=13\}=7. \end{array}$$

Thus we must have

$$w^2=g_{26}(u, v)$$

where $g_{26}(u, v)$ is a weighted homogeneous polynomial of degree 26. (By considering the effect of i^* one sees that the relation cannot have the form $\lambda w^2 = wh_{12}(u, v) + g'_{26}(u, v)$ with $h_{12} \neq 0$ or $\lambda = 0$). Finally check that

$$\begin{aligned} &\#\{u^i v^j | i + 2j = n\} + \#\{wu^i v^j | i + 2j = n - 13\} \\ &\quad = \left[\frac{n}{2}\right] + 1 + \left[\frac{n-13}{2}\right] + 1 \\ &\quad = n + 5 = h^0(C, \mathcal{O}(nP)) \end{aligned}$$

for $n \geqslant 11$.

This means that

$$R = C[u, v, w]/(w^2 - g_{26}(u, v))$$

and that (by Facts 1 and 2)

$$C \simeq \operatorname{Proj}(R(C, K_C)) = \operatorname{Proj}(R(C, (OP)) = \operatorname{Proj}(R(C, P)) = \operatorname{Proj}(R).$$

Also note that C is embedded in $\mathbb{P}(1, 2, 13)$ by the map

$$C[u, v, w]$$

Before proceeding one should note that $g_{26}(0, v) \neq 0$. This is because C is assumed non-singular. To see this, consider the open set $U_{(v)} \subseteq \operatorname{Proj}(R)$ given by the degree 0 elements in the localization $R_{(v)}$ (i.e. the open set where $v \neq 0$). Then

$$\begin{aligned} U_{(v)} &= \operatorname{Spec}\left(C\left[\frac{u^2}{v}, \frac{uw}{v^7}, \frac{w^2}{v^{13}}\right] \Big/ \left(\frac{w^2}{v^{13}} - \frac{g_{26}(u, v)}{v^{13}}\right)\right) \\ &= \operatorname{Spec}(C[x, y, z]/(z - \bar{g}_{13}(x), xz - y^2)) \end{aligned}$$

where $\bar{g}_{13}(x) = \bar{g}_{13}(x, 1)$ is obtained by replacing u^2 by x and v by y in $g_{26}(u, v)$. If $\bar{g}_{13}(0) = 0$ then $(0, 0, 0) \in C$. But then the Jacobi matrix

$$\begin{bmatrix} \dfrac{\partial g}{\partial x} & 0 & 1 \\ z & -2y & x \end{bmatrix}$$

has rank 1 at $(0, 0, 0) \in C$ and so C is singular. Therefore $\bar{g}_{13}(0) \neq 0$ and so $g_{26}(0, v) \neq 0$.

The map

$$C[u^2, v] \hookrightarrow R$$

yields the 2-to-1 covering map

$$\pi\colon \mathrm{Proj}(R) \simeq C \to \mathbb{P}' \simeq \mathrm{Proj}(C[u^2, v]).$$

By the Hurwitz genus formula there are 14 branch points of this map on $\mathbb{P}^1$. These break up into two sets. Thirteen of them given by

$$g_{26}(u, v) = 0.$$

on $\mathbb{P}^1$. And the fourteenth given by

$$u^2 = 0.$$

(This is a fourteenth by the fact that $g_{26}(0, v) \neq 0$.)
All of these remarks give

THEOREM I.A: *Let C be a smooth hyper-elliptic curve of genus 6. Then*

$$C \simeq \mathrm{Proj}(R(C, P)) \simeq \mathrm{Proj}\big(C[u, v, w]/\big(w^2 - g_{26}(u, v)\big)\big)$$

where the grading is

$$\mathrm{weight}(u) = 1$$

$$\mathrm{weight}(v) = 2$$

$$\mathrm{weight}(w) = 13$$

and $g_{26}(u, v)$ is a weighted homogeneous polynomial such that (1) $g_{26}(0, v) \neq 0$ *and* (2) $\bar{g}_{13}(x, y)$ *has* 13 *distinct roots.*

REMARK: When $u^2 = 0$ we have $w^2 = \lambda v^{13}$. This seems to yield two points on C namely

$$(0, 1, \pm\sqrt{\lambda}).$$

However, under the C^* action on $\mathbb{P}(1, 2, 13)$

$$-1 \cdot (0, 1, +\sqrt{\lambda}) = \big((-1)\cdot 0, (-1)^2 \cdot 1, (-1)^{13} \cdot \sqrt{\lambda}\big)$$

$$= (0, 1, -\sqrt{\lambda}).$$

Thus these two points are really the same! This in turn means that $u^2 = 0$ is indeed a branch point.

As the next step in the proof of Theorem I we write down generators and relations for the subring $R(C, 5P) \subseteq R(C, P)$. Denote this subring by $R^{(5)}$. Then $R^{(5)}$ is the graded ring given by

$$R^{(5)}_d = R_{5d} = H^0(C, \mathcal{O}_C(5dP))$$

for each $d \geqslant 0$. It is easy to see by counting dimensions and the remarks on the computation of R that

$$R^{(5)} = C[u^5, u^3v, uv^2, v^5, u^2w, vw]/I$$

where $I = (w^2 - g_{26}(u, v)) \cap R^{(5)}$. Indeed, these monomials clearly generate $R^{(5)}$ in degrees 1, 2, and 3. Furthermore any monomial, m, of degree $5d$ in R can be written as u^iv^j where $i + 2j = 5d$ or as wu^iv^j where $i + 2j = 5d - 13$, since $w^2 = g_{26}(u, v)$. Thus either

$$m = u^iv^j \quad \text{and} \quad 5 | i + 2j \quad \text{or,}$$

$$m = u^2wu^iv^j \quad \text{and} \quad 5 | i + 2j \quad \text{or,}$$

$$m = vwu^iv^j \quad \text{and} \quad 5 | i + 2j.$$

Finally it is easy to see that if $5 | i + 2j$ then u^iv^j is a monomial in u^5, u^3v, uv^2, and v^5.

As to the computation of I let us set

$$\left.\begin{array}{l} x_1 = u^5 \\ x_2 = u^3v \\ x_3 = uv^2 \end{array}\right\} \quad \text{weight } 1$$

$$y = v^5 \qquad \text{weight } 2$$

$$\left.\begin{array}{l} z_1 = u^2w \\ z_2 = vw \end{array}\right\} \quad \text{weight } 3$$

in $R^{(5)}$.

First there are the "Koszul relations" (Table 1). These can be very nicely expressed in the following determinantal form:

$$\text{rank} \begin{bmatrix} x_1 & x_2 & z_1 & x_3^2 \\ x_2 & x_3 & z_2 & y \end{bmatrix} \leqslant 1. \qquad (*)$$

The non-trivial relations in I come from the equation $w^2 - g_{26}(u, v)$ as in Table 2. These last three equations define C as a Weil divisor on the determinantal variety of $\mathbb{P}(1^3, 2, 3^2)$ given by $(*)$. They are obtained from each other by monomial replacement.

TABLE 1

r_1: $x_1x_3 - x_2^2$

r_2: $x_1y - x_2x_3^2$

r_3: $x_2y - x_3^3$

r_4: $x_1z_2 - x_2z_1$

r_5: $x_2z_2 - x_3z_1$

and

r_6: $z_1y - z_2x_3^2$

TABLE 2

r_7: $z_1^2 - u^4g_{26}(u, v) = z_1^2 - f_1(x_1, x_2, x_3, y)$

r_8: $z_1z_2 - u^2vg_{26}(u, v) = z_1z_2 - f_2(x_1, x_2, x_3, y)$

r_9: $z_2^2 - v^2g_{26}(u, v) = z_2^2 - f_3(x_1, x_2, x_3, y)$.

For later reference the form of the $f_j(x_i, y)$ will be examined more closely. By property 1) in Theorem 2 the coefficient of v^{15} in $v^2g_{26}(u, v)$ is non-zero. Since $v^{15} = y^3$ this yields

$$f_3(x_i, y) = \lambda y^3 - f_3'(x_i, y)$$

with $\lambda \neq 0$. Further

$$v^2g_{26}(u, v) = \lambda v^{15} + uv^2h_{25}(u, v)$$

and since $x_3 = uv^2$ this gives

$$f_3(x_i, y) = \lambda y^3 + x_3Q(x_1, x_2, x_3, y)$$

where $Q(x_i, y)$ is a weighted homogeneous polynomial of weight five (namely $Q(x_i, y) = h_{25}(u, v) = g_{26}(u, v) - \lambda v^{13}/u$). Similarly it is easy to see that

$$f_2(x_i, y) = \lambda x_3^2 y^2 + x_2 Q(x_i, y)$$

$$f_1(x_i, y) = \lambda x_3^4 y + x_1 Q(x_i, y)$$

Thus we have

$$C \simeq \mathrm{Proj}(R(C, 5P)) = \mathrm{Proj}(C[x_1, x_2, x_3, y, z_1, z_2]/I)$$

where

TABLE 3

$I =$

$r_1 - r_6$: rank $\begin{bmatrix} x_1 & x_2 & z_1 & x_3^2 \\ x_2 & x_3 & z_2 & y \end{bmatrix} = 1$

and

r_7: $z_1^2 - \lambda x_3^4 y - x_1 Q(x_i, y)$

r_8: $z_1 z_2 - \lambda x_3^2 y^2 - x_2 Q(x_i, y)$

r_9: $z_2^2 - \lambda y^3 - x_3 Q(x_i, y)$

and each f_i is weighted homogeneous of weight 6.

Next the 1st syzygies are computed. The first group of these can again be written in determinantal form,

$$\mathrm{rank} \begin{bmatrix} x_1 & x_2 & z_1 & x_3^2 \\ x_1 & x_2 & z_1 & x_3^2 \\ x_2 & x_3 & z_2 & y \end{bmatrix} \leqslant 2 \qquad (**)$$

$$\mathrm{rank} \begin{bmatrix} x_2 & x_3 & z_2 & y \\ x_1 & x_2 & z_1 & x_3^2 \\ x_2 & x_3 & z_2 & y \end{bmatrix} \leqslant 2.$$

More precisely the eight 3×3 minors of these matrices should each have determinant zero. Expanding across the top rows yields,

$$s_1 \colon x_1 r_3 - x_2 r_2 + x_3^2 r_1 \equiv 0$$

$$s_2 \colon x_2 r_3 - x_3 r_2 + y r_1 \equiv 0$$

$$s_3 \colon x_1 r_5 - x_2 r_4 + z_1 r_1 \equiv 0$$

$$s_4 \colon x_2 r_5 - x_3 r_4 + z_2 r_1 \equiv 0$$

$$s_5 \colon x_1 r_6 - z_1 r_2 + x_3^2 r_4 \equiv 0$$

$$s_6 \colon x_2 r_6 - z_2 r_2 + y r_4 \equiv 0$$

$$s_7 \colon x_2 r_6 - z_1 r_3 + x_3^2 r_5 \equiv 0$$

$$s_8 \colon x_3 r_6 - z_2 r_3 + y r_5 \equiv 0.$$

The second group of syzygies involves the non-trivial relations r_7, r_8, and r_9. They are easily derived from the form of the equations in table 3. Only one of the four will be computed here.

Consider

$$\begin{aligned} x_1 r_8 - x_2 r_7 &= x_1 z_1 z_2 - x_2 z_1^2 - \lambda y x_3^2 \left(s_1 y - x_2 x_3^2 \right) \\ &= z_1 r_4 - \lambda y x_3^2 r_2 \end{aligned}$$

Thus

$$s_9 \colon x_1 r_8 - x_2 r_7 - z_1 r_4 + \lambda y x_3^2 r_2$$

Continuing in a completely analogous fashion yields

$$s_{10} \colon x_2 r_8 - x_3 r_7 - z_1 r_5 + \lambda y x_3^2 r_3 - Q(x_i, y) r_1$$

$$s_{11} \colon x_1 r_9 - x_2 r_8 - z_2 r_4 + \lambda y^2 r_2 + Q(x_i, y) r_1$$

$$s_{12} \colon x_2 r_9 - x_3 r_8 - z_2 r_5 + \lambda y^2 r_3$$

Collecting all the syzygies gives Table 4.

Note, the curve C_0 resides naturally in a weighted projective space,

$$\mathbb{P}(1^3, 2, 3^2) = \operatorname{Proj}(C[x_1, x_2, x_3, y, z_1, z_2])$$

and that the semi-canonical map

$$\varphi_{|L_0|} \colon C_0 \to \mathbb{P}^2$$

is given by the restriction, to C_0, of the projection map in $\mathbb{P}(1^3, 2, 3^2)$

TABLE 4

$$s_1 - s_8:\quad \text{rank}\begin{bmatrix} x_1 & x_2 & z_1 & x_3^2 \\ x_1 & x_2 & z_1 & x_3^2 \\ x_2 & x_3 & z_2 & y \end{bmatrix} \leqslant 2$$

and

$$\text{rank}\begin{bmatrix} x_2 & x_3 & z_2 & y \\ x_1 & x_2 & z_1 & x_3^2 \\ x_2 & x_3 & z_2 & y \end{bmatrix} \leqslant 2$$

s_9: $x_1r_8 - x_2r_7 - z_1r_4 + \lambda yx_3^2r_2$

s_{10}: $x_2r_8 - x_3r_7 - z_1r_5 + \lambda yx_3^2r_3 - Q(x_i, y)r_1$

s_{11}: $x_1r_9 - x_2r_8 - z_2r_4 + \lambda y_2^2 + Q(x_i, y)r_1$

s_{12}: $x_2r_9 - x_3r_8 - z_2r_5 - \lambda y^2r_3$

from the "weighted plane,"

$$V = \{x_i = 0\} \subseteq \mathbb{P}(1^3, 2, 3^2)$$

to the standard plane

$$\mathbb{P}^2 = \{y + z_i = 0\} \subseteq \mathbb{P}(1^3, 2, 3^2).$$

The single point $P \in C_0$ which lies in V is, of course, the base point of $|L_0|$ and corresponds to the special branch point defined by $u^2 = 0$ in the discussion above. (Again at first glance

$$C_0 \cap V = \{(0, 0, 0, 1, 1, 1), (0, 0, 0, 1, -1, -1)\}$$

but these "two" points are identified under the C^* action).

Finally, on to the deformation of R to the semi-canonical ring of a plane quintic. The observation that the semi-canonical map is a projection to $\mathbb{P}^2$ suggests that the deformation should somehow "eliminate" the weighted variables y, z_1, and z_2. (A more specific motivation is given in [Griffin].)

To implement this idea proceed as follows:

r_1: $x_1x_3 - x_2^2$ becomes

R_1: $x_1x_3 - x_2^2 + t^2\lambda y$

r_2: $x_1y - x_2x_3^2$ becomes

R_2: $x_1y - x_2x_3^2 + tz_1$

and

r_3: $x_2 y - x_3^3$ becomes

R_3: $x_2 y - x_3^3 + tz_2$.

These new equations, R_1, R_2 and R_3 are used to "eliminate" y, z_1, and z_2 when $t \neq 0$, and when $t = 0$ $R_i = r_i$. Specifically, they give an embedding

$$\mathbb{P}^2 \hookrightarrow \mathbb{P}(1^3, 2, 3^2)$$

for each non-zero t, namely

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, y, z_1, z_2)$$

where

$$y = -(x_1 x_3 - x_2^2)/\lambda t^2$$

$$z_1 = x_1(x_1 x_3 - x_2^2)/\lambda t^3 + x_2 x_3^2/t$$

and

$$z_2 = x_2(x_1 x_3 - x_2^2)/\lambda t^3 + x_3^3/t.$$

Now consider the syzygy

$$s_1 = x_1 r_3 - x_2 r_2 + x_3^2 r_1 = 0.$$

Replacing r_i by R_i gives

$$\begin{aligned} x_1 R_3 - x_2 R_2 + x_3^2 R_1 &= s_1 + t x_1 z_2 - t x_2 z_1 + t^2 \lambda x_3^2 y \\ &= s_1 + t r_4 + t^2 \lambda x_3^2 y. \end{aligned}$$

To extend s_1 to a syzygy S_1 (which must be done to insure the family C is flat) one simple alteration is necessary,

$$R_4\colon r_4 + t\lambda x_3^2 y.$$

Then

$$\begin{aligned} x_1 R_3 - x_2 R_2 + x_3^2 R_1 &= s_1 + t r_4 + t^2 \lambda x_3^2 y \\ &= s_1 + t R_4. \end{aligned}$$

Note: It was in order to make this last equation work that we added $t^2\lambda y$ to r_1 above, instead of the simpler $t\lambda y$. Thus S_1 is

$$S_1\colon x_1R_3 - x_2R_2 + x_3^2R_1 - tR_4.$$

Clearly $S_1 \to s_1$ as $t \to 0$. Similarly one has,

$$S_2\colon x_2R_3 - x_3R_2 + yR_1 - tR_5$$

with

$$R_5\colon r_5 + t\lambda y^2.$$

Note that in the process of extending s_1 and s_2, the alterations were "forced" on r_4 and r_5.

Only one more such calculation will be shown. Consider the syzygy

$$s_{11}\colon x_1r_9 - x_2r_8 - z_2r_4 + \lambda y^2r_2 + Q(x_i, y)r_1$$

Substituting R_1, R_2, R_4 as above yields

$$x_1r_9 - x_2r_8 - z_2R_4 + \lambda y^2R_2 + Q(x_i, y)R1$$
$$= s_{11} - t\lambda z_2x_3^2y + t\lambda z_1y^2 + t^2\lambda Q(x_i, y)y$$
$$= s_{11} + t\lambda y\left(z_1y - z_2x_3^2 + tQ(x_i, y)\right)$$

so it r_6 becomes

$$R_6\colon z_1y - z_2x_3^2 + tQ(x_i, y)$$

and $r_7 = R_7$, $r_8 = R_8$, and $r_9 = R_9$ are unchanged then s_{11} becomes

$$S_{11}\colon x_1R_9 - x_2R_8 - z_2R_4 + \lambda y^2R_2 + Q(x_i, y)R_1 - t\lambda yR_6.$$

REMARK: It as if by magic that the process of extending the syzygies has "forced" the generic weight five $Q(x_i, y)$ down from the degree 6 relations r_7, r_8, r_9 to the one degree 5 equation R_6 when $t \neq 0$!

Here, then, is the family, $\mathscr{C}$, of smooth plane quintic curves with $C_0 \simeq C$:

$$\mathscr{C} \simeq \operatorname{Proj}(C[x_1, x_2, x_3, y, z_1, z_2]/\ell$$

where

TABLE 5

$\ell =$

R_1: $x_1x_3 - x_2^2 + t^2\lambda y$

R_2: $x_1y - x_2x_3^2 + tz_1$

R_3: $x_2y - x_3^3 + tz_2$

R_4: $x_1z_2 - x_2z_1 + t\lambda x_3^2 y$

R_5: $x_2z_2 - x_3z_2 + t\lambda y^2$

R_6: $z_1y - z_2x_3^2 + tQ(x_i, y)$

R_7: $z_1^2 - \lambda x_3^4 y - x_1Q(x_i, y)$

R_8: $z_1z_2 - \lambda x_3^2 y^2 - x_2Q(x_i, y)$

R_9: $z_2^2 - \lambda y^3 - x_3Q(x_i, y)$

S_1: $x_1R_3 - x_2R_2 + x_3^2R_1 - tR_4$

S_2: $x_2R_3 - x_3R_2 + yR_1 - tR_5$

S_3: $x_1R_5 - x_2R_4 + z_1R_1 - \lambda tyR_2$

S_4: $x_2R_5 - x_3R_4 + z_2R_1 - \lambda tyR_3$

S_5: $x_1R_6 - x_3^2R_4 + z_1R_2 - tR_7$

S_6: $x_2R_6 - yR_4 + z_2R_2 - tR_8$

S_7: $x_2R_6 - x_3^2R_5 + z_1R_3 - tR_8$

S_8: $x_3R_6 - yR_5 + z_2R_3 - tR_9$

S_9: $x_1R_8 - x_2R_7 - z_1R_4 + \lambda yx_3^2R_2$

S_{10}: $x_2R_8 - x_3R_7 - z_1R_5 + \lambda yx_3^2R_3 - Q(x_i, y)R_1 + \lambda tyR_6$

S_{11}: $x_1R_9 - x_2R_8 - z_2R_4 + \lambda y^2R_2 + Q(x_i, y)R_1 - \lambda tyR_6$

S_{12}: $x_2R_9 = x_3R_8 - z_2R_5 + \lambda y^2R_3$

where $Q(x_i, y)$ is a weight 5 homogeneous polynomial.

PROOF OF THEOREM I: Given a smooth, genus 6 hyperelliptic curve, C, one can assume that one of the 14 branch points on $\mathbb{P}^1 \simeq \mathrm{Proj}(C[x, y])$

of the 2-to-1 map

$$C \to \mathbb{P}^1$$

is the point (0, 1). Let $f(x, y)$ be a homogeneous polynomial of degree 13 which vanishes at the other 13 branch points. Take

$$g_{26}(u, v) = f(u^2, v)$$

in Theorem 2 and then set

$$h_{25}(u, v) = \frac{g_{26}(u, v) - \lambda v^{13}}{u}.$$

Finally taking

$$Q(x_i, y) = h_{25}(u, v)$$

in table 5 gives the desired family.

Q.E.D.

it is interesting to write down the single quintic equation (with parameter t) that gives the plane model of the family $\mathscr{C}$. This is done by 'eliminating' y, z_1 and z_2 for $t \neq 0$ and using R_6. The result is:

$$x_1(x_1x_3 - x_2^2)^2 + t^2(x_1x_3 - x_2^2)(2\lambda x_2 x_3^2 - A(x_i)(x_1x_3 - x_2^2))$$
$$+\lambda t^4(\lambda x_3^5 + B(x_i)(x_1x_3 - x_2^2)) - \lambda^2 t^6 C(x_i) = 0 \qquad (***)$$

where

$$Q(x_i, y) = A(x_i)y^2 + B(x_i)y + C(x_i).$$

First, note that the plane model of C_0 is a double conic plus a TANGENT line. It is clear that for $Q(x_i, y)$ sufficiently general (see Theorem 2) $C(x_i)$ is general and thus $(***)$ represents a family of smooth plane quintics for $t \neq 0$.

Second, in this case, it is easy to explain why only even powers of t appear in the family above. This means that $t \to -t$ gives an involution

$$i: \mathscr{C} \to \mathscr{C}$$

on the whole family. To see that there must be such an involution and that it induces the natural one, i, on C, consider the *normal sheaf*. The map

$$\varphi: C_0 \to C_0/i$$

yields on exact sequence,

$$0 \to \Theta_{C_0} \to \varphi_* \Theta_{\mathbb{P}^1} \to N_\varphi \to 0$$

which defines the normal sheaf, N_φ. By [SGAI] the obstruction to extending φ to any deformation of C_0 is an element of

$$H^1(C_0, N_\varphi \otimes J).$$

Here J is an ideal in an Artin ring $\mathscr{R}$, annihilated by $m_{\mathscr{R}}$. In this case N_φ is supported on points and therefore,

$$H^1(C_0, N_\varphi \otimes J) = 0$$

for all such J. So, the map φ *can* be extended to the complete family $\mathscr{C}$. This in turn means that there must be an involution on $\mathscr{C}$ which induces i on C_0.

3. Surfaces

In order to give an explicit algebraic description of the deformation space of numerical quintic surfaces we begin with $S \in$ II. (i.e. K_S has a simple base point [Horikawa]). Then just as in the curve case we can construct

$$R = R(S, K_S) = \bigoplus_{n \geqslant 0} H^0(S, nK_S).$$

By [Mumford] and [Bombieri] one has that R is finitely generated and that in the case at hand

$$S \simeq \mathrm{Proj}(R(S, K_S)).$$

Then we perturb the defining equations in R to obtain the canonical ring of a smooth quintic in $\mathbb{P}^3$. These computations can be made in a purely cohomologica fashion (see [Griffin]) once an explicit realization of S as a ramified double cover of a quadric cone is seen. However, there is a purely algebraic method that uses only the constructions in §2 and the fact that if $C_0 \in |K_s|$ and C_0 is smooth then C_0 is hyper-elliptic, of genus 6, and

$$K_S|_{C_0} \equiv \tfrac{1}{2} K_{C_0} \equiv D_0.$$

To begin recall that the irregularity

$$q_S = 0.$$

Thus the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(K_S) \to \mathcal{O}_C(D_0) \to 0$$

where $D_0 \in |\frac{1}{2}K_C|$ yields,

$$0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}_S(K_S)) \to H^0(\mathcal{O}_C(D_0)) \to 0.$$

By induction and Serre Duality, then,

$$0 \to H^0(S, nK_S) \to H^0(S, (n+1)K_s) \to H^0(C, (n+1)D_0) \to 0$$

for all $n \geqslant 0$. This means there is a degree preserving surjection

$$R(S, K_S) \twoheadrightarrow R(C_0, D_0)$$

whose kernel is a principal ideal, (x_0), with weight $x_0 = 1$.

The map above simply expresses the fact that C_0 is a canonical (i.e. degree one) hyper-plane section of S. So, by suitable choice of generators we can write

$$R(S, K_S) = C[x_0, x_1, x_2, x_3, y, z_1, z_2]/\hat{I}$$

$$\to C[x_1, x_2, x_3, y, z_1, z_2]/I = R(C_0, D_0)$$

That is,

$$C_0 = S \cap \{x_0 = 0\} \subseteq \mathbb{P}(1^4, 2, 3^2).$$

We now proceed to find a nice form for $\hat{I}$. In order to see the point of these manipulations the reader is advised to skip forward to Theorem II for a clear statement of the result before going through the next few pages.

Next, for each relation in table 3 there must be a relation in $R(S, K_S)$. That is, I is obtained from $\hat{I}$ by setting $x_0 = 0$ so in R_S, we have Table 6, in which the a's, b's, c's, d's, and e's are weighted homogeneous polynomials of the appropriate weight.

Consider

$$a = \lambda_0 x_0 + \lambda_1 x_1 - 2\lambda_2 x_2 + \lambda_3 x_3.$$

By a change of coordinates,

$$x_0' = x_0$$

$$x_1' = x_1 - \lambda_3 x_0$$

Table 6

$\hat{I} =$

r_1: $x_1x_3 - x_2^2 = ax_0$

r_2: $x_1y - x_2x_3^2 = b_1x_0$

r_3: $x_2y - x_3^3 = b_2x_0$

r_4: $x_1z_2 - x_2z_1 = c_1x_0$

r_5: $x_2z_2 - x_3z_1 = c_2x_0$

r_6: $z_1y - z_2x_3^2 = dx_0$

r_7: $z_1^2 - \lambda yx_3^4 - x_1Q(x_i, y) = e_1x_0$

r_8: $z_1z_2 - \lambda y^2x_3^2 - x_2Q(x_i, y) = e_x2_0$

r_9: $z_2^2 - \lambda y^3 - x_3Q(x_i, y) = e_3x_0$

$$x_2' = x_2 - \lambda_2x_0$$

$$x_3' = x_3 - \lambda_1x_0$$

r_1 becomes

$$r_1': x_1'x_3' - x_2'^2 = \alpha x_0^2$$

where

$$\alpha = \lambda_0 + \lambda_1\lambda_3 - \lambda_2^2.$$

So we may assume that r_1 has the form

$$r_1: x_1x_3 - x_2^2 = \alpha x_0^2$$

Further, using this relation we may assume that *no* monomial in any of the b's, c's, d's, or e's is divisible by x_2^2. Of course, all of the x_i's in $r_1 - r_9$ must be changed but by altering the b's, c's, d's, and e's the form of the relations is unchanged. Continuing the same vein, replacing y by

$$y \mapsto y - x_0b_{11}$$

where,

$$b_1 = x_1b_{11} + b_{12}$$

and x_1 does not appear in any monomial in b_{12}, yields

$$r_2\colon x_1 y - x_2 x_3^2 = x_0 b_{12}.$$

By similar changes,

$$z_1 \to z_1 + x_0 x_1 m$$

and

$$z_2 \to z_2 + x_0 x_2 m$$

we can arrange that neither x_1 nor x_2 appears in any monomial in c_1. Finally by using r_2, r_3, r_4, and r_5 to reduce d we may assume that none of $x_2 y$, $x_1 y$, $x_1 z_2$, $x_2 z_2$ appear in d.

Now the syzygies in R_S must also descend to those in R. For example s_1 becomes

$$s_1\colon x_1 r_3 - x_2 r_2 + x_3^2 r_1 \in x_0 \hat{I}$$

which implies,

$$x_0\left(x_1 b_2 - x_2 b_1 + x_3^2 a\right) \in x_0 \hat{I}$$

or

$$x_1 b_2 - x_2 b_1 + x_3^2 a \in \hat{I}.$$

Repeating this process for the next eleven syzygies yields Table 7.

Let

$$b_i = \bar{b}_i + x_0 b_i'$$

$$c_i = \bar{c}_i + x_0 c_i'$$

$$d = \bar{d} + x_0 d'$$

$$e_i = \bar{e}_i + x_0 e_i',$$

and remember that $a = \alpha x_0$. By the previous remarks,

$$\bar{b}_1 = \beta x_3^2 + \gamma x_2 x_3$$

for some β, $\gamma \in C$. Reducing s_1 mod x_0 then yields

$$x_1 \bar{b}_2 - \beta x_2 x_3^2 - \gamma x_2^2 x_3 \in I.$$

Thus

$$x_1 \bar{b}_2 - \beta x_1 y - \gamma x_1 x_3^2 \in I$$

TABLE 7

s_1: $x_1 b_2 - x_2 b_1 + x_3^2 a \in \hat{I}$

s_2: $x_2 b_2 - x_3 b_1 + ya \in \hat{I}$

s_3: $x_1 c_2 - x_2 c_1 + z_1 a \in \hat{I}$

s_4: $x_2 c_2 - x_3 c_1 + z_2 a \in \hat{I}$

s_5: $x_1 d - z_1 b_1 + x_3^2 c_1 \in \hat{I}$

s_6: $x_2 d - z_2 b_1 + y c_1 \in \hat{I}$

s_7: $x_2 d - z_1 b_2 + x_3^2 c_2 \in \hat{I}$

s_8: $x_3 d - z_2 b_2 + y c_2 \in \hat{I}$

s_9: $x_1 e_2 - x_2 e_1 - z_1 c_1 + \lambda y x_3^2 b_1 \in \hat{I}$

s_{10}: $x_2 e_2 - x_3 e_1 - z_1 c_2 + \lambda y x_3^2 b_2 - Q(x_i, y) a \in \hat{I}$

s_{11}: $x_1 e_3 - x_2 e_2 - z_2 c_1 + \lambda y^2 b_1 + Q(x_i, y) a \in \hat{I}$

s_{12}: $x_2 e_3 - x_3 e_2 - z_2 c_2 + \lambda y^2 b_2 \in \hat{I}$

(using r_1 and $r_2 \in I$). And hence

$$x_1\left(\bar{b}_2 - \beta y - \gamma x_3^2\right) \in I$$

so

$$\bar{b}_2 - \beta y - \gamma x_3^2 \in I.$$

The only generator of I in degree 2 is r_1 so

$$\bar{b}_2 = \beta y + \gamma x_3^2 + \lambda r_1.$$

If, in the original equations b_2 is replaced by $b_2 - \lambda r_1$, which has no effect on $\hat{I}$, we then have

$$\bar{b}_2 = \beta y + \gamma x_3^2.$$

Next, $\bar{c}_1$ has the form

$$\bar{c}_1 = \epsilon x_3 y - \delta z_1 + \tau z_2.$$

Reducing s_3 mod x_0 yields

$$x_1 \bar{c}_2 - \epsilon x_2 x_3 y - \delta x_2 z_1 - \tau x_2 z_2 \in I.$$

The only degree four relation in I involving x_2x_3y is $x_2x_3y - x_3^4$ which has no x_1 in it. Thus $\epsilon = 0$. Likewise for x_2z_2 the only relation is $x_2z_2 - x_3z_1$. So $\tau = 0$ as well. This leaves

$$\bar{c}_1 = \delta z_1 \quad \text{and} \quad \bar{c}_2 \in \delta z_2 + I.$$

Again after adjusting c_2 by an element of $\hat{I}$ we may conclude that $c_2 = \delta z_2$.

Reducing s_5 mod x_0 yields

$$x_1\bar{d} - \beta x_3^2 z_1 - \gamma x_2x_3z_1 + \delta x_3^2 z_1 \in I$$

Since $x_2z_2 - x_3z_1$ involves no x_1, we must have $\beta = \delta$. This implies

$$x_1\bar{d} - \gamma x_2x_3z_1 \in I$$

or

$$x_1\bar{d} - \gamma x_1x_3z_2 \in I$$

or

$$\bar{d} - \gamma x_3z_2 \in I.$$

Once again, adjusting d by an element of $\hat{I}$ allows $\bar{d}$ to be,

$$\bar{d} = \gamma x_3z_2.$$

It is easy to check that s_2, s_4, s_6, s_7, and s_8 impose no further conditions on $\bar{b}_i$, $\bar{c}_i$, or $\bar{d}$. So far we have,

$$a = \alpha x_0$$

$$b_1 = \beta x_3^2 + \gamma x_2x_3 + x_{01}'$$

$$b_2 = \beta y + \gamma x_3^2 + x_0b_2'$$

$$c_1 = \beta z_1 + x_0c_1'$$

$$c_2 = \beta z_2 + x_0c_2'$$

$$d = \gamma z_2x_3 + x_0d'.$$

Setting

$$b_i' = \bar{b}_i' + x_0b_i''$$

$$c_i' = \bar{c}_i' + x_0 b_i''$$

$$d' = \bar{d}' + x_0 d''$$

and substituting into s_1 in Table 7 yields

$$\beta x_1 y + \gamma x_1 x_3^2 + x_0 x_1 \bar{b}_2'$$
$$+ x_0^2 x_1 b_2'' - \beta x_2 x_3 - \gamma x_2^2 x_3 - x_0 x_2 \bar{b}_1' - x_0^2 x_2 b_1'' + \alpha x_0 x_3^2 \in \hat{I}$$

or

$$x_0\big(\beta b_1 + \gamma\alpha x_0 x_3 + x_1 \bar{b}_2' + x_1 \bar{b}_2' - x_2 \bar{b}_1'$$
$$+ \alpha x_3^2 + x_0 x_1 b_2'' - x_0 x_2 b_1''\big) \in \hat{I}.$$

Thus

$$\beta^2 x_3^2 + \beta\gamma x_2 x_3 + \beta x_0 b_1'$$
$$+ \gamma\alpha x_0 x_3 + x_1 \bar{b}_2' - x_2 \bar{b}_1' + \alpha x_3^2 + x_0 x_q b_2'' - x_0 x_2 b_1'' \in \hat{I}.$$

Reducing mod x_0 gives

$$\big(\beta^2 + \alpha\big) x_3^2 + \beta\gamma x_2 x_3 + x_1 \bar{b}_2' - x_2 \bar{b}_1' \in I,$$

which clearly implies $\beta^2 + \alpha = 0$. Recall x_1 does not appear in b_1 so,

$$\bar{b}_1' = k_{12} x_2 + k_{13} x_3$$

and

$$\bar{b}_2' = k_{21} x_1 + k_{22} x_2 + k_{23} x_3.$$

Thus

$$k_{13} = \beta\gamma$$

$$k_{12} = k_{23} = \delta \ (\text{a new} \quad \delta!)$$

and

$$k_{22} = k_{21} = 0.$$

The same process applied to s_3 gives

$$\beta x_1 z_2 + x_0 x_1 \bar{c}_2' + x_0^2 x_1 c_2'' - \beta x_2 z_1 - x_0 x_2 \bar{c}_1' - x_0^2 x_2 c_1'' + \alpha x_0 z_1 \in \hat{I}$$

or

$$\beta x_0 c_1 + x_0 x_1 \bar{c}'_2 + x_0^2 x_1 c''_2 - x_0 x_2 \bar{c}'_1 - x_0^2 x_2 c''_1 + \alpha x_0 z_1 \in \hat{I}.$$

Hence

$$\beta^2 z_1 + \beta x_0 c'_1 + x_1 \bar{c}'_2 - x_2 \bar{c}'_1 + \alpha z_1 + x_0 (x_1 c''_2 - x_2 c''_1) \in \hat{I},$$

and reducing mod x_0 gives

$$(\beta^2 + \alpha) z_1 + x_1 \bar{c}'_2 - x_2 \bar{c}'_1 \in I$$

or

$$x_1 \bar{c}'_2 - x_2 \bar{c}'_1 \in I. \qquad **$$

By the previous work, neither x_1 nor x_2 appear in $\bar{c}'_1$ so

$$\bar{c}'_1 = k_1 y + k_2 x_3^2.$$

By the relation $**$ one sees that $k_1 = 0$ and then that

$$\bar{c}'_2 = k_2 y = \epsilon y.$$

Repeating this somewhat tedious routine with s_5 in Table 7 yields

$$\gamma x_1 x_3 z_2 + x_1 x_0 \bar{d}' + x_1 x_0^2 d'' - \beta x_3^2 z_1 - \gamma x_2 x_3 z_1 - \beta \gamma x_0 x_3 z_1$$
$$- \delta x_0 x_2 z_1 - x_0'' z_1 b_1'' + \beta x_3^2 z_1 + \epsilon x_0 x_3^4 + x_0^2 x_3^2 c_1'' \in \hat{I}$$

or

$$\gamma x_0 x_3 c_1 + x_0 x_1 \bar{d}' + x_0^2 x_1 d''$$
$$- \beta \gamma x_0 x_3 z_1 - \delta x_0 x_2 z_1 - x_0^2 z_1 b'' + \epsilon x_0 x_3^4 + x_0^2 x_3^2 c_1 \in \hat{I}.$$

Thus,

$$x_0 [\gamma \beta x_3 z_1 + \gamma x_3 x_0 c'_1 + x_1 \bar{d}'$$
$$+ x_0 x_1 d'' - \gamma \beta x_3 z_1 - \delta x_2 z_1 - x_0 z_1 b'' + \epsilon x_3^4 + x_0 x_3^2 c''_1] \in \hat{I}$$

or mod x_0,

$$x_1 \bar{d}' - \delta x_2 z_1 + \epsilon x_3^4 \in I.$$

Hence

$$\bar{d}' = \delta z_2$$

and

$$_{\epsilon} = 0$$

Thus

$$a = -\beta^2 x_0$$

$$b_1 = \beta x_3^2 + \gamma x_2 x_3 + \beta\gamma x_0 x_3 + \delta x_0 x_2 + b_1'' x_0^2$$

$$b_2 = \beta y + \gamma x_3^2 + \delta x_0 x_3 + b_2'' x_0^2$$

$$c_1 = \beta z_1 + c_1'' x_0^2$$

$$c_2 = \beta z_2 + c_2'' x_0^2$$

$$d = \gamma x_3 z_2 + \delta x_0 z_2 + d'' x_0^2.$$

Again s_1 yields,

$$\beta\delta x_2 + b_2'' x_1 - b_1'' x_2 \in I$$

and thus

$$b_2'' = 0$$

$$b_1'' = \beta\delta.$$

Substituting in s_3 gives

$$\beta c_1'' x_0 + c_2'' x_1 - c_1'' x_2 \in \hat{I} \qquad (***)$$

and so

$$\bar{c}_2'' x_1 - \bar{c}_1'' x_2 \in \hat{I}.$$

Now $x_1,\ x_2 \notin c_1$ implies that

$$\bar{c}_1'' = \bar{c}_2'' = 0.$$

From $(***)$ one has,

$$\beta c_1''' x_0^2 + c_2''' x_0 x_1 - c_1''' x_0 x_2 \in \hat{I}.$$

Since τ_1 is the only degree z relation in $\hat{I}$ this means,

$$c_1''' = c_2''' = 0.$$

Finally, s_5 gives.

$$\gamma x_0 x_3 c_1 + \delta x_0^2 c_1 + x_0^2 x_1 d'' - \beta\gamma x_0 x_3 z_1 - \beta\delta x_0^2 z_1 \in \hat{I}$$

or

$$x_0^2 x_1 d'' \in \hat{I}.$$

So by adjusting d as before we may assume that $d'' = 0$.

This completes the determination of a, b_1, b_2, c_1, c_2, and d. It also shows that Table 6 has the form

TABLE 8

$\hat{I} =$

r_1: $x_1 x_3 - x_2^2$

r_2: $x_1 y - (x_2 + \beta x_0)(x_3^2 + \gamma x_0 x_3 + \delta x_0^2) = 0$

r_3: $(x_2 - \beta x_0) y - x_3(x_3^2 + \gamma x_0 x_3 + \delta x_0^2) = 0$

r_4: $x_1 z_2 - (x_2 + \beta x_0) z_1 = 0$

r_5: $(x_2 - \beta x_0) z_2 - x_3 z_1 = 0$

r_6: $z_1 y - z_2(x_3^2 + \gamma x_0 x_3 + \delta x_0^2) = 0$

r_7: $z_1^2 - \lambda y x_3^4 - x_1 Q(x_i, y) = x_0 e_1$

r_8: $z_1 z_2 - \lambda y^2 x_3^2 - x_2 Q(x_i, y) = x_0 e_2$

r_9: $z_2^2 - \lambda y^3 - x_3 Q(x_i, y) = x_0 e_3$

where the e_i's are weight five homogeneous polynomials that satisfy s_9 through s_{12} in Table 7. It is very tedious to determine the actual form of the e_i's and so we leave this task to the patient reader.

Hence,

THEOREM II: *Let S be a numerical quintic surface ($p_g = 4$, $q = 0$, $c_1^2 = 5$) such that $|K_S|$ has a base point (i.e. $S \in II$). Then*

$$R(S, K_S) = C[x_0, x_1, x_2, x_3, y, z_1, z_2]/\hat{I}$$

where $\hat{I}$ is generated by relations

$$r_1 - r_6\text{: rank} \begin{pmatrix} x_1 & x_2 - \beta x_0 & z_1 & x_3^2 + \gamma x_0 x_3 + \delta x_0^2 \\ x_2 + \beta x_0 & x_3 & z_2 & y \end{pmatrix} = 1$$

and r_7, r_8, and r_9 as in Table 8 *above.*

Note: Conversely, if the e_i's are suitable, i.e. they satisfy the syzygies and Table 8 defines a smooth surface, then it is possible to check that this surface is indeed of Type 2. One simply solves the equations in terms of x_0, x_1, x_2, x_3 over the open sets $x_0 \neq 0$, $x_1 \neq 0$, etc. Thus if one has an explicit description of the suitable e_i's (see for example condition 2 in Theorem I.a) one would have, in some sense, a "parametrization" of the deformation space of the type II surfaces.

It is clear that $S \in \mathrm{II}_b$ (i.e. the canonical image of S in $\mathbb{P}^3$ is a *singular* quadric) if and only if $\beta = 0$. In order to show that any such an $S \in \mathrm{II}_b$ occurs as the limit of smooth quintic surfaces in $\mathbb{P}^3$ one simply follows the curve case.

THEOREM III: *Given $S \in II_b$ there exists a family of surfaces, $\mathcal{S}$, whose generic member is a smooth quintic surface in $\mathbb{P}^3$ and such that $S_0 \simeq S$.*

PROOF: First write

$$S \simeq \operatorname{Proj}(R(S, K_S)) \simeq \operatorname{Proj}(C[x_0, x_1, x_2, x_3, y, z_1, z_2]/\hat{I})$$

where

$$\hat{I} = r_1\colon x_1 x_3 - x_2^2$$

$$r_2\colon x_1 y - x_2(x_3^2 + \gamma x_0 x_3 + \delta x_0^2)$$

$$r_3\colon x_2 y - x_3(x_3^2 + \gamma x_0 x_3 + \delta x_0^2)$$

$$r_4\colon x_1 z_2 - x_2 z_1$$

$$r_5\colon x_2 z_2 - x_3 z_1$$

$$r_6\colon z_1 y - z_2(x_3^2 + \gamma x_0 x_3 + \delta x_0^2)$$

$$r_7\colon z_1^2 - \lambda y(x_3 + \gamma x_0 x_3 + \delta x_0^2)^2 - x_1 \hat{Q}(x_i, y) - x_0 \hat{e}_1$$

$$r_8\colon z_1 z_2 - \lambda y^2(x_3^2 + \gamma x_0 x_3 + \delta x_0^2) - x_2 \hat{Q}(x_i, y) - x_0 \hat{e}_w$$

$$r_9\colon z_2^2 - \lambda y^3 - x_3 \hat{Q}(x_i, y) - x_0 \hat{e}_3$$

where $\hat{Q}(x_0, x_1, x_2, x_3, y, z_1, z_2)$ is of weight five and where the $\hat{e}_i$'s are determined via the syzygies $s_9 - s_{12}$ in,

$$s_1\colon x_1r_3 - x_2r_2 + \left(x_3^2 + \gamma x_0x_3 + \delta x_0^2\right)r_1$$

$$s_2\colon x_2r_3 - x_3r_2 + y_1^r$$

$$s_3\colon x_1r_5 - x_2r_4 + z_1r_1$$

$$s_4\colon x_2r_5 - x_3r_4 + z_2r_1$$

$$s_5\colon x_1r_6 - z_1r_2 + \left(x_3^2 + \gamma x_0x_3 + \delta x_0^2\right)r_4$$

$$s_6\colon x_2r_6 - z_2r_2 + yr_4$$

$$s_7\colon x_2r_6 - z_1r_3 + \left(x_3^2 + \gamma x_0x_3 + \delta x_0^2\right)r_5$$

$$s_8\colon x_3r_6 - z_2r_3 + yr_5$$

$$s_9\colon x_1r_8 - x_2r_7 - z_1r_4 + \lambda y\left(x_3^2 + \gamma x_0x_3 + \delta x_0^2\right)r_2$$

$$s_{10}\colon x_2r_8 - x_3r_7 - z_1r_5 + \lambda y\left(x_3^2 + \gamma x_0x_3 + \delta x_0^2\right)r_3 - \hat{Q}(x_i, y)r_1$$

$$s_{11}\colon x_1r_9 - x_2r_8 - z_2r_4 + \lambda y^2r_2 - \hat{Q}(x_i, y)r_1$$

$$s_{12}\colon x_2r_9 - x_3r_8 - z_2r_5 + \lambda y^2r_3.$$

Then set

$$\mathscr{S} = \operatorname{Proj}\left(C[x_0, x_1, x_2, x_3, y, z_1, z_2]/\hat{\ell}\right)$$

where

$$\hat{\ell} = \mathscr{R}_1\colon x_1x_3 - x_2^2 + t^2\lambda y$$

$$\mathscr{R}_2\colon x_1y - x_2\left(x_3^2 + \gamma x_0x_3 + \delta x_0^2\right) + tz_1$$

$$\mathscr{R}F_3\colon x_2y - x_3\left(x_3^2 + \gamma x_0x_3 + \delta x_0^2\right) + tz_2$$

$$\mathscr{R}_4\colon x_1z_2 - x_2z_1 + t\lambda\left(x_3^2 + \gamma x_0x_3 + \delta x_0^2\right)y$$

$$\mathscr{R}_5\colon x_2z_2 - x_3z_1 + t\lambda y^2$$

$$\mathscr{R}_6\colon z_1y - z_2\left(x_3^2 + \gamma x_0x_3 + \delta x_0^2\right) + t\hat{Q}(x_i, y)$$

$$\mathscr{R}_7 = r_7$$

$$\mathscr{R}_8 = r_8$$

$$\mathscr{R}_9 = r_9$$

As before, it is now easy to check that $\mathscr{S}$ is the desired family.

Q.E.D.

REMARK: As in the curve case, one can eliminate the weighted variables to obtain the canonical image of $\mathscr{S}$ in $\mathbb{P}^3$. The image of S_0 is a quadric cone plus a *tangent* plane. The computations mimic the curve case and are left to the reader.

So we have described, algebraically, in the preceeding tables all of the deformation space of the numerical quintics. Component I where $\beta = 0$ and component II where $t = 0$ meet in the locus II_b when $\beta = t = 0$. The only piece of information we have not recovered that Horikawa shows in [Horikawa] is that these components meet transversally. This project is postponed to a later paper.

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