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GIUSEPPE VIGNA SURIA

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q-PSEUDOCONVEX and q-COMPLETE DOMAINS

Giuseppe Vigna Suria

Introduction

The Levi problem was originally posed in the following terms: if D is a domain in $\mathbb{C}^n$ with C^2 boundary which is pseudoconvex is D a domain of holomorphy?

It was then realised that the hypothesis on the boundary can be removed if pseudoconvexity is replaced by completeness, which is a concept that makes sense in any analytic manifold, and the final solution of the Levi problem due to Grauert says that a complete analytic manifold is necessarily Stein [3].

The original spirit of the problem has not been betrayed: domains with C^2 boundary in $\mathbb{C}^n$ are pseudoconvex if and only if they are complete ([4] p. 50).

The same is not true any more if $\mathbb{C}^n$ is replaced by any analytic manifold: a well known example of Grauert provides a subset with C^2 boundary of a complex torus which is pseudoconvex but all holomorphic functions thereon are constant.

In this paper we prove that a q-pseudoconvex open subset of a Stein manifold is necessarily q-complete (the converse is also true, see [2]). This seems to be one of those facts that every complex analyst believes, perhaps for psycological reasons, but no precise reference is, to my knowledge, available and all mathematicians whom I have asked so far don't seem to know how a precise proof should go; the modest aim of this paper is to fill this gap and provide a definite reference.

Most of the ideas in the proof are due to Mike Eastwood to whom I am, once more, deeply grateful.

We briefly recall the basic definitions:

DEFINITION 1: Let D be an open subset of an analytic manifold M of dimension n; we say that D has C^2 boundary if for all $x \in \partial D$ there exists an open neighbourhood U of x and a C^2 function $\varphi : U \to \mathbb{R}$, called *defining function* of D at x s.t. $D \cap U = \{ y \in U \text{ s.t. } \varphi(y) < 0 \}$ and

$d\varphi(x) \neq 0$; in these conditions we can consider the *complex Hessian*

$$\mathscr{H}(\varphi)(x) = \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(x) \right)_{i,j=1}^{n}$$

where $z_1, z_2, \ldots, z_n$ are local holomorphic coordinates at x. The signature of this Hermitian matrix does not depend on the choice of the local holomorphic coordinates but it does depend on φ. However the *Levi form*

$$\mathscr{L}(\varphi)(x) =: \mathscr{H}(\varphi)(x)_{|T_x \partial D},$$

where $T_x\partial D = \{ v = \Sigma_1^n v_i \partial/\partial z_i \in T_x M \text{ s.t. } \Sigma_1^n v_i \partial\varphi/\partial z_i(x) = 0 \}$ is the holomorphic tangent space of ∂D at x, has a signature that depends only on D and x.

If $n(x)$ denotes the number of negative eigenvalues of $\mathscr{L}(\varphi)(x)$ we say that D is *q-pseudoconvex* if $n(x) \leqslant q$ for all $x \in \partial D$.

DEFINITION 2: A complex n-dimensional manifold D is said to be *q-complete* if we can find a *q-plurisubharmonic exaustion function* on D i.e. a C^2 function $\Psi : D \to \mathbb{R}$ s.t.

(1) for all $c \in \mathbb{R}$ the set $B_c = \{ x \in D \text{ s.t. } \Psi(x) < c \}$ is relatively compact in D and

(2) The complex Hessian $\mathscr{H}(\Psi)(x)$ has at least $n - q$ positive eigenvalues for all x in D.

0-pseudoconvex and 0-complete domains are simply called pseudoconvex and complete.

THEOREM: *If D is a domain with C^2 boundary in a Stein manifold M and D is q-pseudoconvex then it is also q-complete.*

PROOF: We shall divide the proof into several steps.

Step 1: As there is always an analytic embedding of M into $\mathbb{C}^N$, for some large N (see [5] p. 359) we can suppose at once that M is an analytic submanifold of $\mathbb{C}^N$. Choose a holomorphic tubular neighbourhood $p : V \to M$ and set $\tilde{D} = p^{-1}(D)$ (cfr. [1] proof of Lemma 1, p. 131). We claim that, after shrinking V if necessary,

(a) $\forall x \in \partial\tilde{D} \cap V$, $\partial\tilde{D}$ is C^2 at x,

(b) If we consider $\tilde{D}$ as an open subset of $\mathbb{C}^N$ then $n(x, \tilde{D}) = n(p(x), D)$, for all $x \in \partial\tilde{D} \cap V$.

Indeed, since the problem is local we can suppose that local coordinates $z_1, z_2, \ldots, z_N$ have been chosen s.t., near x, $M = \{ z \text{ s.t. } z_{N-n+1} = z_{N-n+2} = \ldots = z_N = 0 \}$, $z_1, z_2, \ldots, z_n$ are local coordinates of M at x and $p(z_1, z_2, \ldots, z_N) = (z_1, z_2, \ldots, z_n, 0, \ldots, 0)$.

Let $\tilde{U}$ be a neighbourhood of x in $\mathbb{C}^N$ so small that $z_1, z_2, \dots, z_N$ are defined in $\tilde{U}$ and that there exists a C^2 defining function $\Phi : U = \tilde{U} \cap M \to \mathbb{R}$ for D with $d\Phi(x) \neq 0$ and $\tilde{U} \subseteq V$. By shrinking $\tilde{U}$ if necessary we can also suppose that $\tilde{U} \subseteq p^{-1}(U)$.

Define $\tilde{\Phi} : \tilde{U} \to \mathbb{R}$ by $\tilde{\Phi} = \Phi \circ p$ i.e. $\tilde{\Phi}(z_1, z_2, \dots, z_N) = \Phi(z_1, z_2, \dots, z_n, 0, \dots, 0)$. Then $\tilde{\Phi}$ is a defining function for $\tilde{D}$ at x. Moreover

$$T_x \partial \tilde{D} = \left\{ v \in T_x \mathbb{C}^N \text{ s.t. } \sum_{i=1}^{N} \frac{\partial \tilde{\Phi}}{\partial z_i}(x) v_i = 0 \right\}$$

$$= \left\{ v \in T_x \mathbb{C}^N \text{ s.t. } \sum_{i=1}^{n} \frac{\partial \Phi}{\partial z_i}(p(x)) v_i = 0 \right\} \simeq T_{p(x)} \partial D \times \mathbb{C}^{N-n}$$

where as usual $v = \sum_{i=1}^{N} v_i \partial / \partial z_i$, and

$$\frac{\partial^2 \tilde{\Phi}(x)}{\partial z_i \partial \bar{z}_j} = \begin{cases} \dfrac{\partial^2 \Phi(p(x))}{\partial z_i \partial \bar{z}_j} & \text{if } i, j \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

This proves the claim.

Step 2: So, if we suppose that D is q-pseudoconvex we have that, $\forall x \in V \cap \partial \tilde{D}$, $\partial \tilde{D}$ is C^2 at x and $n(x, \tilde{D}) \leqslant q$.

Consider the function $\rho : \mathbb{C}^N \to \mathbb{R}$ given by

$$\rho(y) = \begin{cases} \operatorname{dist}(y, \partial \tilde{D}) & \text{if } y \in \bar{\tilde{D}} \\ -\operatorname{dist}(y, \partial \tilde{D}) & \text{if } y \in \mathbb{C}^N - \tilde{D}, \end{cases}$$

Where dist denotes the Euclidean distance. Since $\forall x \in \partial \tilde{D}$, $\partial \tilde{D}$ is C^2 at x, we can conclude that there exists a neighbourhood $\tilde{U}'$ of ∂D in $\mathbb{C}^N$ on which ρ is C^2 (by the inverse function theorem).

By shrinking $\tilde{U}'$ if necessary, we can also suppose that $\forall y$ in $\tilde{U}'$ there exists exactly one point $c(y) \in \partial \tilde{D} \cap \tilde{U}'$ which is the closest point to y under the Euclidean distance, that $d\rho(c(y)) \neq 0$ and that $n(c(y), \tilde{D}) \leqslant q$.

Let φ: $\tilde{U}' \cap \tilde{D} \to \mathbb{R}$ be the function $\varphi = \log \rho$; we claim that the Hessian $(\mathscr{H}\varphi)(y)$ has at most q positive eigenvalues $\forall y$.

Indeed suppose that this is false, i.e. there exists a point y in $\tilde{U}' \cap \tilde{D}$ s.t. $(\mathscr{H}\varphi)(y)$ has (at least) $q+1$ positive eigenvalues; the geometric interpretation is: there are linear coordinates $(t_1, t_2, \dots, t_N)$ of $\mathbb{C}^N$ s.t. the Hermitian form given by the matrix

$$(C_{jk})_{j,k=1}^{q+1} = \left(\frac{\partial^2 \varphi(y)}{\partial t_j \partial \bar{t}_k} \right)_{j,k=1}^{q+1}$$

is positive definite on the linear subspace V of $T_y\mathbb{C}^N = \mathbb{C}^N$ spanned by $(\partial/\partial t_1, \partial/\partial t_2, \ldots, \partial/\partial t_{q+1})$.

By Taylor's theorem we have

$$\varphi\left(y + \sum_{j=1}^{q+1} t_j \frac{\partial}{\partial t_j}\right) = \log\rho\left(y + \sum_{j=1}^{q+1} T_j \frac{\partial}{\partial t_j}\right)$$

$$= \log\rho(y) + \operatorname{Re}\left(\sum_{i=1}^{q+1} a_i t_i + \sum_{j,k=1}^{q+1} b_{jk} t_j t_k\right)$$

$$+ \sum_{j,k=1}^{q+1} C_{jk} t_j \bar{t}_k + 0(|t|^2),$$

where $a_i = \frac{1}{2}\partial\varphi/\partial t_i(y)$ and $b_{jk} = \partial^2\varphi(y)/\partial t_j \partial t_k$ are constants, and $0(|t|^2)$ has the property that $\lim_{t\to 0} 0(|t|^2)/|t|^2 = 0$ and so also

$$\lim_{t\to 0} \frac{0(|t|^2)}{\sum_{j,k}^{q+1} C_{jk} t_j \bar{t}_k} = 0.$$

In order to simplify notation omit the limits of the summands and write $A(t) = y + \Sigma t_j \partial/\partial t_j$, $B(t) = \exp(\Sigma a_i t_i + \Sigma b_{jk} t_j t_k)$.

Then the above equality can be written as $\rho(A(t)) - \rho(y)|B(t)| = \{\exp\Sigma C_{jk} t_j \bar{t}_k + 0(|t|^2)) - 1\}\rho(y)|B(t)| = \{\Sigma C_{jk} t_j \bar{t}_k + 0'(|t|^2)\}\rho(y)|B(t)|$, where the last equality is obtained by expanding in Taylor series the function exp and $0'(|t|^2)$ has the same properties as $0(|t|^2)$. Then one has

$$\lim_{t\to 0} \frac{\rho(A(t)) - \rho(y)|B(t)|}{\Sigma C_{jk} t_j \bar{t}_k} = \rho(y),$$

so we can choose $\epsilon > 0$ small enough s.t. $\forall t$, $|t| < \epsilon$, one has

(a) $A(t) \in \tilde{D} \cap \tilde{U}'$ and

(b) $\rho(A(t)) - \rho(y)|B(t)| > \rho(y)/2 \cdot \Sigma C_{jk} t_j \bar{t}_k$.

Set $u = c(y) - y$ and define an analytic function T on the open ball $B_\epsilon = \{t \in \mathbb{C}^{q+1}$ s.t. $|t| < \epsilon\}$:

T: $B_\epsilon \to \mathbb{C}^N$ is given by $T(t) = A(t) + uB(t)$.

We can also suppose that ϵ is so small that $T(t) \in \tilde{U}'$ if $t \in B_\epsilon$. Then it is easy to check, and a picture shows how, that if $t \in B_\epsilon$ one has

(c) $\rho(T(t)) \geqslant \rho(A(t)) - |u||B(t)| \geqslant |u|/2\Sigma C_{jk} t_j \bar{t}_k \geqslant 0$

This in particular proves that $T(t) \in \tilde{D}$ for all $t \in B_\epsilon - \{0\}$, and, since

$\rho(T(0))=\rho(c(y))=0$, 0 is a minimum for the function $\rho \circ T\colon B_\epsilon \to \mathbb{R}$, and so, taking partial derivatives,

$$\frac{\partial \rho \circ T(0)}{\partial t_j}=0 \quad \text{for all} \quad j=1,2,\ldots,q+1.$$

Using the chain rule and the fact that T is analytic we have:
(d) $\sum_{h=1}^{N} \partial\rho/\partial z_h(c(y))\partial T_h/\partial t_j(0)=0$ for $j=1,2,\ldots,q+1$.
In other words the vectors $\partial T/\partial t_j(0)$, $j=1,2,\ldots,q+1$, are in $T_{c(y)}\partial\tilde{D}$.

Moreover, $\forall t$ in $\mathbb{C}^{q+1}$, we have
(e) $\sum_{j,k=1}^{q+1}\partial^2\rho \circ T(0)/\partial t_j\partial \bar{t}_k t_j\bar{t}_k \geqslant |u|/4\sum_{j,k=1}^{q+1}C_{jk}t_j\bar{t}_k$.

To prove this we first observe that it is clearly enough to check it for small $|t|$.

From the above inequality (c), using Taylor series, we deduce

$$\mathrm{Re}\Big(\sum d_{jk}t_jt_k\Big)+\sum\frac{\partial^2\rho \circ T(0)}{\partial t_j\partial\bar{t}_k}t_j\bar{t}_k+0''(|t|^2)\geqslant\frac{|u|}{2}\sum C_{jk}t_j\bar{t}_k,$$

for all $t\in B_\epsilon$, where $d_{jk}=\partial^2\rho \circ T(0)/\partial t_j\partial t_k$ are constants and $0''(|t|^2)$ has the same properties as $0(|t|^2)$.

Then, after reducing ϵ if necessary, we have, $\forall t\in B_\epsilon$,

$$\mathrm{Re}\Big(\sum d_{jk}t_jt_k\Big)+\sum\frac{\partial^2\rho \circ T(0)}{\partial t_j\partial\bar{t}_k}T_j\bar{t}_k\geqslant\frac{|u|}{4}\sum C_{jk}t_j\bar{t}_k.$$

Let $t_j'=e^{i\theta}t_j$ for $0\leqslant\theta\leqslant 2\pi$; writing t' in the above inequality and observing that the second and third term are unchanged under the substitution $t\to t'$, we deduce, $\forall\theta$,

$$\mathrm{Re}\Big(e^{i2\theta}\sum d_{jk}t_jt_k\Big)+\sum\frac{\partial^2\rho \circ T(0)}{\partial t_j\partial\bar{t}_k}t_j\bar{t}_k\geqslant\frac{|u|}{4}\sum C_{jk}\bar{t}_jt_k,$$

and by choosing θ so that the first term is negative we prove the inequality (e).

Using again the chain rule and the fact that T is analytic we have that the Hermitian form

$$\left(\sum_{h,m=1}^{N}\frac{\partial^2\rho(c(y))}{\partial z_h\partial\bar{z}_m}\cdot\frac{\partial T_h}{\partial t_j}(0)\cdot\left(\overline{\frac{\partial^T m}{\partial t_k}(0)}\right)\right)_{j,k=1}^{q+1}$$

is positive definite.

It follows easily that the Hermitian form $(\partial^2\rho(c(y))/\partial z_h\partial\bar{z}_m)_{h,m=1}^N$ is positive definite on the linear subspace V of $T_{c(y)}\partial\tilde{D}$ spanned by the vectors $\partial T/\partial T_j(0)$, $j = 1, 2, \ldots, q+1$; in particular it follows automatically that these vectors are linearly independent, so that $\dim_{\mathbb{C}} V = q+1$; but since $-\rho$ is a defining function for $\overline{D}$ at $c(y)$, we have that $n(c(y), \tilde{D}) \geqslant q+1$ and this contradicts our hypothesis, so that the claim is proved.

Step 3: By restricting φ to $\tilde{U}' \cap D$ we find a C^2 function, called again $\varphi : W = \tilde{U}' \cap D \to \mathbb{R}$ s.t.

(a) $\lim_{y \to \partial D} \varphi(y) = -\infty$,

(b) $(\mathscr{H}\varphi)(y)$ has at most q positive eigenvalues $\forall y$ in W.

Let F be a closed subset of M s.t. $D - W \subseteq \operatorname{int} F \subseteq F \subseteq D$, and let $0 \leqslant \Psi \leqslant 1$ be a C^2 bump function s.t. $\Psi = 0$ on F, $\Psi = 1$ in a neighbourhood of $M - D$, and suppose that F is chosen so that $\varphi(y) \leqslant 0$ for $y \notin F$.

By considering the function $\varphi' = \varphi \cdot \Psi$, we have that

(a) $\lim_{y \to \partial D} \varphi'(y) = -\infty$,

(b) $(\mathscr{H}\varphi')(y)$ has at most q positive eigenvalues $\forall y \in D - F$,

(c) $\varphi' \leqslant 0$.

Now we use the fact that M is Stein and so 0-complete (see [5], lemma p. 358) i.e. there exists a 0-plurisubharmonic exhaustion function $\lambda : M \to \mathbb{R}$.

$\forall n \in Z$, the set $K_n = \{y \in M \text{ s.t. } \lambda(y) \leqslant n\}$ is compact, therefore so is $F \cap K_n$ and there exist constants C_n s.t.

$$C_n(\mathscr{H}\lambda)(y) - (\mathscr{H}\varphi')(y) > 0 \quad \forall y \in F \cap K_n.$$

Now choose a C^2 function $f : \mathbb{R} \to \mathbb{R}$ with the properties

(a) $f' > 0, f'' > 0$ always,

(b) $f'(r) > C_{E(r)+1}, r$, where $E(r)$ denotes the integral part of r.

(c) $f'(r) > C_0 \quad \forall r$, and consider the C^2 function

$$\chi = f \circ \lambda - \Psi' : D \to \mathbb{R}.$$

First we notice that, $\forall c \in \mathbb{R}$, $B_c = \{y \in D \text{ s.t. } \chi(y) \leqslant c\}$ is contained, by the property (c) of φ' in $\{y \in D \text{ s.t. } f \circ \lambda(y) \leqslant c\}$ which is compact by the assumptions on f and λ. Moreover B_c is closed in D and, since $\lim_{y \to \partial D} \varphi'(y) = -\infty$, it is also closed in M. Thus B_c is compact and χ is an exhaustion function.

For all $y \in D$ we have

$$(\mathscr{H}\chi)(y) = f''(\lambda(y)) \cdot A(y) + f'(\lambda(y)) \cdot (\mathscr{H}\lambda)(y) - (\mathscr{H}\varphi')(y),$$

where $A(y) = \left(\partial\lambda/\partial z_i(y) \cdot \overline{\partial\lambda/\partial z_j(y)}\right)_{i,j=1}^{n}$ is a semipositive Hermitian form. If $y \in D - F$ then there exists a linear subspace V of $T_y D$, of dimension $n - q$ where $-(\mathscr{H}\varphi')(y)$ is positive semidefinite. Therefore $(\mathscr{H}\chi)(y)$ is positive definite on V.

If $y \in F$ then either $y \in K_0 \cap F$ in which case

$$(\mathscr{H}\chi)(y) \geqslant f'(\lambda(y))(\mathscr{H}\lambda)(y) - (\mathscr{H}\varphi')(y)$$
$$\geqslant C_0(\mathscr{H}\lambda)(y) - (\mathscr{H}\varphi')(y) > 0,$$

or $y \in (K_{n+1} - K_n) \cap F$ for some integer $n \geqslant 0$, in which case

$$f'(\lambda(y)) > C_{n+1} \qquad \text{and so}$$

$$(\mathscr{H}\chi)(y) > C_{n+1}(\mathscr{H}\lambda)(y) - (\mathscr{H}\varphi')(y) > 0.$$

Therefore χ is also q-plurisubharmonic and we can finally say that the theorem is proved. □

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Giuseppe Vigna Suria
Dip. di Matematica
Fac. di Scienze
Università di Trento
38050 Povo (TN)
Italy