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RAHIM ZAARE-NAHANDI

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SEMINORMALITY OF CERTAIN GENERIC PROJECTIONS

Rahim Zaare-Nahandi

1. Introduction

Seminormality of generic projections has been conjectured by some authors, such as E. Bombieri [4], A. Andreotti and P. Holm [1]. The major objective of this paper has been in this direction. Our main result is the following.

THEOREM: *Let X be a nonsingular projective variety of dimension $r \geqslant 6$ over a field k with* $\mathrm{char}(k) \neq 2, 3$. *Assume that X is suitably embedded. If $\pi: X \to \mathbb{P}_k^m$ is a generic projection where $(3r-3)/2 \leqslant m \leqslant 2r$, then $\pi(X)$ is seminormal.*

For real varieties, a similar result has been proved when $3r/2 \leqslant m \leqslant 2r$ by Andreotti and Holm [1, Theorem 25.1]. By completely different methods the result has been proved for complex varieties in the range $(3r-1)/2 \leqslant m \leqslant 2r$ by Adkins, Andreotti and Leahy [28, Corollary 7.5]. The case $m = [(3r-3)/2]$ is of particular importance. From the geometric point of view, the difficulty is due to appearance of the two following types of singularities: points with several analytic branches where some of them are non-linear, points of $S_1^{(2)}$-type singularity. Our theorem covers this case as well. Adkins has recently proved that generic projections in all range are Lipschitz saturated which is a weaker property than seminormality [29, Theorem 5.1].

Our basic method to check the seminormality of generic projections is Proposition (2.10). At some points we have checked the seminormality directly. A condition on the depth of certain modules which is needed to complete the proof of the main theorem, is proved in the following chapter.

An earlier version of this paper formed part of a Ph.D. thesis, written at the University of Minnesota under the direction of Professor Joel Roberts. I wish to thank him for his encouragement and for many valuable suggestions. The appendix is an unpublished result of Professor Roberts which we have used in this paper several times.

We would like to refer to R. Hartshorne's book of algebraic geometry [8] for the notation and the techniques of modern algebraic geometry, which will be used throughout this paper.

2. Preliminaries and some criteria of seminormality

All rings will be assumed to be commutative with identity. Let A be a ring, let B be an overring of A. The *conductor of A in B* is $(A:B)=\{a\in A : aB\subseteq A\}$ which is also the annihilator of the A-module B/A. Let K be the total ring of quotients of A. We denote by $\bar{A}$ the integral closure of A in K. $(A:\bar{A})$ is called *the conductor of A*. We will let $R(A)$ denote the Jacobson radical of A, i.e., $R(A)$ is the intersection of all maximal ideals of A. Let M be an A-module, for $P\in \operatorname{Spec}(A)$, we denote by M_P the A_p-module $S^{-1}M$, where $S=A-P$. If $m\in M$, we let m_p denote the image of m under the canonical homomorphism $M\to M_P$. We also let $k(P)=A_P/PA_P$.

Definition (2.1): Let $A\subset B$ be an integral extension of rings. We define

$$\underset{B}{+A}=\{b\in B \mid b_P\in A_P+R(B_P),\quad \forall P\in \operatorname{Spec}(A)\}.$$

$\underset{B}{+A}$ is called the *seminormalization of A in B* and if $A=\underset{B}{+A}$ we say that *A is seminormal in B*. If B is the integral closure of A (in its total ring of quotients) we set $+A=\underset{B}{+A}$ and we say that *A is seminormal* if $A=+A$. $+A$ is called the seminormalization of A.

Remark: Some authors for a ring A to be seminormal, besides the condition $A=+A$, require that A to be a Mori ring [7,1.3]. A Mori ring is a ring which is reduced and its integral closure is a finitely generated A-module. Since algebro-geometric rings are Mori rings [27, Vol. I, page 167, Theorem 9, and Vol. II, page 320, Theorem 31], in dealing with varieties the two definitions will coincide.

We recall Traverso's characterization of the operation of seminormalization.

Proposition (2.2): (*Traverso* [24]). *$\underset{B}{+A}$ is the largest subring A' of B containing A such that*:

(1) *For each $P\in \operatorname{Spec}(A)$, there is exactly one $P'\in \operatorname{Spec}(A')$ lying over P, and*

(2) *The canonical homomorphism $k(P)\to k(P')$ is an isomorphism.*

The following version of a result of Greco and Traverso will be used in this paper.

Theorem (2.3): (*Greco and Traverso* [7, *Theorem* 2.6].) *Let B be a finite* *overring of the ring A. Then*:

(*a*) *The following are equivalent*:

(*i*) *A is seminormal in B*;

(*ii*) A_P *is seminormal in* B_P *for all* $P \in \mathrm{Spec}(A)$;

(*iii*) A_m *is seminormal in* B_m *for all maximal ideals m of A*;

(*iv*) A_p *is seminormal in* B_P *for all* $P \in \mathrm{Ass}_A(B/A)$.

(*b*) *Consider the condition*

(*) *every non-zerodivisor of A is a non-zerodivisor in B.*

If (*) *holds and A is* S_1 *then the conditions* (*i*) *to* (*iv*) *are equivalent to*:

(*v*) $\mathrm{nil}(B) \subset A$ *and* A_P *is seminormal in* B_P *whenever* $\mathrm{depth}(A_P) = 1$.

(*c*) *If* (*) *holds and A is* S_2 *then* (*i*) *to* (*v*) *are equivalent to*:

(*vi*) $\mathrm{nil}(B) \subset A$ *and* A_P *is seminormal in* B_P *whenever P has height* 1.

REMARK: The formulation of Greco and Traverso is the same as Theorem (2.3) except that in (b) A is not assumed to be S_1 and the condition $\mathrm{nil}(B) \subset A$ is not imposed in either (v) or (vi). Under such weaker hypothesis the implications (v) → (i) and (vi) → (i) are not valid. As a counterexample let $A = k$ be a field, let $B = k[x]/(x^2)$. Then (v) and (vi) are satisfied but A is not seminormal in B because $\underset{B}{+}A = B$. In the suggested formulation the proof of the mentioned implications follows from the argument given by Greco and Traverso and the following lemma. Observe that Theorem (2.3) applies when B is the normalization of a Mori ring A.

LEMMA (2.4): *Let K be a ring of dimension zero, and let R be a finite overring of K. If* $\mathrm{nil}(R) \subset K$ *then K is seminormal in R.* (*By dimension of a ring we mean its Krull dimension*).

PROOF: Let $C = \underset{R}{+}K$. We show that $C_{\mathrm{red}} = K_{\mathrm{red}}$, then since $\mathrm{nil}(C) \subset K$, by [7, Proposition 2.5, (ii) → (i)] K is seminormal in C, which is not possible unless $C = K$. If $P_1, \dots, P_n$ are the prime ideals of K, then $K_{\mathrm{red}} \cong K_1 \times \dots \times K_n$ where $K_i = K/P_i$. As C is the seminormalization of K, Spec(C) consists of $Q_1, \dots, Q_n$ where Q_i lies over P_i, and since Q_i are also maximal ideals of C, $C/Q_i \cong K_i$. But then $C_{\mathrm{red}} \cong C/Q_1 \times \dots \times C/Q_n \cong K_1 \times \dots \times K_n \cong K_{\mathrm{red}}$.

We will need the following result of Orecchia.

PROPOSITION (2.5): (*Orecchia* [17, *the main result*].) *Let A be a noetherian reduced ring with minimal primes* $P_1, \dots, P_n$. *Let* $A_i = A/P_i$, *and* $I = (A : \Pi_{i=1}^n A_i)$. *Assume that the ideals* $P_i + \bigcap_{i \neq j \leq h} P_j$ ($i, h = 1, \dots, n$) *all are* *unmixed of common pure height whenever* $\bigcap_{i \neq j \leqslant h} P_j \neq \emptyset$. *Then the following are equivalent*:

(*i*) *A is seminormal in* $\prod_{i=1}^{n} A_i$;

(*ii*) *I is a radical ideal in A*;

(*iii*) *For every associated prime P of I, the Zariski tangent space $\Theta_{X,P}$ of $X = \operatorname{Spec}(A)$ at P is the direct sum of the Zariski tangent spaces Θ_{X_i,P_i} of $X_i = \operatorname{Spec}(A_i)$ corresponding to the minimal primes P_i contained in P.*

NOTE: Observe that we have

$$I = \bigcap_{i=1}^{n} \left(P_i + \bigcap_{j \neq i} P_j \right). \tag{2.5.1}$$

We now recall that if R is a noetherian ring, and M is an R-module, an M-regular sequence (or simply M-sequence) of length n is a sequence $a_1, \ldots, a_n$ of elements in R such that a_1 is not a zerodivisor of M, and a_i is not a zerodivisor of $M/(a_1, \ldots, a_{i-1})M$ for $i = 2, \ldots, n-1$, and $(a_1, \ldots, a_n).\ M \neq M$. By $\operatorname{depth}_R(M)$ we mean the length of a maximum M-regular sequence. Also, $\dim_R(M) = \sup \dim(R/P)$, for all $P \in \operatorname{Ass}_R(M)$.

The following result is well-known (c.f. [2, Lemma 1.4], or for similar result [11, page 103, exercise #14]).

LEMMA (2.6): *Let A be a local ring, and let*

$$0 \to L \to M \to N \to 0$$

be an exact sequence of finite A-modules. Then one of the following statements must hold:

(*i*) *depth $L \geqslant$ depth $M =$ depth N*

(*ii*) *depth $M \geqslant$ depth $L = 1 +$ depth N*

(*iii*) *depth $N \geqslant$ depth $L =$ depth M.*

We will need the following result of J.P. Serre [23, Chapter IV, Proposition 12].

PROPOSITION (2.7): *Let (A, m) be a noetherian local ring, let B be a noetherian ring, and let φ: $A \to B$ be a ring homomorphism. Assume that B is a finite A-module via φ, so that B is semilocal. Then* $\operatorname{depth}_A M = \operatorname{depth}_B M$.

NOTE: In the formulation of Serre B is local ring. However almost the same proof applies when B is semilocal.

LEMMA (2.8): *Let R be a Mori ring, let S be the integral closure of R. Assume that S is a Cohen-Macaulay ring. Let C be the conductor of R.*

Assume that all minimal prime ideals of C have common height t. If P is a prime ideal of R containing C, and of height h, then $\operatorname{depth}_{R_p}(R_p) \leqslant h - t + 1$.

PROOF: Consider the exact sequence of R modules $0 \to R \to S \to S/R \to 0$. This gives an exact sequence of R_P modules $0 \to R_P \to S_{R-P} \to (S/R)_{R-P} \to 0$. ($S_{R-P}$ means $(R-P)^{-1}$. S, and so on). Let $A = R_P$, $B = S_{R-P}$. Since S is a Cohen-Macaulay ring, B is a Cohen-Macaulay ring. Since B is the normalization of A, we have $\dim(B) = \dim(A) = \operatorname{height}(P) = h$. Thus $\operatorname{depth}_B(B) = \dim(B) = h$. Since R is a Mori ring, S is a finite R-module, and hence B is a finite A-module. Therefore by Proposition (2.7) we have $\operatorname{depth}_A(B) = h$. We now show that $\operatorname{depth}_A((S/R)_{R-P}) \leqslant h$. Since $C \subseteq P$, P contains some minimal prime ideals of C in R. Let $Q_1, \ldots, Q_n$ be all the minimal prime ideals of C contained in P. By assumption $\operatorname{height}(Q_i) = t$, $i = 1, \ldots, n$. Since S is a finite R-module, the conductor of A in B is $C \cdot A$ [27, Vol. I, page 269]. The minimal primes of $C \cdot A$ are $Q_1 \cdot A, \ldots, Q_n \cdot A$. Since $(S/R)_{R-P} \cong (S_{R-P})/R_P = B/A$, and by definition we have $\dim_A(B/A) = \dim(A/Q_i \cdot A)$, (Krull dimension), and

$$\dim(A/Q_i \cdot A) = \dim(R_P/Q_i R_P) = \dim(R_P) - \operatorname{height}(Q_i R_P)$$

$$= \operatorname{height}(P) - \operatorname{height}(Q_i) = h - t,$$

we obtain that $\operatorname{depth}_A((S/R)_{R-P}) \leqslant \dim_A(B/A) = h - t \leqslant h$. Now applying Lemma (2.6) for the exact sequence $0 \to A \to B \to B/A \to 0$, since $\operatorname{depth}_A(B/A) \leqslant \operatorname{depth}_A(B)$, we obtain

$$\operatorname{depth}_A(A) = 1 + \operatorname{depth}_A(B/A) \leqslant 1 + \dim_A(B/A) = 1 + h - t$$

$$= h - t + 1.$$

COROLLARY (2.9): *Let X be a nonsingular projective variety of dimension r. Let π: $X \to \mathbb{P}_k^m$, $r + 1 \leqslant m \leqslant 2r$, be a generic projection. Let ξ be a general point of $X' = \pi(X)$ of height h, which lies on* $\operatorname{Sing}(X')$. *Then* $\operatorname{depth}(\hat{\mathcal{O}}_{X',\xi}) \leqslant r + h - m + 1$.

PROOF: By [18, Theorem 1], $\operatorname{Sing}(X')$ is equidimensional of dimension $2r - m$. Thus an application of Lemma (2.8) for $t = r - (2r - m) = m - r$ implies the desired inequality.

We will impose the condition of being a Mori ring as part of the definition of seminormality of a ring. We recall that a scheme (locally noetherian) is said to be seminormal if it can be covered by affine open sets whose rings are seminormal. This is equivalent to having all local rings seminormal. Also recall that such local rings are seminormal if and only if their completion are seminormal [12, Theorem 1.20].

PROPOSITION (2.10): *Let X' be a variety of dimension $r \geqslant 2$. Assume that on each irreducible component of* Sing(X') *there is at least one point y such that $\mathcal{O}_{X',y}$ is seminormal. If for every non-generic point ξ of* Sing(X') *we have* depth($\mathcal{O}_{X',\xi}$) $\geqslant 2$, *then X' is a seminormal variety.*

PROOF: By Theorem (2.3) we need to show that $\mathcal{O}_{X',y}$ is seminormal for every closed point $y \in X'$. If y is a nonsingular point, then $\mathcal{O}_{X'y}$ is a regular local ring. Thus assume that $y \in \text{Sing}(X')$. Let $A = \mathcal{O}_{X',y}$. Since X' is a variety, A is a Mori ring. Thus by Theorem (2.3) it is enough to show that A_p is seminormal for all $p \in \text{Spec}(A)$, whenever $\text{depth}(A_p) = 1$. In other words, it is enough to show that $\mathcal{O}_{X',\xi}$ is seminormal whenever $\text{depth}(\mathcal{O}_{X',\xi}) = 1$ for every general point ξ on Sing(X'). By the hypothesis this may happen only whenever ξ is a generic point of Sing(X'). Each irreducible component of Sing(X') contains at least one point y such that $\mathcal{O}_{X',y}$ is seminormal. For any generic point ξ of Sing(X'), $\mathcal{O}_{X',\xi}$ is a localization of $\mathcal{O}_{X',y}$ for some y such that $\mathcal{O}_{X',y}$ is seminormal. Thus $\mathcal{O}_{X',\xi}$ is seminormal.

REMARK: Let $r \geqslant 2$. For generic projections as considered in corollary (2.9), we saw that depth $(\mathcal{O}_{X',\xi}) \leqslant r + h - m + 1$, for every general point ξ of height h which lies on Sing(X'). For $h = m - r$, ξ is a generic point of Sing(X'), and since X' is a variety we have depth $(\mathcal{O}_{X',\xi}) = 1$. If ξ is not a generic point of Sing(X'), then $h \geqslant m - r + 1$ and hence $r + h - m + 1 \geqslant 2$. For X' to be seminormal, by Proposition (2.10) we require that $\text{depth}(\mathcal{O}_{X',\xi}) \geqslant 2$ for all such ξ. This condition is weaker than assuming that X' is S_2 variety. We also require that each component of Sing(X') contains at least one point y such that $\mathcal{O}_{X',y}$ is seminormal. We will check these conditions for certain generic projections. However we first like to give an example to show that for "less generic" maps, it is possible that there is no point y on Sing(X') such that $\mathcal{O}_{X',y}$ is seminormal.

EXAMPLE (2.11): Let X' be the surface given by $f(x, y, z) = 3x^2y - y^3 + (x^2 + y^2)^2 = 0$ in $\mathbb{C}^3$, and let X be the normalization of X'. The singular locus of X' is the "z-axis" $\{x = y = 0\}$. We claim that on no point P on Sing(X'), $\mathcal{O}_{X',P}$ is seminormal. It is enough to check the claim for the origin $P = (0, 0, 0)$, as for any other point Q on Sing(X') we have $\mathcal{O}_{X',Q} \cong \mathcal{O}_{X',P}$. Let $A = \hat{\mathcal{O}}_{X',P} \cong \mathbb{C}[[x, y, z]]/(f)$. Since f does not involve z, and the curve given by f in $\mathbb{C}^2$ has three simple analytic branchesat the point (0, 0), X' has three simple analytic branches at P and hence by (2.5.1) the expression of the conductor of A is the same as the expression of the conductor of $\mathbb{C}[[x, y]]/(f)$. It is clear that the curve f is not seminormal (Fig. 2.11.1), hence by Proposition (2.5), the conductor of $\mathbb{C}[[x, y]]/(f)$ is not a radical ideal of $\mathbb{C}[[x, y]]/(f)$. Thus the conductor

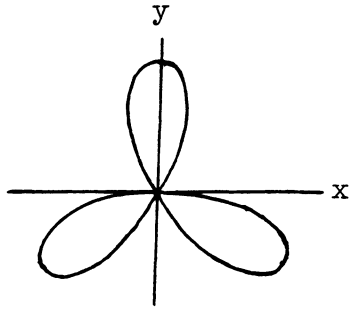

Fig. 2.11.1.

of A is not a radical ideal in A and hence A is not seminormal. Also observe that $\operatorname{depth}(A) = 2$.

We now prove a partial converse of Proposition (2.10).

PROPOSITION (2.12): *Let $X' \subseteq \mathbb{P}_k^m$ be a variety of dimension $r \geqslant 2$, where k is an algebraically closed field. Let $r + 1 \leqslant m \leqslant 2r - 1$. Assume that* $\operatorname{Sing}(X')$ *is equidimensional of dimension* $2r - m$. *If X' is seminormal then* $\operatorname{depth}\mathcal{O}_{X',y}) \geqslant 2$ *for every closed point $y \in X'$.*

PROOF: Let $A = \hat{\mathcal{O}}_{X',y}$ and let $\mathcal{M}$ be the maximal ideal of A. Since $\operatorname{depth}(\mathcal{O}_{X',y}) = \operatorname{depth}(A)$, and A is a complete noetherian local ring, if we show that A has rational normalization (i.e., $A/(\mathcal{M}_1 \cap A) \cong B/\mathcal{M}_1$ for every maximal ideal $\mathcal{M}_1$ of B), and $\operatorname{Spec}(A) - \{\mathcal{M}\}$ is connected, then by a recent result of M. Vitulli [25, Proposition 3.4] we will have $\operatorname{depth}(\mathcal{O}_{X',y}) \geqslant 2$. If B is the normalization of A, and $\mathcal{M}_1$ is a maximal ideal of B, then $\mathcal{M}_1$ lies over $\mathcal{M}$ and $B/\mathcal{M}_1$ is a finite extension of $A/\mathcal{M} = k$. Since k is algebraically closed, $B/\mathcal{M}_1 = k$, thus A has rational normalization. To show that $\operatorname{Spec}(A) - \{\mathcal{M}\}$ is connected, we use a result of R. Hartshorne [9, Proposition 1.1). Let $A = k[[t_1,\dots,t_m]]/P_1 \cap \dots \cap P_n$ for some integer n. It is well-known that the prime ideals $P_1,\dots,P_n$ all have height $m - r$. (For example this follows from more general results in [15]. Because $\mathcal{O}_{X',y}$ is dominated by A in the sense of [15, page 14], and hence the theorem of transition [15, page 64, (19.1)] holds for $\mathcal{O}_{X',y}$ and A, and therefore by [15, page 75, (22.9)], the extension of the zero ideal of $\mathcal{O}_{X',y}$ in A is equidimensional of dimension r). Let $V_i = \operatorname{Spec}(A/P_i)$, $i = 1,\dots,n$. These are the irreducible components of $V = \operatorname{Spec}(A)$. Since $\operatorname{codim}(\{\mathcal{M}\}, V) = r > r - 1$, we need to show that $\operatorname{codim}(V_i \cap V_j, V) \leqslant r - 1$ for $i, j = 1,\dots,n$ and $i \neq j$. This is equivalent to prove that $\sqrt{P_i + P_j} \neq \mathcal{M}$ for all $i \neq j$. But since $a = k[[t_1,\dots,t_m]]/P_1 \cap \dots \cap P_n$ and $k[[t_1,\dots,t_m]]$ is a regular local ring, by a result of J.P. Serre [23, ch. V, Theorem 3] if Q is any minimal prime ideal over $P_i + P_j$, we have

$$\operatorname{height}(Q) \leqslant \operatorname{height}(P_i) + \operatorname{height}(P_j) = 2(m - r).$$

Since $m < 2r$, $2m - 2r < m$ and hence Q is not a maximal ideal of $k[[t_1, \ldots, t_m]]$. Therefore $\mathrm{Spec}(A) - \{\mathscr{M}\}$ is connected.

COROLLARY (2.13): *Let X be a nonsingular variety of dimension $r \geqslant 2$, and let π: $X \to X' \subset \mathbb{P}_k^{2r-1}$ be a finite birational morphism of varieties such that Sing(X') is equidimensional of dimension one. Then X' is seminormal if and only if each irreducible component of Sing(X') contains at least one point y such that $\mathcal{O}_{X',y}$ is seminormal, and for every closed point $z \in$ Sing(X') we have* $\mathrm{depth}(\mathcal{O}_{X',z}) = 2$.

PROOF: By Lemma (2.8), $\mathrm{depth}(\mathcal{O}_{X',z}) \leqslant r - (r-1) + 1 = 2$. Thus the corollary follows from propositions (2.10) and (2.12).

PROPOSITION (2.14): *Let $X \subset \mathbb{P}_k^n$ be a nonsingular variety of dimension r, embedded properly. Let π: $X \to \mathbb{P}^m$ be a generic projection, $m = 2r$ or $m = 2r - 1$. Then $X' = \pi(X)$ is a seminormal variety.*

PROOF: For $m = 2r$, by [18, theorem 2], Sing(X') consists of finitely many points, and if $y \in \mathrm{Sing}(X')$, then $\hat{\mathcal{O}}_{X',y} \cong k[[t_1, \ldots, t_{2r}]]/I$, where $I = (t_1, \ldots, t_r) \cap (t_{r+1}, \ldots, t_{2r})$. Observe that here $\mathrm{depth}(\hat{\mathcal{O}}_{X',y}) = 1$. However $\hat{\mathcal{O}}_{X',y}$ is seminormal by Proposition (2.5), because the conductor is $(t_1, \ldots, t_{2r})/I$ which is a prime ideal of $\hat{\mathcal{O}}_{X',y}$. For $m = 2r - 1$ if $r = 2$, then $2r - 1 = r + 1$ and this case is proved by Bombieri [4]. Thus assume that $r \geqslant 3$. By [18, Theorem 3], Sing(X') is purely of dimension 1, and for all closed points on a dense open subset of Sing(X'), we have $\hat{\mathcal{O}}_{X',y} \cong k[[t_1, \ldots, t_{2r-1}]]/I$, where $I = (t_1, \ldots, t_{r-1}) \cap (t_r, \ldots, t_{2r-r})$. The latter ring is seminormal again by Proposition (2.5). By Proposition (2.10) to prove the seminormality of X' we only need to show that for the remaining points on Sing(X'), we have $\mathrm{depth}(\mathcal{O}_{X',y}) \geqslant 2$. For these points if char(k) $\neq 2$, then $\hat{\mathcal{O}}_{X',y} \cong k[[t_1, \ldots, t_{r-1}, t_1 t_r, \ldots, t_{r-1} t_r, t_r^2]]$, and if char($k$) = 2, then $\hat{\mathcal{O}}_{X',y} = k[[t_1, \ldots, t_{r-1}, t_1 t_r, \ldots, t_{r-1} t_r, t_r^2 + t_r^3]]$. We will show that $\mathrm{depth}(\hat{\mathcal{O}}_{X',y}) \geqslant 2$ by computing $\mathscr{C}$ the conductor of $\hat{\mathcal{O}}_{X',y}$.

Char *(k)* $\neq$ *2*. Let $A = k[[t_1, \ldots, t_{r-1}, t_1 t_r, \ldots, t_{r-1} t_r, t_r^2]]$. The normalization of A is $B = k[[t_1, \ldots, t_r]]$. Let $A_0 = k[[t_1, \ldots, t_{r-1}, t_r^2]]$ and let m be the maximal ideal of A_0. Since B is a finite A_0-module and $B/m \cdot B$ is generated by $\bar{1}$ and $\bar{t}_r$ as a k-vector space, by Nakayama's lemma B is generated by 1, t_r as an A_0-module. The prime ideal $(t_1, \ldots, t_{r-1}) \cdot B$ is contained in the conductor because if $g + h t_r$ is any element of B, where $g, h \in A_0$, then $t_i(g + h t_r) = g t_i + h t_i t_r \in A$, for all $i = 1, \ldots, r-1$. On the other hand since dim(Sing(X')) = 1 and X is nonsingular, $\mathscr{C}$ has height $r - 1$. Therefore $\mathscr{C} = (t_1, \ldots, t_{r-1}) \cdot B$. Now we consider the exact sequence of A-modules $0 \to A \to B \to B/A \to 0$. If we show that $\mathrm{depth}_A(B/A) \geqslant 1$, by Corollary (2.9) and part (ii) of Lemma (2.6) we have $\mathrm{depth}(A) \geqslant 2$. We claim athat t_r^2 is a non-zerodivisor of B/A. let $f = g + h t_r \in B - A$, $g, h \in A_0$. Then $h \notin \mathscr{C}$ because otherwise $f \in A$. Assume that $ft_r^2 \in A$, then $gt_r^2 + ht_r^2 t_r \in A$, thus $ht_r^2 t_r \in A$. Since $ht_r^2 \in A$, we have $ht_r^2 \in \mathscr{C}$. But $h \notin \mathscr{C}$ and $\mathscr{C}$ is a prime ideal of B, therefore $t_r \in \mathscr{C}$. This is a contradiction.

Char (k) = 2. Let $A = k[[t_1,\dots,t_{r-1}, t_1t_r,\dots,t_{r-1}t_r, t_r^2 + t_r^3]]$ and $A_0 = k[[t_1,\dots,t_{r-1}, t_r^2 + t_r^3]]$. Again the normalization of A is $B = k[[t_1,\dots,t_r]]$. By the same argument as above B is generated by 1, t_r as an A_0-module, and $\mathscr{C} = (t_1,\dots,t_{r-1})$. $t_r^2 + t_r^3$ is a non-zerodivisor of B/A, because if $f = g + ht_r$, g, $h \in A_0$, $h \notin \mathscr{C}$ and $f(t_r^2 + t_r^3) \in A$, then $h(t_r^2 + t_r^3)t_r \in \mathscr{C}$, thus $(1 + t_r)ht_r^3 \in \mathscr{C}$, hence $ht_r^3 \in \mathscr{C}$, ($\mathscr{C}$ as an ideal of B), therefore $t_r \in \mathscr{C}$, which is a contradiction.

3. Seminormality of some strongly generic projections

In this chapter we will examine seminormality through the theory of singularity subschemes. For basic properties of singularity subschemes we refer to [21]. Let f: $X \to Y$ be a morphism of nonsingular varieties over a field , with $\dim(X) = r$, $\dim(Y) = m$. Let x be a closed point of X and $y = f(x)$. For $i \geqslant 0$, the first order singularity subscheme of f, $S_i(f)$, is defined by $(i-1)$-st Fitting ideal of $\Omega^1_{X/Y}$, where $\Omega^1_{X/Y}$ is the sheaf of relative Kähler differentials of X over Y. $S_i(f)$ is a closed subscheme of X such that $x \in S_i(f)$ if and only if $\dim_{k(x)}(\Omega^1_{X/Y}(x)) \geqslant i$. If k is algebraically closed and x is a closed point of X, then $x \in S_i(f)$ if and only if the linear map of Zariski tangent spaces $\Theta_{X,x} \to \Theta_{Y,f(x)}$ has rank $\leqslant r - i$, where $r = \dim_k(\Theta_{X,x})$. The higher order singularity subscheme $S_1^{(q)}(f) \subset X - S_2(f)$ is defined by the q-th Fitting ideal of $\mathscr{O}_X$-algebra of q-th order relative principal parts corresponding to f. Then $S_1^{(0)}(f) = X - S_2(f)$. For $q \geqslant 1$ if k is algebraically closed, and if x is a closed point of X, then $x \in S_1^{(q)}(f)$ if and only if:

(i) $\dim_k[m_x/(f^{\#}(m_y)\cdot\mathscr{O}_x + m_x^2)] = 1$, and

(ii) $\dim_k[\mathscr{O}_x/f^{\#}(m_y)\cdot\mathscr{O}_x] \geqslant q + 1$,

where $(\mathscr{O}_x, m_x) = \mathscr{O}_{X,x}$, $(\mathscr{O}_y, m_y) = \mathscr{O}_{Y,y}$ and $f^{\#}$: $\mathscr{O}_y \to \mathscr{O}_x$ is induced by f.

For $d \geqslant 1$, let $\Sigma_d(f) \subset X \times_Y \dots \times_Y X$ (d-fold fibre product) be the complement of the union of all diagonals. If $q_1,\dots,q_d$ are non-negative integers, we regard $S_1^{(q_1)}(f) \times_Y \dots \times_Y S_1^{(q_d)}(f)$ as a subscheme of $X \times_Y \dots \times_Y X$. Then by definition

$$\Sigma_d(f; q_1,\dots,q_d) = \Sigma_d(f) \cap \left(S_1^{(q_1)}(f) \times_Y \dots \times_Y S_1^{(q_d)}(f)\right).$$

On can regard $\Sigma_d(f; q_1,\dots,q_d)$ as consisting of all d-tuples $(x_1,\dots,x_d)$ of distinct closed points of X such that (a) $x_j \in S_1^{(q_j)}(f)$, $j = 1,\dots,d$, and (b) $f(x_1) = \dots = f(x_d)$.

By [21, Theorem A], if X is a nonsingular projective variety of dimension r, and $r + 1 \leqslant m \leqslant 2r$, then there is an embedding $X \subseteq \mathbb{P}_k^n$ for some n such that if π: $X \to \mathbb{P}_k^m$ is a generic projection, then $\Sigma_d(\pi; q_1,\dots,q_d)$ is either empty or of pure dimension $l = dr - (d-1)m$

$-(m-r+1)\Sigma_{j=1}^{d} q_j$. If char$(k) \nmid (q_j+1)$ for all j, and if π is "strongly" generic, then $\Sigma_d(\pi; q_1,\dots,q_d)$ is nonsingular too. By [21, Theorem 12.1] if $x_j \in S_1^{(q_j)}(\pi) - S_1^{(q_j+1)}(\pi)$, $j=1,\dots,d$, and if $X' = \pi(X)$, $y = f(x_j)$, $j = 1,\dots,d$, then identifying $\hat{\mathcal{O}}_{X,x_j}$ by $k[[t_1,\dots,t_r]]$ and $\hat{\mathcal{O}}_{\mathbb{P}^m}$ by $k[[u_1,\dots,u_m]]$, the induced map $\hat{\mathcal{O}}_{\mathbb{P}^m,y} \to \Pi_{j=1}^{d} \hat{\mathcal{O}}_{X,x_j}$ will be identified by $g = (g_1,\dots,g_d)$, where $g_j\colon k[[u_1,\dots,u_m]] \to k[[t_1,\dots,t_r]]$, $j = 1,\dots,d$ are defined as

$$\begin{cases} g_j(u_i) = t_i, \quad i = 1,\dots,r-1 \\ g_j(u_r) = t_r^{q+1} + \Sigma_{\nu=1}^{q-1} t_{q(m-r)+\nu} t_r^{\nu} \\ g_j(u_{r+i}) = \Sigma_{\nu=1}^{q} t_{q(i-1)+\nu} t_r^{\nu}, \quad i = 1,\dots,m-r. \end{cases} \tag{3.0.1}$$

This gives the canonical form of $\hat{\mathcal{O}}_{X',y}$ for those points y such that if $y = \pi(x)$, then $x \notin S_2(\pi)$. By [21, Corollary 7.5] for generic projections $S_2(\pi)$ is either empty or of pure codimension $2(m-r+2)$ in X. We will consider those values of m for which $S_2(\pi)$ is empty, namely $m > (3r-4)/2$. For r even we get $m \geqslant (3r-2)/2$, for r odd we get $m \geqslant (3r-3)/2$.

For $m = r+1$ the assertion is proved by Greco and Traverso when char$(k) = 0$ [7, Theorem 3.7]. The inequalities $m \geqslant r+2$ and $m > (3r-4)/2$ are admissible for $r \geqslant 6$. So we will always assume that $r \geqslant 6$ and $m > (3r-4)/2$.

Observe that by [21, Corollary 1.2] for $q \geqslant 0$, and for strongly generic projections $S_1^{(q)}(\pi)$ is either empty or of pure codimension $q(m-r+1)$,and if char$(k) \nmid (q+1)$, $S_1^{(q)}(\pi)$ is smooth. If r is even and $m \geqslant (3r-2)/2$, then $q(m-r+1) \geqslant (qr)/2$. Thus if $q \geqslant 3$, $S_1^{(q)}(\pi) = \emptyset$. If r is odd and $m \geqslant (3r-3)/2$, then $q(m-r+1) \geqslant q(r-1)/2$. hence for $q \geqslant 3$ again $S_1^{(q)}(\pi) = \emptyset$. Therefore in the range $(3r-4)/2 < m \leqslant 2r$, we will only deal with $S_1^{(1)}(\pi)$ and $S_1^{(2)}(\pi)$. We also will always assume that char$(k) \neq 2, 3$ so that these subschemes are smooth.

Also observe that since $S_2(\pi) = \emptyset$, $S_1^{(1)}(\pi) = S_1(\pi)$ and hence if $x \in X - S_1^{(1)}(\pi)$, the map of Zariski tangent spaces $\Theta_{X,x} \to \Theta_{\mathbb{P}^m,y}$ has rank r, where $y = \pi(x)$. Therefore if $X' = \pi(X)$ is analytically irreducible at y, we have $\mathcal{O}_{X,x} \cong \mathcal{O}_{X',y}$.

We will need the following proposition.

Proposition (3.1): *Let A be a Mori ring. Then A is seminormal if and only if $A[x]$ is seminormal.*

Proof: By a result of Traverso [24, Theorem 3.6], a ring B is seminormal if and only if the canonical homomorphism Pic$(B) \to$ Pic$(B[T])$ is an isomorphism for any finite set of in determinants T, where Pic(B) is the group of isomorphism classes of invertible sheaves on Spec(B) under the operation $\otimes$. (cf. [8, page 143]). Also recall that a Mori ring B is seminormal if $b \in \bar{B}$, b^2, $b^3 \in B$, then $b \in B$, where $\bar{B}$ is the normalization of B [12, Proposition 1.4]. Let A be seminormal. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Pic}(A) & \xrightarrow{f} & \mathrm{Pic}(A[x][T]) \\ g \downarrow & \nearrow h & \\ \mathrm{Pic}(A[x]) & & \end{array}$$

f and g are isomorphisms, thus h is an isomorphism and hence $A[x]$ is seminormal. Conversely assume that $A[x]$ is seminormal. Observe that the normalization of $A[x]$ is $\bar{A}[x]$, where $\bar{A}$ is the normalization of A. Let $b \in \bar{A}$ and b^2, $b^3 \in A$. Then $b \in A[x]$ and hence $b \in A$. Therefore A is seminormal.

THEOREM (3.2): *Let X be a nonsingular projective variety of dimension r, $r \geqslant 6$, embedded appropriately. Then strongly generic projections π: $X \to \mathbb{P}_k^m$ are seminormal for all m, $(3r-3)/2 \leqslant m \leqslant 2r$.*

PROOF: For $m = 2r$ and $m = 2r - 1$ we have already proved this in fact with no restriction on char(k) or r. Thus we assume that $m \leqslant 2r - 2$. We will divide the proof of this theorem into four cases. Case 1; $m \geqslant 3r/2$. Case 2; r odd, $m = (3r-1)/2$. Case 3; r even, $m = (3r-2)/2$. Case 4; r odd, $m = (3r-3)/2$.

Case 1. $m \geqslant 3r/2$. Let $l = \dim \Sigma_d(\pi; q_1, \ldots, q_d)$. If $\Sigma_{j=1}^d q_j \geqslant 1$, then

$$l = dr - (d-1)m - (m-r+1)\Sigma_{j+1}^d q_j$$

$$\leqslant dr - (d-1)(3r/2) - (r+2)/2 = (r/2)(2-d) - 1.$$

Thus for $d \geqslant 2$, $\Sigma_d(\pi; q_1, \ldots, q_d) = \emptyset$. Therefore by the fact that the number of analytic branches at a point $y \in X' = \pi(X)$ and the number of points as $x \in X$ such that $\pi(x) = y$ are equal [16, Theorem 1], X' is analytically irreducible at every point $x \in S_1^{(q)}(\pi)$, $q \geqslant 1$. Since

$$\mathrm{codim}\big(S_1^{(q)}(\pi)\big) = q(m-r+1) \geqslant q(r+2)/2.$$

$S_1^{(2)}(\pi) = \emptyset$ and $\dim(S_1^{(1)}(\pi)) = (r-2)/2$. We will apply Proposition (2.10). By [18, Theorem 1], every irreducible component of Sing(X') contains a dense open subset such that on this set $\hat{\mathcal{O}}_{X',y} \cong k[[u_1, \ldots, u_m]]/I$, where $I = (u_1, \ldots, u_{m-r}) \cap (u_{m-r+1}, \ldots, u_{2(m-r)})$. Thus $\hat{\mathcal{O}}_{X',y}$ is seminormal by Proposition (2.5). Let $x \in S_1^{(1)}(\pi)$ and $y = \pi(x)$, and let $A = \hat{\mathcal{O}}_{X',y}$. By (3.0.1) we have

$$A \cong k\left[\left[t_1,\dots,t_{r-1}, t_1t_r,\dots,t_{m-r}t_r, t_r^2\right]\right].$$

Let $B = K[[t_1,\dots,t_r]]$. Since by [18, Theorem 1] Sing(X') has pure codimension $m-r$ in X', and X is nonsingular, the conductor of A is $\mathscr{C} = (t_1,\dots,t_{m-r})\cdot B$. Thus by similar method as in Proposition (2.14), we have depth$(A) \geqslant 2$. Thus X' is seminormal by Proposition (2.10).

Case 2. r odd, $m = (3r-1)/2$. If $\Sigma_{j=1}^{d} q_j \geqslant 1$, then

$$l \leqslant dr - (d-1)(3r-1)/2 - (r+1)/2 = (r-1)(2-d)/2.$$

Hence $\Sigma_d(\pi; q_1,\dots,q_d) = \emptyset$ for $d \geqslant 3$. For $d = 2$, $l = [1-(q_1+q_2)](r+1)/2$, which is negative if $q_1 + q_2 \geqslant 2$. If $q_1 + q_2 = 1$, then $l = 0$, i.e., there are only finitely many points as y on X' such that X' has two analytic branches at y, and if $\pi^{-1}(y) = \{x_1, x_2\}$, then $x_1 \in S_1^{(1)}(\pi) - S_1^{(2)}(\pi)$ and $x_2 \in X - S_1^{(1)}(\pi)$. By (3.0.1) this means that one of the two analytic branches is simple and another has an ordinary pinch point. Both of these branches are seminormal. If $q_1 = q_2 = 0$, then $l = (r+1)/2$. We will directly prove that $\mathcal{O}_{X',y}$ is seminormal for such closed points $y \in X'$ by using Proposition (2.5). Let $q_1 = 1$, $q_2 = 0$. The integral closure of $\hat{\mathcal{O}}_{X',y}$ is $\hat{\mathcal{O}}_{X,\pi^{-1}(y)} \cong \hat{\mathcal{O}}_{X,x_1} \times \hat{\mathcal{O}}_{X,x_2} \cong k[[t_1,\dots,t_r]] \times k[[t_1,\dots,t_r]]$. If $g = (g_1, g_2)$: $\hat{\mathcal{O}}_{\mathbb{P}^m,y} \to \hat{\mathcal{O}}_{X,x_1} \times \hat{\mathcal{O}}_{X,x_2}$ is the map given in (3.0.1), then $\hat{\mathcal{O}}_{X',y} \cong k[[u_1,\dots,u_m]]/(\operatorname{Ker}(g_1) \cap \operatorname{Ker}(g_2))$. By [21, Theorem 12.1], g_1 and g_2 are defined by:

$$\begin{cases} g_1(u_i) = t_i, & i = 1,\dots,r-1 \\ g_1(u_r) = t_r^2 \\ g_1(u_{r+1}) = t_it_r, & i = 1,\dots,m-r \end{cases}$$

$$\begin{cases} g_2(u_i) = t_i, & i = 1,\dots,m-r \\ g_2(u_{r+1}) = t_{m-r+1+i}, & i = 0,\dots,m-r \\ g_2(u_i) = 0, & i = m-r+1,\dots,r-1 \end{cases}$$

Let $P_1 = \operatorname{Ker}(g_1)$, $P_2 = \operatorname{Ker}(g_2)$. By the appendix P_1 is a prime ideal of $k[[u_1,\dots,u_m]]$ generated by certain elements which only involve $u_1,\dots,u_{m-r}, u_r, u_{r+1},\dots,u_m$. Clearly $P_2 = (u_{m-r+1},\dots,u_{r-1})$. This would imply that $P_1 + P_2$ is a prime ideal of $k[[u_1,\dots,u_m]]$. If $ab \in P_1 + P_2$, we can write $a = a_1 + a_2$ and $b = b_1 + b_2$ where a_1, b_1 do not have any term involving $u_{m-r+1},\dots,u_{r-1}$ and a_2, $b_2 \in P_2$. Thus $ab \in P_1 + P_2$ implies that $a_1b_1 \in P_1 + P_2$. But a_1b_1 does not have any term involving $u_{m-r+1},\dots,u_{r-1}$, thus $a_1b_1 \in P_1$ and hence $a_1 \in P_1$ or $b_1 \in P_1$. Therefore

$a \in P_1 + P_2$ or $b \in P_1 + P_2$. Therefore the conductor $\mathscr{C} = (P_1 + P_2)/(P_1 \cap P_2)$ is a prime ideal of $\hat{\mathcal{O}}_{X',y}$. Since $k[[u_1,\ldots,u_m]]/P_1$ is seminormal by case 1 and $k[[u_1,\ldots,u_m]]/P_2$ is a regular local ring, $\hat{\mathcal{O}}_{X',y}$ is seminormal by Proposition (2.5). If $q_1 = q_2 = 0$, then again by [21, Theorem 12.1], $g = (g_1, g_2)$ will be defined as

$$\begin{cases} g_1(u_i) = t_i, & i = 1,\ldots,r \\ g_1(u_i) = 0, & i = r+1,\ldots,m. \end{cases}$$

$$\begin{cases} g_2(u_i) = 0, & i = 1,\ldots,m-r \\ g_2(u_{m-r+i}) = t_i, & i = 1,\ldots,r. \end{cases}$$

Thus $\hat{\mathcal{O}}_{X',y} \cong k[[u_1,\ldots,u_m]]/I$ where $I = (u_1,\ldots,u_{m-r}) \cap (u_{r+1},\ldots,u_m)$. Since $m - r = (r-1)/2 < r+1$, $\hat{\mathcal{O}}_{X',y}$ is seminormal by Proposition (2.5). For $d = 1$, since $\operatorname{codim}(S_1^{(2)}(\pi)) = 2(m - r + 1) \geqslant r + 1$, we have $S_1^{(2)}(\pi) = \emptyset$. If $y = \pi(x)$ and $x \in S_1^{(1)}(\pi)$, then $\hat{\mathcal{O}}_{X',y}$ is seminormal again as in case 1. If $x \in X - S_1^{(1)}(\pi)$, then $\hat{\mathcal{O}}_{X',y}$ is a regular local ring and hence is seminormal. Therefore X' is a seminormal variety.

Case 3. r even, $m = (3r-2)/2$. If $\Sigma_{j=1}^{d} q_j \geqslant 1$, then $l \leqslant dr - (d-1)(3r-2)/2 - r/2 = [(r\text{-}2)(2-d)/2] + 1$. Thus if $d \geqslant 3$, then $\Sigma_d(\pi;\, q_1,\ldots,q_d) = \emptyset$. Let $d = 2$, then $l = (r/2) + 1 - r(q_1 + q_2)/2 = r[1 - (q_1 + q_2)] + 1$ which is negative if $q_1 + q_2 \geqslant 2$. Let $q_1 + q_2 = 1$, then $l = 1$. Thus $\Sigma_2(\pi;\, 1, 0)$ is one dimensional, and if $(x_1, x_2) \in \Sigma_2(\pi;\, 1, 0)$ and $y = \pi(x_1) = \pi(x_2)$, then $\hat{\mathcal{O}}_{X',y}$ is seminormal. To see this observe that with similar notation as previous cases, here g_2 will be defined as

$$\begin{cases} g_2(u_i) = t_i, & i = 1,\ldots,m-r+1 \\ g_2(u_{r+i}) = t_{m-r+2+i}, & i = 0,\ldots,m-r \\ g_2(u_i) = 0, & i = m-r+2,\ldots,r-1. \end{cases}$$

Thus $P_2 = (u_{m-r+1},\ldots,u_{r-1})$. The rest of the proof of $\hat{\mathcal{O}}_{X',y}$ being seminormal is the same as the proof of similar part in case 2 considering only this change in P_2. If $q_1 = q_2 = 0$ by the same proof as in case 2, $\hat{\mathcal{O}}_{X',y}$ is seminormal. Now assume that $d = 1$. If $y = \pi(x)$ and $x \in S_1^{(1)}(\pi)$, $\hat{\mathcal{O}}_{X',y}$ is seminormal as before. However in this case $\dim(S_1^{(2)}(\pi)) = r - 2(m - r + 1) = 0$. Thus there are finitely many points as y on X' such that if $y = \pi(x)$, $x \in S_1^{(2)}(\pi)$. Since we have proved that $\hat{\mathcal{O}}_{x',y}$ is seminormal for all closed points $y \in X'$ except for points y such that $x = \pi^{-1}(y) \in S_1^{(2)}(\pi)$, and since any localization of a seminormal ring is seminormal, all local rings $\mathcal{O}_{X',\xi}$ are seminormal, where ξ is any general point of X' except these closed points y. Thus by Theorem (2.3) and Proposition (2.12) X' is seminormal if and only if $\operatorname{depth}(\hat{\mathcal{O}}_{X',y}) \geqslant 2$ for these finitely many points.

By (3.0.1) we have

$$\hat{\mathcal{O}}_{X',y} \cong k\left[\left[t_1,\ldots,t_{r-1},\, t_1t_r + t_2t_r^2,\ldots,t_{r-3}t_r + t_{r-2}t_r^2,\, t_{r-1}t_r + t_r^3\right]\right].$$

By Corollary (2.9) the statement (ii) of Lemma (2.6) holds for the corresponding exact sequence of modules, so it is enough to show that

$$\operatorname{depth}_{\hat{\mathcal{O}}_{X',y}}\left(k\left[\left[t_1,\ldots,t_r\right]\right]/\hat{\mathcal{O}}_{X',y}\right) \geqslant 1.$$

Let $A = k[t_1,\ldots,t_{r-1}, t_1t_r + t_2t_r^2,\ldots,t_{r-1}t_r + t_r^3]$ and $B = k[t_1,\ldots,t_r]$. We need to show that $\operatorname{depth}_A(B/A) \geqslant 1$. This is a very delicate work to do and requires more information about the structure of A. In Chapter 4 we will set up the required machinery of the theory of double point schemes to compute the conductor of A and then using this together with some more finer constructions we will prove that $\operatorname{depth}_A(B/A) \geqslant 1$. Therefore X' again would be seminormal. This in particular would imply that the local ring $k[[t_1,\ldots,t_{r-1}, t_1t_r + t_2t_r^2,\ldots,t_{r-1}t_r + t_r^3]]$ is seminormal.

Case 4. r odd, $m = (3r-3)/2$. We will discuss this in two subcases $r \geqslant 9$ and $r = 7$.

(a). $r \geqslant 9$. If $\Sigma_{j=1}^d q_j \geqslant 1$, then $l \leqslant dr - (d-1)(3r-2)/2 = [(r-3)(2-d)/2] + 2$. Since $r \geqslant 9$, for $d \geqslant 3$ we have $l < 0$. For $d = 2$ we have $l = (r+3)/2 - (r-1)(q_1+q_2)/2 = [(r-1)[1-(q_1+q_2)]/2] + 2$, which is again negative for $q_1 + q_2 \geqslant 2$. Let $q_1 + q_2 = 1$, then $l = 2$. Thus $\Sigma_2(\pi;\, 1, 0)$ is two dimensional. The situation is similar to case 3. The map g_2 will be defined as

$$\begin{cases} g_2(u_i) = t_i, & i = 1,\ldots,m-r+2 \\ g_2(u_{r+i}) = t_{m-r+3+i}, & i = 0,\ldots,m-r \\ g_2(u_i) = 0, & i = m-r+3,\ldots,r-1. \end{cases}$$

Thus $P_2 = (t_{m-r+3},\ldots,t_{r-1})$. Therefore similar proof as case 3 works. If $q_1 + q_2 = 0$, again by the proof as case 2, the local rings at corresponding points on X' are seminormal. Assume that $d = 1$. If $y = \pi(x)$ and $x \in S_1^{(1)}(\pi)$, $\hat{\mathcal{O}}_{X',x}$ is seminormal. In this case $\dim(S_1^{(2)}(\pi)) = r - 2(m-r+1) = 1$. If $y = \pi(x)$ and $x \in S_1^{(2)}(\pi)$, by (3.0.1) we have

$$\hat{\mathcal{O}}_{X',y} \cong k\left[\left[t_1,\ldots,t_{r-1},\, t_1t_r + t_2t_r^2,\ldots,t_{r-4}t_r + t_{r-3}t_r^2,\, t_{r-2}t_r + t_r^3\right]\right].$$

Thus $\hat{\mathcal{O}}_{X',y} \cong [[t_1,\ldots,t_{r-2}, t_1t_r + t_2t_r^2,\ldots,t_{r-2}t_r + t_r^3]][[t_{r-1}]]$. Since by case 3 the ring $k[t_1,\ldots,t_{r-2}, t_1t_r + t_2t_r^2,\ldots,t_{r-2}t_r + t_r^3]$ is seminormal, by Proposition (3.1) and [12, Theorem (1.20)] $\hat{\mathcal{O}}_{X',y}$ is seminormal.

(b) $r = 7$, $(m = 9)$. If $\Sigma_{j=1}^d q_j \geqslant 1$, then $l \leqslant 6 - 2d$ which is negative if $d \geqslant 4$. If $d = 3$, then $l = 3 - 3(q_1 + q_1 + q_3)$. Thus if $q_1 + q_2 + q_3 \geqslant 2$, again $l < 0$. For $q_1 + q_2 + q_3 = 1$ we have $l = 0$. Thus there are finitely many points as y on X', such that X' has three analytic branches at y, one branch is an ordinary pinch point and two branches are simple. With the notation introduced in (3.0.1), by [21, Theorem 12.1], the homomorphism g_1, g_2, g_3 on $k[[u_1,\dots,u_9]]$ into $k[[t_1,\dots,t_7]]$ are defined as:

$$\begin{cases} g_1(u_i)=t_i, & i=1,\dots,6 \\ g_1(u_7)=t_7^2 & \\ g_1(u_8)=t_1t_7 & \\ g_1(u_9)=t_2t_7 & \end{cases}$$

$$\begin{cases} g_2(u_i)=t_i, & i=1,2 \\ g_2(u_i)=0, & i=3,4 \\ g_2(u_4+i)=t_{2+i}, & i=1,\dots,5 \end{cases}$$

$$\begin{cases} g_3(u_i)=t_i, & i=1,\dots,4 \\ g_3(u_i)=0, & i=5,6 \\ g_3(u_{6+i})=t_{4+i}, & i=1,2,3. \end{cases}$$

Let $P_i=\mathrm{Ker}(g_i)$, $i=1, 2, 3$. By the appendix we have $P_1=(u_1u_9-u_2u_8, u_1^2u_7-u_8^2, u_2^2u_7^2-u_9^2, u_1u_2u_7-u_8u_9)$. Clearly $P_2=(u_3, u_4)$, $P_3=(u_5, u_6)$. Let $B=k[[u_1,\dots,u_9]]$. Since B/P_i, $i=1, 2, 3$ are seminormal, by Proposition (2.5), to show that $\hat{\mathcal{O}}_{X',y}\cong B/(P_1\cap P_2\cap P_3)$ is seminormal it is enough to show that the ideals P_1+P_2, P_1+P_3, P_2+P_3, $P_1+P_2\cap P_3$, $P_2+P_1\cap P_3$, $P_3+P_1\cap P_2$ are radical ideals of B with common pure height 4. It is clear that the ideals P_1+p_2, P_1+P_3, P_2+P_3 are in fact prime ideals. By the appendix, P_1 has height 2, and by [23, Ch. V, Theorem 3] we have height$(P_1+P_2)\leqslant 4$. Since P_1+P_2 contains a subset of a system of parameters of B which consists of 4 elements, we have height$(P_1+P_2)=4$. Similarly height$(P_1+P_3)=4$. It is also clear that height$(P_2+P_3)=4$. Now we show that the ideal $P_2+P_1\cap P_3$ is radical. Let $f^n\in P_2+P_1\cap P_3$. We can write $f=g+h$ such that $g\in P_2$ and h does not have any term involving u_3, u_4. Then $f^n=gg_1+h^n$ for some $g_1\in B$. Thus $f^n\in P_2+P_2\cap P_3$ implies that $h^n\in P_2+P_1\cap P_3$. Since $P_2=(u_3, u_4)$ and h^n does not involve u_3 and u_4, we have $h^n\in P_1\cap P_3$, thus $h\in P_1\cap P_2$, and hence $f=g+h\in P_2+P_1\cap P_3$. Similarly $P_3+P_1\cap P_2$ is a radical ideal of B. To show that $P_1+P_2\cap P_3$ is a radical ideal consider the homomorphism g_1: $B\to k[[t_1,\dots,t_7]]$, let $Q_2=(t_3, t_4)$, $Q_3=(t_5, t_6)$. Then $P_2=g_1^{-1}(Q_2)$, $P_3=g_1^{-1}(Q_3)$ and hence $g_1^{-1}(Q_2\cap Q_3)=P_2\cap P_3$. Thus g_1 induces a homomorphism $B/(P_2\cap P_3)\to k[[t_1,\dots,t_7]]/(Q_2\cap Q_3)$. The kernel of this map is a radical ideal, because $k[[t_1,\dots,t_7]]/(Q_2\cap Q_3)$ is a reduced ring. But the kernel of this map is $P_1+P_2\cap P_3$. Therefore $P_1+P_2\cap P_3$ is also radical. Any minimal prime ideal over $P_1 + P_2 \cap P_3$ or $P_2 + P_1 \cap P_3$ or $P_3 + P_1 \cap P_2$ has height $\leqslant 4$, and since any of these three ideals contain a 4-element subset of a system of parameters of B, such a minimal prime has height 4. Therefore these ideals all are of pure height 4. Hence Proposition (2.5) now applies to imply seminormality of $\hat{\mathcal{O}}_{X',y}$. The seminormality of the local rings of X' at the other points were covered at part (a).

The proof of Theorem (3.2) is now complete.

4. Calculation of some conductors

In proving the seminormality of strongly generic projections we encountered with the fact that if $y = \pi(x)$ and $x \in S_1^{(1)}(\pi)$, the conductor of $\hat{\mathcal{O}}_{X',y}$ is a prime ideal of the normalization of this ring. We also found an explicit expression for the conductor. It is therefore tempting to ask this question when $x \in S_1^{(2)}(\pi)$.

Let X be a nonsingular projective variety of dimension r. Let r be even and $r \geqslant 6$, and let $m = (3r-2)/2$. We consider a strongly generic projection $\pi: X \to \mathbb{P}_k^m$. Let $X' = \pi(X)$. If $y = \pi(x)$, $x \in S_1^{(2)}(\pi)$ and X' is analytically irreducible at y, then we have

$$\hat{\mathcal{O}}_{X',y} \cong k\left[\left[t_1,\ldots,t_{r-1}, t_1t_r + t_2t_r^2,\ldots,t_{r-1}t_r + t_r^3\right]\right].$$

If r is odd, $r \geqslant 7$ and $m = (3r-3)/2$, then

$$\hat{\mathcal{O}}_{X',y} \cong k\left[\left[t_1,\ldots,t_{r-1}, t_1t_r + t_2t_r^2,\ldots,t_{r-2}t_r + t_r^3\right]\right].$$

Our result will give explicit expressions for the conductors of these rings. We will then prove the result needed in Theorem (3.2).

We need to recall some properties of the blowing up of an affine variety along a closed subscheme of the variety which is a complete intersection.

Let $Y = \operatorname{Spec}(B)$ be a nonsingular affine variety. Let $Z = \operatorname{Spec}(A) \subset Y$ be a nonsingular closed subvariety of Y which is a complete intersection. Thus $A \cong B/I$ where $I = (f_1,\ldots,f_d)$ and $d = \operatorname{Codim}(Z; Y)$. Let $\tilde{Y} = \operatorname{Proj}(\tilde{B})$ be the blowing up of Y along Z, where $B = B \oplus \mathrm{I} \oplus I^2 \oplus \ldots$. Let $\varphi: \tilde{Y} \to Y$ be the canonical projection. The closed subscheme of $\tilde{Y}$, $E = \operatorname{Proj}(\tilde{B}/I \cdot \tilde{B})$ is called the exceptional locus of the blowing up. The underlying topological space of E is $\varphi^{-1}(Z)$. Observe that since

$$\tilde{B}/I \cdot B \cong \tilde{B} \otimes_B B/I = (B \oplus I \oplus \ldots) \otimes_B (B/I)$$

$$\cong (B \otimes_B B/I) \oplus (I \otimes_B B/I) \oplus \ldots \cong B/I \oplus I/I^2 \oplus \ldots = gr_I(B)$$

where $gr_I(B)$ is the associated graded ring of B with respect to I, we have $E = \operatorname{Proj}(gr_I(B))$. The ring $\tilde{B}$, as a graded B-algebra is generated by degree one elements $f_1,\dots,f_d$. Thus $\tilde{Y}=\mathrm{Proj}(\tilde{B})$ is a union of affine open subsets $D_+(f_1),\dots,D_+(f_d)$, where $D_+(f_i)=\mathrm{Spec}(\tilde{B}_{(f_i)})$, and $\tilde{B}_{(f_i)}$ is the degree zero part of $\tilde{B}_{f_i}$, $i=1,\dots,d$. Let us fix $i=1$. Observe that $\tilde{B}_{(f_1)}\cong B[f_2/f_1,\dots,f_d/f_1]$, because the elements of $\tilde{B}_{(f_1)}$ are finite sum of elements of the form h/f_1^m where $h\in I^m$ for some m, thus the elements of $\tilde{B}_{(f_1)}$ are polynomials in $f_2/f_1,\dots,f_d/f_1$. Therefore $D_+(f_1)\cong \mathrm{Spec}(B[f_2/f_1,\dots,f_d/f_1])=\mathrm{Spec}(B[g_1,\dots,g_d])$ where $g_i\in K$, the ring of quotients of B, and $f_1g_i=f_i$, $i=2,\dots,d$.

We now restrict ourselves to a special case. Let $X=\mathbb{A}_k^r$ be the r-dimensional affine space. Let $Y=X\times_k X=\mathbb{A}_k^{2r}$ and let $Z=\Delta_X$ be "the diagonal" in $X\times_k X$. Z is a complete intersection. Let r be even and $r\geqslant 6$, and $m=(3r-2)/2$. Define the morphism $f\colon \mathbb{A}_k^r\to\mathbb{A}_k^m$ by $f(t_1,\dots,t_r)=(u_1,\dots,u_m)$ where

$$\begin{cases} u_i=t_i, & i=1,\dots,r-1\\ u_r=t_{r-1}t_r+t_r^3 & \\ u_{r+1}=t_1t_r+t_2t_r^2 & \\ \vdots & \\ u_m=t_{r-3}t_r+t_{r-2}t_r^2. & \end{cases}$$

Consider the morphism $f\times f\colon \mathbb{A}_k^{2r}\to\mathbb{A}_k^{2m}$ where $(f\times f)(t_1,\dots,t_r,s_1,\dots,s_r)=(u_1,\dots,u_m,v_1,\dots,v_m)$, $v_1,\dots,v_m$ are given in terms of $s_1,\dots,s_r$ in the manner that $u_1,\dots,u_m$ are given in terms of $t_1,\dots,t_r$. $\Delta_{\mathbb{A}^m}\subset\mathbb{A}_k^{2m}$ is given by the equations $u_i-v_i=0$, $i=1,\dots,m$. Thus $(f\times f)^{-1}(\Delta_{\mathbb{A}^m})$ is given by

$$\begin{cases} t_i-s_i=0, & i=1,\dots,r-1\\ t_{r-1}t_r+t_r^3-s_{r-1}s_r-s_r^3=0 & \\ t_1t_r+t_2t_r^2-s_1s_r-s_2s_r^2=0 & \\ \vdots & \\ t_{r-3}t_r+t_{r-2}t_r^2-s_{r-3}s_r-s_{r-2}s_r^2=0. & \end{cases}$$

Now consider $\tilde{Y}$, the blowing up of Y along Δ_X, and the canonical projection $\varphi\colon\tilde{Y}\to Y$. While $Y=\mathrm{Spec}(k[t_1,\dots,t_r,s_1,\dots,s_r])$, $\overline{Y}$ is the union of r affine varieties $D_+(t_i-s_i)$, $i=1,\dots,r$. Let $\xi_i=(t_i-s_i)/(t_r-s_r)$, $i=1,\dots,r-1$. Let $U=D_+(t_r-s_r)$. Then $U=\mathrm{Spec}(k[t_1,\dots,t_r,s_1,\dots,s_r][\xi_1,\dots,\xi_{r-1}])=\mathrm{Spec}(k[t_1,\dots,t_r,s_r,\xi_1,\dots,\xi_{r-1}])$. U is the $2r$-dimensional affine space. $E\cap U$ is given by $t_r-s_r=0$. If J is the ideal defining the subscheme $\varphi^{-1}((f\times f)^{-1}(\Delta_{\mathbb{A}^m}))$, then $J\subseteq(t_r-s_r)$. Let $Z_1\subseteq U$ be the closed subscheme defined by the ideal $(t_r-s_r)^{-1}J$.

Observing that

$$t_{r-1}t_r - s_{r-1}s_r + t_r^3 - s_r^3 = \left[t_{r-1} + t_r^2 + s_r(t_r + s_r) + s_r\xi_{r-1}\right](t_r - s_r),$$

$$t_1t_r - s_1s_r + t_2t_r^2 - s_2s_r^2 = \left[t_1 + t_2t_r + t_2s_r + s_r\xi_1 + s_r^2\xi_2\right](t_r - s_r),$$

$$\vdots,$$

the ideal of $k[t_1,\dots,t_r, s_r, \xi_1,\dots,\xi_{r-1}]$ which defines Z_1 is

$$\left(\xi_1,\dots,\xi_{r-1}, t_{r-1} + t_r^2 + t_rs_r + s_r^2 + s_r\xi_{r-1}, t_1 + t_2t_r + t_2s_r\right.$$
$$\left. + s_r\xi_1 + s_r^2\xi_2,\dots\right),$$

which is the same as

$$J_1 = \left(\xi_1,\dots,\xi_{r-1}, t_{r-1} + t_r^2 + t_rs_r + s_r^2, t_1 + t_2t_r + t_2s_r,\dots,t_{r-3}\right.$$
$$\left. + t_{r-2}t_r + s_{r-2}s_r\right).$$

We claim that $Z_1 \cong \mathbb{A}_k^{(r+2)/2}$. The coordinate ring of Z_1 is $k[t_1,\dots,t_r, s_r, \xi_1,\dots,\zeta_{r-1}]/J_1 \cong k[t_1,\dots,t_r, s_r]/P$ where

$$P = \left(t_{r-1} + t_r^2 + t_rs_r + s_r^2, t_1 + t_2t_r + t_2s_r,\dots,t_{r-3} + t_{r+2}t_r + t_{r-2}s_r\right).$$

But $k[t_1,\dots,t_r, s_r]/P = k[t_2, t_4,\dots,t_{r-2}, t_r, s_r]$ because P is the kernel of the homomorphism $k[t_1,\dots,t_r, s_r] \to k[t_2, t_4,\dots,t_{r-2}, t_r, s_r]$ where $t_2, t_4,\dots,t_r, s_r$ are fixed and

$$\begin{cases} t_1 & \mapsto -t_2t_r - t_2s_r \\ \vdots & \\ t_{r-3} & \mapsto -t_{r-2}t_r - t_{r-2}s_r \\ t_{r-1} & \mapsto -t_r^2 - t_rs_r - s_r^2. \end{cases}$$

Consider the morphism $g\colon Z_1 \to \mathbb{A}_k^r$ where $g(t_1,\dots,t_r, s_r, \xi_1,\dots,\xi_{r-1}) = (t_1,\dots,t_r)$, and let $W = g(Z_1)$. Then the following are known

(a) g is a finite morphism,
(b) W is the set of points of $\mathbb{A}_k^r$ where f is not one-to-one,
(c) Z_1 is the normalization of W, W has an ordinary pinch point at the origin, $S_1^{(1)}(g)$ is smooth.

Notice that by our definition of f, the origin belongs to $S_1^{(2)}(f)$. For these results we refer to [22, in particular Theorem 4.5], together with the following explanations.

Observe that $\varphi^{-1}[(f\times f)^{-1}(\Delta_{\mathbb{A}^m})] = Z(f)\cap E$, where $Z(f)$ is the "double point scheme of f", E is the exceptional locus. The following diagram commutes

$$\begin{array}{ccc} Z(f) & \hookrightarrow & (X\times_k X)^{\sim} \\ g\downarrow & & \downarrow\varphi \\ \mathbb{A}^r_k = X & \xrightarrow{p_2} & X\times_k X = \mathbb{A}^{2r}_k \end{array}$$

where p_2 is the projection to the second factor. On the open set U, introduced before, Z_1 and $Z(f)$ are identical.

Now observing that the subvariety W is given by the prime ideal $P\cap k[t_1,\dots,t_r]$ where $P=(t_{r-1}+t_r^2+t_rs_r+s_r^2,\ t_1+t_2t_r+t_2s_r,\dots,t_{r-3}+t_{r-2}t_r+t_{r-2}s_r)$ in $k[t_1,\dots,t_r,s_r]$, we can state our theorem. In the proof we will use the results discussed above. We also assume that $\mathrm{char}(k)\neq 2, 3$.

THEOREM (4.1): *Let C be the conductor of $A=k[t_1,\dots,t_{r-1},t_1t_r+t_2t_r^2,\dots,t_{r-1}t_r+t_r^3]$. As an ideal of $B=k[t_1,\dots,t_r]$, C is the determinantal ideal of the matrix*

$$M=\begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \\ \vdots & \vdots \\ t_{r-3} & t_{r-2} \\ t_2t_{r-1}+t_1t_r+t_2t_r^2 & -t_1 \\ t_4t_{r-1}+t_3t_r+t_4t_r^2 & -t_3 \\ \vdots & \vdots \\ t_{r-2}t_{r-1}+t_{r-3}t_r+t_{r-2}t_r^2 & -t_{r-3} \end{bmatrix},$$

i.e., the ideal generated by all 2×2 minors of M. C is a prime ideal of B of height $(r-2)/2$.

PROOF: Consider the homomorphism h: $k[u_1,\dots,u_m]\to B$ defined by

$$\begin{cases} h(u_i)=t_i, & i=1,\dots,r-1 \\ h(u_r)=t_{r-1}t_r+t_r^3 & \\ h(u_{r+1})=t_1t_r+t_2t_r^2 & \\ \vdots & \\ h(u_m)=t_{r-3}t_r+t_{r-2}t_r^2. & \end{cases}$$

(Recall, r is even and $m=(3r-2)/2$).

Then $A \equiv k[u_1,\dots,u_m]/\mathrm{Ker}(h)$. Let (M) be the determinantal ideal of M. It is easy to see that $(M) \subseteq C$. By Nakayama's Lemma B is generated by 1, t_r, t_r^2 as an A-module. Thus $b \in C$ if b, bt_r, $bt_r^2 \in A$. Considering the relations

$$\begin{cases} h(u_{r-1}) = t_1 t_r + t_2 t_r^2 \\ h(u_{r+2}) = t_3 t_r + t_4 t_r^2, \end{cases}$$

by Cramer's rule we have

$$\begin{cases} (t_1 t_4 - t_2 t_3) t_r = h(u_4 u_{r+1} - u_2 u_{r+2}) \\ (t_1 t_4 - t_2 t_3) t_r^2 = h(u_1 u_{r+2} - u_3 u_{r+1}). \end{cases}$$

Hence $t_1 t_4 - t_2 t_3 \in C$. Using the expression of $h(u_r)$ together with others, we get the following relations

$$\begin{cases} h(u_2 u_r) = h(u_2 u_{r-1} + u_{r+1}) t_r - h(u_1) t_r^2 \\ \quad = (t_2 t_{r-1} + t_1 t_r + t_2 t_r^2) t_r - t_1 t_r^2 \\ h(u_4 u_r) = h(u_4 u_{r-1} + u_{r+2}) t_r - h(u_3) t_r^2 \\ \quad = (t_4 t_{r-1} + t_3 t_r + t_4 t_r^2) t_r - t_3 t_r^2 \\ \vdots \\ h(u_{r-2} u_r) = h(u_{r-2} u_{r-1} + u_m) t_r - h(u_{r-3}) t_r^2 \\ \quad = (t_{r-2} t_{r-1} + t_{r-3} t_r + t_{r-2} t_r^2) t_r - t_{r-3} t_r^2. \end{cases}$$

Thus by Cramer's rule as shown, we see that $(M) \subseteq C$.

To show the other inclusion we first show that $P \cap k[t_1,\dots,t_r] \subseteq (M)$, where P is the prime ideal introduced before. Consider the inclusion $k[t_1,\dots,t_r] \subset k[t_1,\dots,t_r, s_r]$, and the following change of variables in these two rings

$$\begin{cases} t_{2i} = T_{2i}, & i = 1,\dots,r/2 \\ t_1 = -T_1 - T_2 T_r/2 \\ \vdots \\ t_{r-3} = -T_{r-3} - T_{r-2} T_r/2 \\ t_{r-1} = -T_{r-1} - 3(T_r/2)^2 \\ s_r = S_r - T_r/2. \end{cases}$$

These define isomorphisms of rings. In $k[T_1,\dots,T_r, S_r]$ we have $P =$

$(-T_{r-1}+S_r^2,\ -T_1+T_2S_r,\dots,-T_{r-3}+T_{r-2}S_r)$. Thus if we consider the composition

$$k[T_1,\dots,T_r]\overset{\psi_1}{\hookrightarrow}k[T_1,\dots,T_r,S_r]\overset{\psi_0}{\to}k[T_2,T_4,\dots,T_{r-2},T_r,S_r],$$

where ψ_1 is the inclusion, and ψ_0 is defined by

$$\begin{cases}\psi_0(T_{2i})=T_{2i}, & i=1,\dots,r/2\\ \psi_0(S_r)=S_r\\ \psi_0(T_1)=T_2S_r\\ \vdots\\ \psi_0(T_{r-3})=T_{r-2}S_r\\ \psi_0(T_{r-1})=S_r^2,\end{cases}$$

then $P=\operatorname{Ker}(\psi_0)$, $P\cap k[T_1,\dots,T_r]=\operatorname{Ker}(\psi_0\circ\psi_1)$. Since $\psi_0\circ\psi_1(T_r)=T_r$ and T_r is not used in defining ψ_0 or ψ_1, we can apply the result of the appendix for the homomorphism

$$\varphi_0\colon k[T_1,T_2,\dots,T_{r-1}]\to k[T_2,T_4,\dots,T_{r-2},S_r]$$

where φ_0 is the restriction of $\psi_0\circ\psi_1$. Therefore $\operatorname{Ker}(\varphi_0)$ is generated by $(r-2)^2/4$ elements

$$\begin{cases}T_2T_3-T_4T_1, & T_4T_5-T_6T_3 & ,\dots, & T_{r-4}T_{r-3}-T_{r-2}T_{r-5}\\ T_2T_5-T_6T_1 & \vdots & & \\ \vdots & T_4T_{r-3}-T_{r-2}T_3 & & \\ T_2T_{r-3}-T_{r-2}T_1 & & & \end{cases} \tag{1}$$

$$\{T_2^2T_{r-1}-T_1^2,\quad T_4^2T_{r-1}-T_3^2\quad ,\dots,\quad T_{r-2}^2T_{r-1}-T_{r-3}^2 \tag{2.a}$$

$$\begin{cases}T_2T_4T_{r-1}-T_1T_3, & T_4T_6T_{r-1}-T_3T_5 & ,\dots, & T_{r-4}T_{r-2}T_{r-1}-T_{r-5}T_{r-3}\\ T_2T_6T_{r-1}-T_1T_5 & \vdots & & \\ \vdots & T_4T_{r-2}T_{r-1}-T_3T_{r-3} & & \\ T_2T_{r-2}T_{r-1}-T_1T_{r-3} & & & \end{cases} \tag{2.b}$$

Thus $P\cap k[T_1,\dots,T_r]$ will be generated by these elements in $k[T_1,\dots,T_r]$. Now by changing the variables back to $t_1,\dots,t_r$, we find out that $P\cap k[t_1,\dots,t_r]$ is generated by certain elements in (M). A typical element of group (1) is

$$T_2T_3-T_4T_1=t_2(-t_3-t_4t_r/2)-t_4(-t_1-t_2t_r/2)=t_1t_4-t_2t_3.$$

A typical element of group (2.a) is

$$T_4^2T_{r-1} - t_3^2 = t_4^2(-t_{r-1} - 3t_r^2/2) - (-t_3 - t_4t_r/2)^2$$
$$= -t_3^2 - t_4(t_4t_{r-1} + t_3t_r + t_4t_r^2).$$

A typical element of group (2.b) is

$$T_4T_6T_{r-1} - T_3T_5$$
$$= t_4t_6(-t_{r-1} - 3t_r^2/2) - (-t_3 - t_4t_r/2)(-t_5 - t_6t_r/2)$$
$$= -t_3t_5 - t_4(t_6t_{r-1} + t_6t_r^2) - (t_3t_6 + t_4t_5)t_r/2$$
$$= [-t_3t_5 - t_4(t_6t_{r-1} + t_5t_r + t_6t_r^2)] - (t_3t_6 - t_4t_5)t_r/2.$$

The first two elements are minors of M and the third element is generated by two minors of M. Therefore we have

$$P \cap k[t_1,\dots,t_r] \subseteq (M).$$

Now we have $P \cap k[t_1,\dots,t_r] \subseteq (M) \subseteq C$. By [18, Theorem 1], any minimal prime over C has height $m - r = (r-2)/2$. By the result of the appendix $P \cap k[t_1,\dots,t_r]$ has height$(r-2)/2$. Therefore we have $C = (M) = P \cap k[t_1,\dots,t_r]$. The proof of Theorem (4.1) is now complete.

COROLLARY (4.2): *Let A, B and C be as in Theorem* (4.1). *Then* $C = \mathrm{Ann}_A(\overline{t_r})$, *where $\overline{t_r}$ denotes the class of t_r in B/A.*

PROOF: B is an A-module generated by 1, t_r, t_r^2. Thus $C = \mathrm{Ann}_A(\overline{t_r}) \cap \mathrm{Ann}_A(\overline{t_r^2})$. Let $a \in A$ and $at_r \in A$. Then since A is a ring, $a^2t_r \in A$ and $a^2t_r^2 \in A$. Thus $a^2 \in C$, but C is a prime ideal, so we have $a \in C$. Conversely assume that $a \in A$ and $at_r^2 \in A$. Then $a^2t_r^2 \in A$ and $a^2t_r^4 \in A$. Let $v = t_{r-1}t_r + t_r^3$, $(v \in A)$. Then $t_r^3 = v - t_{r-1}t_r$ and $t_r^4 = vt_r - t_{r-1}t_r^2$. Since $a^2t_r^4 \in A$, we have $a^2vt_r - t_{r-1}(a^2t_r^2) \in A$. Since $a^2t_r^2 \in A$, we get $a^2vt_r \in A$. We also have $a^2vt_r^2 \in A$. Therefore $a^2v \in C$, i.e., $a^2t_r(t_{r-1} - t_r^2) \in C$. C is a prime ideal of B and $t_r \notin C$, $t_{r-1} - t_r^2 \notin C$, thus we get $a \in C$.

We now prove the result which we used in Theorem (3.2).

THEOREM (4.3): *As in Theorem* (4.1), *let* $A = k[t_1,\dots,t_{r-1},\ t_1t_r + t_2t_r^2,\dots,t_{r-1}t_r + t_r^3]$, $B = k[t_1,\dots,t_r]$. *Then*

$$\mathrm{depth}_A(B/A) \geqslant 1.$$

PROOF: Recall that we have assumed that r is even, $r \geqslant 6$, and $m = (3r - 2)/2$. Let $v = t_{r-1}t_r + t_r^3$, $w_1 = t_1t_r + t_2t_r^2, \dots, w_{m-r} = t_{r-3}t_r + t_{r-2}t_r^2$. Let $A_0 = k[t_1, \dots, t_{r-1}, v]$. We will find a "nice" generating set for A as an A_0-module, and we then will use them to prove the theorem. Let K_0 and K be fields of quotients of A_0 and B respectively. K is an algebraic extension of K_0 of degree three. Thus every element of K satisfies a monic polynomial of degree three in $K_0[x]$. However B is integral over A_0, so for each $i = 1, \dots, m - r$, w_i would satisfy a monic polynomial in $A_0[x]$. By the uniqueness of the minimal polynomial of w_i in $K_0[x]$, the minimal monic polynomial of w_i in $A_0[x]$ is the same as the minimal polynomial of w_i in $K_0[x]$. In fact by direct calculation one can find the minimal polynomial of w_i in $A_0[x]$. For example consider $w_1 = t_1t_r + t_2t_r^2$. By expanding $(t_1t_r + t_2t_r^2)^3$, $(t_1t_r + t_2t_r^2)^2$ and multiplying out of $(t_{r-1}t_r + t_r^3)(t_1t_r + t_2t_r^2)$ we find that w_1 satisfies the following polynomial in $A_1[x]$

$$x^2 = 2t_2t_{r-1}x^2 - \left(3t_1t_2v - t_2^2t_{r-1}^2 - t_1^2t_{r-1}\right)x$$

$$-t_1^3v - t_1t_2^2t_{r-1}v - t_2^3v^2 = 0.$$

The elements $w_2, \dots, w_{m-r}$ satisfy similar polynomials of degree three in $A_0[x]$. This proves that the elements

$$w_1^{i_1} \cdot w_2^{i_2} \cdot \dots \cdot w_{m-r}^{i_{m-r}}, \mathrm{w}, \qquad 0 \leqslant \mathrm{i}_1, \mathrm{i}_2, \dots, \mathrm{i}_{m-r} \leqslant 2 \tag{4.3.1}$$

generate A as an A_0-module. We will produce another set of generators for A. To demonstrate the techniques we first consider the "key" case $r = 6$, $m = 8$. Let

$$z_1 = (t_2v - t_1t_5)t_2t_6 + t_1^2t_6^2$$

$$z_2 = (t_4v - t_3t_5)t_4t_6 + t_3^2t_6^2$$

$$z_{12} = (t_2v - t_1t_5)t_4t_6 + t_1t_3t_6^2.$$

We claim that 1, w_1, w_2, z_1, z_2, z_{12} generate A as an A_0-module. Observe that

$$\begin{cases} w_1^2 - 2t_1t_2v + t_2t_5w_1 = z_1 \\ w_2^2 - 2t_3t_4v + t_4t_5w_2 = z_2 \\ w_1w_2 - (t_1t_4 + t_2t_3)v + t_2t_5w_2 = z_{12} \end{cases}$$

$$\begin{cases} w_1^2w_2 - 2t_1t_2vw_2 + t_2t_5w_1w_2 = z_1w_2 \\ w_1w_2^2 - 2t_3t_4vw_1 + t_4t_5w_1w_2 = z_2w_1 \\ w_1^2w_2^2 - 2t_3t_4vw_1^2 + t_4t_5w_1^2w_2 = z_2w_1^2. \end{cases}$$

These relations imply that $1, w_1, w_2, z_1, z_2, z_{12}, z_1w_2, z_2w_1, z_2w_1^2$ generate A as an A_0-module. We show that the last three elements are extra. Replacing t_6^3 by $v - t_5t_6$ and t_6^4 by $vt_6 - t_5t_6^2$ in the product $z_1w_2 = [(t_2^2v - t_1t_2t_5)t_6 + t_1^2t_6^2](t_3t_6 + t_4t_6^2)$, we get $z_1w_2 = (t_1^2t_3v + t_2^2t_4v^2 - t_1t_2t_4t_5v) + (-t_1^2t_3t_5 - t_2^2t_4t_5v + t_1t_2t_4t_5^2 + t_1^2t_4v)t_6 + (t_3t_2^2v - t_1t_2t_3t_5 - t_1^2t_4t_5)t_6^2$.
Now replacing $t_2t_6^2$ by $w_1 - t_1t_6$ and $t_1^2t_6^2$ by $z_1 - (t_2v - t_1t_5)t_2t_6$ we get

$$z_1w_2 = \left(t_1^2t_3v + t_2^2t_4v^2 - t_1t_2t_4t_5v\right) + \left(t_1t_4 - t_2t_3\right)t_1vt_6 - t_2t_3vw_1$$
$$- t_1t_3t_5w_1 - t_4t_5z_1.$$

Finally since $(t_1t_4 - t_2t_3)t_6 = t_4w_1 - t_2w_2$, we have

$$z_1w_2 = \left(t_1^2t_3v + t_2^2t_4v^2 - t_1t_2t_4t_5v\right) + \left(t_1t_4v + t_2t_3v - t_1t_3t_5\right)w_1$$
$$- t_1t_2vw_2 - t_4t_5z_1.$$

A similar computation shows that

$$z_2w_1 = \left(t_1t_3^2v + t_2t_4^2v^2 - t_2t_3t_4t_5v\right) + \left(t_1t_4v - t_2t_3v - t_1t_3t_5\right)w_2$$
$$+ t_3t_4vw_1 - t_2t_5z_2.$$

Thus z_1w_2 and z_2w_1 are extra. We also have

$$z_2w_1^2 = (z_2w_1)w_1 = \left(t_1t_3^2v + t_2t_4^2v^2 - t_2t_3t_4t_5\right)w_1$$
$$+ \left(t_1t_4v - t_2t_3v - t_1t_3t_5\right)\left(w_1w_2\right) + t_3t_4vw_1^2 - t_2t_5z_2w_1.$$

Therefore by above calculations $z_2w_1^2$ is also extra. The set $\{1, w_1, w_2, z_1, z_2, z_{12}\}$ is the favorable generating set for A as an A_0-module. Observe that the coefficient of t_6^2 in each of the elements $w_1, w_2, z_1, z_2, z_{12}$ does not involve t_5. This is the main point which makes the proof of the theorem work. Incidently in this particular but important case of $r = 6$, it seems that the set $\{1, w_1, w_2, z_1, z_2, z_{12}\}$ is a minimal generating set for A as an A_0-module. To check this for example one needs to show that $\dim_k(A/(m_{A_0} \cdot A)) \geq 6$, where m_{A_0} is the maximal ideal of A_0. However we will not use this.

For general r, consider $w_1, \ldots, w_{m-r}$ as before and let

$$z_i = \left(t_{2i}v - t_{2i-1}t_{r-1}\right)t_{2i}t_r + t_{2i-1}^2t_r^2, \qquad 1 \leq i \leq m - r,$$

$$z_{ij} = \left(t_{2i}v - t_{2i-1}t_{r-1}\right)t_{2j}t_r + t_{2i-1}t_{2j-1}t_r^2, \qquad 1 \leq i < j \leq m - r.$$

We first claim that $1, w_1, \ldots, w_{m-r}, z_1, \ldots, z_{m-r}, z_{ij}, w_1^{l_1} \cdot w_2^{l_2} \cdot \ldots . w_{m-r}^{l_{m-r}};$

$1 \leqslant i < j \leqslant m-r$, $0 \leqslant i_1, \ldots, i_{m-r} \leqslant 1$, $i_1 + \ldots + i_{m-r} \geqslant 3$, generate A as an A_0-module. The relations

$$w_i^2 - 2t_{2i-1}t_{2i}v + t_{2i}t_{r-1}w_1 = z_i, \qquad i = 1, \ldots, m-r,$$

show that in the list (4.3.1), $w_1^2, \ldots, w_{m-r}^2$ can be replaced by $z_1, \ldots, z_{m-r}$. The relations

$$w_i w_j - \left(t_{2i-1}t_{2j} + t_{2i}t_{2j-1}\right)v + t_{2i}t_{r-1}w_j = z_{ij}, \qquad 1 \leqslant 2 < j \leqslant m-r,$$

imply that in the list (4.3.1) $w_i w_j$ can be replaced by z_{ij}. The relations

$$\begin{aligned} z_1 w_j = &\left(t_1^2 t_{2j-1}v + t_2^2 t_{2j}v^2 - t_1 t_2 t_{2j} t_{r-1} v\right) \\ &+ \left(t_1 t_{2j} v + t_2 t_{2j-1} v - t_1 t_{2j-1} t_{r-1}\right) w_1 \\ &- t_1 t_2 v w_j - t_{2j} t_{r-1} z_1, \quad z_1 w_j^2 = \left(z_1 w_j\right) w_j, \qquad 2 \leqslant j \leqslant m-r \end{aligned}$$

would imply that in the list (4.3.1) all the elements

$$w_1^{i_1} \cdot w_2^{i_2} \cdot \ldots \cdot w_{m-r}^{i_{m-r}}, \qquad i_1 + \ldots + i_{m-r} \geqslant 3, \qquad i_l = 2$$

for some l, are extra. To show this we may assume that $i_1 = 2$ and $i_3, \ldots, i_s = 1$ or 2 for some s, and $i_{s+1} = \cdots = i_{m-r} = 0$. Then $w_1^2 \cdot w_2^{i_2} \cdot \cdots \cdot w_s^{i_s} = 2t_1 t_2 v w_2^{i_2} \ldots w_s^{i_s} - t_2 t_{r-1} w_1 w_2^{i_2} \ldots w_s^{i_s} + z_1 w_2^{i_2} \ldots w_s^{i_s}$. Replacing $z_1 w_2$ or $z_1 w_2^2$ (according to whether $i_1 = 1$ or 2), from relations above in $z_1 w_2^{i_2} \ldots w_s^{i_s}$ and repeating this process for $z_1 w_3^{i_3}, \ldots, z_1 w_s^{i_s}$ we find that $w_1^2 w_2^{i_2} \ldots w_s^{i_s}$ is generated by elements of the form $w_1^{j_1} \cdot w_2^{j_2} \ldots w_s^{j_s}$ and z_1 where $j_1, \ldots, j_s = 0, 1, 2$ but the number of j_l's with $j_l = 2$ is less than the number of i_l's with $i_l = 2$ in the monomial $w_1^{i_1} \ldots w_s^{i_s}$ we had started with. Thus by induction on the number of i_s's with $i_s = 2$ we find that $w_1^{i_1} \ldots w_s^{i_s}$ is generated by $w_1^{j_1} \ldots w_s^{j_s}$'s with $j_l = 0, 1$, $j_1 + \ldots + j_s \geqslant 3$ and $z_1, \ldots, z_s$. Therefore the elements $w_1^{i_1} \ldots w_{m-r}^{i_{m-r}}$ with $i_1 + \ldots + i_{m-r} \geqslant 3$, $i_l = 2$ for some l, are extra. The second thing we do, we replace the elements $w_1^{i_1} \ldots w_{m-r}^{i_{m-r}}$, $0 \leqslant i_1, \ldots, i_{m-r} \leqslant 1$, $i_1 + \ldots + i_{m-r} \geqslant 3$ by another element of the form gt_r where $g \in A_0$. This will also be proved by induction on the number of $w_j^{i_j}$'s with $i_j = 1$, (this number is $\geqslant 3$ by assumption). For the first step observe that by

$$w_1 w_2 = \left(t_1 t_4 + t_2 t_3\right) v + t_2 t_{r-1} w_2 + z_{12}$$

we have

$$w_1 w_2 w_3 = \left(t_1 t_4 + t_2 t_3\right) v w_3 + t_2 t_{r-1} w_2 w_3 + z_{12} w_3,$$

and we also have

$$z_{12}w_3 = \left(t_2t_4t_6v^2 - t_1t_4t_6t_{r-1}v + t_1t_3t_5v\right)$$
$$+\left(-t_2t_4t_6t_{r-1}v - t_1t_4t_6t_{r-1}^2 + t_1t_3t_5t_{r-1}\right)t_r$$
$$+\left(t_2t_4t_5v - t_1t_4t_5t_{r-1} - t_1t_3t_5t_{r-1}\right)t_r^2.$$

Since $t_4t_r^2 = w_2 - t_3t_r$ and $t_6t_r^2 = w_3 - t_5t_4$, we see that $w_1w_2w_3$ can be replaced by gt_r with $g \in A_0$. For inductive step it is enough to show that if $g_1t_r \in A$ with $g_1 \in A_0$, then $g_1t_rw_1$ can be replaced by g_2t_r with $g_2 \in A_0$. We have

$$g_1t_rw_1 = g_1t_2v - g_1t_1t_2t_r + g_1t_1t_r^2.$$

We can impose the condition that g_1 does not have constant term or a term which is only a power of v as part of our induction hypothesis. Consider each monomial in g_1, if it has a factor of t_{2_j} for some j, replace $t_{2_j}t_r^2$ by $w_j - t_{2_j-1}t_r$, if it has a factor of t_{2_j-1} for some j, replace $t_{2j-1}t_1t_r^2$ by $z_{1_j} - (t_2v - t_1t_{r-1})t_{2_j}t_r$. Thus $g_1t_rw_1$ can be replaced by g_2t_r where $g_2 \in A_0$ and g_2 satisfies the imposed condition on g_1.

We have now produced a generating set for A as an A_0-module which was desired. Namely 1, $w_1,\dots,w_{m-r}$, $z_1,\dots,z_{m-r}$, z_{ij}, $1 \leqslant i < j \leqslant m-r$ and some elements of the form gt_r with $g \in A_0$.

We now prove that $\operatorname{depth}_A(B/A) \geqslant 1$. Observe that since B is generated by 1, t_r, t_r^2 as an A_0-module, by a result due to J.P. Serre [23, Ch. IV, Proposition 22], B is a free A_0-module of rank three. We show that t_{r-1} is a non-zerodivisor of B/A. Let $f \in B - A$ and $ft_{r-1} \in A$. For convenience let us denote all the non-constant elements we just obtained as a generating set for A as an A_0-module, by w_i, $i = 1,\dots,n$. Then $ft_{r-1} = \alpha_0 + \alpha_1w_1 + \dots + \alpha_nw_n$ with $\alpha_0,\dots,\alpha_n \in A_0$. We may assume that $\alpha_0,\dots,\alpha_n \in k[t_1,\dots,t_{r-2}, v]$, because if $\alpha_i = \beta_i + t_{r-1}\gamma_i$ with $\beta_i \in k[t_1,\dots,t_{r-2}, v]$ and $\gamma_i \in A_0$. then $[f - (\gamma_0 + \gamma_1w_1 + \dots + \gamma_nw_n)]t_{r-1} = \beta_0 + \beta_1w_1 + \dots + \beta_nw_n \in A$, and if we show that $f - (\gamma_0 + \dots + \gamma_nw_n) \in A$, then $f \in A$. Thus assume that $\alpha_0, \alpha_1,\dots,\alpha_n \in k[t_1,\dots,t_{r-2}, v]$. Since the coefficient of t_r^2 in each w_i only involves $t_1,\dots,t_{r-2}$, the coefficient of t_r^2 in the sum $\alpha_0 + \alpha_1w_1 + \dots + \alpha_nw_n$ only involves $t_1,\dots,t_{r-2}$, v. Now let $f = g_0 + gt_r + ht_r^2$ with g_0, g, $h \in A_0$. We may assume that $g_0 = 0$, because $(f - g_0)t_{r-1} \in A$ and if $f - g_0 \in A$, then $f \in A$. Thus we have

$$ft_{r-1} = (gt_{r-1})t_r + (ht_{r-1})t_r^2$$
$$= \alpha_0 + (\alpha_1t_1 + \alpha_2t_3 + \dots)t_r + (\alpha_1t_2 + \alpha_2t_4 + \dots)t_r^2.$$

Since 1, t_r, t_r^2 is a free basis for B as an A_0-module, we have $\alpha_0 = 0$ and

$\alpha_1 t_2 + \alpha_2 t_4 + \alpha_3 t_6 + \ldots = h t_{r-1}$. Since the L.H.S. only involves $t_1, \ldots, t_{r-2}, v$, and B is a free A_0-module, no element of A_0 is a zerodivisor of B/A_0, thus we have $h = 0$. Therefore $g t_{r-1} \in \mathrm{Ann}_A(\overline{t_r})$. But by Corollary (4.2) $\mathrm{Ann}_A(\overline{t_r}) = C$, the conductor of A in B. By Theorem (4.1) C is a prime ideal of B and $t_{r-1} \notin C$, thus $g \in C$. Therefore $f = g t_r \in A$, as desired. The proof of Theorem (4.3) is now complete.

COROLLARY (4.4): *Let X be a nonsingular projective variety of dimension r, $r \geqslant 6$. Let π: $X \to \mathbb{P}_k^m$ be a strongly generic projection. Let $X' = \pi(X)$ be analytically irreducible at $y = \pi(x)$, where $x \in S_1^{(2)}(\pi)$, (x, y closed points). Let $\mathscr{C}$ be the conductor of $\hat{\mathcal{O}}_{X',y}$. Consider $\mathscr{C}$ as an ideal of $k[[t_1, \ldots, t_r]]$. If r is even and $m = (3r-2)/2$, then $\mathscr{C} = (M_1)$, and if r is odd and $m = (3r-3)/2$, then $\mathscr{C} = (M_2)$, where (M_1) and (M_2) are determinantal ideals of the following matrices in $k[[t_1, \ldots, t_r]]$*

$$M_1 = \begin{bmatrix} t_1 & t_2 \\ \vdots & \vdots \\ t_{r-3} & t_{r-2} \\ t_2 t_{r-1} + t_1 t_r + t_2 t_r^2 & -t_1 \\ \vdots & \vdots \\ t_{r-2} t_{r-1} + t_{r-3} t_r + t_{r-2} t_r^2 & -t_{r-3} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} t_1 & t_2 \\ \vdots & \vdots \\ t_{r-4} & t_{r-3} \\ t_2 t_{r-2} + t_1 t_r + t_2 t_r^2 & -t_1 \\ \vdots & \vdots \\ t_{r-3} t_{r-2} + t_{r-4} t_r + t_{r-3} t_r^2 & -t_{r-4} \end{bmatrix}.$$

PROOF: Using the notation of Theorem (4.1) we have $\mathscr{C} = C \cdot k[[t_1, \ldots, t_r]]$. Now the corollary follows from (3.0.1) and Theorem (4.1) for both cases of r even and r odd.

REMARKS: In Theorem (4.1) we showed that C is a prime ideal of $k[t_1, \ldots, t_r]$. Thus by known results [27, vol. II, page 320, Theorem 32], $\mathscr{C}$ is a radical ideal of $k[[t_1, \ldots, t_r]]$. This also follows from Theorem (3.2), because the conductor of a seminormal ring is radial in the normalization of the ring.

Also note that to prove that some determinantal ideal of a matrix with entries in $B = k[t_1, \ldots, t_r]$ is a prime ideal of B in general is a difficult task. In Theorem (4.1) we proved that (M) is a prime ideal of B. A direct computation shows that if we discard some of the last $r/2$ rows (not all of them), the determinantal ideal of the remaining matrix will no longer be a prime ideal of B. There are beautiful results concerning the determinantal ideals by many authors as J.A. Eagon, D.G. Northcott and M. Hochster. (See the references of [10].)

Appendix

The following unpublished result due to Joel Roberts, was originally given to serve as examples of prime ideals in power series rings which require a large number of generator (cf. [14, remark 4]). In this thesis we have used this result several times. To make it a convenient reference we will rewrite this result.

Construction. Let k be a field, $r \geqslant 2$ an integer. Let $A_0 = k[t_1,\ldots,t_{r-1}, u_1,\ldots,u_{r-1}, v]$ and $B = k[t_1,\ldots,t_r]$ be polynomial rings, and let $A = k[[t_1,\ldots,t_{r-1}, u_1,\ldots,u_{r-1}, v]]$ and $B = k[[t_1,\ldots,t_r]]$ be formal power series rings. Let φ_0: $A_0 \to B_0$ and φ: $A \to B$ be determined by:

$$t_i \mapsto t_i \qquad i = 1,\ldots,r-1 \tag{1.1}$$

$$u_i \mapsto t_i t_r \qquad i = 1,\ldots,r-1 \tag{1.2}$$

$$v \mapsto t_r^2. \tag{1.3}$$

Let $P_0 = \mathrm{Ker}(\varphi_0)$ and $P = \mathrm{Ker}(\varphi)$. Since B_0 and B are integral over $\varphi_0(A_0)$ and $\varphi(A)$ respectively, it follows that P_0 and P are prime ideals of height $r-1$.

Geometric discussion. Let k be algebraically closed and let $V = \mathrm{Spec}(A_0/P_0)$. Then φ_0 defines a finite birational morphism π: $\mathbb{A}_k^r \to V$, so that $\mathbb{A}_k^r$ is the normalization of V. Then $V \subset \mathbb{A}_k^{2r-1}$, and the singular locus of V is the line where all coordinates except v are zero. At all points except the origin, V looks analytically like two r-dimensional subspaces of $\mathbb{A}_k^{2r-1}$ meeting transversally. It follows that the Zariski tangent space of V at any singular point has dimension $2r-1$. From this and the fact that π^{-1} (origin in $\mathbb{A}_k^{2r-1}$) = (origin in $\mathbb{A}_k^r$), we find:

$$P_0 \cdot A = P \tag{2}$$

P does not contain any element of order 1. (3)

Of course these results continue to hold if we drop the assumption $k = \bar{k}$.

A subset of a minimal generating set. It is clear that P_0 contains the following elements:

$$t_i u_j - t_j u_i \qquad 1 \leqslant i < j \leqslant r-1 \tag{4.1}$$

$$t_i^2 v - u_i^2 \qquad 1 \leqslant i \leqslant r-1 \tag{4.2a}$$

$$t_i t_j v - u_i u_j \qquad 1 \leqslant i < j \leqslant r-1. \tag{b}$$

Using (3) and the fact that these elements are linearly independent modulo m^3 (where m = maximal ideal of A), we see that they form a subset of a minimal generating set of P. In this way we have shown that a minimal generating set of P must have at least $(r-1)^2$ elements.

We claim that (4.1) and (4.2) generate P.

Proof of the claim. By (2), it suffices to show that the elements in question generate P_0. Let I be the ideal generated by these elements. We observe that $\varphi_0(A_0)$ is generated by monomials; therefore $\text{Ker}(\varphi_0)$ is spanned as k-vector space by the elements $M - N$, where M and N are monomials such that $\varphi_0(M) = \varphi_0(N)$. Therefore it will suffice to show that $M - N \in I$.

Let $f_1 g_1 v^a - f_2 g_2 v^b \in \text{Ker}(\varphi_0)$, where f_1, f_2 are monomials in $u_1, \ldots, u_{r-1}$, and g_1, g_2 are monomials in $t_1, \ldots, t_{r-1}$. We must show that $f_1 g_1 v^a - f_2 g_2 v^b \in I$. We reduce to the case where degree $(f_1) \leqslant 1$ and degree $(f_2) \leqslant 1$ by using the fact that $u_i u_j - t_i t_j v \in I$ for all i, j. In this case we may also assume that degree(f_1) = degree(f_2) and $a = b$, because $\varphi_0(f_1 g_1 v^a)$ and $\varphi_0(f_2 g_2 v^b)$ must have the same degree in t_r. If $f_1 = f_2 = 1$, or if $f_1 = f_2 = u_i$, we must have $g_1 = g_2$. Finally if $f_1 = u_i$ and $f_2 = u_j$ with $i \neq j$, we observe that $t_i \mid g_2$, $t_j \mid g_1$. We then use the fact that $t_i u_j - t_j u_i \in I$ to conclude that $f_1 g_1 v^a - f_2 g_2 v^b \in I$.

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Department of Mathematics
Faculty of Science
University of Tehran
Tehran 14
Iran