COMPOSITIO MATHEMATICA

I. MOERDIJK

Spaced spaces

SPACED SPACES

I. Moerdijk

Contents

Introduction

The model theory of intuitionistic higher order logic, as e.g. described in [1], makes it possible to interpret intuitionistic mathematics in categories of sheaves. This has proved to be particularly useful in algebra (see [6]). Here "internal" mathematical objects correspond to structures which are familiar from the theory of sectional representations. For example, a ring (= commutative ring with 1)-object in the category $Sh(X)$ of Set-valued sheaves on a topological space X is the same as a sheaf of rings on X, or a sheaf of sets on X with a continuous ring structure on the stalks. In order words, doing ring theory *internally* in the category $Sh(X)$ coincides with studying *ringed spaces* with base space X.

In this paper, we will discuss what happens when one replaces "ring" by "topological space" in the above: we will deal with the question of

what topological space-objects in categories of sheaves look like, and we will consider the relation between such topological space-objects and sheaves with values in the category of topological spaces and continuous functions.

Our approach will be rather category-theoretic: we will concentrate on equivalence of categories-theorems (representation theorems), and on adjunctions between categories. By doing this, we hope to provide some general background to the models for intuitionistic topology of the kind discussed by Grayson [4], for example. In intuitionistic topology (or, topology in sheaves) one finds a lot of pathological situations which are due to a lack of points. Many of the pathologies disappear when one does intuitionistic topology *without* points, that is, *locale theory*. In this sense, locale theory seems to be the proper way of doing topology when one's underlying logic is intuitionistic logic. Still, in this paper we will not consider locales, but study models for intuitionistic topology in the more traditional sense of [9], [4], where a *space* is a *set of points* with some additional structure.

Let us briefly outline how this paper is organized. We assume the reader to be acquainted with the basis facts of general topology and category theory, and to have a thorough understanding of the model theory of intuitionistic (higher order) logic as described in [1]. This paper [1] will be the starting point for the first part, where we consider external representations of topological space objects in categories of sheaves. We prove Stout's representation theorem, and derive Fourman's representation theorem for sober spaces as an easy corollary. In the second part, we will consider the relation between the category $TOP(Sh(X))$ of topological space objects in $Sh(X)$, and the category $Sh(X, TOP)$ of sheaves on X with values in the category of topological spaces and continuous functions. We prove a general adjunction theorem, and we show that in the case of a locally compact zero-dimensional base-space X, $TOP(Sh(X))$ is (equivalent to) a reflective subcategory of $Sh(X, TOP)$. In the third part, we consider change of base space, and investigate some of the structure of the category of "spaced spaces", which is defined as an analogon of the category of ringed spaces, or the category of geometric spaces, familiar from algebraic geometry.

This paper has quite a long history. A first version was written in the fall of 1980 as [5]. The main reason for the delay in producing the present version was that the central representation theorem (Theorem 5, Part 1, Section 1 below) turned out to have been proved independently, but much earlier, by Stout (cf. [7]). I would like to thank L.N. Stout for bringing the existence of [7] to my attention. Also, I would like to thank professors van Dalen and Troelstra for helpful comments on the earlier version just mentioned, and for encouraging me to write the present version.

PART 1. TOPOLOGICAL SPACES IN SHEAVES

1. Representation of topological spaces

Throughout this paper, we will use the well-known representation theorem for sheaves saying that a sheaf A over a topological space T is representable as the sheaf of continuous sections of a local homeomorphism $E_A \overset{p}{\to} T$. This correspondence is an equivalence of categories. Natural transformations $A \overset{\tau}{\to} B$ from one sheaf A over T to another are represented by continuous maps $E_A \overset{f}{\to} E_B$ over T; i.e. τ acts on sections by just composing with f. (For more details, see e.g. [1], [2].) Given this representation, we may either think of sheaves as Set-valued functors, or as local homeomorphisms. Since in the definition of a sheaf over a space T we need only refer to the lattice $\mathcal{O}(T)$ of open subsets of T, we will assume that the base-space T is sober, whenever this is convenient.

If $E \overset{p}{\to} T$ is a local homeomorphism, an open neighbourhood (nbd) U_e of a point $e \in E$ will be called *small* if $p \upharpoonright U_e$: $U_e \to p(U_e) \subseteq T$ is a homeomorphism. By definition, the small neighbourhoods form a basis for E.

Let us now turn to topological spaces in sheaves. Using the interpretation of higher-order logic in sheaves as presented in [1], we can define a topology on a sheaf A over T as a subobject $\mathcal{O}(A)$ of $\mathcal{P}(A)$ such that

$$Sh(T) \vDash \emptyset \in \mathcal{O}(A) \wedge A \in \mathcal{O}(A) \wedge \forall U, V \in \mathcal{O}(A)(U \cap V \in \mathcal{O}(A))$$

$$\wedge \forall \mathcal{V} \subseteq \mathcal{O}(A)(\bigcup \mathcal{V} \in \mathcal{O}(A)) \quad (*)$$

as in the classical case. Our aim in this section is to give an external representation of topological space-objects in $Sh(T)$.

1.1. LEMMA: *Let A be a sheaf on T. Then global elements of $\mathcal{P}(A)$ in $Sh(T)$ correspond to open subsets of E_A, with equality given by*

$$\llbracket O = O' \rrbracket = \bigcup \{ U \in \mathcal{O}(T) \mid p^{-1}(U) \cap O = p^{-1}(U) \cap O' \}$$

for open subsets O, O' of E_A.

PROOF: A global element of $\mathcal{P}(A)$ is a (strict and extensional) predicate $A \to \mathcal{O}(T)$ ([1]). An open $O \subseteq E_A$ defines such a predicate P_O by $P_O(a) = a^{-1}(O)$. (Here we identify elements of A with sections of the representing local homeomorphism $E_A \overset{p}{\to} T$). Conversely, a predicate P: $A \to \mathcal{O}(T)$

defines an open set

$$O_p = \bigcup \left\{ U \subseteq E_A | U \quad \text{is small and} \quad P\left((p \restriction U)^{-1} \right) = p(U) \right\}.$$

To prove the correspondence, we show that

1. $P_{O_p} = P$ for all predicates P on A
2. $O_{P_O} = O$ for all opens $O \in E_A$.

ad 1. ($\supseteq$) Take $a \in A$. We have to show that $P(a) \subseteq \bigcup \{ a^{-1}(U) | U$ is small and $P((p \restriction U)^{-1}) = p(U) \}$. Take $t \in P(a)$, and let U be a small nbd of $a(t)$ such that $p(U) \subseteq P(a)$ (p is continuous). Since $p \restriction U$: $U \to p(U)$ is a homeomorphism, we derive that $a \restriction p(U) = (p \restriction U)^{-1}$, and hence $P((p \restriction U)^{-1}) = P(a \restriction p(U)) = P(a) \cap p(U) = p(U)$. ($\subseteq$) Take $a \in A$ and U a small open subset of E_A. We have to show that if $P((p \restriction U)^{-1}) = p(U)$, then $a^{-1}(U) \subseteq P(a)$. So suppose $P((p \restriction U)^{-1}) = p(U)$. Since $a^{-1}(U) = [\![a = (p \restriction U)^{-1}]\!]$, we obtain $P(a) \cap a^{-1}(U) = P(a \restriction [\![a = (p \restriction U)^{-1}]\!]) = P((p \restriction U)^{-1} \restriction [\![a = (p \restriction U)^{-1}]\!]) = p(U) \cap a^{-1}(U) = a^{-1}(U)$. Hence $a^{-1}(U) \subseteq P(a)$.

ad 2. We have to show that for an open $O \subseteq E_A$,

$$O = \bigcup \left\{ U | U \text{ is small and } \left((p \restriction U)^{-1} \right)^{-1}(O) = (p \restriction U)(O) = p(U) \right\}.$$

But $(p \restriction U)(O) = p(U)$ iff $p(O \cap U) = p(U)$, iff $U \subseteq O$, so this is immediate from the fact that the small open sets form a basis. $\square$

The following lemma is an analog of 8.12(i) of [1].

1.2. LEMMA: *Let τ be an internal topology on a sheaf A over T (i.e. τ is a subsheaf of $\mathscr{P}(A)$ satisfying the definition ($*$) given above). Then every element of τ is the restriction of a global element of τ.*

PROOF: As usual, τ may also be regarded as a predicate on $\mathscr{P}(A)$, and for $O \in \mathcal{O}(E_A)$ we write $\tau(O) = [\![O \in \tau]\!]$. Now let (O, U) be a section of τ over $U \in \mathcal{O}(T)$, i.e. $O \in \mathcal{O}(E_A)$ is a global element of $\mathscr{P}(A)$ (cf. Lemma 1) with $O(a) := a^{-1}(O) \subseteq U$ for all $a \in A$, and $U \in \mathcal{O}(T)$ is such that $U \subseteq \tau(O)$. Consider the predicate $u : \mathscr{P}(A) \to \mathcal{O}(T)$ defined by

$$u(O') = U \cap [\![O = O']\!]$$

$$\left(= \bigcup \{ W \subseteq U | O \cap p^{-1}(W) = O' \cap p^{-1}(W), \quad W \in \mathcal{O}(T) \} \right).$$

Then $[\![u(O') \to \tau(O')]\!] = T$ for every $O' \in \mathcal{O}(E_A)$, since

$$u(O') = U \cap [\![O = O']\!] \subseteq \tau(O) \cap [\![O = O']\!] \subseteq \tau(O')$$

(the last inclusion by extensionality of τ). Hence since by definition

$$\models \forall O \in \mathscr{P}(A)(u(O) \to \tau(O)) \to \tau(\textstyle\bigcup u),$$

we get that $[\![\bigcup u \in \tau]\!] = T$, i.e. $\bigcup \{O' \cap p^{-1}u(O')|O' \in \mathcal{O}(E_A)\} = \bigcup \{O' \cap p^{-1}(\bigcup\{W \subseteq U|O \cap p^{-1}(W) = O' \cap p^{-1}(W)\})|O' \in \mathcal{O}(E_A)\} = O \cap p^{-1}(U)$ is a global section of τ. $\square$

1.3. REMARK: Let us write Γ for the global sections functor. Then the proof just given shows that for any internal topology τ on A,

$$U \subseteq \tau(O) \quad \text{iff} \quad O \cap p^{-1}(U) \in \Gamma\tau,$$

for any $U \in \mathcal{O}(T)$ and $O \in \mathcal{O}(E_A)$. In particular, $p^{-1}(U) \in \Gamma\tau$ for every $U \in \mathcal{O}(T)$.

1.4. DEFINITION: Let $E \xrightarrow{p} T$ be a local homeomorphism. A *p-topology* on E is a topology Γ on E which is coarser then the original topology on E (i.e. $\Gamma \subseteq \mathcal{O}(E)$), and makes p continuous (in the sense that $p^{-1}(U) \in \Gamma$ for every $U \in \mathcal{O}(T)$).

1.5. THEOREM: *Let A be a sheaf over T, represented by the local homeomorphism $E_A \xrightarrow{p} T$. Then internal topologies on A correspond to (external) p-topologies on E_A.*

PROOF: An internal topology is a predicate τ on $\mathscr{P}(A)$ satisfying the definition $(*)$ given above. τ is determined by its restriction $\tau: \mathcal{O}(E_A) \to \mathcal{O}(T)$, which must be an extensional function. Let $\Gamma\tau$ be the set of global elements of τ. We claim that:

(1) $\Gamma\tau$ is a topology on E_A which makes p continuous. The latter part of (1) follows from Remark 1.3. As for the first part, clearly $\emptyset \in \Gamma\tau$ and $E_A \in \Gamma\tau$. Also, if $O, O' \in \Gamma\tau$, then because $\tau(O) \cap \tau(O') \subseteq \tau(O \cap O')$, $O \cap O' \in \Gamma\tau$. Finally, if $\mathcal{U} \subseteq \Gamma\tau$, consider the (global) predicate $u: \mathcal{O}(E_A) \to \mathcal{O}(T)$ defined by

$$u(O) = \bigcup \{U \in \mathcal{O}(T)|O \cap p^{-1}(U) \in \mathcal{U}\}.$$

By definition, $\models \forall O(u(O) \to \tau(O)) \to \tau(\bigcup u)$. But $\bigcup u$ corresponds to the open subset $\bigcup \mathcal{U}$ of E_A (since $[\![a \in \bigcup u]\!] = \bigcup \{u(O') \cap [\![a \in O']\!]|O' \in \mathcal{O}(E_A)\} = a^{-1}(\bigcup \mathcal{U})$), and for all $O \in \mathcal{O}(E_A)$, $u(O) \subseteq \tau(O)$ (since $u(O) =$

$\bigcup\{U \in \mathcal{O}(T)|O \cap p^{-1}(U) \in \mathcal{U}\} \subseteq \bigcup\{U \in \mathcal{O}(T)|O \cap p^{-1}(U) \in \Gamma\tau\}$; using Remark 3, $O \cap p^{-1}(U) \in \Gamma\tau$ implies $U \subseteq \tau(O)$, so $u(O) \subseteq \tau(O)$). Combining these, we get $\tau(\bigcup u) = T$, so $\bigcup\mathcal{U} \in \Gamma\tau$.

Conversely, let $\Gamma \subseteq \mathcal{O}(E_A)$ be a topology on E_A making p continuous. Define a predicate $\tau_\Gamma: \mathscr{P}(a) \to \mathcal{O}(T)$ by setting, for $O \in \mathcal{O}(E_A)$,

$$\tau_\Gamma(O) = \bigcup\{U|O \cap p^{-1}(U) \in \Gamma\} \quad (= \text{“}[\![O \in \tau_\Gamma]\!]\text{”})$$

(by Remark 3, we are forced to do so). Then

(2) τ_Γ is an internal topology on A. To show (2), first note that clearly $[\![\emptyset \in \tau_\Gamma]\!] = [\![E_A \in \tau_\Gamma]\!] = T$, and for all globals $O, O' \in \mathcal{O}(E_A)$, $[\![O \in \tau_\Gamma]\!] \cap [\![O' \in \tau_\Gamma]\!] \subseteq [\![O \cap O' \in \tau_\Gamma]\!]$. It is slightly less trivial that τ_Γ is internally closed under unions. Take a global predicate $u: \mathcal{O}(E_A) \to \mathcal{O}(T)$ on $\mathscr{P}(A)$; we have to show that

$$\models \forall O \in \mathscr{P}(A)(u(O) \to \tau_\Gamma(O)) \to \tau_\Gamma(\bigcup u).$$

For sections a of $E_A \xrightarrow{p} T$, $[\![a \in \bigcup u]\!] = [\![\exists O \in u \cdot a \in O]\!] = \bigcup\{a^{-1}(O) \cap u(O)|O \in \mathcal{O}(E_A)\} = a^{-1}(\bigcup\{O \cap p^{-1}u(O)|O \in \mathcal{O}(E_A)\})$. Now suppose $W \subseteq [\![u(O) \to \tau_\Gamma(O)]\!]$ for all $O \in \mathcal{O}(E_A)$. We claim that $W \subseteq \tau_\Gamma(\bigcup u)$. $W \subseteq [\![u(O) \to \tau_\Gamma(O)]\!]$ means that $W \in u(O) \subseteq \bigcup\{U|O \cap p^{-1}(U) \in \Gamma\}$. Hence for all $O \in \mathcal{O}(E_A)$, $O \cap p^{-1}(W \cup u(O)) \in \Gamma$ (since Γ makes p continuous). So

$$\bigcup\{O \cap p^{-1}(W \cap u(O))|O \in \mathcal{O}(E_A)\}$$

$$= p^{-1}(W) \cap \bigcup\{O \cap p^{-1}u(O)|O \in \mathcal{O}(E_A)\} \in \Gamma.$$

But this says that

$$W \subseteq \tau_\Gamma(\bigcup\{O \cap p^{-1}u(O)|O \in \mathcal{O}(E_a)\}) = \tau_\Gamma(\bigcup u),$$

by definition of τ_Γ. This shows that τ_Γ is an internal topology.

To prove the correspondence, we show that

(1) for any internal topology τ, $\tau_{\Gamma_\tau} = \tau$
(2) for any p-topology Γ on E_A, $\Gamma_{\tau_\Gamma} = \Gamma$.

(2) is trivial: $O \in \Gamma_{\tau_\Gamma} \Leftrightarrow \tau_\Gamma(O) = T \Leftrightarrow O \cap p^{-1}(T) \in \Gamma \Leftrightarrow O \in \tau$. For (1) note that because Γ is an internal topology, it suffices by Lemma 1.2 to show that $\Gamma_{\tau_{\Gamma_\tau}} = \Gamma_\tau$. But this follows from (1). This completes the proof of the theorem. $\square$

In the next section, we will reformulate this correspondence as an equivalence of categories. Let us first look at two kinds of internal spaces.

1.6. EXAMPLES: (a) Let $Y \overset{g}{\to} T$ be a continuous function from a topological space Y to the base space T. The sheaf of sections of g carries a natural internal topology τ induced by the topology of Y: global elements of τ are the predicates P_O, $O \in \mathcal{O}(Y)$, where for a section $U \overset{a}{\to} Y$ of g, $P_O(a) := a^{-1}(O)$. It is readily checked that this indeed defines a topology. This internal space is called the *space of sections* of $Y \overset{g}{\to} T$, and is denoted by g_T.

(b) As a special case of (a), let X be a topological space, and consider the internal space of sections of the projection $X \times T \overset{\pi}{\to} T$. This space is denoted by X_T, and is called the *constant space associated with X*.

Below, we shall return to the constructions of internal spaces from external ones, and investigate their categorical properties.

2. The category of internal topological spaces

We mentioned above that a sheaf A over T can be represented as a sheaf of sections of a local homeomorphism $E_A \overset{p}{\to} T$, and that sheaf-maps (natural transformations) $A \to B$ correspond to continuous functions $E_A \to E_B$ over T. Now suppose we have two sheaves A and B over T, equipped with internal topologies τ and σ respectively. An (internal) continuous function $f: (A, \tau) \to (B, \sigma)$ is a sheaf-map $A \overset{f}{\to} B$ such that $\vDash \forall O \in \mathcal{P}(B)(O \in \sigma \to f^{-1}(O) \in \tau)$, as usual. Let $E_A \overset{\hat{f}}{\to} E_B$ be the representation of f. Then (identifying elements of A and B with sections of $E_A \overset{p}{\to} T$, $E_B \overset{q}{\to} T$, etc) we find the following correspondence.

2.1. LEMMA: *A sheaf-map $f: (A, \tau) \to (B, \sigma)$ is internally continuous in $Sh(T)$ iff its representation $\hat{f}$ is continuous w.r.t. $\Gamma\tau$ on E_A and $\Gamma\sigma$ on E_B.*

PROOF: f and $\hat{f}$ are related through $f(a) = \hat{f} \circ a$ for all sections a of p. Further, for a global element O of $\mathcal{P}(B)$, $[\![a \in f^{-1}(O)]\!] = [\![f(a) \in O]\!] = a^{-1}(\hat{f}^{-1}(O))$. Therefore (using Remark 3 of Section 1), if $\hat{f}$ is continuous (w.r.t. $\Gamma\sigma$, $\Gamma\tau$), then if O is a global element of $\mathcal{P}(B)$,

$$U \subseteq [\![O \in \sigma]\!] \Rightarrow O \cap q^{-1}(U) \in \Gamma\sigma$$

$$\Rightarrow \hat{f}^{-1}(O \cap q^{-1}(U)) = \hat{f}^{-1}(O) \cap p^{-1}(U) \in \Gamma\tau$$

$$\Rightarrow U \subseteq [\![\hat{f}^{-1}(O) \in \tau]\!]$$

so f is internally continuous.

Conversely, if f is internally continuous, then (using $[\![a \in f^{-1}(O)]\!] = a^{-1}(\hat{f}^{-1}(O))$ as above) $O \in \Gamma\sigma$ implies $\hat{f}^{-1}(O) \in \Gamma\tau$.   □

Putting Theorem 5 of the preceding section and this lemma together, we obtain an equivalence of categories. Let $TOP(Sh(T))$ denote the category of internal topological spaces and (global) continuous functions in $Sh(T)$, and let $SPSP(T)$ ("spaced spaces, with base space T") denote the category whose objects are of the form $(E, \Gamma) \overset{p}{\to} T$, where E is a topological space, $p\colon E \to T$ is a local homeomorphism, and Γ is a p-topology on E, and whose arrows from an object $(E, \Gamma) \overset{p}{\to} T$ to an object $(F, \Delta) \overset{q}{\to} T$ are functions $f\colon E \to F$ over T which are continuous w.r.t. *both* topologies on E, F.

2.2. THEOREM: *The categories $TOP(Sh(T))$ and $SPSP(T)$ are equivalent.*
                                                                         □

Let us return to spaces of sections (1.6.) for a moment. If we apply the representation to constant spaces X_T we get that these are represented by structures $(X_T, \mathcal{O}(X_T)) \overset{\pi}{\to} T$, where X_T is an external topological space (note that we use X_T to denote two different things, an internal space and an external one!), $X_T \overset{\pi}{\to} T$ a local homeomorphism, and $\mathcal{O}(X_T)$ a π-topology on X_T. The external space X_T is calculated in the standard way (cf. [2]). Let us quickly recall some details. X_T has as its set of points

$$\left\{ \langle [f]_t, t \rangle | t \in T,\ U \overset{f}{\to} X \text{ a cts function on some open nbd } U \text{ of } t \right\}$$

(here $[f]_t$ denotes the equivalence-class of f with respect to local equality of functions at t). The mapping $\pi\colon X_T \to X$ is defined by setting $\pi\langle [f]_t, t \rangle = t$. The topology on X_T making π a local homeomorphism has as basic opens the sets $[f, U]$, for U open $\subseteq T$ and $U \overset{f}{\to} X$ continuous, where
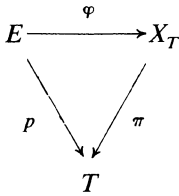
$$[f, U] = \{ \langle [f]_t, t \rangle | t \in U \}.$$

The π-topology $\mathcal{O}(X_T)$ on X_T has as opens the sets

$$U^* = \{ \langle [f]_t, t \rangle | (f(t), t) \in U \}$$

for U on open subset of the product $X \times T$. It is readily checked that this is indeed a π-topology.

Now internal continuous functions from a space $(E, \Gamma) \overset{p}{\to} T$ to a constant space X_T are commutative diagrams

with φ being a continuous function w.r.t. both topologies. With such a φ we can associate a function $E \overset{\varphi'}{\to} X$ which is continuous w.r.t. the topology Γ on E by composing φ with the evaluation-function $X_T \overset{\epsilon}{\to} X$ defined by $\epsilon(\langle [f]_t, t \rangle) = f(t)$. It is not difficult to see that this correspondence

$$\varphi \mapsto \varphi' := \epsilon \circ \varphi$$

is bijective.

If we let U be the forgetful functor from $SPSP(T)$ to the category TOP (of (external) topological spaces and continuous functions), which associates to an internal space $(E, \Gamma) \overset{p}{\to} T$ the external space (E, Γ) which has E as its set of points and Γ as its topology, and if we let C: $TOP \to SPSP(T)$ be the constant-space embedding $X \mapsto X_T$, then the correspondence $\varphi \mapsto \varphi'$ amounts to part (a) of the following theorem. Part (b) is proved similarly. Here S: $(TOP \downarrow T) \to SPSP(T)$ is the space of sections-functor $(Y \overset{g}{\to} T) \mapsto g_T$.

2.3. THEOREM: (a) *TOP is equivalent to a reflective subcategory of $SPSP(T)$, and we have an adjunction U–$|C$.*

(b) *we have an adjunction U'–$|S$, where U' is the obvious forgetful functor $((E, \Gamma) \overset{p}{\to} T) \mapsto (U((E, \Gamma) \overset{p}{\to} T) \overset{p}{\to} T)$.*

In fact, connections between TOP and $SPSP(T)$ as expressed in this theorem can be formulated somewhat more generally, by taking yet another look at internal spaces. In order to make things work, we will for the remainder of this section assume *that the base space T is always sober*.

Recall (cf. [1]) that if Ω is a locale (*cHa*, frame), a *point* or superfilter of Ω is a subset $F \subseteq \Omega$ with $\bot \notin F$, $\top \in F$, $U \wedge V \in F \Leftrightarrow U \in F$ and $V \in F$ (i.e. F is a filter) and moreover, for any $\Delta \subseteq \Omega$, $\bigvee \Delta \in F \Leftrightarrow \exists U \in \Delta \ U \in F$. By $pt(\Omega)$ we denote the space of points of Ω with the canonical topology.

If $E = ((E, \Gamma) \overset{p}{\to} T)$ is an internal space in $Sh(T)$, we get a continuous function $E \overset{j}{\to} pt(\Gamma)$ from the embedding $\Gamma \hookrightarrow \mathcal{O}(E)$.

2.4. LEMMA: *Let p and j be as above. In the diagram*
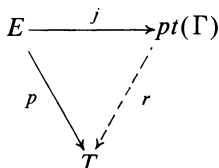
there exists a unique factorization r of p through j.

PROOF: Since $j^{-1}: \Gamma \hookrightarrow \mathcal{O}(E)$ is injective, i.e. a monomorphism of locales, j is an epimorphism of sober spaces, so uniqueness of r is evident. As for its existence, define for a superfilter $x \in pt(\Gamma)$,

$$r^*(x) = \{ W \in \mathcal{O}(T) | p^{-1}(W) \in x \}.$$

It is easy to see that $r^*(x)$ is a superfilter in $\mathcal{O}(T)$, and consequently, there is a unique point $r(x) \in T$ such that $r^*(x) = \{ W \in \mathcal{O}(T) | r(x) \in W \}$. This defines a function $pt(\Gamma) \overset{r}{\to} T$, which is continuous, since $r^{-1}(W) = \{ x \in pt(\Gamma) | r(x) \in W \} = \{ x \in pt(\Gamma) | p^{-1}(W) \in x \} = p^{-1}(W) \in \Gamma$. Finally, $r \circ j = p$, for $r(j(e)) = r(\{0 \in \Gamma | e \in 0\}) = \{ W | p^{-1}(W) \ni e \} = p(e)$ (identifying real points of sober spaces and superfilters). $\square$

2.5. LEMMA: *If f is an internal continuous function from* $(E, \Gamma) \overset{p}{\to} T$ *to* $(F, \Delta) \overset{q}{\to} T$, *then there exists a unique continuous function g making the diagram below commute.*
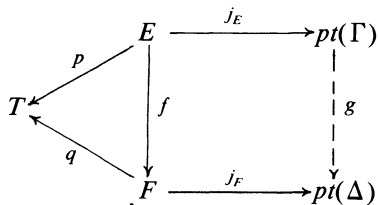
PROOF: The proof is similar to that of 2.4.   □

Let *SOB* denote the category of sober spaces and continuous func-
tions. The preceding two lemmas then tell us that $TOP(Sh(T))$ is
equivalent to a subcategory of $(SOB \downarrow T)^{\rightarrow}$. (Note that when $E \xrightarrow{p} T$ is a
local homeomorphism, and T is sober, so is E.)

2.6. COROLLARY: *$TOP(Sh(T))$ is equivalent to the full subcategory of
$(SOB \downarrow T)^{\rightarrow}$ consisting of commuting triangles*

$$
\begin{array}{ccc}
E & \xrightarrow{\quad j \quad} & X \\
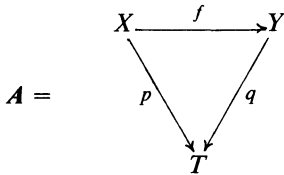 & \searrow{\scriptstyle p} \quad {\scriptstyle q}\swarrow & \\
 & T &
\end{array}
$$
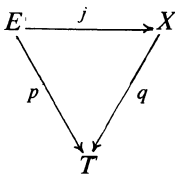
*with p a local homeomorphism, and j a function which separates opens (i.e.
if U, V are open subsets of X, then $U = V \Leftrightarrow j(E) \cap U = j(E) \cap V$).*   □

Conversely, every object of $(SOB \downarrow T)^{\rightarrow}$ defines an internal space, as
in the proof of the following theorem.

2.7. THEOREM: *$TOP(Sh(T))$ is equivalent to a coreflective subcategory of
$(SOB \downarrow T)^{\rightarrow}$.*

PROOF: The coreflector C maps a triangle

$$
A = \quad
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
 & \searrow{\scriptstyle p} \quad {\scriptstyle q}\swarrow & \\
 & T &
\end{array}
$$

on the internal space $(E, \Gamma) \xrightarrow{\bar{p}} T$, where $E \xrightarrow{\bar{p}} T$ is the local homeomor-
phism representing the sheaf of sections of p, and $\Gamma \subseteq \mathcal{O}(E)$ is the
topology determined by $f^{-1}(\mathcal{O}(Y)) = \{ f^{-1}(U) | U \in \mathcal{O}(Y) \}$. The counit

$CA \overset{\epsilon_A}{\to} A$ of the adjunction is the isomorphism ϵ of sections

$$E \xrightarrow{\quad \epsilon \quad} X$$

with maps $\bar{p}$ and p to T.

which is easily seen to factor through $pt(\Gamma) \to Y$.   □

3. Internal sober spaces

Recall (cf. for example [1]) that a topological space X is called *sober* if every superfilter on $\mathcal{O}(X)$ is principal (i.e., is of the form $\{O \in \mathcal{O}(X) | x \in O\}$ for some $x \in X$). In this section we will prove Fourman's representation theorem for internal sober spaces (cf. [1], §8). This is straightforward, using the representations of internal spaces discussed above. Again, we will throughout this section assume that the base space T is sober.

Let us first look at what superfilters of opens are in an internal space $(E, \Gamma) \overset{p}{\to} T$. Let f be a predicate on opens, or equivalently, an extensional function $\Gamma \overset{f}{\to} \mathcal{O}(T)$ (extensional here means that if $O, O' \in \Gamma$ and $p^{-1}(W) \cap O = p^{-1}(W) \cap O'$, then $W \cap f(O) = W \cap f(O')$, for each $W \in \mathcal{O}(T)$). Trivially, we can calculate that (1) and (2) are equivalent, for $U \in \mathcal{O}(T)$.

(1) $U \subseteq \llbracket f \text{ is a superfilter} \rrbracket$
(2) f satisfies the following three conditions (in addition to being extensional)
    (i) $U \subseteq f(E)$
    (ii) $U \cap f(G) \cap f(H) = U \cap f(G \cap H)$, for $G, H \in \Gamma$
    (iii) for each predicate u on Γ, i.e. extensional $u: \Gamma \to \mathcal{O}(T)$,
        $U \cap f(\bigcup_{O \in \Gamma}(O \cap p^{-1}u(O))) \subseteq \bigcup_{O \in \Gamma}(u(O) \cap f(O))$.

(Note that from (ii) it follows that in (iii) we may equivalently require that $U \cap f(\bigcup_{O \in \Gamma}(O \cap p^{-1}(O))) = U \cap \bigcup_{O \in \Gamma}(u(O) \cap f(O))$.)

3.1. LEMMA: *Let* $f_U: \Gamma \to \mathcal{O}(U)$ *be the composite* $\Gamma \overset{f}{\to} \mathcal{O}(T) \overset{-\cap U}{\to} \mathcal{O}(U)$. *Then condition* (2) *above is equivalent to* f_U *being an* $\wedge\vee$*-map such that* $f_U \circ p^{-1} = id_{\mathcal{O}(U)}$.

PROOF: ($\Rightarrow$) Assume the properties in (2). Since f is extensional, we get that $W \subseteq f_U p^{-1}(W)$ for every $W \subseteq U$. And conversely, applying (iii) to

the predicate u: $\Gamma \to \mathcal{O}(T)$, $u(O) = p(O) \cap W$, it is clear that $f_U p^{-1}(W) \subseteq W$. Thus $f_U p^{-1} = id_{\mathcal{O}(U)}$. From (i) and (ii) it is clear that f_U preserves $\top$ and $\wedge$. Also, if $\mathcal{U} \subseteq \Gamma$, we can show that $f_U(\bigcup \mathcal{U}) = \bigcup\{ f_U(G) | G \in \mathcal{U} \}$ as follows. (Of course, $\supseteq$ is clear). By applying (iii) to the predicate u: $\Gamma \to \mathcal{O}(U)$ defined by

$$u(O) = \bigcup\{ W \in \mathcal{O}(T) | O \cap p^{-1}(W) \subseteq \text{ some } G \in \mathcal{U} \}$$

we derive that

$$U \cap f_U\big(\bigcup\{ O \cap p^{-1}(W) | O \in \Gamma, W \in \mathcal{O}(T),$$

$$O \cap p^{-1}(W) \subseteq \text{ some } G \in \mathcal{U}\}$$

$$= U \cap \bigcup\{ f(O) \cap W | O \in \Gamma, W \in \mathcal{O}(T),$$

$$O \cap p^{-1}(W) \subseteq \quad \text{some} \quad G \in \mathcal{U}\}.$$

By extensionality of f, the left hand side of this equality is equal to $f_U(\bigcup \mathcal{U})$, and the right hand side is equal to $\bigcup\{ f_{U_1}(G) | G \in \mathcal{U} \}$. ($\Leftarrow$) Suppose f_U is an $\wedge\bigvee$-map such that $f_U \circ p^{-1} = id$. Evidently, f is extensional, and satisfies conditions (i) and (ii) of (2). For (iii), $U \cap f_U(\bigcup_{O \in \Gamma}(O \cap p^{-1}u(O))) = U \cap \bigcup_{O \in \Gamma}(f_U(O) \cap f_U(p^{-1}u(O))) = U \cap \bigcup_{O \in \Gamma}(f_U(O) \cap u(O))$, for any predicate u: $\Gamma \to \mathcal{O}(T)$. □

3.2. COROLLARY: *An internal space* (*cf. Corollary* 2.6)
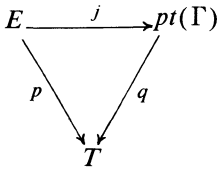
is internally sober iff j induces an isomorphism from sections of p to sections of q (by $a \mapsto j \circ a$).

PROOF: If $(E, \Gamma) \overset{p}{\to} T$ is internally sober, sections of p over $U \in \mathcal{O}(T)$ correspond to $\wedge\bigvee$-maps f^*: $\Gamma \to \mathcal{O}(U)$ such that $f^* \circ p^{-1} = id_{\mathcal{O}(U)}$, by Lemma 3.1 ($s$: $U \to E$ corresponds to s^{-1}: $\Gamma \to \mathcal{O}(U)$ of course). But $\wedge\bigvee$-maps $\Gamma \overset{f^*}{\to} \mathcal{O}(U)$ with $f^* \circ p^{-1} = id$ correspond by duality to continuous sections f: $U \to pt(\Gamma)$ satisfying $\forall W \in \mathcal{O}(T)(p^{-1}(W) \in f(t) \Leftrightarrow t \in W)$, which precisely says that f is a section of q (by the definition of q as in 2.4). □

3.3. COROLLARY: (*cf*. [1], §8). *Every internal sober space in* $Sh(T)$ *is a space of sections* g_T *for some map* $Y \overset{g}{\to} T$ *of sober spaces.*   □

4. Pseudometric and metric spaces in sheaves

Topological space-objects in sheaves are not as nicely behaved as algebraic objects, like commutative rings. In the algebraic case one may regard an internal ring-object, for example, equivalently as a sheaf of rings, or as a sheaf of sets with a continuous ring-structure on the stalks (cf. the Introduction). What remains of the first correspondence will be discussed in Part 2. The second correspondence has no topological analog: in fact it is quite easy to construct examples which show that (contrary to the algebraic situation) the topology Γ of an internal space $(E, \Gamma) \overset{p}{\to} T$ is *not* determined by the subspace topologies it induces on the stalks $p^{-1}(t)$, $t \in T$. In this section, which is meant to provide some more examples of internal topological spaces, it will be seen that "being determined by the structure on the stalks" is regained if one considers a not-strictly topological structure like that of a pseudometric space.

Recall that the real numbers-object in $Sh(T)$ is the constant space $\mathbb{R}_T$ (cf. [1], for example). Hence, by the previous results, sheaf-maps f from a sheaf A over T to $\mathbb{R}_T$ correspond to continuous functions $f: E_A \to \mathbb{R}$, where $E_A \overset{p}{\to} T$ is the étale-space representing A. Henceforth, the real numbers-object in a sheaf will always be denoted by "$\mathscr{R}$", while "$\mathbb{R}$" refers to the external reals. Using the interpretation of higher order logic [1], it makes sense to define a (pseudo-)metric as in the classical case:

4.1. DEFINITION: Let A be a sheaf over T. A sheaf-map $d: A \times A \to \mathscr{R}$ is a *pseudometric on* A if

1. $\forall a, b \ d(a, b) \geqslant 0$, and $\forall a \ d(a, a) = 0$
2. $\forall a, b \ d(a, b) = d(b, a)$
3. $\forall a, b, c \ d(a, b) \leqslant d(a, c) + d(c, b)$

are all valid (quantification is over "elements" of A, i.e. sections of the representing local homeomorphism $E_A \overset{p}{\to} T$). d is called a *metric* if in addition

4. $\forall a, b \ d(a, b) = 0 \to a = b$.

The following proposition says what pseudometrics are from an external point of view.

4.2. PROPOSITION: *A sheaf map* d: $A \times A \to \mathcal{R}$ *is a pseudometric iff the corresponding function* d: $E_A \times E_A \to \mathbb{R}$ *is externally a continuous pseudometric on fibers.*

(*Explanation.* Recall that the correspondence of continuous functions $E_B \xrightarrow{f} X$ and sheaf-maps f: $B \to X_T$, for a sheaf B and a space X, is given through

$$f(e) = \pi_1\Big(f\big((p \restriction U_e)^{-1}\big)(p(e))$$

$$f(a) = \lambda t \cdot \langle f \circ a(t), t \rangle,$$

where U_e is a small nbd of $e \in E_B$, and a is a section of $E_B \xrightarrow{p} T$.

If $Y \xrightarrow{g} T$ is a continuous function, a function d: $Y \times_T Y \to \mathbb{R}$ is called a continuous pseudometric on fibers if d is continuous, and for all $t \in T$, the restriction of d to $p^{-1}(t) \times p^{-1}(t)$ is a pseudometric on $p^{-1}(t)$.)

PROOF: d corresponds to d, i.e. for sections a, b: $U \to E_A$ of p, and $t \in \mathcal{U}$, $d(a, b)(t) = d(a(t), b(t))$. Clearly, if d is a pseudometric on fibers, d is an internal pseudometric. Conversely, if d is an internal pseudometric, we find that d is a pseudometric on fibers by considering, for points e, $e' \in E_A$ with $p(e) = p(e')$, the sections $(p \restriction U_e)^{-1}, (p \restriction U_{e'})^{-1}$, where U_e, $U_{e'}$ are small nbds of e and e' such that $p(U_e) = p(U_{e'})$. □

Being a metric is not a "fiberwise" property:

4.3. PROPOSITION: *A sheaf map* d: $A \times A \to \mathcal{R}$ *is a metric* (*internally*) *iff the corresponding function* d: $E_A \times_T E_A \to \mathbb{R}$ *is a continuous pseudometric on fibers with the additional property* $\operatorname{Int} d^{-1}(\{0\}) \subseteq \Delta$ ($\Delta = \Delta_E$ *is the diagonal* $\{\langle e, e \rangle | e \in E\} \subseteq E \times_T E$).

PROOF: By 5.2 it suffices to show that

$$\vDash \forall a, b\big(d(a, b) = 0 \to a = b\big) \quad \text{iff} \quad \operatorname{Int} d^{-1}(\{0\}) \subseteq \Delta.$$

($\Rightarrow$) If $\langle e, e' \rangle \in E \times_T E$ is in $\operatorname{Int} d^{-1}(\{0\})$, then there are small nbds U_e of e and $U_{e'}$ of e' such that $p(U_e) = p(U_{e'})$ and $U_e \times_T U_{e'} \subseteq d^{-1}(0)$. Thus for all $t \in p(U_e) = p(U_{e'})$, $d((p \restriction U_e)^{-1}, (p \restriction U_{e'})^{-1})(t) = d((p \restriction U_e)^{-1}(t),$ $(p \restriction U_{e'})^{-1}(t)) = 0$, so $p(U_e) \subseteq [\![(p \restriction U_e)^{-1} = (p \restriction U_e)^{-1}]\!]$.
($\Leftarrow$) Take sections a and b over $U \in \mathcal{O}(T)$, and take an open $W \subseteq [\![d(a, b) = 0]\!] = \operatorname{Int}(\{t \in U | d(a(t), b(t)) = 0\})$. Then for all $t \in W$, $\langle a(t), b(t) \rangle \in \operatorname{Int} d^{-1}(0)$, so $a(t) = b(t)$. Hence $W \subseteq [\![a = b]\!]$. □

If A is a sheaf on T, and d: $A \times A \to \mathcal{R}$ is a pseudometric, the internal topology τ_d on d is defined by setting for global predicates $O \in \mathcal{O}(E_A)$ on T,

$$[\![O \in \tau_d]\!] = [\![\forall a \in O \exists \epsilon > 0 \forall a' \in A \, d(a, a') < \epsilon \to a' \in O]\!]$$

as in the classical case. Since the usual proof that this defines a topology is contructively valid, we get that τ_d is indeed an internal topology. What is its external representation? Let U be a small open subset of E_A, and let ϵ: $p(U) \to \mathbb{R}^+ = (0, \to)$ be a continuous function. The *continuous ball* around U of radius ϵ, $B(U, \epsilon)$, is the set of points $e \in p^{-1}p(U)$ such that $\exists e' \in p^{-1}p(e) \cap U d(e, e') < \epsilon(p(e))$.

4.4. PROPOSITION: *Let a be a sheaf on T, $E_A \xrightarrow{p} T$ its representation. Let d be an internal pseudometric on A represented by d: $E_A \times_T E_A \to \mathbb{R}$. Then the internal topology induced by d corresponds to (in the sense of 1.5) the topology on E_A having the set of continuous balls $B(U, \epsilon)$, with U small and $p(U) \xrightarrow{\epsilon} \mathbb{R}^+$ continuous, as a basis.*

PROOF: We have to show that the global internal opens are exactly those generated by the basis defined in the proposition. Let $O \in \mathcal{O}(E_A)$. By definition, $O \in \Gamma \tau_d$ iff

(1) for all sections $U \xrightarrow{a} E_A$ of p and for all $t \in U \cap a^{-1}(O)$ there exist a nbd $W_t \subseteq U$ and a continuous ϵ: $W_t \to \mathbb{R}^+$ such that for all sections b: $W_t \supseteq W \to E_A$, $\{t \in W | d(a(t), b(t)) < \epsilon(t)\} \subseteq b^{-1}(O)$.

We will show that (1) is equivalent to

(2) $\forall e \in O \exists$ small nbd U_e of $e \exists \epsilon$: $U_e \to \mathbb{R}^+$: $B(U, \epsilon) \subseteq O$.

(1) $\Rightarrow$ (2). Take $e \in O$. Using (1) we find a nbd $W_{p(e)}$ and a continuous $\epsilon > 0$ satisfying the condition in (1). Now let U_e be a small nbd of e with $p(U_e) \subseteq W_{p(e)}$, and let $\epsilon' = \epsilon \restriction p(U_e)$. Then $B(U_e, \epsilon') \subseteq O$. For if $e' \in p^{-1}p(U_e)$ with $e' \in B(U_e, \epsilon')$, take a small nbd $U_{e'}$ of e' such that $p(U_{e'}) \subseteq p(U_e)$, and consider the section $(p \restriction U_{e'})^{-1}$. Since $e' \in B(U_e, \epsilon')$, $d(e', e'') < \epsilon(p(e'))$ for some $e'' \in p^{-1}p(e') \cap U_e$, so $p(e') \in \{t | d(p \restriction U_{e'})^{-1}(t), (p \restriction U_e)^{-1}(t)) < \epsilon(t)\} \subseteq (p \restriction U_{e'})(O)$, so $e' \in O$.

(2) $\Rightarrow$ (1). Take a section $U \xrightarrow{a} E_A$ of p, and choose $t_0 \in U \cap a^{-1}(O)$. Let $U_{a(t_0)}$ be a small nbd of $a(t_0)$, and let ϵ: $p(U_{a(t_0)}) \to \mathbb{R}^+$ be such that $B(U, \epsilon) \subseteq O$ (by (2)). Then if b: $W \to E_A$ is another section of p, with $W \subseteq p(U_e)$, $d(a(t), b(t)) < \epsilon(t)$ implies $b(t) \in O$ (for $t \in W$), by definition of $B(U, \epsilon)$. $\square$

PART 2. SPACES IN SHEAVES AND SHEAVES OF SPACES

In this part, we will point out some connections between internal topological spaces $(E, \Gamma) \overset{p}{\to} T$ and *TOP*-valued sheaves which have as their underlying sheaf of sets the sheaf of sections of $E \overset{p}{\to} T$. As has been remarked before, these connections are not as nice as in the case of internal algebraic objects. The main reason for this is, of course, that the theory of topology is not a geometric theory, it is not even a first-order theory. The fact that the structure of internal spaces is not determined by the stalks is related to this. Another fact that makes things look not very hopeful is that *TOP*-valued sheaves are rather awkward objects. For if F is a *TOP*-valued sheaf on T, the space FU, $U \in \mathcal{O}(T)$, must be toplogically embedded

$$FU \hookrightarrow \Pi_i FU_i$$

in the product $\Pi_i FU_i$ for every cover $\{U_i\}_i$ of U, as follows immediately from the definition as given, for example, in [3].

   In the first section of this part, we will prove a general adjunction theorem relating the category $TOP(Sh(T))$ of internal topological spaces in $Sh(T)$ and the category $Sh(T, TOP)$ of sheaves on T with values in the category TOP of topological spaces. In the second section, we will consider the special case of a locally compact zero-dimensional base-space. Such spaces are precisely the Stone-spaces of Boolean rings (not necessarily having a unit).

1. A general adjunction theorem

Let T denote the base space. We consider assignments

$$B: U \mapsto B_U$$

of a set of subsets B_U of U to each open $U \subseteq T$, such that

   (i) (*monotone*) For U and V open subsets of T, and K an arbitrary subset of T, if $K \subseteq V \subseteq U$ then $K \in B_V$ iff $K \in B_U$.
   (ii) (*compact*) If $K \in B_U$ and $\{U_i\}_i$ is an open cover of U, then there are finitely many $U_{i_1}, \ldots, U_{i_n}$ and sets

$$K_{i_1} \in B_{U_{i_1}}, \ldots, K_{i_n} \in B_{U_{i_n}} \quad \text{such that} \quad K = \bigcup_{j=1}^{n} K_{i_j}.$$

From now on, we will arbitrarily fix such a monotone and compact assignment B. Relative to this assignment B, we may then define a

functor

$$F_{(B)} : TOP(Sh(T)) \to Sh(T, TOP)$$

as follows. For an internal space $A = ((E_A, \Gamma) \xrightarrow{p} T)$ we define a sheaf $F_{(B)}(A)$ with the same underlying set-valued sheaf as A (that is, sections of p): on each $A(U) = \{$sections of p over $U\}$, $U \in \mathcal{O}(T)$, we define a topology by taking as subbasic opens the sets

$$[K, O] = \{ a \in A(U) \mid a(K) \subseteq O \}$$

for $K \in B_U$ and $0 \in \Gamma$. Let us verify that this is well-defined:

1.1. LEMMA: (a) $F_{(B)}(A)$ is a sheaf of topological spaces on T
              (b) $F_{(B)}$ is a functor from $TOP(Sh(T))$ to $Sh(T, TOP)$.

PROOF: (a) The monotonicity condition (i) above makes restrictions $\rho = \rho^U_V : A(U) \to A(V)$, for $U \supseteq V$, continuous. Also, if $U = \bigcup_i U_i$, the canonical inclusion

$$A(U) \xhookrightarrow{g} \Pi_i A(U_i)$$

is a topological embedding: we only have to show that the image of a sub-basic open $[K, O]$ in $A(U)$ is open in $g(A(U))$. But by the compactness property (ii) above, if $(b_i)_i = g(a) \in g([K, O])$. then there are U_{i_j}, and $K_{i_j} \in B_{U_{i_j}}, j = 1, \ldots, n$, such that $K = K_{i_1} \cup \ldots \cup K_{i_n}$, so $b_{i_j} \in [K_{i_j}, O]$ for $j = 1, \ldots, n$, and

$$\bigcap_{j=n}^n \Pi_{i_j}^{-1}\big([K_{i_j}, O]\big) \cap g(A(U)) \subseteq g([K, O]),$$

showing that $g([K, O])$ is open in $g(A(U))$.

(b) If $((E_A, \Gamma) \xrightarrow{p} T) \xrightarrow{f} ((E_B, \Delta) \xrightarrow{q} T)$ is an (internal global) continuous map, i.e. $qf = p$, and f is continuous w.r.t. both topologies, then each component of f as a natural transformation $f: FA \to FB$ is continuous: this is obvious, because $f_U^{-1}([K, O]) = [K, f^{-1}(O)]$. $\square$

Going in the other direction, suppose we are given a TOP-valued sheaf A with underlying étale-space $E_A \xrightarrow{p} T$, and define a collection Γ_A of open

subsets of E_A by setting for $O \in \mathcal{O}(E_A)$,

$O \in \Gamma_A \Leftrightarrow$ for each $U \in \mathcal{O}(T)$ and each $K \in B_U$,

$[K, O]$ is open in $A(U)$.

We then have

1.2. LEMMA: (a) Γ_A is a p-topology on $E_A \overset{p}{\to} T$

(b) The assignment $A \mapsto ((E_A, \Gamma_A) \overset{p}{\to} T)$ defines a functor $L_{(B)}: Sh(T, TOP) \to TOP(Sh(T))$.
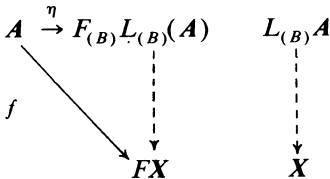
PROOF: (a) If Γ_A is a topology, it obviously makes p continuous. So let's show it is a topology. Clearly, $\emptyset$ and E_A are in Γ_A. Also, if $O, O' \in \Gamma_A$, then $O \cap O' \in \Gamma_A$, since $[K, O \cap O'] = [K, O] \cap [K, O']$. For unions, suppose $O_i \in \Gamma_A$ for each $i \in I$, and take $U \in \mathcal{O}(T)$, $K \in B_U$. If $a(K) \subseteq \bigcup_i O_i$, then $K \subseteq \bigcup_i a^{-1}(O_i)$, so there are $K_1 \in B_{a^{-1}(O_{i_1})}, \dots, K_n \in B_{a^{-1}(O_{i_n})}$ such that $K_1 \cup \dots \cup K_n = K$, and thus $a \in \bigcap_{j=1}^n [K_j, O_{i_j}] \subseteq [K, \bigcup_i O_i]$. This shows that $\bigcup_i O_i \in \Gamma_A$ whenever each of the $O_i \in \Gamma_A$.

(b) $L_{(B)}$ is a functor, for if $f: A \overset{\cdot}{\to} B$ and all components f_U are continuous, then f is continuous as a map $L_{(B)}(A) \to L_{(B)}(B)$: write $L_{(B)}(A) = (E_A, \Gamma_A) \overset{p}{\to} T$, $L_{(B)}(B) = (E_B, \Gamma_B) \overset{q}{\to} T$, and take $O \in \Gamma_B$, i.e. $\forall U \in \mathcal{O}(T) \forall K \in B_U [K, O]$ is open in $B(U)$. Then $f^{-1}(O)$ also has this property, since for such U and K,

$$\{ a \in A(U) | a(K) \subseteq f^{-1}(O)\} = f_U^{-1}(\{ b \in B(U) | b(K) \subseteq O\}).$$

1.3. THEOREM: $L_{(B)}$ is left-adjoint to $F_{(B)}$.   □

PROOF: Let $\eta_A: A \overset{\cdot}{\to} F_{(B)} L_{(B)}(A)$ be the identity, for each TOP-valued sheaf A. The components $(\eta A)_U$ are all continuous, as is easily verified. To prove the adjunction, take a TOP-valued sheaf A, an internal space $X = (E, \Delta) \overset{q}{\to} T$, and a natural transformation $f: A \overset{\cdot}{\to} L_{(B)}(X)$, such that each component f_U is continuous.

$$\begin{array}{ccccc}
A & \overset{\eta}{\to} & F_{(B)} L_{(B)}(A) & & L_{(B)} A \\
 & & \big\downarrow & & \big\downarrow \\
f & & & & \\
 & \searrow & & & \\
 & & FX & & X
\end{array}$$

It suffices to show that f is continuous as a morphism $L_{(B)}(A) \to X$. So take $O \in \Delta$ – we have to show that $f^{-1}(O)$ is open in $L_{(B)}(A)$. To this end, pick $U \in \mathcal{O}(T)$ and $K \in B_U$. Then $[K, f^{-1}(O)] = f_U^{-1}([K, O])$, which

is open in $A(U)$ by continuity of f_U.   □

1.4. EXAMPLES: Given $(E, \Gamma) \overset{p}{\to} T$ we may equip the sections of p over an open subset of T with the familiar pointwise-convergence topology (with respect to Γ), or with the compact-open topology (with respect to Γ). These are two extreme cases of the definition of $F_{(B)}$: let B_V = finite subsets of V, B_V = compact subsets of V, respectively.

2. Internal spaces over a locally compact zero-dimensional base space

Suppose that the base space T has a basis of compact open-and-closed sets. We may then define a monotone and compact assignment B by setting

$$B_U = \{ K \subseteq U \,|\, K \text{ is compact and open} \},$$

for each $U \in \mathcal{O}(T)$. We keep this B fixed throughout this section, and we write L for $L_{(B)}$, F for $F_{(B)}$.

2.1. LEMMA: *For T and B as above, for any internal topological space $A = (E_A, \Gamma) \overset{p}{\to} T$, the counit ϵ_A: $LFA \to A$ is a (global internal) homeomorphism.*

PROOF: Let Γ' be the p-topology of LFA. Thus

$O' \in \Gamma' \Leftrightarrow$ for each open U, each compact clopen $K \subseteq U$, each $a \in A(U)$ with $a(K) \subseteq O'$ we can find compact clopens $K_1, \ldots, K_n$, and opens $O_1, \ldots, O_n \in \Gamma$ such that $a \in \bigcap_{i=1}^n [K_i, O_i] \subseteq [K, O']$.

of course, $\Gamma \subseteq \Gamma'$ (i.e. ϵ is continuous). Also $\Gamma' \subseteq \Gamma$, for if $y \in O' \in \Gamma'$, then (since Γ' is a p-topology) we may find a compact clopen nbd G of $p(y)$ and a section a over G running through y, with $a(G) \subseteq O'$. By definition of Γ', we find K_i, and $0_i \in \Gamma$ as above, such that

$$a \in \bigcap_{i=1}^n [K_i, O_i] \subseteq [K, O'],$$

and we may assume that the K_i are mutually disjoint. It then follows easily from properties of the base space that $y \in O_i \cap p^{-1}(K_i) \subseteq O'$ (if $p(y) \in K_i$), or that $y \in p^{-1}(B \setminus \bigcap_{i=1}^n K_i) \subseteq O'$, showing that O' is open in Γ.   □

This lemma tells us that if T is locally compact and zero-dimensional, $TOP(Sh(T))$ is equivalent to a full reflective subcategory of $Sh(T, TOP)$. To describe this subcategory, let us try to characterize the image of the functor F.

If A is a *TOP*-valued sheaf, and U is an open subset of T, we call a subset $W \subseteq A(U)$ *extensional* if sections which are locally an element of W are already an element of W. More precisely, $W \subseteq A(U)$ is extensional iff

> $\forall a \in A(U)$, if $\exists$ cover $\{B_i\}_i$ of U such that for all i there exists a $b \in W$ with $\rho^U_{B_i}(b) = \rho^U_{B_i}(a)$, then $a \in W$.

Note that the extensional subsets are closed under finite intersections. Call a *TOP*-valued sheaf A over T *extensional* if for each open subset U of T, $A(U)$ has an open basis consisting of extensional sets. (Clearly, this is equivalent to saying that each $A(G)$, G a compact clopen subset of T, has a basis consisting of extensional sets, since if $\{G_i\}_i$ is a disjoint cover of U by clopen compact sets, the canonical map $A(U) \to \Pi_i A(G_i)$ is a homeomorphism.)

Also note that the sets $[K, O]$ occurring in the definition of the functor $F_{(B)}$ are extensional, hence each *TOP*-valued sheaf of the from $F_{(B)}(X)$, X an internal space, is extensional. In fact, the converse holds also:

2.2. LEMMA: *For an extensional TOP-valued sheaf A on T, the unit $\eta_A: A \to FLA$ is an isomorphism, i.e. each component $(\eta_A)_U$ is a homeomorphism.*

PROOF: It suffices to show this for each compact clopen U, since these form a basis for the topology on T. Now first note that if U is covered by disjoint compact clopen sets U_i, $i \in I$, then $g: A(U) \to \Pi_i A(U_i)$, $g = \langle \rho^U_{U_i} \rangle_i$, is a homeomorphism (provided $A(U) \neq \emptyset$), and in particular, if $V \subseteq U$ are both compact clopen, $\rho^U_V: AU \to AV$ is an open surjection.

We show that $(\eta_A)_U$ is an *open* mapping, if U is compact clopen. Recall that the subbase of $FLA(U)$ consists of the sets $[K, O]$, K compact clopen $\subseteq U$, and $O \in \mathcal{O}(E_A)$ such that for each compact clopen K', and each open $U' \supseteq K'$, $[K', O]$ is open in AU. Now take $a \in A(U)$, and let W_a be an extensional nbd of a in $A(U)$. We have to find $K_1, \ldots, K_n$ (compact clopen) and $O_1, \ldots, O_n$ such that $a \in \bigcap_{i=1}^n [K_i, O_i] \subseteq W_a$, and each O_i open in LA. In fact one O suffices: let

$$O = \{ e \in p^{-1}(U) | \exists b \in W_a \ e \in \mathrm{range}(b) \}.$$

Then O is open in E_A, and $a \in [U, O] \subseteq W_a$ since W_a is extensional. So the proof is complete if we show that O is open in LA, i.e. that for $V \in \mathcal{O}(T)$ and $K \subseteq V$ compact clopen, $[K, O]$ is open in $A(V)$. But we may assume that $K \subseteq U$, since $p(O) = U$, so it follows by extensionality that $[K, O] = (\rho^V_K)^{-1} \rho^U_K(W_a)$. $\square$

Putting these facts together, we have the following theorem.

2.3. THEOREM: *Let T be a compact zero-dimensional space. Then the category $TOP(Sh(T))$ is equivalent to the category of extensional TOP-valued sheaves of T, and this category is a full reflective subcategory of $Sh(T, TOP)$.*   □

PART 3. CHANGE OF BASE SPACE

In this part, we will no longer consider topological spaces over a fixed base space T, but we will jump from one base space to another. This is possible on the basis of an extension of the change-of-base adjunction for categories of sheaves to categories of topological spaces in sheaves, which is presented in Section 1. In a next section we then define the category *SPSP* of spaced spaces, analogous to the definition of the category of ringed spaces in algebraic geometry. This category is shown to be a topological category in the sense of Wyler [10]. The other two sections of this chapter are concerned with some structural properties of this category. Factorization properties are considered in Section 3, while in the final section limits and colimits are discussed.

1. Direct and inverse images

1.1. INVERSE IMAGES: Let $E = ((E, \Gamma) \xrightarrow{p} T)$ be a topological space in $Sh(T)$, and let $S \xrightarrow{f} T$ be a continuous map. We define the *inverse image* $f^*(E)$ of E to be the internal space $((f^*(E), f^*(\Gamma)) \xrightarrow{f^*(p)} S)$ in $Sh(S)$, where $f^*(E) \xrightarrow{f^*(p)} S$, the "underlying set" of the space, is defined in the standard way (cf. [1], [2]), that is, the following diagram is a pullback in *TOP* (hence $f^*(p)$ is a local homeomorphism)

$$
\begin{array}{ccc}
f^*(E) & \longrightarrow & E \\
\downarrow f^*(p) & & \downarrow p \\
S & \xrightarrow{\ f\ } & T
\end{array}
$$

and $f^*(\Gamma)$ is the product topology on $f^*(E)$ having as a basis for the open sets the set $\{O \times_T U \,|\, O \in \Gamma, \ U \in \mathcal{O}(S)\}$.   □

1.2. DIRECT IMAGES: Let $E = ((E, \Gamma) \xrightarrow{p} S)$ be a topological space in $Sh(S)$, and again let $S \xrightarrow{f} T$ be a continuous function. Write $E(U)$ for

the set of sections $U \xrightarrow{a} E$ of p over U, for $U \in \mathcal{O}(S)$. We define an internal space $(f_*(E), f_*(\Gamma)) \xrightarrow{f_*(p)} T$, the *direct image* of E, as follows. The "underlying set" is again defined as usual, i.e. the stalks $f_*(p)^{-1}(t)$ of $f_*(E)$ are given by

$$f_*(p)^{-1}(t) = \varinjlim_{t \in U} E(f^{-1}(U)),$$

while the topology on $f_*(E)$ is defined by taking as basic open sets $a_U = \{\langle [a]_t, t \rangle | t \in U\}$, for $U \in \mathcal{O}(T)$ and $a \in E(f^{-1}(U))$. ($[a]_t$ denotes the equivalence class of a in $\varinjlim_{t \in V} E(f^{-1}(V))$.) This makes $f_*(E) \xrightarrow{f_*(p)} T$ into a local homeomorphism. The topology $f_*(\Gamma)$ on $f_*(E)$ is defined by taking as an open basis the set $\{G_* | G \in \Gamma\}$, where for $a \in E(f^{-1}(U))$ and $t \in U$, we define

$$\langle [a]_t, t \rangle \in G_* \Leftrightarrow \text{for some nbd } W_t \subseteq U \text{ of } t,$$
$$f^{-1}(W_t) \subseteq a^{-1}(G).$$

It is easy to see that $\{G_* | G \in \Gamma\}$ is indeed a base for a topology (since $G_* \cap H_* = (G \cap H)_*$ for $G, H \in \Gamma$) and that is a $f_*(p)$-toplogy. $\square$

Our next aim is to show that the adjunction $f^* \dashv f_*$ ([1], [2]) remains valid, if we regard f^* and f_* as functors $SPSP(S) \rightleftarrows SPSP(T)$ instead of functors $Sh(S) \rightleftarrows Sh(T)$ on the underlying sheaves.

1.3. LEMMA: *Let* $S \xrightarrow{f} T$ *be a continuous function. Then* f_* *and* f^* *preserve internal continuity. Hence we have a pair of functors* $SPSP(S) \overset{f^*}{\underset{f_*}{\rightleftarrows}} SPSP(T)$.

PROOF: (a) f^* is a functor: Suppose φ is an internal continuous function $((E, \Gamma) \xrightarrow{p} T) \to ((E', \Gamma') \xrightarrow{p'} T)$, that is, $E \xrightarrow{\varphi} E'$ is a function over T, continuous w.r.t. both topologies. $f^*(\varphi)$ is the unique function making the diagram below commute.
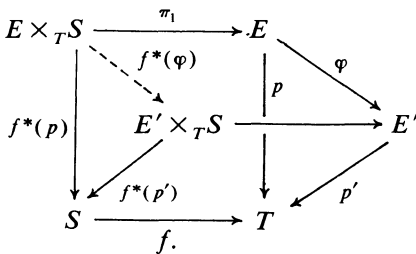
Now take a basic open subset $O' \times_T U$ in $E' \times_T S$. Then $f^*(\varphi)^{-1}(O' \times _T U) = \{(e, s) \mid p(e) = f(s) \text{ and } \varphi(e) \in O', s \in U\} = \varphi^{-1}(O') \times_T U$. Hence $f^*(\varphi)$ is internally continuous.

(b) f_* is a functor: Again take an internally continuous function φ,

$$(E, \Gamma) \xrightarrow{\ \varphi\ } (E', \Gamma')$$
$$p \searrow \quad \swarrow p'$$
$$S$$

$f_*(\varphi)$ is defined on sections by the components $f_*(\varphi)_U: f_*(E)(U) \to f_*(E')(U)$ by composing with φ. That is, $f_*(\varphi)_U$ is the function $a \mapsto \varphi \circ a: E(f^{-1}(U)) \to E'(f^{-1}(U))$. Now take a basic open set G'_* in $f_*(E', \Gamma')$. Then for a point $\langle [a]_t, t \rangle \in f_*(E)$, with $a \in E(f^{-1}(U))$ and $t \in U$,

$$\langle [a]_t, t \rangle \in f_*(\varphi)^{-1}(G'_*)$$

$$\Leftrightarrow \langle [\varphi \circ a]_t, t \rangle \in G'_*$$

$$\Leftrightarrow \exists \text{nbd } W_t \subseteq U \text{ with } f^{-1}(W) \subseteq a^{-1}\varphi^{-1}(G')$$

$$\Leftrightarrow \langle [a]_t, t \rangle \in \varphi^{-1}(G')_*.$$

so $f_*(\varphi)^{-1}(G'_*) = \varphi(G')_*$, which is in $f_*(\Gamma)$.   □

1.4. LEMMA: *Let* $S \xrightarrow{f} T$ *be continuous, and let* $E \in SPSP(T), F \in SPSP(S)$. *Then the unit* η_E *and the co-unit* ε_F *of the adjoint pair* f_*, f^*: $Sh(S) \rightleftarrows Sh(T)$ *are internally continuous.*

PROOF: Let $E = ((E, \Gamma) \xrightarrow{p} T)$. Then the unit $\eta_E = \eta_{(E \xrightarrow{p} T)}$ is defined by associating with a section a of p over U the section $s \mapsto \langle a(f(s)), s \rangle$ of $f^*(p)$ over $f^{-1}(U)$. In other words, if $e \in E$ and U_e is a small nbd of e,

$$\eta_E(e) = \langle \left[\lambda s \cdot \langle (p \upharpoonright U_e)^{-1}(f(s)), s \rangle \right]_{p(e)}, p(e) \rangle.$$

To see that η_E is continuous, choose a basic open set G_* in $f_* f^*(E)$, and write $G = \bigcup_{i \in I}(O_i \times_T U_i)$, with $O_i \in \Gamma$ and $U_i \in \mathcal{O}(S)$. It then is almost immediate from the definition of η_E above that if $e \in E$ and U_e is

a small nbd of e, we find for all $e' \in U_e$,

$$\eta_E(e') \in G_* \Leftrightarrow \text{for some nbd } W_{p(e')} \subseteq p(U_e) \text{ of } p(e'),$$

$$\forall s \in f^{-1}(W_{p(e')}) \exists i \in I \; s \in f^{-1}p(U_e \cap O_i) \cap U_i,$$

or equivalently,

$$\eta_E(e') \in G_* \Leftrightarrow e' \in p^{-1} \operatorname{Int}\left[T \backslash f\left(S \backslash \bigcup_{i \in I} \left(f^{-1}p(U_e \cap 0_i) \cap U_i\right)\right)\right].$$

Hence, η_E is continuous.

(b) Take F to be $((F, \Delta) \overset{q}{\to} S)$. Recall that a point of $f^*f_*(G)$ is a pair (x, s) with $x = \langle [a_x]_{f(s)}, f(s)\rangle \in f_*(E)$, and $s \in S$. Say $a_x : f^{-1}(U) \to E$ is a section of p over $f^{-1}(U)$, and $f(s) \in U \subseteq T$. Such a point $\langle x, s \rangle$ is mapped by ϵ_F on $a_x(s)$. Hence if $G \in \Gamma$ is a nbd of $\epsilon_F(\langle x, s\rangle)$ and $U_{a_x(s)}$ is a small nbd of $a_x(s)$ with $p(U_{a_x(s)}) \subseteq f^{-1}(U)$, we get that

$$\epsilon_F\left(G_* \times_T p\left(U_{a_x(s)}\right)\right) \subseteq G,$$

so ϵ_F is continuous. $\square$

1.5. THEOREM: *The functor* $SPSP(S) \overset{f_*}{\to} SPSP(T)$ *is right-adjoint to the functor* $SPSP(T) \overset{f^*}{\to} SPSP(S)$, *for every continuous function* $S \overset{f}{\to} T$.

PROOF: This is immediate from the preceding two lemmas. The adjunction $Sh(T) \overset{f_*}{\underset{f^*}{\leftrightarrows}} Sh(S)$ has as bijection of morphisms

$$Sh(T)(F, f_*(G)) \to Sh(S)(f^*(F), G)$$

$$\varphi \mapsto \epsilon_G \circ f^*(\varphi)$$

with inverse

$$Sh(S)(f^*(F), G) \to Sh(T)(F, f_*(G))$$

$$\psi \mapsto f_*(\psi) \circ \eta_F.$$

From Lemmas 1.3 and 1.4 we get that this bijection restricts to a bijection of continuous functions between internal spaces. $\square$

2. The category of spaced spaces

In this section we define the category $SPSP$ of spaced spaces, or, of topological spaces in sheaves. The definition is based on the adjunction of the preceding section. We will show that this category $SPSP$ is a topological category on the category Sh of sheaves in the sense of [10], and also in the sense of [8].

2.1. DEFINITION OF $SPSP$: The category $SPSP$ has as its objects topological spaces in sheaves; these will be identified with structures of the form

$$\boldsymbol{E} = (E, \Gamma) \xrightarrow{p} T$$

where E and T are topological spaces, p is a local homeomorphism, and Γ is a p-topology on E. An arrow in the category from a space $\boldsymbol{E} = (E, \Gamma) \xrightarrow{p} S$ to a space $\boldsymbol{F} = (F, \Delta) \xrightarrow{q} T$ is a pair $\boldsymbol{f} = (f, f^+)$, with $S \xrightarrow{f} T$ a continuous function, and f^+ an internal continuous function (in $Sh(T)$) from $\boldsymbol{F}$ to $f_*(\boldsymbol{E})$; that is, f^+ is a function $(F, \Delta) \to (f_*(E), f_*(\Gamma))$ over T which is continuous w.r.t. both topologies. The composition of (f, f^+): $((E, \Gamma) \xrightarrow{p} R) \to ((F, \Delta) \xrightarrow{q} S)$ and (g, g^+): $((F, \Delta) \xrightarrow{q} S) \to ((G, \Sigma) \xrightarrow{r} T)$ is defined to be the pair $(g \circ f, g_*(f^+) \circ g^+)$,

$$(G, \Sigma) \xrightarrow{g^+} g_*(F, \Delta) \xrightarrow{g_*(f^+)} g_* f_*(E, \Gamma) \simeq (gf)_*(E, \Gamma).$$

(Note that, trivially, (by the pullback lemma) inverse images commute with composition, i.e. $(gf)^* \simeq f^* g^*$; hence also, by the adjunction of Theorem 1.5, $g_* f_* \simeq (gf)_*.$) □

2.2. EXAMPLE: For a fixed topological space X, we have a functor X_-: $TOP \to SPSP$, associating with a continuous function $S \xrightarrow{f} T$ the map $X_S \xrightarrow{(f, f^+)} X_T$ with underlying function $f: S \to T$, and with sheaf-map $X_T \xrightarrow{f^+} f_*(X_S)$ having components $(f^+)_U: \mathscr{C}(U, X) \to \mathscr{C}(f^{-1}(U), X)$ defined by $a \mapsto a \circ f$.

This definition is also functorial in X, as follows easily from the results in Part 1 (cf. Part 1, Theorem 2.3), and the adjunctions of Part 1, Section 2, can be lifted to adjoints for the functors $X_-: TOP \to SPSP$ and $(-)_-$: $TOP \times TOP \to SPSP$. □

Let us now turn to Wyler's notion of top-category ([10], §2). We first

have to define the category Sh of sheaves. Objects of Sh are local homeomorphisms $E \xrightarrow{p} S$, and an arrow in Sh from $A = (E \xrightarrow{p} S)$ to $B = (F \xrightarrow{q} T)$ is a pair (f, f^+), with $f : S \to T$ a continuous function, and $f^+ : B \to f_*(A)$ a sheaf-map in $Sh(T)$ or equivalently, f^+ is a continuous function $F \to f_*(E)$ over T (f_* now denotes direct image of $sheaves$).

Obviously, we have a forgetful functor $U : SPSP \to Sh$. For each local homeomorphism $E \xrightarrow{p} S$, let $\tau(E \xrightarrow{p} S)$ be the complete lattice of p-topologies on E, ordered by inclusion. Note that (infinite) meets in this lattice are just intersections. Now given a Sh-morphism $f = (f, f^+) : (E \xrightarrow{p} S) \to (F \xrightarrow{q} T)$, we obtain a function f^τ from q-topologies on F to p-topologies on E, which assigns to a q-topology on F the finest p-topology on E which makes $f^*(F) \xrightarrow{\hat{f}^+} E$ continuous ($\hat{f}^+$ is the adjunct of f^+); in other words, for $U \subseteq E$,

$$ U \in f^\tau(\Gamma) \Leftrightarrow \hat{f}^{+\,-1}(U) \in f^*(\Gamma). $$

It is not hard to see that f^τ preserves intersections. Thus we obtain

2.3. PROPOSITION: *SPSP is a top-category over Sh.*   □

In [8], G. Strecker has formulated a criterion for concrete categories to be called *topological*. The following proposition expresses that *SPSP* can be called *topological* over *Sh*.

2.4. PROPOSITION: *Let* $\{F_i\}_{i \in I} = \{((F_i, \Delta_i) \xrightarrow{q_i} T_i)\}_{i \in I}$ *be a collection of internal spaces, and let* $A = (E \xrightarrow{p} S)$ *be a local homeomorphism. Then any family of Sh-morphisms* $\{A \xrightarrow{f_i} UF_i\}$ *has a U-initial lift.*

Before proving the proposition, let us explain what it means. It says that there exists a p-topology Γ on E making all the f_i continuous (i.e. morphisms of $SPSP$), and which has the additional property that whenever $G = ((G, \Sigma) \xrightarrow{r} R)$ is an internal space and $h : UG \to A$ is a Sh-morphisms such that each $f_i \circ h : G \to F_i$ is continuous (an $SPSP$-morphism), then $h : G \to (A, \Gamma) = ((E, \Gamma) \xrightarrow{p} S)$ is continuous.

PROOF: Consider the adjuncts $\hat{f}_i^+ : f_i^*(F_i) \to E$ of the morphisms $\{f_i\}$,

and define Γ by setting, for U open $\subseteq E$,

$$U \in \Gamma \Leftrightarrow \forall i \in I(\hat{f}_i^+)^{-1}(U) \in f_i^*(\Delta_i).$$

Γ is easily seen to be the required p-topology. $\hspace{2cm}$ □
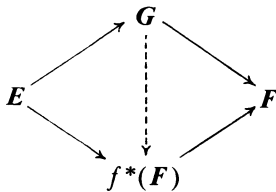
3. Factorizations in *SPSP*

As a consequence of the adjunction proved in Section 1, we obtain factorization theorems for morphisms of spaced spaces, similar to the case of ringed spaces.

$\hspace{1cm}$ Let $E = ((E, \Gamma) \overset{p}{\to} S)$ and $F = ((F, \Delta) \overset{q}{\to} T)$ be spaced spaces, and let f: $E \to F$ be a map in *SPSP*. We will first consider factorizations $f = g \circ h$, where either h has as underlying map the identity on S (3.1) or g has as underlying map the identity on T (3.2, 3.3).

3.1. PROPOSITION: *Let $E \overset{f}{\to} F$ be as above. Then f has a unique factorization*

$$E \overset{i}{\to} f^*(F) \overset{e}{\to} F$$

with $i = (i, i^+)$ and i is the identity on S, and $e = (e, e^+)$ with e^+ being the unit η_F: $F \to f_* f^*(F)$. Moreover, this factorization is universal among factorizations $E \overset{g}{\to} G \overset{h}{\to} F$ of f which have $g = id_S$ as underlying map of $g = (g, g^+)$. That is, for such a factorization there exists a unique SPSP-morphism $G \to f^*(F)$ making the diagram

commute.

PROOF: This is almost immediate from the adjunction of Theorem 1.5. $\hspace{0.5cm}$ □

$\hspace{1cm}$ Factorizations $E \to G \to F$ of $E \to F$ which have id_T as underlying map of $G \to F$ split into two types, related to subspaces and quotient-spaces, respectively.

If $(E, \Gamma) \overset{p}{\to} T$ is a space in $Sh(T)$, a *subspace* of it in $Sh(T)$ is a subsheaf of $E \overset{p}{\to} T$ with the subspace topology inherited from Γ. An internal continuous function f in $Sh(T)$, as in

$$
\begin{array}{ccc}
(E, \Gamma) & \overset{f}{\longrightarrow} & (F, \Delta) \\
& \searrow_{p} \quad \swarrow_{q} & \\
& T &
\end{array}
\tag{1}
$$

can be factored as

$$
\begin{array}{ccccc}
(E, \Gamma) & \longrightarrow & (f(E), \Delta_{f(E)}) & \hookrightarrow & (F, \Delta) \\
& \searrow_{p} & \downarrow {\scriptstyle q \restriction f(E)} & \swarrow_{q} & \\
& & T & &
\end{array}
\tag{2}
$$

where $f(E)$ has the subspace topology inherited from F, which makes $q \restriction f(E)$ into a local homeomorphism since f is necessarily an open map $E \to F$; further, $\Delta_{f(E)}$ is the subspace topology on $f(E)$ inherited from Δ. It is clear that $(f(E), \Delta_{f(E)}) \overset{q \restriction f(E)}{\to} T$ is the *smallest subspace* of $(F, \Delta) \overset{q}{\to} T$ in $TOP(Sh(T))$ through which f factors.

An internal continuous function also has a quotient-factorization: let f be as in (1) above, and let $K(f) = \{(e, e') \in E \times E \mid fe = fe'\}$ be the kernel-relation on E induced by f. Since f is a local homeomorphism as a function $E \to F$, the projection $E/_{K(f)} \overset{\bar{p}}{\to} T$ from the space $E/_{K(f)}$ with the quotient-topology is again a local homeomorphism, where $\bar{p}$ is the unique function making the diagram (3) commute (with π the quotient-projection).

$$
\begin{array}{ccc}
E & \overset{\pi}{\longrightarrow} & E/_{K(f)} \\
& \searrow_{p} \quad \swarrow_{\bar{p}} & \\
& T &
\end{array}
\tag{3}
$$

The quotient-topology $\Gamma/_{K(f)}$ on $E/_{K(f)}$ inherited from Γ,

$$\Gamma/_{K(f)} = \left\{ U \in E/_{K(f)} | U \text{ is open and } \pi^{-1}(U) \in \Gamma \right\},$$

is a $\bar{p}$-topology on $E/_{K(f)}$. Thus we obtain a factorization

$$(4)$$

of f. $(E/_{K(f)}, \Gamma/_{K(f)}) \xrightarrow{\bar{p}} T$ is the *largest quotient* of (E, Γ) through which f factors.

Applying the adjunction Theorem 1.5 to these two factorizations, we obtain Propositions 3.2 and 3.3.

3.2. PROPOSITION: *Let $E = ((E, \Gamma) \xrightarrow{q} S)$ and $F = ((F, \Delta) \xrightarrow{r} T)$ be spaced spaces, and let $f: E \to F$ be an SPSP-morphism. Then there exists a unique factorization $E \xrightarrow{i} f(F) \xrightarrow{a} F$ of f such that*

   (*i*) $a = (a, a^+)$ *has as underlying function* $a = id_T$,
   (*ii*) i^+ *is a topological embedding (in $TOP(Sh(T))$), and such that the following universality-property holds:*

   (*iii*) *if $E \xrightarrow{g} G \xrightarrow{b} F$ is another factorization of f with $b = id_T$ and g^+ a topological embedding in $TOP(Sh(T))$, then there exists a unique SPSP-morphism $G \to f(F)$ such that the diagram below commutes.*

3.3. PROPOSITION: *Let f be as above. Then there exists a unique factorization $E \xrightarrow{r} (F)_f \xrightarrow{p} F$ of f such that*

   (*i*) $p = (p, p^+)$ *has as underlying map* $p = id_T$
   (*ii*) p^+ *is an internal quotient-mapping in $TOP(Sh(T))$ and such that the following universality-property holds:*

   (*iii*) *if $E \xrightarrow{g} G \xrightarrow{b} F$ is another factorization of f with $b = id_T$ underlying*

b and *b*$^+$ *an internal quotient-mapping, then there exists a unique SPSP-morphism* $(F)_f \to G$ *such that the diagram*
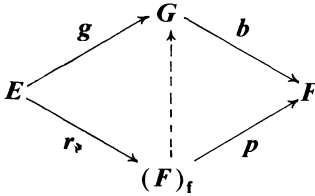
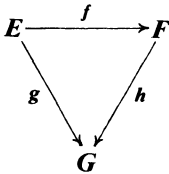*commutes.*   □

Another consequence of the change of base-adjunction, including 3.1 as a special case, is the following. Suppose we have internal spaces $E = ((E, \Gamma) \overset{p}{\to} S)$ and $F = ((F, \Delta) \overset{q}{\to} T)$, and let $f = (f, f^+)$ be a morphism $E \to F$. If

$$S \overset{g}{\to} R \overset{h}{\to} T$$

is a factorization of f, we say that this factorization can be *lifted* if there exists an internal space $G = ((G, \Sigma) \overset{r}{\to} R)$ over R, and there are g^+ and h^+ such that

commutes, with $g = (g, g^+)$, $h = (h, h^+)$ morphisms in *SPSP*. G is called a *lifting* of the factorization $h \circ g$.

3.4. PROPOSITION: *Let* $E \overset{f}{\to} F$ *be as above. Then every factorization* $S \overset{g}{\to} R \overset{h}{\to} T$ *of* $S \overset{f}{\to} T$ *can be lifted. Among the liftings of this factorization,* $g_*(E)$ *is initial, and* $h^*(F)$ *is terminal.*

PROOF: The proof is a straightforward application of 1.5.   □

4. Limits and colimits of spaced spaces

In the construction of equalizers in *SPSP*, we will need coequalizers in $TOP(Sh(T))$, for a fixed base space T. These coequalizers may be constructed as in the proof of the following lemma.

4.1. LEMMA: *Let* T *be a fixed base space. Then* $TOP(Sh(T))$ *has coequalizers.*

PROOF: Let $(E, \Gamma) \xrightarrow{p} T$ and $(F, \Delta) \xrightarrow{q} T$ be two internal spaces, and suppose that we have two morphisms f and g which are internally continuous,

We construct a quotient of (F, Δ) in the standard way. For $x, y \in F$, define

$$x \sim y \quad \text{iff} \quad \exists e \in E(x = f(e) \& y = g(e), \quad \text{or}$$

$$x = g(e) \& y = f(e)).$$

Then let

$$x \sim^n y \quad \text{iff} \quad \exists z_1, \ldots, z_n \quad x \sim z_1 \sim .. \sim z_n \sim y,$$

and define the equivalence relation $\approx$ by

$$x \approx y \quad \text{iff} \quad \exists n \, x \sim^n y, \quad \text{or} \quad x = y.$$

Let $F/_\approx$ be the quotient space of F (w.r.t. the topology on F which makes q a local homemorphism), and let $\Delta/_\approx$ be the quotient-topology on $F/_\approx$ w.r.t. the topology Δ on F. Write π for the canonical projection $F \to F/_\approx$. Clearly, $\approx$ only identifies points which are in the same fiber of q, i.e. $x \approx y$ implies $qx = qy$. Hence we can find a map $\bar{q} : F/_\approx \to T$ such that $\bar{q} \circ \pi = q$.

The only perhaps non-trivial thing that has to be shown is that $\bar{q} : F/_\approx \to T$ is a local homemorphism. This is seen as follows. First, $\bar{q}$ is clearly continuous. $\bar{q}$ is also an open map, for if $W \subseteq F/_\approx$ is open, then $\bar{q}(W) = q\pi^{-1}(W)$ is an open subset of T, since q is open. To show that $\bar{q}$ is a homeomorphism when restricted to sufficiently small open sets, first note that π is an open map. For if $W \subseteq F$ is open, then

$$\pi^{-1}\pi(W) = \bigcup_{n \in \omega} W^n,$$

where $W^0 = W$, and $W^{n+1} = \{x | \exists y \in W^n x \sim y\}$. Since $W^{n+1} = gf^{-1}(W^n) \cup fg^{-1}(W^n)$ is open whenever W^n is (f and $g : E \to F$ are open) we find that $\pi^{-1}\pi(W)$ is open in F; hence $\pi(W)$ is open in $F/_\approx$. It is then easy to see that if U_x is a small nbd of $x \in F$, $\bar{q} \upharpoonright \pi(W_x) : \pi(W_x) \to \bar{q}\pi(U_x) = q(U_x)$ is $1 - 1$, hence a homeomorphism. $\square$

4.2. PROPOSITION: *SPSP has equalizers.*

PROOF: Let $\mathbf{E} = ((E, \Gamma) \overset{p}{\to} S)$ and $\mathbf{F} = ((F, \Delta) \overset{q}{\to} T)$ be two internal spaces, and let $\mathbf{f} = (f, f^+)$ and $\mathbf{g} = (g, g^+)$ be morhpisms from $\mathbf{E}$ to $\mathbf{F}$. The equalizer of $\mathbf{f}$ and $\mathbf{g}$ will be an internal space over the base space R, where

$$R \overset{j}{\hookrightarrow} S \underset{g}{\overset{f}{\rightrightarrows}} T$$

is an equalizer diagram in *TOP*. First construct the inverse image $j^*(E) = (j^*(E, \Gamma) \overset{j^*(p)}{\to} R)$ of E. We then have two morphisms from F to $f_* j_* j^*(E) \simeq g_* j_* j^*(E)$ over T, namely

$$(F, \Delta) \overset{f^+}{\to} f_*(E, \Gamma) \overset{f_*(\eta_E)}{\to} f_* j_* j^*(E, \Gamma)$$

and

$$(F, \Delta) \overset{g^+}{\to} g_*(E, \Gamma) \overset{g_*(\eta_E)}{\to} g_* j_* j^*(E, \Gamma).$$

Taking transposed morphisms along the adjunction (Theorem 1.5) we obtain two morphisms in *TOP(Sh(R))*:

$$j^*f^*(F) \overset{j^*(\hat{f}^+)}{\to} j^*(E), \quad \text{and} \quad j^*g^*(F) \overset{j^*(\hat{g}^+)}{\to} j^*(E).$$

Now let H be the coequalizer of $j^*(\hat{f}^+)$ and $j^*(\hat{g}^+)$ in *TOP(Sh(R))*; that is, $H = (j^*(E, \Gamma)/_\approx \overset{j^*(p)}{\to} R)$ is the quotient of $j^*(E)$ as constructed in Lemma 4.1. And let

$$j^+ : E \to j_* H$$

be the transposed of the canonical projection $j^*(E) \overset{\pi}{\to} H$.

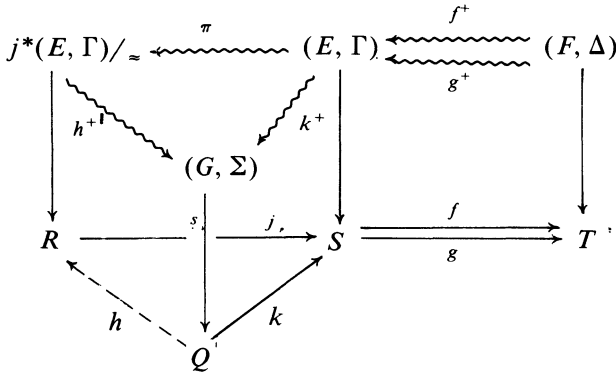   To see that

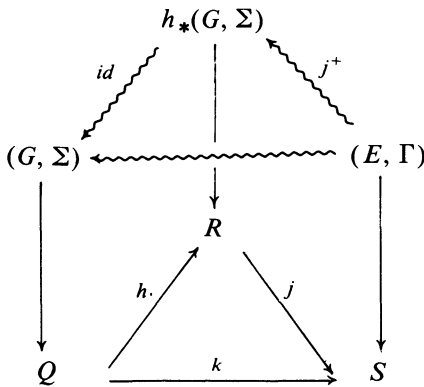$$H \overset{j}{\to} E \underset{g}{\overset{f}{\rightrightarrows}} F$$

is an equalizer diagram in *SPSP*, choose any internal space $G = ((G, \Sigma) \overset{s}{\to} Q)$, and let $\mathbf{k} = (k, k^+)$ be a morphism $G \to E$ such that $\mathbf{f} \circ \mathbf{k} = \mathbf{g} \circ \mathbf{k}$.

   (In the diagram below, a wavy arrow $(E, \Gamma) \rightsquigarrow (G, \Sigma)$ over $Q \overset{k}{\to} S$

indicates a sheaf-morphism $(E, \Gamma) \to k_*(G, \Sigma)$. Similarly for the other wavy arrows.)

Then $f \circ k = g \circ k$, so k factors through j by a unique $Q \xrightarrow{h} R$. This factorization can be lifted, as in the diagram

To show the existence of a (unique) morphism $j^*(E, \Gamma)/_{\approx} \to k_*(G, \Sigma)$, it suffices to show that the two compositions

$$j^*f^*(F, \Delta) \xrightarrow{j^*(\hat{f}^+)} j^*(E, \Gamma) \xrightarrow{\hat{h}^+} h_*(G, Z) \tag{1}$$

and

$$j^*g^*(F, \Delta) \xrightarrow{j^*(\hat{g}^+)} j^*(E, \Gamma) \xrightarrow{\hat{h}^+} h_*(G, \Sigma) \tag{2}$$

are equal, since

$$f*f*(F, \Delta) = j*g*(F, \delta) \underset{j*(\hat{g}^+)}{\overset{j*(\hat{f}^+)}{\rightrightarrows}} j*(E, \Gamma) \overset{\pi}{\to} j*(E, \Gamma)/_{\approx}$$

is a coequalizer in $TOP(Sh(R))$. Taking transposed morphisms of (1) and (2), we see that (1) = (2) iff

$$k*f*(F) \overset{k*(\hat{f}^+)}{\to} k*(E) \overset{\hat{k}^+}{\to} G$$

equals

$$k*f*(F) \overset{k*(\hat{g}^+)}{\to} k*(E) \overset{\hat{k}^+}{\to} G.$$

But this is immediate from the fact that $f \circ k = g \circ k$. So the equality of (1) and (2) is established.   □

To construct the binary product, take two internal spaces $E = ((E, \Gamma) \overset{p}{\to} S)$ and $F = ((F, \Delta) \overset{q}{\to} T)$. The product $E \times F$ will be an internal space over the base-space $S \times T$. First we define a presheaf P on $S \times T$ by letting $P(U \times V)$ be the disjoint union of the set of sections of p over U and the set of sections of q over V, i.e.

$$P(U \times V) = E(U) \amalg F(V).$$

P as defined on canonical basic open subsets of $S \times T$ is separated, hence by glueing together elements of the $P(W)$ (W a basic open) we obtain its sheafification $\tilde{P}$. The stalkspace of $\tilde{P}$ is denoted by $E \otimes F$, and the local homeomorphism by $E \otimes F \overset{p \otimes q}{\to} S \times T$. The projections π_1^+ and π_2^+ are the adjuncts of the morphisms

$$\hat{\pi}_1^+ : \pi_1^* \left(E \overset{p}{\to} S \right) \to \left(E \otimes F \overset{p \otimes q}{\to} S \times T \right)$$

$$\hat{\pi}_2^+ : \pi_2^* \left(F \overset{q}{\to} S \right) \to \left(E \otimes F \overset{p \otimes q}{\to} S \times T \right).$$

$\hat{\pi}_1^+$ is defined on sections as follows. Note that

$$
\begin{array}{ccc}
E \times T & \xrightarrow{\ \pi\ } & E \\
\downarrow {\scriptstyle p \times 1} & & \downarrow {\scriptstyle p} \\
S \times T & \xrightarrow{\ \pi\ } & S
\end{array}
$$

is pullback. So a section of $\pi_1^*(E \xrightarrow{p} T)$ over a basic open $U \times V$ corresponds to a continuous function $U \times V \xrightarrow{q} E$ such that $p \circ a = \pi_1$. $(\hat{\pi}_1^+)_{U \times V}(a)$ has to be a section of $p \otimes q$ over $U \times V$; that is, it must locally be either a section of $E \xrightarrow{p} S$ or a section of $F \xrightarrow{q} T$. It is possible to define such a section, since the function $U \times V \xrightarrow{a} E$ locally only depends on the first coordinate! To see this, choose a point $(s_0, t_0) \in U \times V$, and let $U_e \subseteq E$ be a small nbd of $e = a(s_0, t_0)$. Choose nbds U_{s_0} and V_{t_0} of s_0 and t_0 such that $a(U_{s_0} \times V_{t_0}) \subseteq U_e$. Then

$$\forall s \in U_{s_0} \exists ! x \in U_e \, p(x) = s$$

hence $a \upharpoonright (U_{s_0} \times V_{t_0})$ only depends on s, that is,

$$\forall s \in U_{s_0} \forall t, t' \in U_{t_0} a(s, t) = a(s, t').$$

Therefore we can define $(\hat{\pi}_1^+)_{U \times V}(a)$ locally as the function $U \to E$ whose restriction to U_{s_0} is defined by

$$\left(\hat{\pi}_1^+\right)_{U \times V}(a)(s) = a(s, t_0), \quad s \in U_{s_0}.$$

The definition of $\hat{\pi}_2^+$ is similar. Finally, let $\Gamma \otimes \Delta$ be the finest $p \otimes q$-topology on $E \otimes F$ making both $\hat{\pi}_1^+$ and $\hat{\pi}_2^+$ continuous (cf. Proposition 2.4). This completes the construction of the product and its projections.

To see that it has the required universal property, take a space $G = ((G, \Sigma) \xrightarrow{r} R)$, together with morphisms $f = (f, f^+)$ and $g = (g, g^+)$ from G to E and from G to F. Then f and g define a continuous function $R \xrightarrow{(f, g)} S \times T$, while the function $(f, g)^+ : E \otimes F \to (f, g)_*(G)$ is defined on sections by the components

$$(f, g)_{U \times V}^+ : (E \otimes F)_{U \times V} \to (f, g)_*(G)(U \times V),$$

or equivalently,

$$P(U \times V) \to G\left(f^{-1}(U) \cap g^{-1}(V)\right)$$

by setting for a section $a \in E(U)$,

$$(f, g)^+_{U \times V}(a) = f^+_U(a) \restriction g^{-1}(V)$$

and for a section $b \in F(V)$,

$$(f, g)^+_{U \times V}(b) = g^+_V(b) \to f^{-1}(U).$$

This defines a morphism $(f, g) = ((f, g), (f, g)^+)$. It is straightforward to check that (f, g) is internally continuous, and that it is the unique one with $\pi_1 \circ (f, g) = f$, and $\pi_2 \circ (f, g) = g$.

Thus we have proved

4.3. PROPOSITION: *SPSP has binary products.*   □

Clearly, *SPSP* has a terminal object. The construction of products over an arbitrary index set proceeds completely analogous to that of binary products, and is omitted. Summarizing the discussion of limits, we get

4.4. THEOREM: *SPSP is a small-complete category.*   □

The construction of colimits is in most respects dual to the construction of limits, but easier. We will just briefly sketch the construction of coequalizers and (binary) coproducts in *SPSP*.

The coequalizer of a pair of morphisms $E \underset{g}{\overset{f}{\rightrightarrows}} F$ (with $E = ((E, \Gamma) \overset{p}{\to} S)$

and $F = ((F, \Delta) \overset{q}{\to} T))$ is constructed by first taking the coequalizer of f and g in *TOP*, say $S \underset{g}{\overset{f}{\rightrightarrows}} T \overset{c}{\to} R$. We then construct the direct image

$c_*(F) = c_*(F, \Delta) \overset{c_*(q)}{\to} R$ over R, and let $(K, \Sigma) \to R$ be the equalizer of

$c_*(F, \Delta) \overset{c_*(f^+)}{\to} c_* f_*(E, \Gamma)$ and $c_*(F, \Delta) \overset{c_*(g^+)}{\to} c_* g_*(E, \Gamma)$ in

$TOP(Sh(R))$. (To construct the latter, you need to observe that if $E \underset{g}{\overset{f}{\rightrightarrows}} F$ are any two morphisms of sheaves over R, the set $\{e \mid fe = ge\}$ is an open subset of E.)
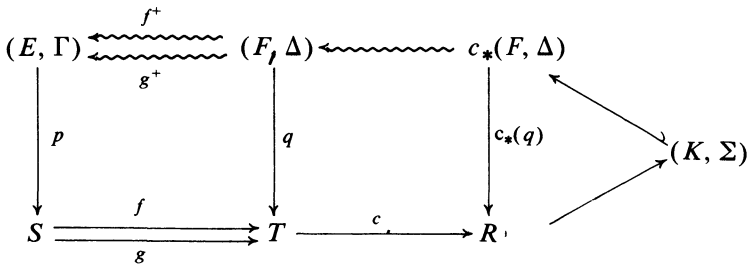
The coproduct $E \otimes F$ of $E = (E,\, \Gamma) \overset{p}{\to} S)$ and $F = ((F,\, \Delta) \overset{q}{\to} T)$ is the sheaf $E \otimes F$ over the topological sum $S \otimes T$ which has a sections over an open $W \subseteq S \otimes T$ the set of pairs $(a,\, b)$ with a a section of E over $W \cap S$, and b a section of F over $W \cap T$; that is,

$$E \otimes F(W) = E(W \cap S) \times F(W \cap T).$$

The injections $\mathbf{i}_k = (i_k^+,\, i_k)$ $(k = 1,\, 2)$ are defined as follows. $i_1 \colon S \hookrightarrow S \otimes T$ and $i_2 \colon T \hookrightarrow S \otimes T$ are the ordinary topological embeddings, and $i_1^+ \colon E \otimes F \to i_{1*}(E,\, \Gamma)$ is defined on sections by the components

$$(i_1^+)_W \colon E \otimes S(W) \to i_{1*}(E)(W) = E(W \cap S)$$

which are the ordinary projections. The $p \otimes q$-topology on $E \otimes F$ is the coarsest topology making both injections continuous.

If $G = ((G,\, \Sigma) \overset{r}{\to} R)$ is another space, and we have morphisms $f \colon E \to G$ and $g \colon F \to G$, the unique morphism $[f,\, g] = ([f,\, g],\, [f,\, g]^+) \colon E \otimes F \to G$ such that $[f,\, g] \circ i_1 = f$, $[f,\, g] \circ i_2 = g$ has as base-function the function $[f,\, g] \colon S \otimes T \to T$, and as sheaf-morphism

$$[f,\, g]^+ \colon (G,\, \Sigma) \to [f,\, g]_*(E \otimes F).$$

The morphism which is defined on sections by the components

$$[f,\, g]_U^+ \colon GU \to E(f^{-1}(U)) \times F(g^{-1}(U))$$

which are just the product-maps $\langle f_U^+,\, g_U^+ \rangle$.

Arbitrary coproduct are constructed similarly. In this way we obtain

4.5. Theorem: *SPSP is a small-cocomplete category.*   □

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Department of Mathematics
University of Amsterdam
Roetersstraat 15
1018 WB Amsterdam
The Netherlands