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FRAME FLOWS ON MANIFOLDS WITH PINCHED NEGATIVE CURVATURE

M. Brin * and H. Karcher **

PART I. ERGODIC COMPONENTS AND THE TRANSITIVITY GROUP

Section 1

Let $M = M^n$ be a smooth Riemannian manifold, T_1M be the manifold of unit tangent vectors and $St_k(M)$, $k = 1, 2, \ldots, n$, be the manifold of ordered orthonormal k-frames over M, $St_k(M) = \{(x, v_1, v_2, \ldots, v_k) | x \in M, v_i \in T_xM, \|v_i\| = 1, v_i \perp v_j \text{ for } i \neq j, 1 \leqslant i, j \leqslant n\}$, $St_1(M) = T_1M$. The *geodesic flow* g^t on M acts in T_1M in the following way: if $v \in T_1M$, then g^tv is the result of the parallel translation of v along the geodesic determined by v at distance t. Let μ be the Riemannian volume on M and λ_1 be the Lebesgue measure on the unit sphere S^{n-1} in T_xM. The natural product measure $d\nu_1 = d\mu \times d\lambda_1$ on T_1M is preserved under g^t. If M is a compact manifold of strictly negative curvature, then the geodesic flow is ergodic with respect to the Liouville measure ν_1 (see [1]). The *frame flow* F_k^t, $k = 1, 2, \ldots, n$, on M is a natural extension of the geodesic flow. It acts in the space $St_k(M)$; given a frame $w = (x, v_1, \ldots, v_k)$ we define F_k^tw as the result of the parallel translation of w along the geodesic determined by (x, v_1) at distance t. It follows directly from the definition that $F_1^t = g^t$. Let λ_k denote the natural Lebesgue measure on the Stiefel manifold $St_k = St_k^n$ of orthonormal k-frames in $\mathbb{R}^n$ with support at the origin. Then the frame flow F_k^t preserves the product measure $d\nu_k = d\mu \times d\lambda_k$. A natural problem emerges to find out when F_k^t is ergodic (with respect to ν_k). It turns out that, even if M is compact and negatively curved, F_k^t may have first integrals (see [4], [5]). On the other hand, sometimes the frame flow on such an M is ergodic.

(1) For any compact negatively curved M of odd dimension different from 7 the frame flow is ergodic (and Bernoulli) for every k (see [4]).

(2) For any compact M the set of negatively curved metrics on M with ergodic F_k^t (for every k) is open and dense in the space of all C^3-metrics

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on M of negative curvature (see [2], [4]). This open set of metrics contains the metric of constant negative curvature, if such a metric exists [5].

In this paper we prove the following theorem.

1.1. THEOREM: *Let M be a compact connected manifold with a C^3-Riemannian metric of negative variable sectional curvature K, $-\Lambda^2 < K < -\lambda^2$. Suppose the dimension of M is even and different from* 8 *and* $\lambda/\Lambda > 0.93$.

Then F_k^t is ergodic and Bernoulli for $k = 1, 2, \ldots, \dim M - 1$.

1.2. REMARK: The space of n-frames on an orientable n-manifold M is not connected, it has two components. Thus, the flow F_n^t must have at least two ergodic components, and the ergodicity of F_n^t on each of them is equivalent to the ergodicity of F_{n-1}^t. On the other hand, if M is not orientable, the ergodicity of F_{n-1}^t implies the ergodicity of F_n^t. Therefore, from now on, when dealing with F_n^t, we will consider only positively oriented frames.

1.3. CONJECTURE: If $\lambda/\Lambda > \frac{1}{2}$, then F_k^t is ergodic and Bernoulli, $k = 1, 2, \ldots, \dim M - 1$.

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Section 2

Each space $St_k(M)$ is fibered over the manifold of unit tangent vectors T_1M and the fiber is St_{k-1}^n. The space $St_k(M)$ is also fibered over $St_l(M)$ for every $l < k$, the fiber being St_{k-l}^n. It is easy to see that the frame flow preserves the structure of all these fiber bundles. Let P_{kl} denote the natural projection from $St_k(M)$ onto $St_l(M)$, $l < k$, then

$$p_{kl} \circ F_k^t = F_l^t \circ p_{kl} \quad \text{for every} \quad t.$$

That means that each flow F_k^t, $k = 1, 2, \ldots, n-2$ is a factor of F_{n-1}^t. So, if F_{n-1}^t is ergodic or Bernoulli, then the same is true for F_k^t, $k = 1, 2, \ldots, n-2$. The structure group of the bundle $p: St_{n-1}(M) \to T_1M = St_1(M)$ is (a subgroup of) $SO(n-1)$. The group $SO(n-1)$ acts naturally in the fibers of this bundle by rotations in the $(n-1)$-dimensional plane perpendicular to the first vector of the frame. Since the parallel translation along a geodesic commutes with rotations in the perpendicular plane, we have for every $g \in SO(n-1)$, $w \in St_{n-1}(M)$ and real t

$$g \circ F_{n-1}^t w = F_{n-1}^t w \circ g.$$

That means that F^t_{n-1} is a $SO(n-1)$-extension of the geodesic flow $g^t = F^t_1$ (see [3]). Suppose now that M has negative sectional curvature. Consider two tangent vectors $\omega_1 = (x_1, v_1)$ and $\omega_2 = (x_2, v_2)$ which define asymptotic geodesics, i.e., x_1 and x_2 belong to the same horosphere S, v_1 and v_2 are perpendicular to S, and the distance in the induced metric in T_1M between $g^t\omega_1$ and $g^t\omega_2$ tends to 0 as $t \to \infty$. Let d be the distance in T_1M and the induced distances in $St_k(M)$.

2.1. DEFINITION: *Denote by* $\omega_1^\perp \subset T_{x_1}M$ *and* $\omega_2^\perp \subset T_{x_2}M$ *respectively the* $(n-1)$*-planes perpendicular to* v_1 *and* v_2*. For every vector* $v \in \omega_1^\perp$ *there exists a unique vector* $P_{\omega_1\omega_2}v \in \omega_2^\perp$ *such that* $d(F_2^t(\omega_1, v), F_2^t(\omega_2, P_{\omega_1\omega_2}v)) \to 0$ *as* $t \to \infty$*. The linear isometry* $P_{\omega_1\omega_2}: \omega_1^\perp \to \omega_2^\perp$ *is called a positive horospherical translation.*

In a similar manner, if $-v_1$ and $-v_2$ define asymptotic geodesic, we define the corresponding negative horospherical translation.

2.2. DEFINITION: *Fix a unit tangent vector* $\omega = (x, v)$, $v \in T_xM$, *and consider a finite sequence* $\omega_i \in T_1M$, $i = 1, 2, \dots, m$, *such that* $\omega_1 = \omega_2 m = \omega$ *and for every* $i = 1, 2, \dots, m-1$ *the pair* (ω_i, ω_{i+1}) *defines a positive or negative horospherical translation. The composition of these translations* $P_{\omega_1\omega_2\dots\omega_m}$ *is a rotation in* $\omega^\perp$*. All rotations we can obtain in this way form a subgroup* H_ω *of* $SO(n-1)$ *called the transivity group at* ω.

The transitivity group H_ω acts in the unit sphere S^{n-2} of ω. Consider the bundle $p: St_{n-1}(M) \to St_1(M)$ with fiber $SO(n-1)$ and the induced action of H in $p^{-1}(\omega)$.

2.3. PROPOSITION (see [4]): *If* $\overline{H}_\omega$ *acts transitively in* $p^{-1}(\omega)$ *then* F^t_{n-1} *is ergodic and is a K-flow* ($\overline{H}_\omega$ *is the closure of* H_ω).

2.4. REMARK: The same flow F^t_{n-1} is a group extension of the geodesic flow g^t which is Bernoulli (see [12]). Therefore (see [13]), the K-property for F^t_{n-1} implies the Bernoulli property (actually mixing is enough).

2.5. REMARK: Since the sectional curvature of M is negative, for every ω, $\omega' \in T_1M$ there are $\omega_1, \omega_2, \dots, \omega_m \in T_1M$ such that $\omega_1 = \omega$, $\omega_2 = \omega'$ and for every $i = 1, 2, \dots, m-1$ the pair ω_i, ω_{i+1} defines two geodesics asymptotic in the positive or negative direction (see [1]). Hence, for every ω, $\omega' \in T_1M$ the transitivity groups H_ω and $H_{\omega'}$ are conjugate in $SO(n-1)$.

2.6. PROPOSITION (see [5]): *If* $\overline{H}_\omega$ *acts transitively on* S^{n-2}, *n is even and* $n \neq 8$, *then* $\overline{H}_\omega$ *acts transitively in* $p^{-1}(\omega)$.

Thus, to prove Theorem 1.1 it suffices to show that under the pinching assumption of the theorem the transitivity group acts transitively on the sphere S^{n-2}. It is enough to prove this statement for the universal cover $\tilde{M}$ of M. Fix a unit tangent vector $\omega = (x, v) \in T_1\tilde{M}$ and consider the geodesic γ defined by this vector. Denote by ω_∞^+ and ω_∞^- the two ends of γ on the absolute and let S^+ and S^- be the two horospheres passing through x with centers at ω_∞^+ and ω_∞^- (see Fig. 1a). Denote by S_t^+ and S_t^- the two families of concentric horospheres centered at ω_∞^+ and ω_∞^-. The parameter t is chosen in such a way that $S_0^+ = S^+$, $S_0^- = S^-$, $S_t^+ \cap \gamma \to \omega_\infty^-$ as $t \to \infty$, $S_t^- \cap \gamma \to \omega_\infty^+$ as $t \to \infty$ and t is the arclength along γ. Let $\alpha_t \in S_t^+ \cap S_t^-$, then the isosceles triangle $(\omega_\infty^+, \alpha_t, \omega_\infty^-)$ has a unique triple of coherent midpoints (x, O^+, O^-) so that every two points belong to the same horosphere (or sphere) centered at the opposite vertex (see fig. 1b).

2.7. LEMMA: *The set $S_t^+ \cap S_t^-$ is homeomorphic to the sphere S^{n-2} for every t, $0 < t \leqslant \infty$.*

PROOF: Since the curvature is negative the statement of the lemma is true for any finite positive t. For $t = \infty$ the statement of the lemma was proved independently by Eberlein [8] and Im Hof [10].

Let now $\alpha_\infty \in S_\infty^+ \cap S_\infty^-$ (see Fig. 1c). By construction x is a midpoint of this triangle, denote the other two midpoints by O^+ and O^-. Consider now the tangent unit vectors $\omega_1 = \omega$ at x, ω_2 at O^+, ω_3 at O^- and ω_4 at x as shown in Fig. 2. Every two points (ω_i, ω_{i+1}), $i = 1, 2, 3$, define asymptotic geodesics, but $\omega_1 = -\omega_4$. Therefore, to get an element of the transitivity group at ω in this way we must consider two such points at ∞, α_∞^1 and α_∞^2 and the composition of the parallel translations along the triangles $(x, \omega_\infty^+, \alpha_\infty^1, \omega_\infty^-, x)$ and $(x, \omega_\infty^+, \alpha_\infty^2, \omega_\infty^-, x)$ gives an element $h(\alpha_\infty^1, \alpha_\infty^2)$ of H. Let

$$V = \bigcup_{\alpha_\infty^1 \in S_\infty^+ \cap S_\infty^-} h(\alpha_\infty^1, \alpha_\infty^2).$$

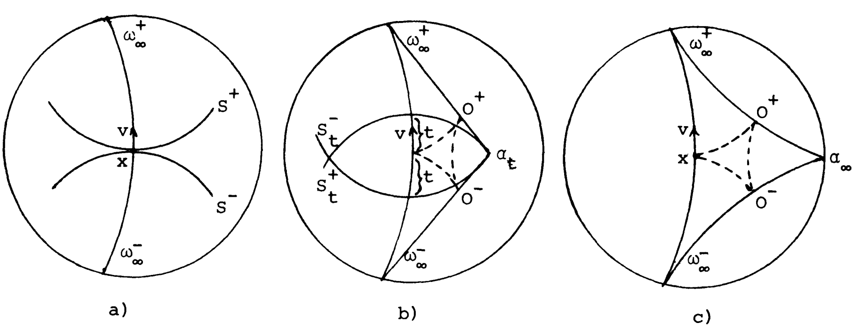

Figure 1

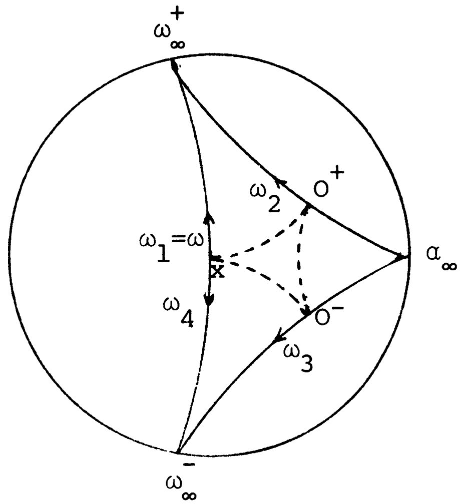

Figure 2

To prove that H_ω is transitive on S^{n-2} we are going to fix $u \in \omega^\perp$ of norm 1 and to show that under the pinching assumption the set $V \cdot u$ contains an open neighborhood on the sphere S^{n-2}. Denote by W the set of elements of $SO(n-1)$ which correspond to the parallel translations along all possible triangles $(x, \omega_\infty^+, \alpha_\infty^1, \omega_\infty^-, x)$. It is obvious that $V \cdot u$ contains an open neighborhood on S^{n-2} if and only if $W \cdot u$ does.

Consider now the case of constant negative curvature (see Fig. 3). Join x and α_∞^1 by a geodesic. The tangent vector v_1 to this geodesic at x is perpendicular to ω. Therefore, the set $S_\infty^+ \cap S_\infty^-$ can be identified with the

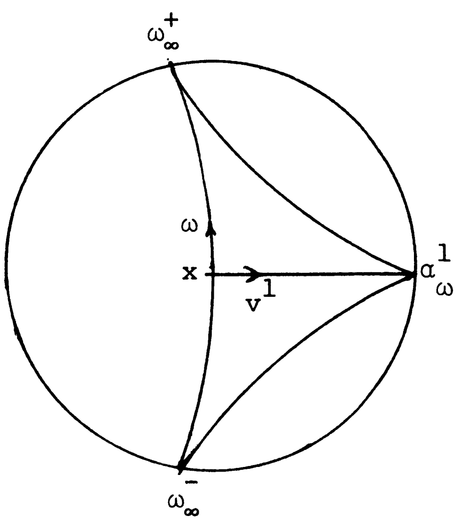

Figure 3

unit sphere in $\omega^\perp$. The parallel translation along the triangle $(x, \omega^+_\infty, \alpha^1_\infty, \omega^-_\infty, x)$ induces an isometry $\mathrm{Is}(v^1)$ of $\omega^\perp$.

2.8. PROPOSITION: *If the curvature is constant, then* $\mathrm{Is}(v^1)$ *is the reflection of* $\omega^\perp$ *with respect to the superplane perpendicular to* v_1.

PROOF: It is clear that v^1 is reflected by $\mathrm{Is}(v^1)$, and $\mathrm{Is}(v^1)$ is identical in the subspace perpendicular to the plane $(\omega = v, v^1)$.

Thus, if the curvature is constant, then by translating any given vector $u \in S^{n-2}$ along different triangles we get the whole sphere S^{n-2}. Now we treat the case of variable curvature as a "perturbation" of the case of constant curvature. Though the situation may deteriorate and we may not be able to get the whole sphere S^{n-2}, we still expect to get an open neighborhood, provided the "perturbation" is small, i.e., the curvature is close to constant.

Given an infinite triangle $(x, \omega^+_\infty, \alpha^1_\infty, \omega^-_\infty, x)$ on $\tilde{M}$ we will construct now the corresponding triangle for the constant curvature case. Take the geodesic connecting α^1_∞ and x and project its tangent vector v^1 on $\omega^\perp$. The projection w is certainly a non-zero vector in $\omega^\perp$ because α^1_∞ does not coincide neither with ω^+_∞ nor with ω^-_∞. Denote $w^1 = w/\|w\| = \pi(\alpha^1_\infty)$. So, we have a projection π from $S^+_\infty \cap S^-_\infty$ on S^{n-2}. The last one can be identified with the unit sphere in $\omega_c^\perp$, where ω_c is a fixed tangent vector to the unit n-ball with the hyperbolic metric. Let $P(\alpha^1_\infty)$ denote the isometry of $\omega^\perp$ induced by the parallel translation along the original triangle and let $P_c(\alpha^1_\infty)$ denote the corresponding reflection in $\omega_c^\perp$ (see Proposition 2.8).

2.9. PROPOSITION: *If the angular distance* $d_a(P(\alpha^1_\infty)u, P_c(\alpha^1_\infty)u)$ *on* S^{n-2} *is strictly less than* $\pi/2$ *for every* $\alpha^1_\infty \in S^+_\infty \cap S^-_\infty$, *then the set* $W \cdot u$ *(i.e., the set of all translations of u along all possible triangles) contains an open neighborhood of* $-u$ *on* S^{n-2}.

PROOF: We have a mapping $R_c\colon S^{n-2} \to S^{n-2}$ which comes from the constant curvature case, $R_c(v)$ is the reflection of u with respect to the superplane perpendicular to v. Note that, if n is even, R_c is a mapping of degree O. Another mapping $R\colon S^{n-2} \to S^{n-2}$ sends α^1_∞ to the parallel translation of u along the triangle $(x, \omega^+_\infty, \alpha^1_\infty, \omega^-_\infty)$. Consider the following diagram

$$\begin{array}{ccc} S^+_\infty \cap S^-_\infty = S^{n-2} & \xrightarrow{\quad\pi\quad} & S^{n-2} \\ & {\scriptstyle R}\searrow \qquad \swarrow{\scriptstyle R_c} & \\ & S^{n-2} & \end{array}$$

It does not commute, but by assumption the distance between $R(\alpha)$ and $R_c(\pi(\alpha))$ is less than $\pi/2$.

2.10. LEMMA: *The projection π is homotopic to the identity.*

PROOF: For every set $S_t^+ \cap S_t^-$ consider the corresponding projection π_t on S^{n-2}. When t is small π_t is homotopic to the identity, and it changes continuously. So, π is also homotopic to the identity.

Suppose now that u corresponds to the north pole of the sphere and let $B_\tau = \{z \in S^{n-2} | d_a(u, z) \leqslant \tau\}$. The image $R_c(B_{\pi/4})$ is the lower semisphere, $R_c(\partial B_{\pi/4})$ is the equator. Let us now approximate π by a smooth mapping $\tilde{\pi}$ such that the distance between R and $R_c \circ \tilde{\pi}$ is still less than $\pi/2$. Take the height function h, $h(u) = 1$, $h(-u) = -1$. According to the Sard theorem for almost every number β the preimage $\tilde{\pi}^{-1} \circ h^{-1}(\beta) = \tilde{\pi}^{-1}(\partial B_\beta)$ is a collection $A_i(\beta)$ of embedded spheres S^{n-3}. Let d be the maximal distance between $R(\alpha)$ and $R_c \circ \tilde{\pi}(\alpha)$. Choose a "good" β such that $d/2 < \beta \leqslant \pi/4$. Since $\tilde{\pi}$ is homotopic to the identity, there is a preimage $A_j(\beta)$ which is mapped one-to-one and with degree 1 onto ∂B_β. The set $A_j(\beta)$ divides S^{n-2} into two parts one of which is mapped onto int B_β by $\tilde{\pi}$. Consider now $R(A_j(\beta))$ and its projection $m(R(A_j(\beta)))$ onto $B_{2\beta}$ along the meridians. The composition $m \circ R|_{A_j(\beta)}$ is $\pi/2$-close to $\tilde{\pi} \circ R_c$, so it must have the same degree. Therefore, the south pole with an open neighborhood belongs to the image of R. The proposition is proved.

In the next sections we prove that under the pinching assumption of Theorem 1.1 the angular distance between $R(\alpha)$ and $R_c(\pi(\mathrm{A}))$ is strictly less than $\pi/2$. That will mean that the condition of Proposition 2.9 is satisfied and will finish the proof of Theorem 1.1.

PART II. PARALLEL TRANSLATION AROUND INFINITE TRIANGLES

In the hyperbolic space all triangles with three infinite vertices are congruent and the parallel translation around such a triangle rotates every vector in the plane of the triangle by $180°$ and is the identity for vectors perpendicular to the plane (see Proposition 2.8). We shall prove that this is approximately true in general, the error depending on the curvature bounds $-\Lambda^2 \leqslant K \leqslant -\lambda^2$. Thus, for a curvature sufficiently close to a constant the condition of Proposition 2.9 is satisfied.

Part II is organized as follows. In Section 3 we reformulate the Aleksandrov-Toponogov comparison theorems for infinite triangles, obtain auxiliary bounds for lengths, angles and areas and introduce the approximate plane of a triangle (see 3.3.7). In Section 5 we use the estimates for asymptotic Jacobi fields derived in Section 4 to replace the parallel translation around an infinite triangle by a special parallel

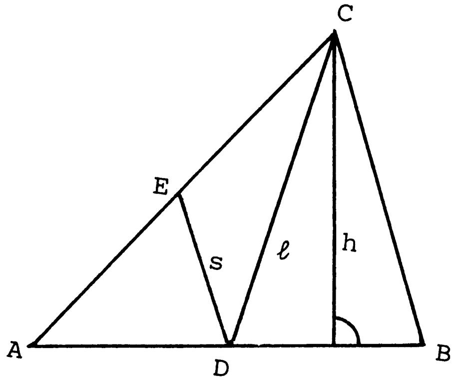

Figure 4

translation around the finite triangle formed by the midpoints (see Fig. 11). The difference between these two translations is bounded by const $\cdot$ $(\Lambda/\lambda - 1)$. Finally, for vectors in the approximate plane of the triangle and for vectors perpendicular to it we use a global construction to produce vector fields which are almost parallel along the edges of the midpoint triangle. Thus, we exhibit a map which rotates the approximate plane of the triangle by 180°, is the identity for perpendicular vectors and differs from the parallel translation around the infinite triangle by an error which tends to 0 when the ratio Λ/λ tends to 1. Bounds for this error for vectors in the approximate plane of the infinite triangle and perpendicular to it are obtained in 5.2.2 and 5.2.4 respectively.

§3. Comparison results for infinite triangles

3.1. Reformulation of the Aleksandrov-Toponogov angle comparison results in terms of triangle secants

Let M be a simply connected Riemannian manifold with curvature bounds $-\Lambda^2 \leqslant K \leqslant -\lambda^2$. Let ABC be a triangle in M with points $D \in AB$ and $E \in AC$ (see Fig. 4). Denote by l, s and h correspondingly the lengths of the secants CD, DE and the height dropped from C on AB. Subindices Λ and λ denote the corresponding quantities in the constant curvature comparison triangles with the same edgelengths. All heights are assumed interior. Let α, β, γ be the angles. The angle comparison results

$$\alpha_\Lambda \leqslant \alpha \leqslant \alpha_\lambda, \quad \beta_\Lambda \leqslant \beta \leqslant \beta_\lambda, \quad \gamma_\Lambda \leqslant \gamma \leqslant \gamma_\lambda \tag{3.1.1}$$

imply immediately

$$l_\Lambda \leqslant l \leqslant l_\lambda, \quad s_\Lambda \leqslant s \leqslant s_\lambda, \quad h_\Lambda \leqslant h \leqslant h_\lambda. \tag{3.1.2}$$

REMARK: As usual, the lower bounds also hold without sign restrictions on the curvature, while the upper bounds are not true for arbitrary triangles.

3.2. Extension to infinite triangles

Since infinite triangles are obtained as limits of finite triangles with lengths of secants and angles converging, one has immediately the extensions of 3.1.1 and 3.1.2 to infinite triangles, except that we have to agree what the corresponding points on the infinite edges of the triangles in M, M_Λ, M_λ are.

3.2.1. In the case of one or two infinite vertices we parametrize the infinite edges in such a way that they are asymptotic at the infinite vertices. Let a, b, c be the edges of the triangle. By choosing $a(0)$ correctly one also finds in the case of three infinite vertices a unique parametrization of the edges a, b, c such that they are pairwise asymptotic (a_+ with b_-, etc.); the points $a(0)$, $b(0)$, $c(0)$ (or O_1, O_2, O_3) are called the midpoints of the infinite edges.

3.3. Explicit formulas

Triangles with **one infinite vertex** in the hyperbolic plane are determined by two of the four quantities a, γ, β ($\geqslant \gamma$), u“ = ”$d(C, \infty) - d(B, \infty)$ through the following formulas (obtained by putting $u := b - c$ in a finite triangle and taking the limit

$$\left\{\begin{matrix} b \to \infty \\ u = \text{const} \end{matrix}\right\}, \quad \text{see Fig. 5)}.$$

$$e^u = \cosh a - \sinh a \cdot \cos \beta = (\cosh a - \sinh a \cdot \cos \gamma)^{-1}$$

$$e^u = \frac{\sin \beta}{\sin \gamma}$$

$$1 = -\cos \beta \cdot \cos \gamma + \sin \beta \cdot \sin \gamma \cdot \cosh a$$

$$\sinh h = \sinh a \cdot \sin \gamma. \qquad (3.3.1)$$

These have to be used together with 3.1.1 and 3.1.2, for example

$$u = 0 \quad \text{implies} \quad \operatorname{tgh} \lambda \frac{a}{2} = \cos \beta_\lambda \leqslant \cos \beta_\lambda \leqslant \cos \beta_\Lambda = \operatorname{tgh} \Lambda \frac{a}{2},$$

$$\beta = \frac{\pi}{2} \quad \text{implies} \quad \cosh \lambda a \leqslant e^{\lambda u}, \quad e^{\Lambda u} \leqslant \cosh \Lambda a. \qquad (3.3.2)$$

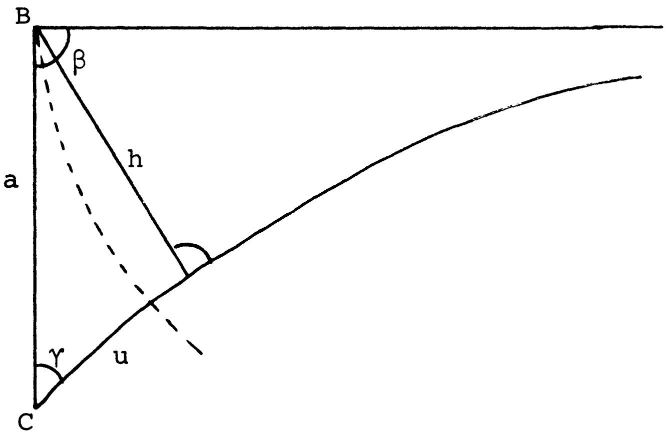

Figure 5. Conventions $\beta \geqslant \gamma$
$u := d(C, \infty) - d(B, \infty)$, - - - - - - horocircles (or horospheres)

Triangles with **two infinite vertices** in the hyperbolic plane are determined by the angle α at the finite vertex or by the length u (see Fig. 6, the dotted curves indicate horospheres) through the formulas

$$e^{u/2} = \sin\frac{\alpha}{2}, \quad e^{u/2} = \cosh h, \quad \cosh h \cdot \sin\frac{\alpha}{2} = 1. \tag{3.3.3}$$

Combining again with 3.1.1 and 3.1.2 we have in general

$$e^{-\Lambda(u/2)} = \sin\frac{\alpha_\Lambda}{2} \leqslant \sin\frac{\alpha}{2} \leqslant \sin\frac{\alpha_\lambda}{2} = e^{-\lambda(u/2)}$$

$$e^{\Lambda(u/2)} = \cosh \Lambda h_\Lambda, \quad h_\Lambda \leqslant h \leqslant h_\lambda, \quad \cosh \lambda h_\lambda = e^{\lambda(u/2)}. \tag{3.3.4}$$

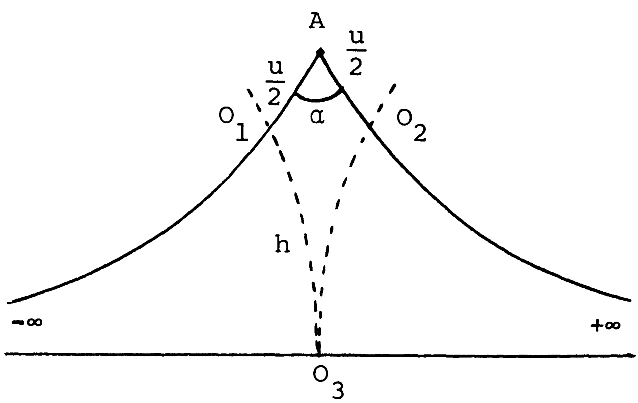

Figure 6. Midpoint O_3 of $+\infty$ and $-\infty$ defined by condition $d(A_1, +\infty) - d(A, -\infty) = 0$.

Application of the above inequalities to triangles with three vertices at infinity now gives the information which we need. O_i's are the midpoints (3.2.1) of the edges. In the hyperbolic plane we have (see Fig. 7):

3.3.5. $\alpha = \frac{1}{2}\pi$, $u = \ln 2$; $\sinh h = 1 = \sinh 2s \cdot \cos\frac{1}{2}\sigma$; on the other hand, $\cos 60° = \cosh s \cdot \sin\frac{1}{2}\sigma$, Hence, $\sinh s = \frac{1}{2}$, $e^s = \frac{1}{2}(\sqrt{5}+1)$, $\sin\gamma = \cos\beta = 1/\sqrt{5} = \operatorname{tgh} s$; V is in the plane spanned by E_+ and N, the "plane of the (totally geodesic) triangle".

This implies estimates which are sharp for $\lambda = \Lambda$:

3.3.6. $(1/\Lambda)\ \log\frac{1}{2}(\sqrt{5}+1) =: s_\Lambda \leqslant s \leqslant s_\lambda := (1/\lambda)\ \log\frac{1}{2}(\sqrt{5}+1)$, (see 3.1.2) $(1/\Lambda)\log 2 =: u_\Lambda \leqslant u \leqslant u_\lambda := (1/\Lambda)\log 2$.
(Take in 3.3.4 the limit of $h - \frac{1}{2}u$ as A goes to infinity),
$(\lambda/\sqrt{5}\,\Lambda)\ \cos\beta^* := \operatorname{tgh}\lambda s_\Lambda \leqslant \cos\beta \leqslant \operatorname{tgh}\Lambda s_\lambda =: \cos\beta_* \leqslant \Lambda/\lambda\sqrt{5}$, (see 3.3.2)
$2^{-\Lambda/2\lambda} = e^{-\Lambda(u_\lambda/2)} \leqslant \sin\frac{1}{2}\alpha \leqslant e^{-\lambda(u_\Lambda/2)} = 2^{-\lambda/2\Lambda}$, (see 3.3.4)
or $\cos\alpha^* := 1 - 2\cdot 2^{-\lambda/\Lambda} \leqslant \sin(\frac{1}{2}\pi - \alpha) \leqslant 1 - 2\cdot 2^{-\Lambda/\lambda} =: \cos\alpha_*$.

The estimates for α in 3.3.6 justify

3.3.7. DEFINITION: *The approximation plane of the triangle* $\infty_1\infty_2\infty_3$ *at* O_1 *is defined as the span of* N *and* E_+; N *is called the approximate normal of the edge at* O_1. (*Fig.* 7 *below illustrates these notions.*)

In the following we need as a measure of the planarity of the infinite triangle $\infty_1\infty_2\infty_3$ a bound ϵ^* for the angle between the halfplanes: positive span $(\pm E_+, N)$ and positive span $(\pm E_+, V)$ (see Fig. 8).

For this purpose choose Y at O_1 such that $Y \perp N$, $Y \perp E_+$, dim span$(E_+, N, V, Y) = 3$, $\sphericalangle(V, Y) \leqslant \frac{1}{2}\pi$.

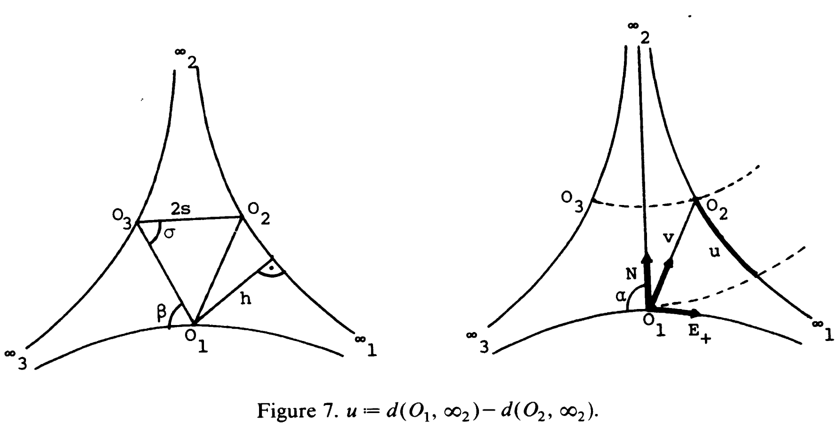

Figure 7. $u := d(O_1, \infty_2) - d(O_2, \infty_2)$.

Clearly $\frac{1}{2}\pi - \sphericalangle(V, Y)$ is the angle between the halfplanes and we shall derive the upper bound ϵ^* by estimating $\sphericalangle(V, Y)$ from below in the triangle $O_2O_1\infty_4$ (∞_4 is the infinite point defined by Y).

First, in the triangles $\infty_1O_1\infty_2$, $\infty_1O_1\infty_4$, $\infty_2O_1\infty_4$ we have that the broken ways from ∞ to ∞ via O_1 are longer by u_1, u_2, u_3 than the direct line from ∞ to ∞ with

$$\frac{1}{\Lambda}\log 2 = u_\Lambda \leqslant u_i \leqslant u_\lambda = \frac{1}{\lambda}\log 2,$$

for $i = 2, 3$ because of 3.3.4, for u_1 because of 3.3.6.

Let $\infty_4\infty_1$ (resp. $\infty_4\infty_2$) be parametrized asymptotically to the edges of $\infty_1\infty_2\infty_3$ at ∞_1 (resp. at ∞_2). Then O_1 is at parameter $-u_1$ from ∞_2 and at $-u_2$ from ∞_4; the parametrization $\infty_2\infty_4$ is ahead of the parametrization of $O_1\infty_4$ by $u_1 + u_2 - u_3$. We therefore shift the origin O_2 by $v := \frac{1}{2}(u_1 + u_2 - u_3)$ towards ∞_1 to obtain the origin O_{124} of the coherent parametrization of $\infty_1\infty_2\infty_4$. Then, again by 3.3.6, $u_\Lambda \leqslant u_4 := d(O_{124}, \infty_4) - d(O_{124}, \infty_2) \leqslant u_\lambda$; similarly 3.1.2 says that $d(O_2, \infty_4) - d(O_2, \infty_1)$ is bounded by the corresponding difference in constant curva-

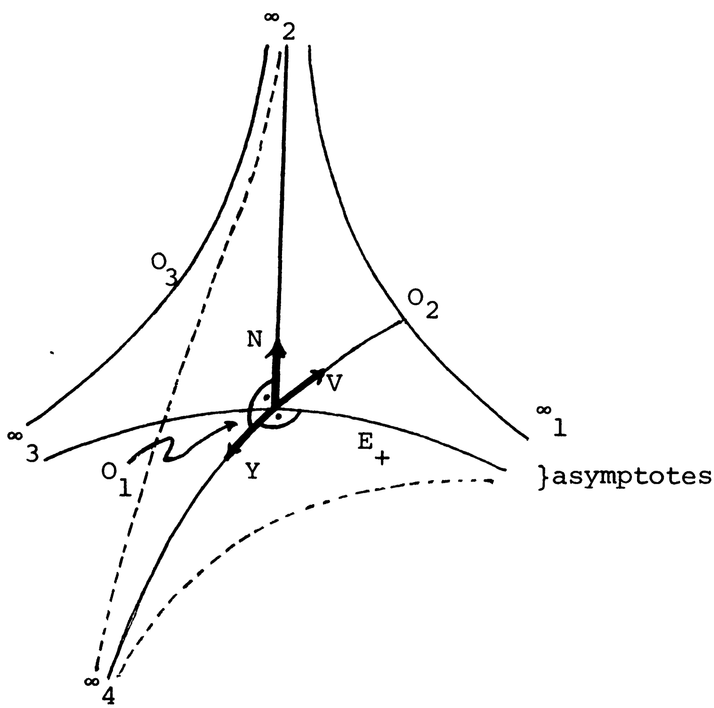

Figure 8. $Y \perp N, E_+$, $\sphericalangle(V, Y) \leqslant \frac{1}{2}\pi$, dim span$(E_+, N, V, Y) = 3$, $\infty_2\infty_4$ and $O_1\infty_4$ are not asymptotically parametrized.

ture, the latter being obtained from 3.3.2 (see Fig. 9):

$$\frac{1}{\lambda}\log\cosh\lambda v \leqslant x^- := d(O_2, \infty_4) - d(O_{124}, \infty_4)$$

$$\leqslant \frac{1}{\Lambda}\log\cosh\Lambda v.$$

This gives the needed bound for the triangle $O_2O_1\infty_4$:

$$d(O_2, \infty_4) - d(O_1, \infty_4) = (u_4 + x^-) - (u_2 - v) \tag{3.3.8}$$

$$\geqslant \left(\frac{3}{2\Lambda} - \frac{1}{\lambda}\right)\log 2 + \frac{1}{\lambda}\log\cosh\left(\left(\frac{\lambda}{\Lambda} - \frac{1}{2}\right)\log 2\right) =: w_* \quad (3.3.8)$$

which implies with 3.1.1, 3.3.1 (and the choice of Y).

$$0 \leqslant \cos\sphericalangle(O_2O_1\infty_4) \leqslant \frac{\cosh 2\Lambda s_\lambda - e^{\Lambda w_*}}{\sinh 2\Lambda s_\lambda} =: \sin\epsilon^*. \tag{3.3.9}$$

3.4. Area bounds

If a surface formed by a family of geodesics is inscribed into any triangle T, then the area of this "ruled" surface is bounded from above by the area of the triangle T_λ with the same edgelengths in the hyperbolic plane

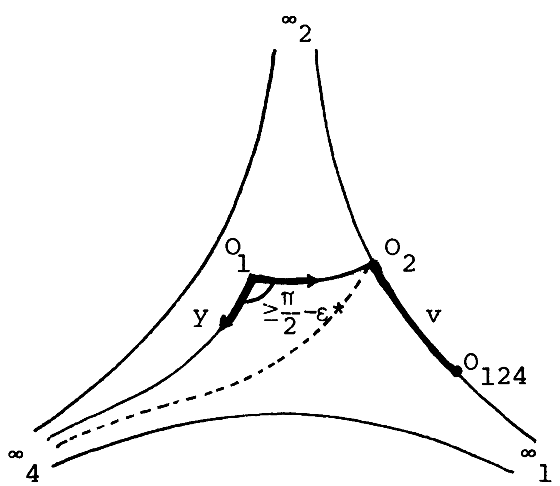

Figure 9. $2s_\Lambda \leqslant d(O_1, O_2) \leqslant 2s_\lambda$, is the origin for $O_{\infty_1\infty_2\infty_4}$, $d(O_2, O_{124}) = v = \frac{1}{2}(u_1 + u_2 - u_3)$, $d(O_2, \infty_4) - d(O_1, \infty_4) \geqslant w_*$.

Table 1

$\frac{\lambda}{\Lambda}$	α_*	α^*	β_*	β^*	$6\cdot\epsilon^*$ (3.3.9)	$\frac{3}{5}\pi(\frac{\Lambda^2}{\lambda_2}-1)$ (5.1.5)	d* (5.2.1)	$6\cdot(\kappa-\lambda)2s_\lambda$ (3.3.6, 4.4.2)
0.8								
0.91	86.20°	93.69°	61.02°	65.68°	63.84°	22.42°	0.134	16.75°
0.93	87.09°	92.85°	61.59°	65.17°	49.93°	16.87°	0.128	12.68°
0.95	87.947°	92.021°	62.144°	64.675°	35.88°	11.68°	0.123	8.82°
0.99	89.6°	90.4°	63.19°	63.68°	7.27°	2.19°	0.114	1.68°
1.00	90°	90°	63.435°	63.435°	O	O	0.112	O

of curvature $-\lambda^2$, [2], [7, p. 106]. This result extends immediately to triangles with one, two or three infinite vertices.

3.4.1. Example: Let T have one infinite vertex and $u=0$ (see 3.3.2), then:

$$\text{area (ruled surface inscribed in } T) \leqslant \frac{\pi - 2\beta_\lambda(a)}{\lambda^2},$$

if in addition $a \leqslant 2s_\lambda$ (see 3.3.6), then $\cos \beta_\lambda(2s_\lambda) = 1/\sqrt{5}$, hence

$$\text{area} \leqslant \frac{\pi - 2\cdot 63,435^\circ}{\lambda^2} \leqslant \frac{0.3\pi}{\lambda^2}.$$

Remark: The slightly weaker inequality – area $\leqslant (\pi - 2\beta_\Lambda(\mathrm{a}))/\lambda^2$ – follows directly from 3.1.1 and the Gauss Bonnet theorem for the spanned surface.

§ 4. Asymptotic Jacobi fields

Throughout this section the curvature bounds $-\Lambda^2 \leqslant K \leqslant -\lambda^2 < 0$ are assumed. Since asymptotic Jacobi fields can be obtained as limits of Jacobi fields with a fixed $J(0)$ and with the first zero further and further away one has the following well known consequence of the classical Raugh estimates (see [9]).

$$|J(0)|e^{-\Lambda s} \leqslant |J(s)| \leqslant |J(0)|e^{-\lambda s},$$

$$|J'(s)| \leqslant \frac{\Lambda^2}{\lambda}|J(s)|, \quad (0 \leqslant s \leqslant \infty). \tag{4.1}$$

In the hyperbolic plane we also have $J'(s) + J(s) = 0$, the following generalizes this relation.

4.2. Put $\kappa^2 = \frac{1}{2}(\Lambda^2 + \lambda^2)$ and let P be a unit length parallel field along a geodesic c. Then for any asymptotic Jacobi field J along c

$$\langle J'(s) + \kappa \cdot J(s), P\rangle \leqslant |J(s)| \cdot (\kappa - \lambda),$$

which has the immediate consequences

$$|J'(s) + \kappa \cdot J(s)| \leqslant |J(s)| \cdot (\kappa - \lambda),$$

$$\lambda \cdot |J(s)| \leqslant \frac{\langle J', J\rangle}{|J|}(s) \leqslant |J'(s)| \leqslant |J(s)| \cdot (2\kappa - \lambda),$$

$$|(J')^{\perp}(s)| \leqslant |J| \cdot (\kappa - \lambda) \quad (\text{“} \perp \text{” means perpendicular to } J),$$

$$|\mathrm{tg}\sphericalangle(-J(s), J'(s))| \leqslant \frac{\kappa - \lambda}{\lambda}, \quad |\sin\sphericalangle(-J, J')| \leqslant \frac{\kappa - \lambda}{\kappa}.$$

PROOF: Since for an asymptotic Jacobi field no point along c is distinguished, it suffices to prove the inequality at $s = 0$. For any parallel field P we have from 4.1

$$\lim_{s \to \infty} \langle e^{-\kappa s} J'(s) + \kappa \cdot e^{-\kappa s} J(s), P(s)\rangle = 0,$$

hence,

$$\langle J'(0) + \kappa \cdot J(0), P\rangle = \int_0^{\infty} \langle -e^{-\kappa s} J'' + \kappa^2 e^{-\kappa s} J, P\rangle \mathrm{d}s.$$

We substitute $J'' = -R(J, c')c'$ and use

$$|R(J, c')c' + \kappa^2 J| \leqslant |J| \cdot \max(\kappa^2 - \lambda^2, \Lambda^2 - \kappa^2)$$

together with (4.1) to obtain the main inequality:

$$\langle J'(0) + \kappa J(0), P\rangle \leqslant \int_0^{\infty} (\kappa^2 - \lambda^2)|J(0)|\, e^{-(\kappa+\lambda)s} \cdot |P| \mathrm{d}s.$$

For the angle estimate note that

$$|\mathrm{tg}\sphericalangle(-J, J')| \leqslant \frac{|J' + \kappa \cdot J|}{|J'|}, \quad |\sin\sphericalangle(-J, J')| \leqslant \frac{|J' + J|}{|\kappa J|}.$$

We illustrate the obtained information by the following

4.2. COROLLARY: *The second fundamental form S of a horosphere satisfies*

$$|S + \kappa id| \leqslant \kappa - \lambda,$$

and its intrinsic curvature K_H is bounded by

$$-\Lambda^2 + \lambda^2 \leqslant K_H \leqslant -\lambda^2 + (2 - \lambda)^2 \leqslant 2\Lambda(\Lambda - \lambda).$$

PROOF: Let $J(0)$ be a vector tangent to a horosphere, if one extends $J(0)$ to an asymptotic Jacobi field, then the second fundamental form is given by $S \cdot J(0) = J'(0)$. The Gauss equations give the bounds for the intrinsic curvature.

4.4. COROLLARY: *A surface spanned by asymptotic geodesics through γ is "almost plane" along γ, see* (4.4.2) *for estimates.*

REMARK: This fact will be used to produce almost parallel vector fields with known global properties.

To be more precise and to prove the corollary we introduce the following notation (see Fig. 10).

4.4.1. Let $s \to c(s, t)$ be a family of unit speed asymptotic geodesics along another unit speed geodesic $\gamma(t)$; in particular $\lim_{s \to \infty} d(c(s, t), c(s, \tau))$

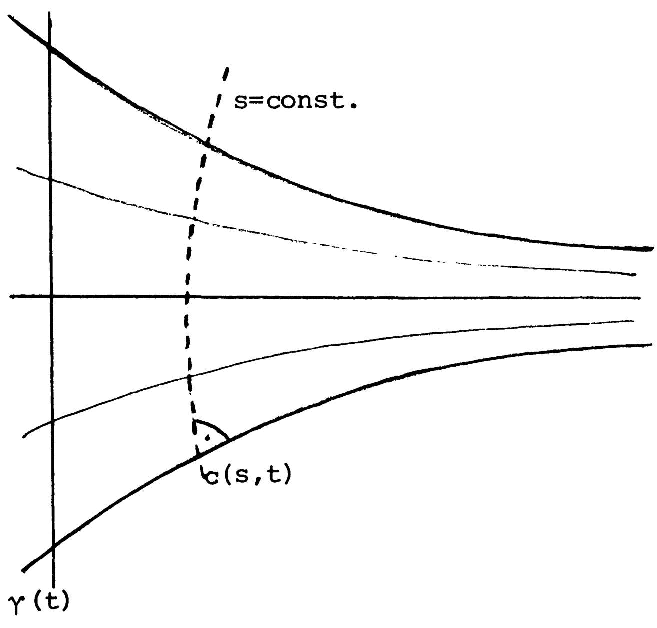

Figure 10

$=0$ and $\gamma(t)=c(\sigma(t),t)$, where σ is a Busemann function for the asymptote class, taken along γ. Let $c'(s,t):=\partial/(\partial s)c(s,t)$ be the unit tangent field of the asymptotes, $\dot c(s,t):=\partial/(\partial t)c(s,t)\perp c'$ is an asymptotic Jacobi field along each of the asymptotes. Since $\dot\gamma(t)=c'(\sigma(t),t)\cdot\dot\sigma+\dot c(\sigma(t),t)$, we have $|\dot c|^2=1-\dot\sigma^2=1-\langle\dot\gamma,c'(\sigma(t),t)\rangle^2$, $D/(\mathrm{d}t)(c'(\sigma(t),t))=D/(\partial s)c'(\sigma(t),t)\cdot\dot\sigma+D/(\partial t)c'=D/(\partial s)\dot c$. Let $V(t):=(c'-\langle c',\dot\gamma\rangle\dot\gamma)/(|c'-\langle c',\dot\gamma\rangle\gamma|)=(c'-\langle c',\dot\gamma\rangle\dot\gamma)/(|\dot c|)$ be the component of c' perpendicular to $\dot\gamma$ and normalized to unit length.

$$\left|\frac{D}{\mathrm{d}t}V\right|\leqslant\kappa-\lambda\leqslant\frac{\Lambda^2-\lambda^2}{4\lambda}\tag{4.4.2}$$

(independent of the angle between γ and the asymptotes).

PROOF: The components perpendicular to $\dot\gamma$ (etc.) are indicated by $\perp\dot\gamma$ (etc.). After a short computation we find (using 4.4.1):

$$\left|\frac{D}{\mathrm{d}t}V\right|^2\cdot|\dot c|^2=\left|\frac{D}{\partial t}c'\right|^2-\left\langle\frac{D}{\partial t}c',\dot\gamma\right\rangle^2-\left\langle\frac{D}{\partial t}c',\frac{(c')^{\perp\dot\gamma}}{|(c')^{\perp\dot\gamma}|}\right\rangle^2$$

$$=\left|\left(\frac{D}{\partial s}\dot c\right)^{\perp\dot c}\right|^2,$$

hence with 4.2

$$\left|\frac{D}{\mathrm{d}t}V\right|\leqslant\frac{|(J')^{\perp}|}{|J|}\leqslant\kappa-\lambda.$$

§ 5. Parallel translation around a cusp

5.1. The adapted connection

We consider again the situation described in 4.4.1. Since the area of the surface spanned by the asymptotes is finite (3.4) we can parallel translate a vector along one asymptote to infinity and back along another one. Also we can split the tangent bundle of M along c.

$$TM|_c=\mathbb{R}\cdot c'\oplus(c')^{\perp};\quad E:=(c')^{\perp}.\tag{5.1.1}$$

We have a natural connection for the bundle E:

5.1.2. DEFINITION: *For a section Y in E put*

$$D^E Y := (DY)^{\perp c'} = E\text{-component of } DY.$$

5.1.3. $D_{c'} Y = D^E_{c'} Y$, since $\langle Y, c' \rangle = 0$ implies $\langle D_{c'} Y, c' \rangle = 0$.

Thus, the two derivatives agree in the direction c', but the advantage of D^E is its small curvature:

$$|R^E(c', \dot{c})Y| \leqslant \tfrac{2}{3}(\Lambda^2 - \lambda^2)|c' \wedge \dot{c}| \cdot |Y|. \tag{5.1.4}$$

PROOF: Assume $(D/(\partial s))Y = 0 = (D^E/(\partial s))Y$. Then

$$R^E(c', \dot{c})Y = \frac{D}{\partial s}\left(\frac{D}{\partial t} Y - \langle \frac{D}{\partial t} Y, c' \rangle c'\right)$$

$$= R(c', \dot{c})Y - \langle R(c', \dot{c})Y, c' \rangle c' = \left(R(c', \dot{c})Y\right)^{\perp c'}$$

Now $|R(u, v)w + ((\Lambda^2 + \lambda^2)/2)(\langle v, w \rangle u - \langle u, w \rangle v))| \leqslant \frac{2}{3}(\Lambda^2 - \lambda^2)|u \wedge v| \cdot |w|$ [7, p. 91] completes the proof.

5.1.5. *D^E-translation around a cusp.* Assume that the baseline γ of the cusp in 4.4.1 has length $\leqslant 2s_\lambda$ (see 3.3.6), since we want to deal with cusps obtained by connecting two midpoints of the edges of an infinite triangle. Then use 3.4.1 and 5.1.4 to obtain:
The D^E-translation around such a cusp differs from the identity at most by [7, p. 92]

$$\|R^E\| \cdot (\text{area of the cusp}) \leqslant \tfrac{1}{5}\pi\left(\frac{\Lambda^2}{\lambda^2} - 1\right).$$

REMARK: If the curvature is constant, $\lambda = \Lambda$, then the D-translation around an infinite triangle is same as the D^E-translation around the triangle connecting its midpoints.

5.2. Almost D^E-parallel vector fields

Clearly, the usefulness of (5.1.5) depends on being able to produce almost D^E-parallel fields. We deal separately with the two cases where the vectors are approximately in the plane of the triangle (see 3.3.7) and perpendicular to it, respectively.

5.2.1. Let $W(t) := (\dot{\gamma} - \langle \dot{\gamma}, c' \rangle c')/(|\dot{\gamma} - \langle \dot{\gamma}, c' \rangle c'|) = \dot{c}/|\dot{c}|$ be the component of $\dot{\gamma}$ perpendicular to c' and normalized to unit length. This vector field starts and ends approximately in the plane of the triangle (3.3.9) and it is almost D^E-parallel:

$$\left|\frac{D^E}{dt} W(t)\right| \leqslant |\dot{\sigma}| \cdot (\kappa - \lambda),$$

where

$$\int_0^{2s} |\dot{\sigma}(t)| dt \leqslant 2\Big(\max_\gamma \sigma - \min_\gamma \sigma\Big) \qquad \text{(the convexity of } \sigma)$$

$$\leqslant 2\frac{1}{\Lambda} \log \cosh\left(\frac{\Lambda}{\lambda} \cdot \log\tfrac{1}{2}(\sqrt{5}+1)\right) =: \frac{2d^*}{\Lambda}.$$

PROOF: The last inequality follows from 3.3.2 with $\beta = \frac{1}{2}\pi$, $a \leqslant s_\lambda$ (3.3.6) and $u = \max_\gamma \sigma - \min_\gamma \sigma$. For the differentiation we use again the formulas in 4.4.1:

$$|\dot{c}| \cdot \frac{D^E}{dt} W(t) = -\dot{\sigma} \frac{D}{\partial t} c' + \left\langle W, \dot{\sigma} \frac{D}{\partial t} \sigma' \right\rangle \cdot W$$

$$= -\dot{\sigma}\left(\frac{D}{\partial s} \dot{c}\right)^{\perp \dot{c}},$$

and the estimate follows from 4.2, since $\dot{c}$ and $(D/\partial s)\dot{c}$ are the value and the derivative of an asymptotic Jacobi field.

5.2.2. THEOREM: *Let, as in* 3.3.7, span(N, E_+) *be the approximate plane of an infinite triangle at the midpoint* (3.2.1) *of an edge*; *let* $N^\perp$ *be the normal to the edge in this plane and let* $PN^\perp$ *be obtained by the parallel translation of* $N^\perp$ *around the infinite triangle. Then the following dimension independent estimate* (*which is sharp for* $\lambda = \Lambda$) *holds*:

$$|\sphericalangle(PN^\perp, -N^\perp)| \leqslant 6\epsilon^* + 6\frac{\kappa - \lambda}{\Lambda} d^* + 0.6\pi\left(\frac{\Lambda^2}{\lambda^2} - 1\right),$$

here ϵ^* *is taken from* 3.3.9 *and* d^* *from* 5.2.1.

PROOF: The third error term results from changing the parallel translation around the infinite triangle to the D^E-translation around the midpoint triangle (see 5.1.5). The error ϵ^* from 3.3.9 occurs twice at each midpoint since the edges of the midpoint triangle may not lie in the approximate plane of the infinite triangle (3.3.7). The last error results from 5.2.1 along each secant since we are using only approximately

D^E-parallel fields to get from one midpoint to another one. Finally note, that along each secant we switch from the normal pointing toward the opposite vertex to the normal pointing away from it.

5.2.3. To construct almost D^E-parallel fields approximately perpendicular to W (see 5.2.1) let X be a D-parallel unit field along γ, $X \perp \gamma$, $X(0) \perp c'$. Define

$$z(t) := \frac{X - \langle X, c' \rangle c'}{|X - \langle X, c' \rangle c'|}(t).$$

Put $\eta := (\kappa - \lambda)\, \mathrm{e}^{(\kappa/\Lambda)d^*}$ where d^* is taken from 5.2.1, and assume the curvature bounds are such that $2s_\lambda \cdot \eta < 1$ (see 3.3.6 for s_λ). Then

$$\left|\frac{D^E}{\mathrm{d}t} z(t)\right| \leqslant (2\kappa - \lambda) \cdot \eta t \cdot (1 - \eta^2 t^2)^{-1/2},$$

$$\int_0^{2s} \left|\frac{D^E}{\mathrm{d}t} z\right| \mathrm{d}t \leqslant (\kappa - \lambda) \cdot (2\kappa - \lambda) 4 s_\lambda^2\, \mathrm{e}^{(\kappa/\Lambda)d^*}.$$

REMARK: The last estimate is still sharp at $\lambda = \Lambda$, but the factor after $(\kappa - \Lambda)$ comes from rather rough estimates in the following proof.
PROOF: Recall $(D/\mathrm{d}t)\, c'(\sigma(t), t) = (D/\partial s)\, \dot{c}(\sigma(t), t)$ from 4.4.1 to get

$$\frac{D^E}{\mathrm{d}t} z = -\frac{\langle X, c' \rangle}{(1 - \langle X, c' \rangle^2)^{1/2}} \left(\frac{D}{\partial s} \dot{c}\right)^{\perp X}$$

where

$$\left|\frac{D}{\partial s} \dot{c}\right| \leqslant (2\kappa - \lambda)|\dot{c}| \leqslant 2\kappa - \lambda \quad \text{(see 4.2, 4.4.1)}.$$

Also, $\langle X, c' \rangle(t)$ ($= 0$ at $t = 0$) grows slowly along γ (use $\langle X, \dot{c} \rangle = -\dot{\sigma} \cdot \langle X, c' \rangle$ from 4.4.1), thus,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle X, c' \rangle \mathrm{e}^{-\kappa\sigma(t)}\right)\right| = \mathrm{e}^{-\kappa\sigma(t)} \left|\left\langle X, \frac{D}{\partial s} \dot{c} + \kappa \cdot \dot{c} \right\rangle\right|$$

$$\leqslant \mathrm{e}^{-\kappa\sigma(t)} (\kappa - \lambda) \cdot |\dot{c}| \quad \text{(see 4.2)},$$

$$|\langle X, c' \rangle(t)| \leqslant (\kappa - \lambda) \cdot \mathrm{e}^{\kappa \cdot (\sigma_{\max} - \sigma_{\min})} \cdot t \leqslant (\kappa - \lambda)\, \mathrm{e}^{(\kappa/\Lambda)d^*} t \quad \text{(see 5.2.1)}.$$

This proves the bound for $(D^E/\mathrm{d}t)\, z$ and together with

$$\int_0^{2s_\lambda} \frac{\eta t}{\sqrt{1-\eta^2 t^2}}\,\mathrm{d}t = \frac{1}{\eta}\left(1 - \sqrt{1-\eta^2 4 s_\lambda^2}\right) \leqslant \eta \cdot 4 s_\lambda^2$$

gives the total rotation of $z(t)$ against a D^E-parallel field (starting from $z(0)$) along γ.

5.2.4. THEOREM: *In the situation described in* 5.2.2 *let X be a vector which at the midpoint O_1 of an infinite edge is perpendicular to the approximate plane of the triangle* (span(N, E_+), *see* 3.3.7). *Then the parallel translation PX of X around the infinite triangle is very close to X*:

$$|\sphericalangle(X, PX)| \leqslant 6\epsilon^* + 0.6\left(\frac{\Lambda^2}{\lambda^2} - 1\right) + 6\cdot(\kappa - \lambda)2s_\lambda$$

$$+ 3(\kappa-\lambda)(2\kappa-\lambda)4s_\lambda^2\, e^{(\kappa/\lambda)d^*}.$$

PROOF: Let X also denote the vector field which is parallel along the halfedges of the infinite triangle and continuous at O_2 and O_3. For a global control of X construct a pyramid over O_1, O_2, O_3 by joining the point X_∞ – which is determined by $X(O_1)$ at infinity (see Fig. 11) – with the edges of the midpoint triangle. Consider the initial tangents of the asymptotic geodesic forming the faces of the pyramid, take the components of these tangents perpendicular to the corresponding cusp of the infinite triangle, normalize the resulting vector field to length 1 and denote it by Y_i along the i-th cusp. Then:

$$|\sphericalangle(Y_i(O_{i+1}), Y_{i+1}(O_{i+1}))| \leqslant 2\epsilon^* \quad \text{(from 3.3.9, indices mod 3)},$$

$$\left|\frac{D}{\mathrm{d}t} Y_i\right| \leqslant 2\cdot(\kappa - \lambda) \quad \text{(from 4.4.2, each of the two surfaces spanned}$$

by asymptotes which meet along O_iO_{i+1} is almost plane in the sense of 4.4).

Next, let V_i be D-parallel along O_iO_{i+1} starting with $V_i(O_i) = Y_i(O_i)$, then $(D^E/\mathrm{d}t)\, V_i$ is bounded by 5.2.3 and the parallel translation around the cusp from O_i to ∞_i to O_{i+1} differs from the D^E-translation along O_iO_{i+1} by the error of 5.1.5. Therefore we control the angle between X and Y_i

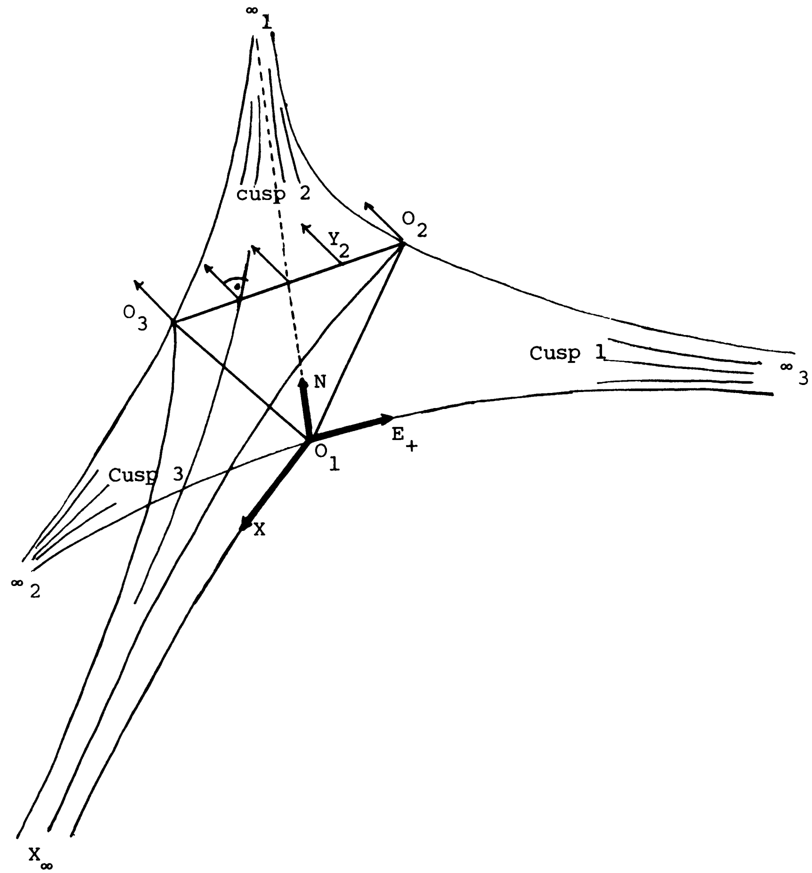

Figure 11. An infinite triangle with the midpoint triangle $O_1O_2O_3$ and the pyramid over it determined by $X \perp N$, E^+.

around the infinite triangle:

$$|\sphericalangle(X(O_1), Y_1(O))| \leqslant \epsilon^*$$

$$|\sphericalangle(X(O_2), Y_1(O_2))| \leqslant \epsilon^* + 2 \cdot \text{error } (4.4.2) + \text{error } (5.2.3) + \text{error } (5.1.5).$$

When switching from Y_1 to Y_2 at O_2 we make another error $\leqslant 2\epsilon^*$, then along O_2O_3 we again pick up $2 \cdot$ error (4.4.2) + error (5.2.3) + error (5.1.5); the same again for Y_3 along O_3O_1 and a final ϵ^* between $X(O_1)$ and $Y_3(O_1)$. This proves 5.2.4.

Table 2 shows that under the pinching assumption $\lambda/\Lambda \geqslant 0.93$ the condition of Proposition 2.9 is satisfied, which proves Theorem 1.1.

Table 2. (The error $6 \cdot \epsilon^*$ is more than $\frac{1}{2} \cdot$ error (5.2.4))

$\frac{\lambda}{\Lambda}$	Error (5.2.2)	$3 \cdot (\kappa - \lambda)(2\kappa - \lambda) 4 s_\lambda^2 e^{(\kappa/\Lambda)d^*}$ (5.2.3)	Error (5.2.4)
0.8			
0.91	88.37°	10.08°	113.09°
0.93	68.37°	7.44°	86.91°
0.95	48.62°	5.04°	61.41°
0.99	9.66°	0.912°	12.05°
1.00	0	0	0

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