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THE RANK OF ÉTALE COHOMOLOGY OF VARIETIES OVER p-ADIC OR NUMBER FIELDS

C. Soulé

Let K be a number field or its completion at a finite prime, p a rational prime, and V a smooth and proper variety over K. In this paper we study the corank of the groups $H^{\ell}(V, \mathbb{Q}_p/\mathbb{Z}_p(i))$, $\ell \geqslant 0$, $i \in \mathbb{Z}$, of étale cohomology of V with coefficients $\mathbb{Q}_p/\mathbb{Z}_p$ twisted i-times by the cyclotomic character.

If $\bar{V}$ denotes the extension of V to an algebraic closure $\bar{K}$ of K (imbedded in the field $\mathbb{C}$ of complex numbers), Grothendieck showed that the group $H^{\ell}(\bar{V}, \mathbb{Q}_p/\mathbb{Z}_p(i))$ is equal to the ordinary (singular) cohomology group $H^{\ell}(\bar{V}(\mathbb{C}), \mathbb{Q}_p/\mathbb{Z}_p)$ of the set of complex points of $\bar{V}$ with constant coefficients $\mathbb{Q}_p/\mathbb{Z}_p$. Therefore the computation of the cohomology of V is a problem of Galois descent, which is of arithmetic nature. We cannot solve this problem in general, but what we show is that *for all but finitely many* integers i, the corank of $H^{\ell}(V, \mathbb{Q}_p/\mathbb{Z}_p(i))$ admits a simple ("geometric") expression.

In the global case this result was suggested by conjectures in algebraic K-theory due to D. Quillen and A. Beilinson (which are shown here to be compatible, cf. I.7.). In the local case the Hodge-Tate decomposition of the cohomology of $\bar{V}$ ([19], [2]) tells us which values of i it is sufficient to avoid.

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1. The global case

1.1. Statement of the result

1.1.1. Notations

Let p be a rational prime, and A an abelian group. We denote by $A[p]$ (resp. A/p) the kernel (resp. cokernel) of the multiplication by p in A.

Let $\mathrm{Div}(A)$ be the maximal divisible subgroup of A. When A is a p-torsion group, we define its dimension (or corank) to be the dimension of the vector space $\mathrm{Div}(A)[p]$ over the field $\mathbb{F}_p$ of order p:

$$\dim A = \dim_{\mathbb{F}_p} \mathrm{Div}(A)[p].$$

We also denote by

$$A^{\wedge} = \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$$

the Pontryagin dual of A.

Given a scheme X such that all its residue fields contain $1/p$ and an integer $n \geqslant 1$, we denote by μ_{p^n} the étale sheaf over X over p^n-th roots of unity and

$$\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$$

their projective limit. When M is an étale sheaf of p-torsion groups over X we define the Tate twists $M(i)$, $i \in \mathbb{Z}$, of M to be

$$M(0) = M$$

$$M(i) = M(i-1)_{\mathbb{Z}_p} \otimes \mathbb{Z}_p(1) \quad \text{when} \quad i \geqslant 1,$$

$$M(i) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(-i), M) \quad \text{when} \quad i < 0.$$

For instance, when $i \in \mathbb{Z}$, the sheaf $\mathbb{Q}_p/\mathbb{Z}_p(i)$ is the inductive limit of the finite sheaves $(\mathbb{Z}/p^n)(i)$, the transition maps being induced by the standard inclusions $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1}$. The fact that inductive limits preserve exact sequences leads us to take $\mathbb{Q}_p/\mathbb{Z}_p(i)$ as coefficients (instead of $\mathbb{Z}_p(i)$) in the Theorem 1 below.

1.1.2. Let K be a number field, $\bar{K}$ an algebraic closure of K, G the Galois group $G = \mathrm{Gal}(\bar{K}/K)$. Let V be a proper smooth variety defined over K, and $\bar{V} = V \otimes_K \bar{K}$ the variety over $\bar{K}$ obained by extending scalars.

For any archimedean prime v of K, let K_v be the completion of K at v, and choose an imbedding v: $K_v \to \mathbb{C}$ of K_v into the complex numbers. Let $V_v = V \otimes_K K_v$ and $V_v(\mathbb{C})$ be the set of complex points of the K_v-variety V_v. For any integer $k \geqslant 1$, we denote by $H^{\ell-1}(V_v(\mathbb{C}), \mathbb{R})$ the group of ordinary (singular) cohomology of the topological space $V_v(\mathbb{C})$ with real coefficients. The Galois group $\mathrm{Gal}(\mathbb{C}/K_v)$ acts upon $H^{\ell-1}(V_v(\mathbb{C}), \mathbb{R})$ and for any integer $i \in \mathbb{Z}$ we define $H^{\ell-1}(V_v(\mathbb{C}), \mathbb{R})^{(i-1)}$ to be $H^{\ell-1}(V_v(\mathbb{C}), \mathbb{R})$ when $K_v \simeq \mathbb{C}$, and the eigenspace of eigenvalue $(-1)^{(i-1)}$ of the generator of $\mathrm{Gal}(\mathbb{C}/K_v)$ when $K_v \simeq \mathbb{R}$.

If S_∞ is the set of archimedean primes of K, $\ell \geqslant 1$ and $i \in \mathbb{Z}$, we define

$$n_{i,k} = \sum_{v \in S_\infty} \dim_{\mathbb{R}} H^{\ell-1}(V_v(\mathbb{C}), \mathbb{R})^{(i-1)}.$$

On the other hand we consider the groups of étale cohomology

$$H^{\ell}(V, \mathbb{Q}_p/\mathbb{Z}(i)) = \varinjlim_n H^{\ell}(V, \mathbb{Z}/p^n(i))$$

defined in 1.1.1.

THEOREM 1: *Let V be as above, $\ell \geqslant 1$ an integer, and p a rational prime.*
(i) *For any integer $i \in \mathbb{Z}$ we have* $\dim H^{\ell}(V, \mathbb{Q}_p/\mathbb{Z}_p(i)) \geqslant n_{i,\ell}$.
(ii) *For almost all integer $i \in \mathbb{Z}$ we have* $\dim H^{\ell}(V, \mathbb{Q}_p/\mathbb{Z}_p(i)) = n_{i,\ell}$.

1.2. To prove Theorem 1 we first show that we can assume $K = \mathbb{Q}$. For let π: $V \to \operatorname{Spec} K$ be the morphism of schemes defining V and $\tilde{V}$ the variety over $\mathbb{Q}$ obtained by composing π with the map

$$\operatorname{Spec} K \to \operatorname{Spec} \mathbb{Q}.$$

Since the étale cohomology does not depend on the field of definition we have

$$H^{\ell}(V, \mathbb{Q}_p/\mathbb{Z}_p(i)) = H^{\ell}(\tilde{V}, \mathbb{Q}_p/\mathbb{Z}_p(i)).$$

On the other hand the numbers $n_{i,\ell}$ are the same for V/K and $\tilde{V}/\mathbb{Q}$. To see this, for any imbedding σ: $K \to \mathbb{C}$, denote by $V \otimes_\sigma \mathbb{C}$ the complex variety obtained from V by extending scalars to $\mathbb{C}$ through σ. We have

$$\tilde{V} \underset{\mathbb{Q}}{\otimes} \mathbb{C} = V \underset{K}{\otimes} \left(K \underset{\mathbb{Q}}{\otimes} \mathbb{C} \right) \simeq \prod_{\sigma} \left(V \underset{\sigma}{\otimes} \mathbb{C} \right)$$

where the product is taken over all imbeddings of K into $\mathbb{C}$. A complex valuation v of K gives two imbeddings σ_v and $c\sigma_v$ of K_v into $\mathbb{C}$ (where c denotes the complex conjugation) and complex conjugation acting upon $\tilde{V} \otimes_{\mathbb{Q}} \mathbb{C}$ permutes the two factors $V \otimes_{\sigma_v} \mathbb{C}$ and $V \otimes_{c\sigma_v} \mathbb{C}$. When v is a real valuation, there is only one imbedding σ_v: $K_v \to \mathbb{C}$ attached to it, and the complex conjugation acting upon $\tilde{V} \otimes_{\mathbb{Q}} \mathbb{C}$ induces the action of $\operatorname{Gal}(\mathbb{C}/K_v)$ on the factor $V \otimes_{\sigma_v} \mathbb{C}$. Therefore, for any $i \in \mathbb{Z}$, we have

$$H^{\ell-1}(\tilde{V}(\mathbb{C}), \mathbb{R})^{(i-1)} \simeq \sum_{v \in S_\infty} H^{\ell-1}(V_v(\mathbb{C}), \mathbb{R})^{(i-1)}.$$

1.3. From now on we will assume $K = \mathbb{Q}$. Let S be a finite set of rational primes containing p and such that V has good reduction outside S. That is to say, if we call $\mathbb{Z}_S$ the ring of rational numbers which are units outside S, there exist a smooth proper scheme $\mathscr{V}$ over $\mathrm{Spec}(\mathbb{Z}_S)$ with generic fiber $\mathscr{V} \otimes_{\mathbb{Z}_S} \mathbb{Q}$.

We fix an integer $k \geqslant 1$ and let $M = H^{k-1}(\bar{V}, \mathbb{Q}_p/\mathbb{Z}_p)$. The module M is a discrete module under the Galois group $G = \mathrm{Gal}(\bar{K}/K)$, and we can also consider M as an étale sheaf over $\mathrm{Spec}(\mathbb{Q})$ [5].

Let j: $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z}_S)$ be the canonical inclusion and let j_*M be the direct image of M.

PROPOSITION 1:

(i) *For any* $i \in \mathbb{Z}$ *we have* $\dim H^r(\mathrm{Spec}\,\mathbb{Q}, M(i)) = 0$ *when* $r \geqslant 3$ *and* $\dim H^1(\mathrm{Spec}\,\mathbb{Q}, M(i)) \geqslant \dim H^1(\mathrm{Spec}\,\mathbb{Z}_S, j_*M(i))$.

(ii) *Let* $i \in \mathbb{Z}$ *be an integer such that* $k \neq 2i-1$ *and* $H^2(\mathrm{Spec}\,\mathbb{Z}_S, j_*M(i))$ *is finite. Then for any integer* $r \geqslant 0$ *we have*

$$\dim H^r(\mathrm{Spec}\,\mathbb{Q}, M(i)) = \dim H^r(\mathrm{Spec}\,\mathbb{Z}_S, j_*M(i)).$$

PROOF: It follows from [21], Theorem 3.1 c) that $\dim H^r(\mathrm{Spec}\,\mathbb{Q}, M(i)) = 0$ when $r \geqslant 3$. Let M_n be the finite module $H^{k-1}(\bar{V}, \mathbb{Z}/p^n)$, so that

$$H^k(\mathrm{Spec}\,\mathbb{Q}, M(i)) = \varinjlim_n H^k(\mathrm{Spec}\,\mathbb{Q}, M_n(i)) \quad \text{and}$$

$$H^k(\mathrm{Spec}\,\mathbb{Z}_S, M(i)) = \varinjlim_n H^k(\mathrm{Spec}\,\mathbb{Z}_S, j_*M_n(i)).$$

The Leray spectral sequence for the functor j_* and the sheaf $M_n(i)$ has E_2-term

$$E_2^{rs} = H^r(\mathrm{Spec}\,\mathbb{Z}_S, R^s j_* M_n(i))$$

and converges to $H^{r+s}(\mathrm{Spec}\,\mathbb{Q}, M_n(i))$.

For any prime l not in S, let $\mathbb{F}_l$ be the field with l elements and

$$i_l: \mathrm{Spec}\,\mathbb{F}_l \to \mathrm{Spec}\,\mathbb{Z}_S$$

the canonical inclusion. When $s \geqslant 1$ the morphism of étale sheaves over $\mathrm{Spec}\,\mathbb{Z}_S$

$$R^s j_* M_n(i) \to \bigoplus_{l \notin S} i_{l*} i_l^* R^s j_* M_n(i)$$

is an isomorphism, as can be seen by looking at the geometric fibers.

Since i_l is a finite morphism we get ([5], II.3., Proposition 3.6.) $H^r(\mathrm{Spec}\,\mathbb{Z}_S, i_{l*}i_l^*R^sj_*M_n(i)) \simeq H^r(\mathrm{Spec}\,\mathbb{F}_l, i_l^*R^sj_*M_n(i))$. Let $\mathbb{Q}_l$ be the field of l-adic numbers and $\mathbb{Q}_l^{nr}$ its maximal unramified extension. Then the sheaf $i_l^*R^sj_*M_n(i))$, considered as a $\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)$-module, is isomorphic to $H^s(\mathbb{Q}_l^{nr}, M_n(i))$, where $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l^{nr})$ acts upon M_n through its inclusion in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The smooth and proper base change theorem ([5], .3., Theorem 3) asserts that the action of the inertia group $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l^{nr})$ on M_n is trivial, and that there is an isomorphism of $\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)$-modules between M_n and $H^{\ell-1}(\overline{W}_l, \mathbb{Z}/p^n)$, where $W_l = \mathscr{V} \otimes_{\mathbb{Z}_S} \mathbb{F}_l$ is the reduction of $\mathscr{V}$ modulo l and $\overline{W}_l = W_l \otimes_{\mathbb{Z}_S} \overline{\mathbb{F}}_l$ is the extension of W_l to an algebraic closure of $\mathbb{F}_l$. Therefore we get isomorphisms of $\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)$-modules

$$H^s(\mathbb{Q}_l^{nr}, M_n(i)) \simeq H^s(\mathbb{Q}_l^{nr}, \mu_{p^n}) \otimes M_n(i-1)$$

$$= \begin{cases} H^{\ell-1}(\overline{W}_l, \mathbb{Z}/p^n)(i-1) & \text{when } s=1 \\ 0 & \text{when } s>1 \end{cases}$$

(compare [16], III.1.3.).

Since $cd_p(\mathbb{F}_l) = 1$, the Leray spectral sequence for j_* and $M_n(i)$ gives an exact sequence

$$0 \to H^1(\mathrm{Spec}\,\mathbb{Z}_S, j_*M_n(i)) \to H^1(\mathrm{Spec}\,\mathbb{Q}, M_n(i))$$

$$\to \bigoplus_{l \notin S} H^0(\mathbb{F}_l, H^{\ell-1}(\overline{W}_l, \mathbb{Z}/p^n)(i-1)) \to H^2(\mathrm{Spec}\,\mathbb{Z}_S, j_*M_n(i))$$

$$\to H^2(\mathrm{Spec}\,\mathbb{Q}, M_n(i)) \to \bigoplus_{l \notin S} H^1(\mathbb{F}_l, H^{\ell-1}(\overline{W}_l, \mathbb{Z}/p^n)(i-1)).$$

Taking the inductive limits over n of these exact sequences we see that to get Proposition 1 it will be enough to show the following

LEMMA 1: *When $2i \neq \ell + 1$ the groups*

$$H^0(\mathbb{F}_l, H^{\ell-1}(\overline{W}_l, \mathbb{Q}_p/\mathbb{Z}_p)(i-1)) \quad \text{and}$$

$$H^1(\mathbb{F}_l, H^{\ell-1}(\overline{W}_l, \mathbb{Q}_p/\mathbb{Z}_p)(i-1))$$

are finite.

PROOF OF LEMMA 1: Since $\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)$ is isomorphic to the profinite completion of $\mathbb{Z}$, when T is a torsion discrete $\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)$-module, the group $H^0(\mathbb{F}_l, T)$ is equal to the group of invariants $T^{\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)}$ of T by $\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)$, and $H^1(\mathbb{F}_l, T)$ is equal to the group of coinvariants $T_{\mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)}$.

Since the cohomology of $\overline{W}_l$ with $\mathbb{Z}_p$-coefficients is finitely generated, we have

$$\operatorname{Div} H^{\ell-1}\left(\overline{W}_l, \mathbb{Q}_p/\mathbb{Z}_p\right) = H^{\ell-1}\left(\overline{W}_l, \mathbb{Z}_p\right) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

where $H^{\ell-1}(\overline{W}_l, \mathbb{Z}_p) = \varprojlim_n H^{\ell-1}(\overline{W}_l, \mathbb{Z}/p^n)$. Let $\phi \in \operatorname{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)$ be the arithmetic Frobenius. It will be enough to show that the endomorphism of $H^{\ell-1}(\overline{W}_l, \mathbb{Q}_p(i-1)) = H^{\ell-1}(\overline{W}_l, \mathbb{Z}_p) \otimes \mathbb{Q}_p(i-1)$ induced by ϕ has no fixed vector. But the Weil conjectures proved by Deligne [4] tell us that any eigenvalue λ of ϕ acting upon $H^{\ell-1}(\overline{W}_l, \mathbb{Q}(i-1))$ is an algebraic number whose archimedean absolute values are equal to $l^{(-\ell+1+2(i-1))/2}$. Therefore λ is different from 1 when $2i \neq \ell + 1$. Q.E.D.

1.4. PROPOSITION 2: *Let M be as above. Then, for any $i \in \mathbb{Z}$,*

$$\sum_{r=0}^{2} (-1)^r \dim H^r(\operatorname{Spec} \mathbb{Z}_S, j_* M(i)) = -n_{i,\ell}$$

PROOF: We shall extend to our situation the arguments in [14], 4, Theorem 6. Let $\chi = \sum_{r=0}^{2}(-1)^r \dim H^r(\operatorname{Spec} \mathbb{Z}_S, j_* M(i))$. The module M is the direct sum of its divisible subgroup $A = \operatorname{Div}(M)$ with a finite group, and the cohomology groups of Spec $\mathbb{Z}_S$ with finite coefficients are finite, therefore we have

$$\chi = \sum_{r=0}^{2} (-1)^r \dim H^r(\operatorname{Spec} \mathbb{Z}_S, j_* A[i]).$$

Let $\mathbb{Q}_S \subset \overline{\mathbb{Q}}$ be the maximal extension of $\mathbb{Q}$ which is unramified outside S (and infinity), and $G_S = \operatorname{Gal}(\mathbb{Q}_S/\mathbb{Q})$ its Galois group over $\mathbb{Q}$. Then, given any finite $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module F, we have

$$H^r(\operatorname{Spec} \mathbb{Z}_S, j_* F) \simeq H^r\left(G_S, F^{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_S)}\right).$$

As we saw when proving Proposition 1, the module $j_* A(i)$ is unramified outside S, therefore

$$H^r(\operatorname{Spec} \mathbb{Z}_S, j_* A(i)) \simeq H^r(G_S, A(i)).$$

Now, the exact sequence of coefficients

$$0 \to A(i)[p] \to A(i) \overset{\times p}{\to} A(i) \to 0$$

gives a long exact sequence of cohomology groups:

$$0 \to H^0(G_S, A(i)[p]) \to H^0(G_S, A(i)) \xrightarrow{\times p} H^0(G_S, A(i))$$

$$\to H^1(G_S, A(i)[p]) \to \ldots \to H^2(G_S, A(i)) \xrightarrow{\times p} H^2(G_S, A(i))$$

$$\to H^2(G_S, A(i))/p \to 0.$$

From this we deduce:

$$\chi = \sum_{r=0}^{2} (-1)^r \dim_{\mathbb{F}_p} H^r(G_S, A(i)[p]) - \dim_{\mathbb{F}_p} H^2(G_S, A(i))/p.$$

By a theorem of Tate ([20], Theorem 2) we get

$$\sum_{r=0}^{2} (-1)^r \dim_{\mathbb{F}_p} H^r(G_S, A(i)[p])$$

$$= \dim_{\mathbb{F}_p} H^0(\mathbb{R}, A(i)[p]) - \dim_{\mathbb{F}_p} A(i)[p].$$

Furthermore, since the maps $H^3(G_S, A(i)[p]) \to H^3(\mathbb{R}, A(i)[p])$ and $H^3(G_S, A(i)) \to H^3(\mathbb{R}, A(i))$ are isomorphisms ([21], Thm. 3.1.c)) we see that the map

$$H^2(G_S, A(i))/p \to H^2(\mathbb{R}, A(i))/p = H^2(\mathbb{R}, A(i))$$

is an isomorphism. So we get

$$\chi = \dim_{\mathbb{F}_p} H^0(\mathbb{R}, A(i)[p]) - \dim A - \dim_{\mathbb{F}_p} H^2(\mathbb{R}, A(i))$$

$$= \dim H^0(\mathbb{R}, A(i)) + \dim_{\mathbb{F}_p} H^0(\mathbb{R}, A(i))/p$$

$$- \dim A - \dim_{\mathbb{F}_p} H^2(\mathbb{R}, A(i)).$$

Since $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$, the numbers

$$\dim_{\mathbb{F}_p} H^1(\mathbb{R}, A(i)[p])$$

$$= \dim_{\mathbb{F}_p} H^0(\mathbb{R}, A(i))/p + \dim_{\mathbb{F}_p} H^1(\mathbb{R}, A(i))$$

and

$$\dim_{\mathbb{F}_p} H^2(\mathbb{R}, A(i)[p]) = \dim_{\mathbb{F}_p} H^1(\mathbb{R}, A(i)) + \dim_{\mathbb{F}_p} H^2(\mathbb{R}, A(i))$$

are equal. Therefore we get

$$\chi = \dim H^0(\mathbb{R}, A(i)) - \dim A = -\dim H^0(\mathbb{R}, A(i-1)).$$

On the other hand, we have isomorphism of Gal($\mathbb{C}/\mathbb{R}$)-modules ([5], V, Cor. 3.3., and IV, Thm. 6.3.):

$$A = \operatorname{Div} H^{\ell-1}\left(\bar{V}, \mathbb{Q}_p/\mathbb{Z}_p\right) = \operatorname{Div} H^{\ell-1}\left(V \underset{\mathbb{Q}}{\otimes} \mathbb{C}, \mathbb{Q}_p/\mathbb{Z}_p\right)$$

$$= \operatorname{Div} H^{\ell-1}\left(V(\mathbb{C}), \mathbb{Q}_p/\mathbb{Z}_p\right) = H^{\ell-1}(V(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

(since the groups of ordinary cohomology if $V(\mathbb{C})$ with constant coefficients $\mathbb{Z}$ are finitely generated). So we have

$$\chi = -\dim H^0\left(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), H^{\ell-1}(V(\mathbb{C}), \mathbb{Z})(i-1) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right)$$

$$= -\dim_{\mathbb{R}} H^0\left(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), H^{\ell-1}(V(\mathbb{C}), \mathbb{R})(i-1)\right) = -n_{i,\ell}.$$

Q.E.D.

1.5. PROPOSITION 3: *For all but finitely many $i \in \mathbb{Z}$ the groups $H^0(\operatorname{Spec} \mathbb{Z}_S, j_* M(i))$ and $H^2(\operatorname{Spec} \mathbb{Z}_S, j_* M(i))$ are finite.*

PROOF:

(a) Let l be a rational prime outside S. As we saw in Proposition 1

$$H^0(\operatorname{Spec} \mathbb{Z}_S, j_* M(i)) = H^0(\operatorname{Spec} \mathbb{Q}, M(i)) \subset H^0(\mathbb{Q}_l, M(i))$$

$$= H^0\left(\mathbb{F}_l, H^{\ell-1}\left(\overline{W}_l, \mathbb{Q}_p/\mathbb{Z}_p(i)\right)\right),$$

and these groups are finite when $2i \neq \ell + 1$.

(b) Let $\mathbb{Q}(\mu_{p^n}) \subset \overline{\mathbb{Q}}$ be the cyclotomic extension of $\mathbb{Q}$ obtained by adding the p^n-th roots of unity, $n \geqslant 1$, and $\mathbb{Q}(\mu_{p^\infty}) = \bigcup_{n \geqslant 1} \mathbb{Q}(\mu_{p^n})$ be the maximal p-cyclotomic extension of $\mathbb{Q}$. The extension $\mathbb{Q}(\mu_{p^n})$ of $\mathbb{Q}$ is unramified outside p, therefore $\mathbb{Q}(\mu_{p^\infty})$ is contained in $\mathbb{Q}_S$ (since $p \in S$). Let

$$H = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}_\infty), \Delta = \mathrm{Gal}\left(\mathbb{Q}(\mu_{p^2})/\mathbb{Q}\right),$$

$$\Gamma = \mathrm{Gal}\left(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_{p^2})\right)$$

and $G_S' = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}(\mu_{p^2}))$.

The order of Δ is $p(p-1)$. The composite of the corestriction map

$$H^2(G_S, M(i)) \to H^2(G_S', M(i))$$

with the transfer map

$$H^2(G'_S, M(i)) \to H^2(G_S, M(i))$$

is the product by the cardinal of Δ. Therefore the kernel of

$$H^2(G_S, M(i)) \to H^2(G'_S, M(i))$$

is contained in $H^2(G_S, M(i))[p]$, which is a finite group. In fact it is a quotient of

$$H^2(G_S, (M/A)(i)[p]) \oplus H^2(G_S, A(i))[p],$$

where $A = \operatorname{Div} M$, and the cohomology groups of G_S with finite coefficients are finite. Therefore, to prove that the groups

$$H^2(\operatorname{Spec} \mathbb{Z}_S, j_* M(i)) = H^2(G_S, M(i))$$

are finite for almost all i, it will be enough to show that $H^2(G'_S, M(i))$ is finite (for almost all i).

For this let us consider the extension of groups

$$1 \to H \to G'_S \to \Gamma \to 1.$$

The Hochschild-Serre spectral sequence deduced from it has E_2-term

$$E_2^{rs} = H^r(\Gamma, H^s(H, M(i)))$$

and converges to $H^{r+s}(G'_S, M(i))$.

The cyclotomic character κ: $\operatorname{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \to \mathbb{Z}_p^*$ is an isomorphism and induces an isomorphism between Γ and the multiplicative group $1 + p^2\mathbb{Z}_p$, i.e. with the additive group $\mathbb{Z}_p$. Therefore $cd_p\Gamma = 1$. On the other hand Ferrero and Washington proved that the μ-invariant of $\mathbb{Q}(\mu_{p^\infty})$ is zero [6], and Iwasawa deduced from this that $cd_p H_p = 1$, where H_p is the maximal pro-p quotient group of H ([8], Theorem 2). Therefore $cd_p H = 1$ and $E_2^{rs} = 0$ when $r \geqslant 2$ or $s \geqslant 2$. So we have

$$H^2(G'_S, M(i)) = H^1(\Gamma, H^1(H, M(i))) = H^1(H, M)(i)_\Gamma$$

(since the action of H on the roots of unity is trivial).

Let $X = H^1(H, M)^\wedge$ be the Pontryagin dual of $H^1(H, M)$. We want to show that $X(-i)^\Gamma$ is finite for almost all i. Call

$$\Lambda = \mathbb{Z}_p[[\Gamma]] = \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$$

the $\mathbb{Z}_p$-algebra of the pro-p-group Γ. Choosing a generator γ_0 of Γ gives an isomorphism between Λ and the ring of powers series $\mathbb{Z}_p[[T]]$, which sends γ_0 to $1+T$. The module X is a Λ-module.

We first notice that X is a noetherian Λ-module. In fact Λ is a local domain, compact for the topology defined by its maximal ideal (p, T), and X is a compact Λ-module therefore ([7], 1.1.) it is enough, by Nakayama's lemma, to show that $X_\Gamma \otimes \mathbb{Z}/p$ is finite. But

$$(X_\Gamma \otimes \mathbb{Z}/p)^\wedge = \hat{X}^\Gamma[p] = H^1(H, M)^\Gamma[p].$$

Since M is the direct sum of its divisible subgroup $\mathrm{Div}(M)$ with a finite group, we have to show that $H^1(H, \mathrm{Div}\, M)^\Gamma[p]$ is finite; but this is a quotient of $H^1(G_S', M[p])$ (by the Hochschild-Serre spectral sequence), therefore it is finite.

The Proposition 3 will then be a consequence of the following

LEMMA 2: *Let X be a noetherian Λ-module. Then $X(-i)^\Gamma$ is finite for almost all integers $i \in \mathbb{Z}$.*

PROOF: It was proved by Iwasawa [7] that X is pseudo-isomorphic to a finite product of Λ-modules of the type $\Lambda/(f(T))$, where $f(T)$ is a polynomial in $\mathbb{Z}_p[T]$. Let $c = \kappa(\gamma_0) \in \mathbb{Z}_p^*$ be the image of γ_0 by the cyclotomic character. Then, when twisting the Λ-module $\Lambda/(f(T))$, we get

$$(\Lambda/(f(t)))(-i) \simeq \Lambda/\big(f\big(c^i(1+T)-1\big)\big).$$

The euclidean algorithm shows that $(\Lambda/(f(T)))(-i)^\Gamma$ is finite whenever $f=0$ or $f(c^i-1) \neq 0$ (cf. [12], Lemma 4.2). A nonzero polynomial having only finitely many roots, the Lemma 2 follows.

1.6. PROOF OF THEOREM 1: The Hochschild-Serre spectral sequence attached to the Galois covering $\bar{V} \to V$ has E_2-term

$$E_2^{rs} = H^r\big(\mathbb{Q}, H^s\big(\bar{V}, \mathbb{Q}_p/\mathbb{Z}_p(i)\big)\big)$$

and converges to $H^{r+s}(V, \mathbb{Q}_p/\mathbb{Z}_p(i))$. From Proposition 1 and 3 we know that, for almost all i, we have

$$\dim H^r\big(\mathbb{Q}, H^s\big(\bar{V}, \mathbb{Q}_p/\mathbb{Z}_p(i)\big)\big)$$

$$= \dim H^r\big(\mathrm{Spec}\,\mathbb{Z}_S, j_* H^s\big(\bar{V}, \mathbb{Q}_p/\mathbb{Z}_p(i)\big)\big)$$

and that this dimension is zero for $r \neq 1$. From Proposition 2 we get

$$\dim H^1\big(\mathbb{Q}, H^{\ell-1}\big(\bar{V}, \mathbb{Q}_p/\mathbb{Z}_p(i)\big)\big) = n_{i,\ell}.$$

Therefore, for almost all $i \in \mathbb{Z}$,

$$\dim H^{\ell}\big(V, \mathbb{Q}_p/\mathbb{Z}_p(i)\big) = \dim E_2^{1,\ell-1}$$

$$= \dim H^1\big(\mathbb{Q}, H^{\ell-1}\big(\bar{V}, \mathbb{Q}_p/\mathbb{Z}_p(i)\big)\big) = n_{i,\ell}.$$

This proves (ii). To prove (i) we notice that for any $i \in \mathbb{Z}$ we have

$$\dim H^{\ell}\big(V, \mathbb{Q}_p/\mathbb{Z}_p(i)\big) \geqslant \dim E_2^{1,\ell-1}$$

$$\geqslant \dim H^1(\operatorname{Spec} \mathbb{Z}_S, j_* M(i)) \geqslant n_{i,\ell}.$$

1.7. Connection with algebraic K-theory

The statement of Theorem 1 was motivated by the following conjectures in algebraic K-theory.

Let V be as in Theorem 1, let $K_m(V)$, $m \geqslant 0$, be the higher algebraic K-groups of the variety V [13], let $K_m(V; \mathbb{Z}/p^n)$ be the groups of algebraic K-theory with coefficients $\mathbb{Z}/p^n$ of V [3], and let

$$K_m\big(V; \mathbb{Q}_p/\mathbb{Z}_p\big) = \varinjlim_n K_m\big(V; \mathbb{Z}/p^n\big).$$

Let $Gr_\gamma^i K_m(V; \mathbb{Q}_p/\mathbb{Z}_p)$ be the i-th quotient of the γ-filtration of the K-theory of V [11].

The theory of p-adic Chern classes [16] gives morphisms

$$c_{i,\ell}\colon Gr_\gamma^i K_{2i-\ell}\big(V, \mathbb{Q}_p/\mathbb{Z}_p\big) \to H^{\ell}\big(V, \mathbb{Q}_p/\mathbb{Z}_p(i)\big), \quad i \geqslant 1,$$

and Quillen's conjecture [13] would imply that the kernel and the cokernel of $c_{i,\ell}$ have dimension zero when i is big enough.

On the other hand Beilinson [1] and Karoubi [9] defined transcendental Chern classes

$$\rho_{i,\ell}\colon \mathbb{R} \underset{\mathbb{Z}}{\otimes} Gr_\gamma^i K_{2i-\ell}(V) \to \bigoplus_{v \in S_\infty} H^{\ell-1}(V_v(\mathbb{C}), \mathbb{R})^{(i-1)}, \quad i \geqslant \ell,$$

which Beilinson expects to be isomorphisms when i is big enough [18].

Assuming that $K_m(V)$ does not contain any divisible subgroup, the Bockstein exact sequences

$$0 \to K_m(V) \otimes \mathbb{Z}/p^n \to K_m(V; \mathbb{Z}/p^n) \to K_{m-1}(V)[p^n] \to 0$$

will show that

$$\dim Gr_\gamma^i K_{2i-\ell}(V; \mathbb{Q}_p/\mathbb{Z}_p) = \dim_{\mathbb{R}} \mathbb{R} \underset{\mathbb{Z}}{\otimes} Gr_\gamma^i K_{2i-\ell}(V).$$

Therefore Theorem 1 expresses the compatibility of Beilinson's conjecture with Quillen's one.

1.9. It is hard to decide for which values of i the equality in Theorem 1 holds. For instance, assume V is a point. Then it is proved in [17], using algebraic K-theory, that

$$\dim H^\ell(K, \mathbb{Q}_p/\mathbb{Z}_p(i)) = n_{i,\ell} \quad \text{when} \quad i \geqslant 2.$$

When $i = 0$, the Leopoldt's conjecture asserts that

$$\dim H^1(K, \mathbb{Q}_p/\mathbb{Z}_p) = n_{0,1} + 1$$

([14], §7, Lemma 1). The case $i < 0$ is not completely understood [14].

2. The local case

2.1. Statement of the result

2.1.1. Notations

Let K be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, $[K:\mathbb{Q}_p]$ its degree, $\bar{K}$ an algebraic closure of K, C the completion of $\bar{K}$, $G = \mathrm{Gal}(\bar{K}/K)$ the Galois group of $\bar{K}$ over K, O_K the ring of integers of K, and k its residue field.

Assume V is a smooth proper variety over K with good reduction, i.e. such that there exists a smooth proper scheme $\mathscr{V}$ over Spec O_K whose generic fiber $\mathscr{V} \otimes_{O_K} K$ is isomorphic to V. We call $W = \mathscr{V} \otimes_K k$ the special fiber of $\mathscr{V}$.

Recall that when X is a scheme whose residual characteristics are different from p we define

$$H^\ell(X, \mathbb{Z}_p(i)) = \varprojlim_n H^\ell(X, \mathbb{Z}/p^n(i))$$

and

$$H^{\ell}(X, \mathbb{Q}_p(i)) = H^{\ell}(X, \mathbb{Z}_p(i)) \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p.$$

2.1.2. Hodge-Tate decomposition

The action of $G = \mathrm{Gal}(\overline{K}/K)$ upon $\overline{K}$ extends to C by continuity. Let G act upon the module $H^{\ell}(\overline{V}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$ diagonally. Call s.s. $(H^{\ell}(\overline{V}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C)$ the semi-simplification of this $\mathbb{Q}_p[G]$-module. In [19] (§4.1., Remark), Tate conjectured that there exists a direct sum decomposition ("Hodge-Tate decomposition")

$$\text{s.s. } H^{\ell}(\overline{V}, \mathbb{Q}_p) \underset{\mathbb{Q}_p}{\otimes} C \simeq \bigoplus_{j=0}^{\ell} C(-j)^{m_{j,\ell}}$$

where $m_{j,\ell}$ are positive integers, $C(-j)$ are the Tate twists of C, and the isomorphism above is a G-isomorphism.

This conjecture was recently proved by S. Bloch and K. Kato [2] when V is "ordinary" in the following sense:

DEFINITION: The variety V is called *ordinary* when the De Rham cohomology groups $H^q(W, d\Omega^j_{W/k})$ of its reduction W with coefficients the image of $\Omega^j_{W/k}$ by the De Rham differential are zero for all positive integers q and j.

2.1.3. THEOREM 2: *Let p be any prime, let V be a smooth proper variety with good reduction over a p-adic field K, and let* $\overline{V} = V \otimes_K \overline{K}$.

(i) *When l is a prime different from p and* $\ell \neq 2i-1, 2i, 2i+1$, *the groups* $H^{\ell}(V, \mathbb{Q}_l(i))$ *are zero.*

(ii) *For all* $i \in \mathbb{Z}$ *we have*

$$\dim_{\mathbb{Q}_p} H^{\ell}(V, \mathbb{Q}_p(i)) \geq [K: \mathbb{Q}_p] \times \dim_{\mathbb{Q}_p} H^{\ell-1}(\overline{V}, \mathbb{Q}_p).$$

Equality holds for almost all $i \in \mathbb{Z}$.

(iii) *When the cohomology groups of* $\overline{V}$ *in dimensions* ℓ, $\ell - 1$ *and* $\ell - 2$ *have a Hodge-Tate decomposition (cf.* 2.1.2) *and* $i < 0$ *or* $i > \ell$, *we have*

$$\dim_{\mathbb{Q}_p} H^{\ell}(V, \mathbb{Q}_p(i)) = [K: \mathbb{Q}_p] \dim_{\mathbb{Q}_p} H^{\ell-1}(\overline{V}, \mathbb{Q}_p).$$

(iv) *When V is projective and ordinary, and* $\ell \neq 2i-1, 2i, 2i+1$, *we have*

$$\dim_{\mathbb{Q}_p} H^{\ell}(V, \mathbb{Q}_p(i)) = [K: \mathbb{Q}_p] \times \dim_{\mathbb{Q}_p} H^{\ell-1}(\overline{V}, \mathbb{Q}_p).$$

2.2. The equalities of Theorem 2 will be deduced from the following:

PROPOSITION 4: *Assume that*

$$H^{\ell}(\overline{V},\mathbb{Q}_l)(i)^G = H^{\ell-1}(\overline{V},\mathbb{Q}_l)(i)^G = H^{\ell-1}(\overline{V},\mathbb{Q}_l(i-1))_G$$

$$= H^{\ell-2}(\overline{V},\mathbb{Q}_l(i-1))_G = 0.$$

Then

$$\dim_{\mathbb{Q}_l} H^{\ell}(V,\mathbb{Q}_l(i)) = \begin{cases} 0 & \text{when} \quad l \neq p \\ [K:\mathbb{Q}_p] \times \dim_{\mathbb{Q}_p} H^{\ell-1}(\overline{V},\mathbb{Q}_p) \\ \quad \text{when} & l = p. \end{cases}$$

PROOF: The Hochschild-Serre spectral sequence attached to the Galois covering $\overline{V} \to V$ has E_2-term

$$E_2^{rs} = H^r(K, H^s(\overline{V}, \mathbb{Z}/l^n(i)))$$

and converges to $H^{r+s}(V, \mathbb{Z}/l^n(i))$. Since all the groups above are finite, one can perform a projective limit of such spectral sequences when n varies (the Mittag-Leffler property holds). We get

$$E_2^{rs} = H^r(K, H^s(\overline{V}, \mathbb{Z}_l)(i)) \to H^{r+s}(V, \mathbb{Z}_l(i)).$$

The duality theorem for finite G-modules ([15], II.5.2., Theorem 2) gives

$$H^2(K, H^s(\overline{V}, \mathbb{Z}/l^n)(i)) = H^0(K, H^s(\overline{V}, \mathbb{Z}/l^n)(i)^{\wedge}(1))^{\wedge}$$

$$= H^s(\overline{V}, \mathbb{Z}/l^n)(i-1)_G,$$

and taking a projective limit we get

$$H^2(K, H^s(\overline{V}, \mathbb{Z}_l)(i)) = H^s(\overline{V}, \mathbb{Z}_l)(i-1)_G.$$

The hypotheses of the Proposition imply that

$$E_2^{0,\ell} \otimes \mathbb{Q}_l = H^0(K, H^{\ell}(\overline{V}, \mathbb{Z}_l(i))) \otimes \mathbb{Q}_l = 0,$$

$$E_2^{0,\ell-1} \otimes \mathbb{Q}_l = 0,\ E_2^{2,\ell-1} \otimes \mathbb{Q}_l = H^{\ell-1}(\overline{V}, \mathbb{Q}_l(i-1))_G = 0$$

$$E_2^{2,\ell-2} \otimes \mathbb{Q}_l = H^{\ell-2}(\overline{V}, \mathbb{Q}_l(i-1))_G = 0.$$

On the other hand let M be any $\mathbb{Z}_l[[G]]$-module which is finitely generated over $\mathbb{Z}_l$, and F its quotient by its torsion subgroup. We have (as in Proposition 2)

$$\begin{aligned}\chi(M) &= \sum_{r=0}^{2} (-1)^r \dim_{\mathbb{Q}_l} H^r(K, M) \otimes \mathbb{Q}_l \\ &= \sum_{r=0}^{2} (-1)^r \dim_{\mathbb{Z}_l} H^r(K, F) \\ &= \sum_{r=0}^{2} (-1)^r \dim_{\mathbb{F}_l} H^r(K, F/lF),\end{aligned}$$

and by [15] (II.5.7, Theorem 5) we get $\chi(M)=0$ when $l \neq p$ and

$$\begin{aligned}\chi(M) &= -\left[K\colon \mathbb{Q}_p\right] \times \dim_{\mathbb{F}_l}(F/lF) \\ &= -\left[K\colon \mathbb{Q}_p\right] \times \dim_{\mathbb{Q}_p} M \otimes \mathbb{Q}_p \quad \text{when} \quad p = l.\end{aligned}$$

Here we get

$$\begin{aligned}&\dim_{\mathbb{Q}_l} H^{\ell}(V, \mathbb{Q}_l(i)) = \dim_{\mathbb{Q}_l} H^1\big(K, H^{\ell-1}(\bar{V}, \mathbb{Q}_l(i))\big) \\ &= -\chi\big(H^{\ell-1}(\bar{V}, \mathbb{Z}_l(i))\big) = \begin{cases} 0 & \text{when} \quad l \neq p \\ \left[K\colon \mathbb{Q}_p\right] \times \dim_{\mathbb{Q}_p} H^{\ell-1}\big(\bar{V}, \mathbb{Q}_p\big) & \\ \quad \text{when} \quad \ell = p. & \end{cases}\end{aligned}$$

2.3. PROOF OF (i): Assume $l \neq p$. We want to show that the hypotheses of Proposition 4 are satisfied when $i \neq 2k-1, 2k, 2k+1$. Since V has good reduction, we get, by the smooth and proper base change theorem [5], an isomorphism of G-modules

$$H^{\ell}\big(\bar{V}, \mathbb{Q}_l(i)\big) \simeq H^{\ell}\big(\overline{W}, \mathbb{Q}_l(i)\big)$$

where $\overline{W} = W \otimes \bar{k}$, $\bar{k}$ is an algebraic closure of k, and G acts on $\overline{W}$ through its projection onto $\mathrm{Gal}(\bar{k}/k)$. The Weil conjectures then imply that

$$H^{\ell}\big(\overline{W}, \mathbb{Q}_l(i)\big)^{\mathrm{Gal}(\bar{k}/k)} = H^{\ell}\big(\overline{W}, \mathbb{Q}_l(i)\big)_{\mathrm{Gal}(\bar{k}/k)} = 0$$

when $\ell \neq 2i$ (cf. Proposition 1 and Lemma 1). Therefore, when $k = 2i-1$, $2i$, $2i+1$, the hypotheses of Proposition 4 hold.

2.4. PROOF OF (ii): From the Hochschild-Serre spectral sequence considered in Proposition 4 we get, for all $i \in \mathbb{Z}$,

$$\dim_{\mathbb{Q}_p} H^{\ell}\big(V, \mathbb{Q}_p(i)\big) \geqslant \dim_{\mathbb{Q}_p} E_2^{1,\ell-1} \otimes \mathbb{Q}_p \geqslant \chi\Big(H^{\ell-1}\big(\overline{V}, \mathbb{Z}_p(i)\big)\Big)$$

$$= \big[K: \mathbb{Q}_p\big] \times \dim_{\mathbb{Q}_p} H^{\ell-1}\big(\overline{V}, \mathbb{Q}_p\big).$$

To see that equality holds for almost all $i \in \mathbb{Z}$, let $K(\mu_{p^\infty}) = \bigcup_{n \geqslant 1} K(\mu_{p^n})$ be the maximal p-cyclotomic extension of K, $\Gamma = \mathrm{Gal}(K(\mu_{p^\infty})/K)$ its Galois group over K, and $H = \mathrm{Ker}(G \to \Gamma)$.

Fixing an integer $\ell \geqslant 0$, let $X = H^{\ell}(\overline{V}, \mathbb{Z}_p)^H$. It is a Λ-module, where $\Lambda = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$ (cf. Proposition 3 and Lemma 2). Furthermore X is finitely generated as a $\mathbb{Z}_p$-module, therefore it is pseudo-isomorphic to a finite product of Λ-modules of the type $\mathbb{Z}_p[[T]]/(f(T))$ where $f(T)$ is a nonzero polynomial. But

$$\big(\Lambda/(f(T))(i)\big)^{\Gamma} = \big(\Lambda/f(c^i(1+T)-1)\big)^{\Gamma}$$

is finite when $f(c^i - 1) \neq 0$. Therefore

$$X(i)^{\Gamma} = H^{\ell}\big(\overline{V}, \mathbb{Z}_p(i)\big)^{G}$$

is finite for almost all i. A similar proof gives that $H^{\ell}(\overline{V}, \mathbb{Z}_p(i))_G$ is finite for almost all i.

2.5. PROOF OF (iii): Assume that $H^{\ell}(\overline{V}, \mathbb{Q}_p)$ admits a Hodge-Tate decomposition

$$H^{\ell}\big(\overline{V}, \mathbb{Q}_p\big) \underset{\mathbb{Q}_p}{\otimes} C \simeq \bigoplus_{j=0}^{\ell} C(-j)^{m_{j,\ell}}$$

i.e.

$$H^{\ell}\big(\overline{V}, \mathbb{Q}_p(i)\big) \underset{\mathbb{Q}_p}{\otimes} C \simeq \bigoplus_{j=0}^{\ell} C(i-j)^{m_{j,\ell}}.$$

The group of invariants $H^{\ell}(\overline{V}, \mathbb{Q}_p(i))^G$ is a $\mathbb{Q}_p$-vector space contained in $(H^{\ell}(\overline{V}, \mathbb{Q}_p(i)) \otimes C)^G$, and Tate proved in [13] (Theorem 2) that when $n \neq 0$ we have $C(n)^G = 0$. Therefore, when $i < 0$ or $i > \ell$, we have $H^{\ell}(\overline{V}, \mathbb{Q}_p(i))^G = 0$.

On the other hand there are isomorphisms of C-vector spaces with semilinear G-action:

$$\text{s.s. Hom}_{\mathbf{Q}_p}\left(H^{\ell}\left(\bar{V}, \mathbf{Q}_p(i)\right), \mathbf{Q}_p\right) \underset{\mathbf{Q}_p}{\otimes} C$$

$$\simeq \text{Hom}_C\left(\text{s.s. } H^{\ell}\left(\bar{V}, \mathbf{Q}_p(i)\right) \underset{\mathbf{Q}_p}{\otimes} C, C\right)$$

$$\simeq \bigoplus_{j=0}^{\ell} \text{Hom}_C\left(C(i-j), C\right)^{m_{j,\ell}} \simeq \bigoplus_{j=0}^{\ell} C(j-i)^{m_{j,\ell}}.$$

So, when $i<0$ or $i>\ell$, we have

$$\text{Hom}_{\mathbf{Q}_p}\left(H^{\ell}\left(\bar{V}, \mathbf{Q}_p(i)\right), \mathbf{Q}_p\right)^G = \text{Hom}_{\mathbf{Q}_p}\left(H^{\ell}\left(\bar{V}, \mathbf{Q}_p(i)\right)_G, \mathbf{Q}_p\right) = 0.$$

From this we deduce that when $i<0$ or $i>\ell$ the hypotheses of Proposition 4 are satisfied.

2.6. Proof of (iv): By Proposition 4 it will be enough to show that

$$H^{\ell}\left(\bar{V}, \mathbf{Q}_p(i)\right)^G = H^{\ell}\left(\bar{V}, \mathbf{Q}_p(i)\right)_G = 0 \quad \text{when} \quad \ell \neq 2i.$$

But Bloch and Kato proved in [2] that there exists a filtration $F^*H^{\ell}(\bar{V}, \mathbf{Q}_p)$ of $H^{\ell}(\bar{V}, \mathbf{Q}_p)$ whose corresponding graded module is $gr^*H^{\ell}(\bar{V}, \mathbf{Q}_p) = \oplus_{j=0}^{\ell} H^{n-j,j}(-j)$, where $(-j)$ is the twisting by the p-cyclotomic character, and $H^{n-j,j}$ is a module under $\text{Gal}(\bar{k}/k)$ upon which G acts via its projection to $\text{Gal}(\bar{k}/k)$. More precisely, writing $\bar{W} = W \otimes_k \bar{k}$, let $H^{\text{n}}_{\text{crys}}(\bar{W}/\mathbf{W}(\bar{k}))$ be its crystalline cohomology and F: $\bar{W} \to \bar{W}$ be the absolute Frobenius (which raises the coordinates over $\bar{k}$ to the p-th power). Then

$$H^{n-j,j} = \left(H^n_{\text{crys}}\left(\bar{W}/\mathbf{W}(\bar{k})\right) \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p\right)^{[F=p^j]}$$

is the kernel of $F - p^j$ acting upon the crystalline cohomology of $\bar{W}$.

Let p^s be the order of the residue field k, let f_g: $W \to W$ be the geometric Frobenius of W (which raises the coordinates over k to the p^s-th power), and let $f_a \in \text{Gal}(\bar{k}/k)$ be the arithmetic Frobenius (sending $x \in \bar{k}$ to x^{p^s}). If we call $\bar{f}_g = f_g \otimes id$: $\bar{W} \to \bar{W}$ (resp. $\bar{f}_a = id \otimes f_a$: $\bar{W} \to \bar{W}$) the tensor product of f_g (resp. f_a) with the identity on $\bar{k}$ (resp. W), we have the following equalities

$$F^s = \bar{f}_g \circ \bar{f}_a = \bar{f}_a \circ \bar{f}_g$$

and since $\bar{f}_a$ and $\bar{f}_g$ commute with F they act upon $H^{n-j,j}$. Since $H^{n-j,j}$ is contained in the crystalline cohomology of $\overline{W}$, the Weil conjectures in crystalline cohomology [10] tell us that the eigenvalues of $\bar{f}_g$ acting on $H^{n-j,j}$ are algebraic numbers whose archimedean absolute values are equal to $p^{sn/2}$.

Since $F = p^j$ on $H^{n-j,j}$, the equality $p^{sj} = \bar{f}_g \circ \bar{f}_a$ shows that $\bar{f}_a$ can have a fixed vector in $H^{n-j,j}$ only when $n = 2j$. Let $\tau \in G$ be a lifting of $f_a \in \mathrm{Gal}(\bar{k}/k)$ such that τ lies in the kernel of the p-cyclotomic character. Then the endomorphism $\tau - 1$ of $H^{n-j,j}(i-j)$ has trivial kernel and cokernel unless $n = 2j$.

On the other hand, if γ_0 is a generator of $\mathrm{Gal}(K(\mu_{p^\infty})/K)$, the endomorphism $\gamma_0 - 1$ of $H^{j,j}(i-j)$ has trivial kernel and cokernel unless $i = j$.

From this we conclude that $H^{\ell}(\overline{V}, \mathbb{Q}_p(i))^G = H^{\ell}(\overline{V}, \mathbb{Q}_p(i))_G = 0$ unless $\ell = 2i$. Q.E.D.

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