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ELLIPSOIDS AND HYPERBOLOIDS WITH INFINITE FOCI

Hans-Christoph Im Hof

Introduction

In an earlier paper it was shown that the family of horospheres through two points of a simply connected complete riemannian manifold of nonpositive curvature is homeomorphic to a sphere (Im Hof [5]). This result was obtained by investigating the families of confocal ellipsoids and confocal hyperboloids with respect to two given points as foci.

Here we will investigate two families of hypersurfaces, called *tubes* and *slices*, which can be viewed as ellipsoids and hyperboloids with respect to two infinite points as foci. Our setting will be a simply connected complete riemannian manifold of dimension n and sectional curvature bounded from above by a negative constant. Fixing two infinite points, represented by two Busemann functions, we define tubes and slices as the level sets of the sum and the difference, respectively, of the chosen Busemann functions.

Here are our main results.

THEOREM 2.4: *Tubes are diffeomorphic to* $S^{n-2} \times \mathbb{R}$; *slices are diffeomorphic to* $\mathbb{R}^{n-1}$.

COROLLARY 3.10: *The set of infinite points of a slice is homeomorphic to* S^{n-2}.

The proofs are similar to the ones in [5], however some modifications are necessary in order to deal with infinite foci. In the case of finite foci we used polar coordinates around one focus, and so we were led to consider angles at this focus. Here it is natural to use "polar coordinates" around an infinite focus, but since at an infinite point there is no angle measurement we have to replace angles by distances on horospheres. In our arguments we will make ample use of comparison theorems for Jacobi fields and for triangles with infinite vertices. These theorems allow us to transform exact calculations in the hyperbolic plane of constant curvature into estimates valid for given situations in the manifold under consideration. The necessary preliminaries are summarized in the first chapter. The second chapter contains a general study of tubes and slices, and in the third chapter we will investigate the behavior of a slice at infinity. At the end we add some remarks concerning the case of one finite and one infinite focus.

The present paper answers a question which has its origin in the work of M. Brin and H. Karcher on the ergodicity of the frame flow on negatively curved manifolds of even dimensions. Corollary 3.10 was independently proved by P. Eberlein. The paper was conceived at the Sonderforschungsbereich "Theoretische Mathematik" in Bonn and worked out at the IHES in Bures-sur-Yvette with support from the Stiftung Volkswagenwerk. We gratefully thank all these institutions for their hospitality and support.

1. Preliminaries

Infinite points and horospheres (cf. Eberlein-O'Neill [2], § 1–§ 3)

Throughout this paper M denotes an n-dimensional simply connected complete riemannian manifold of sectional curvature K satisfying $K \leqslant -1$. An *infinite point* of M is by definition a class of asymptotic geodesic rays. We will write $\gamma(\infty) = z$ to indicate that the geodesic ray γ belongs to the asymptoticity class z. The set of infinite points of M is denoted by $M(\infty)$, and $\overline{M}$ stands for the union $M \cup M(\infty)$. Our assumptions on M imply that any two points of $\overline{M}$ can be joined by a geodesic unique up to parametrization. The set $\overline{M}$ carries a topology with respect to which it is homeomorphic to the closed ball D^n, while $M(\infty)$ is homeomorphic to the boundary sphere S^{n-1}.

Let $d(p, q)$ denote the distance of two points $p, q \in M$. The function a: $M \times M \times M \to \mathbb{R}$ defined by $a(p, m, q) = d(m, q) - d(m, p)$ has a continuous extension to $M \times \overline{M} \times M$, which is given by the expression $a(p, z, q) = f(q) - f(p)$, where f is a Busemann function at the infinite point z (Eberlein [1], Proposition 2.6).

Horospheres with center $z \in M(\infty)$ are by definition level sets of Busemann functions at z. Since Busemann functions are C^2 (see Proposition 1.2 below), horospheres are C^2 submanifolds, and so we can speak of the induced metric on a given horosphere.

Stable Jacobi fields

A Jacobi field Y along a geodesic γ is called *stable* if $\|Y(t)\|$ is bounded for $t \geqslant 0$. From Heintze-Im Hof [4] we quote the following results.

PROPOSITION 1.1 ([4], Thm. 2.4): *Let Y be a stable Jacobi field along a geodesic γ and assume Y is perpendicular to $\dot{\gamma}$. Then $\|Y(t)\| \leqslant \|Y(0)\| e^{-t}$ for $t \geqslant 0$.*

PROPOSITION 1.2 (Eberlein, unpublished; [4], Prop. 3.1): *Let f be a Busemann function at z. Then f is C^2,* (grad f)(p) $= -\dot{\gamma}(0)$, *and* ∇_v grad $f = -Y'(0)$, *where γ denotes the unit speed geodesic with $\gamma(0)=p$ and $\gamma(\infty)=z$, and Y denotes the stable Jacobi field along γ with $Y(0)=v$.*

Infinite triangles

In this section we will compare triangles in M with triangles in the hyperbolic plane H^2 of constant sectional curvature -1. A *simply infinite triangle* Δ is determined by two vertices p, $q \in M$ and one vertex $z \in M(\infty)$. Its parameters are $d(p, q)$, $a(p, z, q)$, and the angles α and β at p and q, respectively. If $a(p, z, q) = 0$, then Δ is called isosceles.

Let Δ_0 be a simply infinite triangle in H^2 determined by its vertices p_0, $q_0 \in H^2$ and $z_0 \in H^2(\infty)$, and consider its respective parameters $d(p_0, q_0)$, $a(p_0, z_0, q_0)$, α_0, and β_0. The relations among these parameters are governed by hyperbolic trigonometry. We recall the basic formula. Under the assumption $\alpha_0 = \pi/2$, the remaining parameters of Δ_0 satisfy the relations

$$\cosh(d(p_0, q_0)) = \exp(a(p_0, z_0, q_0)) = (\sin \beta_0)^{-1}.$$

For triangles in M we have the following comparison theorems.

PROPOSITION 1.3: *Let Δ and Δ_0 be simply infinite triangles in M and H^2, respectively.*

(*i*) *If $d(p, q) = d(p_0, q_0)$ and $\alpha = \alpha_0$, then $a(p, z, q) \geqslant a(p_0, z_0, q_0)$ and $\beta \leqslant \beta_0$.*

(*ii*) *If $d(p, q) = d(p_0, q_0)$ and $a(p, z, q) = a(p_0, z_0, q_0)$, then $\alpha \leqslant \alpha_0$ and $\beta \leqslant \beta_0$.*

PROOF: See Heintze-Im Hof [4], Propositions 4.4 and 4.5. (Observe that Proposition 4.4 of [4] contains a misprint. The inequalities for β should be reversed.)

A *doubly infinite triangle* Δ is determined by one vertex $p \in M$ and two vertices $x, y \in M(\infty)$. Its parameters are the angle α at p and its *detour*, which we define as follows. We choose Busemann functions f at x and g at y, normalized such that $f + g = 0$ on the geodesic joining x and y. Then we define detour$(\Delta) = f(p) + g(p)$. For any two sequences $\{x_i\} \subset M$ and $\{y_i\} \subset M$ converging to x and y, respectively, we have

$$\text{detour}(\Delta) = \lim_{i \to \infty} (d(x_i, p) + d(p, y_i) - d(x_i, y_i)).$$

Another useful parameter of Δ is its *height*, which is defined to be the length of the perpendicular from the finite vertex to the opposite side.

Let Δ_0 denote a doubly infinite triangle in H^2 with vertices $p_0 \in H^2$ and $x_0, y_0 \in H^2(\infty)$. Then we have the relations

$$\cosh(\text{height}(\Delta_0)) = \exp(\text{detour}(\Delta_0)/2) = (\sin(\alpha_0/2))^{-1}.$$

For triangles in M we have the following comparison theorems.

PROPOSITION 1.4: *Let Δ and Δ_0 be doubly infinite triangles in M and H^2, respectively.*

(*i*) *If* $\text{height}(\Delta) = \text{height}(\Delta_0)$, *then* $\text{detour}(\Delta) \geqslant \text{detour}(\Delta_0)$ *and* $\alpha \leqslant \alpha_0$.

(*ii*) *If* $\text{detour}(\Delta) = \text{detour}(\Delta_0)$, *then* $\text{height}(\Delta) \leqslant \text{height}(\Delta_0)$ *and* $\alpha \leqslant \alpha_0$.

PROOF: (i) is an immediate consequence of Proposition 1.3 (i) applied to the two rectangular simply infinite triangles obtained from Δ by dropping the perpendicular from p to the opposite side.

The first estimate in (ii) follows from (i) and the fact that $\text{detour}(\Delta_0)$ is monotone increasing as a function of $\text{height}(\Delta_0)$. In order to get the estimate for α we approximate Δ by finite isosceles triangles $\Delta(t)$. Let γ and μ denote the unit speed geodesic rays with $\gamma(0) = \mu(0) = p$, $\gamma(\infty) = x$, and $\mu(\infty) = y$, respectively. Then we define $\Delta(t)$ to be the triangle with vertices p, $\gamma(t)$, and $\mu(t)$. We compare Δ and $\Delta(t)$ to triangles Δ_0 and $\Delta_0(t)$ as follows. By Δ_0 we denote a doubly infinite triangle in H^2 with detour $(\Delta_0) = \text{detour}(\Delta)$. Let α_0 be the angle of Δ_0 at the finite vertex. By $\Delta_0(t)$ we denote an isosceles triangle in H^2 whose sides have the same length as those of $\Delta(t)$. Let α_t be the angle of $\Delta_0(t)$ opposite to the basis.

From the angle comparison theorem for finite triangles, applied to $\Delta(t)$ and $\Delta_0(t)$, we obtain $\alpha \leqslant \alpha_t$. We claim that $\lim_{t\to\infty} \alpha_t = \alpha_0$. Consider the number $d(t) = 2t - d(\gamma(t), \mu(t))$. It plays the rôle of a detour in both triangles, $\Delta(t)$ and $\Delta_0(t)$. By the approximation of Δ by $\Delta(t)$ we have $\lim_{t\to\infty} d(t) = \text{detour}(\Delta)$. The assumption $\text{detour}(\Delta) = \text{detour}(\Delta_0)$ then implies $\lim_{t\to\infty} d(t) = \text{detour}(\Delta_0)$, therefore the triangles $\Delta_0(t)$ approximate Δ_0, thus $\lim_{t\to\infty} \alpha_t = \alpha_0$. We conclude that $\alpha \leqslant \alpha_0$.

COROLLARY 1.5: *Let Δ be a doubly infinite triangle in M, then*

$$\sin(\alpha/2) \leqslant \exp(-\text{detour}(\Delta)/2).$$

The ball topology and the half space topology

There are different ways of extending the manifold topology of M to a topology on $\overline{M}$. Using the exponential map at a point $p \in M$ and polar coordinates in the tangent space T_pM, we get a diffeomorphism

$$\lambda_p\colon S_pM \times (0, \infty) \to M\setminus\{p\}.$$

(Here S_pM denotes the unit sphere in T_pM.) The map λ_p extends to a bijective map

$$\lambda_p\colon S_pM \times (0,\infty] \to \overline{M}\setminus\{p\},$$

and by requiring this map to be a homeomorphism we induce a topology on $\overline{M}\setminus\{p\}$, which extends to $\overline{M}$ in the obvious way. Equipped with this topology $\overline{M}$ becomes homeomorphic to the closed ball. Although the construction depends on the choice of $p \in M$, the resulting topology turns out to be independent of p (Eberlein-O'Neill [2], § 2). We will call this topology the *ball topology*, in contrast to the half space topology which we define next.

We fix a point $z \in M(\infty)$ and try to construct an analogous map λ_z. Since there is no unit tangent sphere at z, we have to use one of the horospheres centered at z as a substitute. Choose a Busemann function f at z and consider the horosphere $F = f^{-1}(0)$. Let $\pi\colon M \to F$ denote the projection along geodesic rays belonging to z. Then the map $\lambda_z\colon F \times (-\infty, +\infty) \to M$ is defined by $\lambda_z(\pi(m), f(m)) = m$. This map is a homeomorphism (Eberlein-O'Neill [2], Proposition 3.4), and it has an obvious extension to a bijective map $\lambda_z\colon F \times (-\infty, +\infty] \to \overline{M}\setminus\{z\}$. Requiring this map to be a homeomorphism we induce a topology on $\overline{M}\setminus\{z\}$, which extends to a topology on $\overline{M}$ by using horoballs centered at z (i.e., sets of the form $\{m \in M;\ f(m) < c\}$) as basic neighborhoods of z. This topology is called the *half space topology* of $\overline{M}$. It depends on the choice of $z \in M(\infty)$, and in fact, this point plays a special rôle in $M(\infty)$. With respect to the half space topology, $M(\infty)$ decomposes into the connected components $M(\infty)\setminus\{z\}$, homeomorphic to $\mathbb{R}^{n-1}$, and the singleton $\{z\}$. This reflects the basic difference between the unit disc model and the upper half plane model for H^2.

Later we will use a uniform structure on $\overline{M}$ which induces the half space topology. We choose a Busemann function f at z and fix the horosphere $F = f^{-1}(0)$. On F we have the riemannian metric induced by the metric of M. Let d_F denote the corresponding distance function on F. We extend d_F to a quasi-distance on $\overline{M}\setminus\{z\}$ by defining $d_F(p, q) = d_F(\pi(p), \pi(q))$, where π denotes the extended projection $\pi\colon \overline{M}\setminus\{z\} \to F$. Now two points $p, q \in \overline{M}\setminus\{z\}$ are said to be close if $d_F(p, q)$ is small and the numbers $f(p)$, $f(q)$ are close in $[-\infty, -\infty]$, and a point $p \in \overline{M}\setminus\{z\}$ is close to z if $f(p)$ is close to $-\infty$. The use of this uniform structure for the half space topology makes our estimates more transparent. However, we are interested in results expressible in terms of the ball topology. The two are related by the following proposition.

PROPOSITION 1.6: *On* $\overline{M}\setminus\{z\}$ *the half space topology coincides with the ball topology.*

PROOF: At finite points the two topologies coincide with the ordinary topology of M. The equivalence at infinite points different from z is settled by Lemma 1.8 below.

DEFINITION 1.7: Let x be a point of $M(\infty)\setminus\{z\}$. A *truncated p-cone neighborhood* of x is a set of the form

$$T_p(\epsilon, r) = \left\{ m \in \overline{M};\ \sphericalangle_p(m, x) < \epsilon,\ d(m, p) > r \right\}.$$

Here $p \in M$ and $\sphericalangle_p(m, x)$ denotes the angle at p subtended by m and x.
A *truncated z-cone neighborhood* of x is a set of the form

$$T_z(\epsilon, r) = \left\{ m \in \overline{M};\ d_F(m, x) < \epsilon, f(m) > r \right\}.$$

LEMMA 1.8: *Each* $T_z(\epsilon, r)$ *contains a suitable* $T_p(\delta, s)$, *and vice versa.*

PROOF: Observe that we are free to choose any point $p \in M$ to represent the ball topology. A convenient choice for p is the point where the geodesic line through x and z intersects F. Now let $T_z(\epsilon, r)$ be given. Let γ be a geodesic ray starting at p and including an angle δ with the axis of $T_z(\epsilon, r)$. Let α be the projection of γ on F. Then length$(\alpha) \leqslant (1 + \cos\delta)^{-1}\sin\delta$ (Heintze-Im Hof [4], Corollary 4.8), hence for δ sufficiently small we have $T_p(\delta, 0) \subset T_z(\epsilon, 0)$. After suitable truncation we get $T_p(\delta, s) \subset T_z(\epsilon, r)$.

Conversely, let $T_p(\epsilon, r)$ be given. The construction of a suitable $T_z(\delta, s)$ contained in $T_p(\epsilon, r)$ is completely analogous to the proof of Lemma 2.8 in Eberlein-O'Neill [2], except for one modification. Statement (1) of [2] has to be replaced by the statement:

For s sufficiently large, $f(m) \geqslant s$ implies $\sphericalangle_m(p, z) < \epsilon/3$. But this follows from Proposition 1.3 (i) and trigonometry in H^2.

2. Tubes and slices

From now on we fix two distinct points $x, y \in M(\infty)$ and a unit speed geodesic σ with $\sigma(-\infty) = y$ and $\sigma(+\infty) = x$. Up to the choice of $\sigma(0)$ this geodesic is uniquely determined. The points x and y give rise to the following functions.

DEFINITION 2.1: Let f: $M \to \mathbb{R}$ and g: $M \to \mathbb{R}$ denote Busemann functions at x and y, respectively, normalized such that $f(\sigma(0)) = g(\sigma(0)) = 0$. Then we define $e = \frac{1}{2}(f + g)$ and $h = \frac{1}{2}(g - f)$.

The functions e and h are defined and C^2 throughout M. They give rise to two families of hypersurfaces.

DEFINITION 2.2: For $t \geqslant 0$ we define the *tube* E_t as level set $e^{-1}(t)$; for $s \in \mathbb{R}$ we define the *slice* H_s as level set $h^{-1}(s)$. The families $\{E_t;\ t \geqslant 0\}$ and $\{H_s;\ s \in \mathbb{R}\}$ can be regarded as the families of confocal *ellipsoids* and *hyperboloids* with respect to the two given infinite points x, y as foci.

The basic properties of e and h are summarized in the following proposition.

PROPOSITION 2.3:

(*i*) *The function e assumes its minimal value* 0 *on the set* $\sigma(\mathbb{R})$; *outside* $\sigma(\mathbb{R})$ *it has no critical points.*

(*ii*) *The function h has no critical points in M.*

PROOF: (i) If $m = \sigma(s)$ for $s \in \mathbb{R}$, then $e(m) = 0$. If $m \notin \sigma(\mathbb{R})$, then m and σ span a doubly infinite triangle Δ of positive height. From Proposition 1.4 (i) and an argument in H^2 it follows that Δ has positive detour as well. Since detour $(\Delta) = 2e(m)$, this implies $e(m) > 0$.

Now assume that m is a critical point for e, i.e., (grad e)$(m) = 0$. Since the geodesic joining x and y is unique up to parametrization, this implies $m \in \sigma(\mathbb{R})$.

(ii) Clearly grad h never vanishes.

From Proposition 2.3 we immediately conclude that the sets $E_t (t > 0)$ and H_s are $(n-1)$-dimensional submanifolds of M. In the following theorem we determine their differential types.

THEOREM 2.4: *For* $t > 0$ *the submanifolds* E_t *are diffeomorphic to* $S^{n-2} \times \mathbb{R}$; *for* $s \in \mathbb{R}$ *the submanifolds* H_s *are diffeomorphic to* $\mathbb{R}^{n-1}$.

PROOF: Assume $t > 0$. We claim that the restriction e_s of e to H_s is a Morse function on H_s with one critical point, and that the restriction h_t of h to E_t has no critical points. Since grad e and grad h are always perpendicular to each other, we have grad e_s = grad e on H_s and grad h_t = grad h on E_t. Therefore e_s has exactly one critical point, viz. the point $\sigma(s)$, which is the minimal point for e_s, and h_t has no critical point at all.

We have to show that $\sigma(s)$ is nondegenerate. For $v \in T_{\sigma(s)}H_s$ we have

$$\nabla_v \operatorname{grad} e = \nabla_v^s \operatorname{grad} e_s + \lambda \operatorname{grad} h,$$

where ∇^s denotes covariant differentiation on H_s, and $\lambda \in \mathbb{R}$ depends on v. By Proposition 1.2 we have

$$2\, \nabla_v \operatorname{grad} e = Y'(s) - X'(s),$$

where X and Y are Jacobi fields along σ satisfying $X(s) = Y(s) = v$, and such that X is stable in the direction of $\sigma(+\infty)$ and Y is stable in the direction of $\sigma(-\infty)$. In particular, ∇_v grad e is normal to $\dot{\sigma}$, hence $\lambda = 0$. Now ∇_v^s grad $e_s = 0$ implies $X'(s) = Y'(s)$, thus $X = Y$, and X is stable in both directions. Therefore $X = 0$, and hence $v = 0$.

Applying the Morse Lemma to the critical point $\sigma(s)$ of e_s, we find that for small $t > 0$ the set $e_s^{-1}(t)$ is diffeomorphic to S^{n-2}. Since e_s has no further critical points, the same conclusion holds for all $t > 0$. Observing the equality $e_s^{-1}(t) = h_t^{-1}(s) = H_s \cap E_t$, we finally conclude that H_s and E_t are diffeomorphic to $\mathbb{R}^{n-1}$ and $S^{n-2} \times \mathbb{R}$, respectively.

3. Behavior at infinity

In this chapter we will investigate the behavior of a slice at infinity. We begin by extending the function h to $\overline{M}$.

LEMMA 3.1: *The function h: $M \to \mathbb{R}$ has a continuous extension to a function h: $\overline{M} \to \overline{\mathbb{R}}$, where $\overline{M}$ carries the ball topology and $\overline{\mathbb{R}}$ denotes $\mathbb{R} \cup \{\pm\infty\}$.*

PROOF: In a first step we extend h to a continuous function h: $\overline{M}\backslash\{x, y\} \to \mathbb{R}$. Let F and G denote the horospheres $f^{-1}(0)$ and $g^{-1}(0)$, and let π and τ denote the projections of $\overline{M}\backslash\{x, y\}$ onto F and G, respectively. Then for any point $m \in M$ with $f(m) > 0$ and $g(m) > 0$ we have

$$2h(m) = g(m) - f(m) = d(m, \tau(m)) - d(m, \pi(m))$$
$$= a(\pi(m), m, \tau(m)).$$

For $z \in M(\infty)\backslash\{x, y\}$ we now define

$$2h(z) = a(\pi(z), z, \tau(z)).$$

We have to show that $\{h(m_i)\}$ converges to $h(z)$ for any sequence $\{m_i\} \subset M$ converging to z. Since $f(m_i)$ and $g(m_i)$ tend to $+\infty$, we have $f(m_i) > 0$, $g(m_i) > 0$ for i large. Therefore $2h(m_i) = a(\pi(m_i), m_i, \tau(m_i))$. By the continuity of π, τ and a, this in fact tends to $a(\pi(z), z, \tau(z)) = 2h(z)$.

Now we extend h to $\overline{M}$ by defining $h(x) = +\infty$ and $h(y) = -\infty$. We claim that this extension is continuous at x and y. Let s be any real number and let G denote the horosphere $g^{-1}(s + \delta)$ for some $\delta > 0$. Since h is continuous on G, there is a neighborhood U of $\sigma(s + \delta)$ in G, on which h is greater than s, and since h is monotone increasing on all geodesics emanating from y, it exceeds s on the whole truncated y-cone neighborhood determined by U. This shows the continuity of h at x.

Now we consider the vector field $Z = \text{grad}\, e/\|\text{grad}\, e\|^2$, which is defined on $M \setminus \sigma(\mathbb{R})$. For each point m outside $\sigma(\mathbb{R})$ we denote by μ_m the maximal integral curve of Z with the parametrization determined by $\mu_m(e(m)) = m$.

LEMMA 3.2:
(*i*) *Whenever* $\mu_m(t)$ *is defined, we have* $e(\mu_m(t)) = t$ *and* $h(\mu_m(t)) = h(m)$.
(*ii*) *The domain of* μ_m *is the interval* $(0, \infty)$.
(*iii*) *The image of* μ_m *is contained in* $H_{h(m)}$.

The proofs are as in Im Hof [5]. Observe that in (ii) we need the compactness of the sublevel sets $H_s \cap \{e \leqslant c\}$ for $s \in \mathbb{R}$ and $c > 0$.

Now we fix the tube E_1, and for $t > 0$ we define φ_t: $E_1 \to M$ by $\varphi_t(m) = \mu_m(t)$. Obviously the image $\varphi_t(E_1)$ is contained in E_t. The following lemma again is proved as in [5].

LEMMA 3.3: *For* $t > 0$ *the maps* φ_t: $E_1 \to E_t$ *are diffeomorphisms satisfying* $h \circ \varphi_t = h$.

We will study the limit of the family $\{\varphi_t\}$ for t tending to ∞. For this purpose we have to admit $\overline{\mathrm{M}}' = \overline{\mathrm{M}} \setminus \{\mathrm{x}, \mathrm{y}\}$ as range for the maps φ_t. Observe that on $\overline{\mathrm{M}}'$ the half space topology based at x or y coincides with the ball topology.

PROPOSITION 3.4: *The family of maps* φ_t: $E_1 \to \overline{M}'$ *converges locally uniformly on* E_1 *as t tends to* ∞.

PROOF: For any $c > 0$ we introduce the compact sets $\overline{M}(c) = \{m \in \overline{M};\ |h(m)| \leqslant c\}$ and $E_1(c) = E_1 \cap \overline{M}(c)$. Our claim will be proved by showing that for all $c > 0$ the restricted maps φ_t: $E_1(c) \to \overline{M}(c)$ converge uniformly on $E_1(c)$ as t tends to ∞. In order to check this convergence we will estimate expressions of the form $f(\mu_m(t))$ and $d_F(\mu_m(t), \mu_m(t'))$, where F is a suitable horosphere centered at x.
1. *Step*: Since $\mu_m(t) \in E_t$, we have

$$f(\mu_m(t)) \geqslant t - c \tag{1}$$

for all $m \in E_1(c)$. Therefore $f(\mu_m(t))$ tends to ∞ with t, uniformly in $E_1(c)$.
2. *Step*: We fix a point $m \in E_1(c)$. For shortness we will write $\mu = \mu_m$. Let F denote the horosphere $f^{-1}(-c)$ and π the projection of $\overline{M} \setminus \{x\}$ onto F. Consider the curve $\alpha = \pi \circ \mu$: $(0, \infty) \to F$. Then $d_F(\mu(t), \mu(t'))$

$= d_F(\alpha(t), \alpha(t'))$, hence

$$d_F(\mu(t), \mu(t')) \leqslant \text{length}(\alpha \mid [t, t']). \tag{2}$$

3. *Step*: We will estimate $\|\dot{\alpha}\|$ by using comparison theory. Fix $t > 0$ and consider the differential $\pi_*\colon T_{\mu(t)}M \to T_{\alpha(t)}F$. This map decomposes into the linear projection $\rho\colon T_{\mu(t)}M \to T_{\mu(t)}F_t$ and the map $\pi_* \mid T_{\mu(t)}F_t\colon T_{\mu(t)}F_t \to T_{\alpha(t)}F$. Here F_t denotes the horosphere through $\mu(t)$ with center x.

Now look at the doubly infinite triangle spanned by σ and $\mu(t)$, and denote by 2ω the angle at $\mu(t)$. By Proposition 1.2 this is also the angle subtended by grad f and grad g at $\mu(t)$. Therefore $\|(\text{grad } e)(\mu(t))\| = \cos\omega$, hence $\|\dot{\mu}(t)\| = (\cos\omega)^{-1}$, and so $\|\rho(\dot{\mu}(t))\| = \|\dot{\mu}(t)\| \sin\omega = \tan\omega$.

Corollary 1.5 implies $\sin\omega \leqslant e^{-t}$, thus

$$\|\rho(\dot{\mu}(t))\| \leqslant e^{-t}(1 - e^{-2t})^{-1/2}. \tag{3}$$

The effect of $\pi_* \mid T_{\mu(t)}F_t$ is expressed in terms of Jacobi fields. Let Y be the stable Jacobi field along the geodesic from $\mu(t)$ to x with initial value $Y(0) = \rho(\dot{\mu}(t))$. By Lemma 3.2 (i), $f(\mu(t)) = e(\mu(t)) - h(\mu(t)) = t - h(m)$, hence $\dot{\alpha}(t) = Y(t - h(m) + c)$, and Proposition 1.1 implies

$$\|\dot{\alpha}(t)\| \leqslant \|\rho(\dot{\mu}(t))\| e^{-(t - h(m) + c)}.$$

Using (3) and observing that $|h(m)| \leqslant c$ we obtain

$$\|\dot{\alpha}(t)\| \leqslant e^{-2t}(1 - e^{-2t})^{-1/2}. \tag{4}$$

4. *Step*: Integrating (4) and observing (2) we get $d_F(\mu(t), \mu(t')) \leqslant 1 - (1 - e^{-2t})^{1/2}$ for $t' > t > 0$, hence

$$d_F(\mu(t), \mu(t')) < \epsilon \tag{5}$$

for $t' > t$, t sufficiently large, and $m \in E_1(c)$. (Recall $\mu = \mu_m$ and $F = f^{-1}(-c)$.) From (1) and (5) we conclude that $\lim_{t\to\infty} \mu_m(t)$ exists in $\overline{M}(c)$, uniformly for $m \in E_1(c)$.

DEFINITION 3.5: We define $\mu_m(\infty) = \lim_{t\to\infty} \mu_m(t)$ and $\varphi(m) = \lim_{t\to\infty} \varphi_t(m) = \mu_m(\infty)$ for $m \in E_1$.

Obviously $\varphi(m) \in M(\infty)$. Our next aim is to show that φ is a homeomorphism of E_1 onto $M'(\infty) = M(\infty)\setminus\{x, y\}$. We start with two lemmata which are used to prove that φ is injective.

LEMMA 3.6: *Fix $m \in E_1$ and let γ be the geodesic joining x to $\mu_m(\infty)$. Then $d(\mu_m(t), \gamma) \leqslant 1$ for $t > 0$.*

PROOF: Assume $h(m) = s$ and fix $t > 0$. Let F be the horosphere through $\mu_m(t)$ centered at x, and consider the curve $\alpha = \pi \circ \mu_m | [t, \infty]$, where π denotes the projection of $\overline{M} \setminus \{x\}$ onto F. Then $\alpha(\infty)$ is a point of γ, and hence $d(\mu_m(t), \gamma) \leqslant \text{length}(\alpha)$.

As in the proof of Proposition 3.4 we estimate $\|\dot{\alpha}\|$ by comparison theory, and by integration we obtain

$$\text{length}(\alpha) \leqslant e^t - (e^{2t} - 1)^{1/2} \leqslant 1.$$

LEMMA 3.7: *Fix two points $m_1, m_2 \in E_1$ with $h(m_1) = h(m_2)$ and consider $\mu_i = \mu_{m_i}$ for $i = 1, 2$. Then $d(\mu_1(t), \mu_2(t)) \geqslant 2t - k$ for $t \geqslant 1$ and a suitable constant k depending on $d(m_1, m_2)$.*

PROOF: Let $c(t)$ denote the distance $d(\mu_1(t), \mu_2(t))$. For a fixed $t > 0$, let β: $[0, c(t)] \to M$ be the geodesic segment with $\beta(0) = \mu_1(t)$ and $\beta(c(t)) = \mu_2(t)$, and write $v_1 = -\dot{\beta}(0)$ and $v_2 = \dot{\beta}(c(t))$ for shortness. We have (cf. Gromoll et al. [3], p. 251)

$$c'(t) = \langle \dot{\mu}_1(t), v_1 \rangle + \langle \dot{\mu}_2(t), v_2 \rangle. \tag{1}$$

We begin by estimating the terms $\langle (\text{grad } e)(\mu_i(t)), v_i \rangle$ for $i = 1, 2$. By definition

$$2\langle (\text{grad } e)(\mu_i(t)), v_i \rangle$$
$$= \langle (\text{grad } f)(\mu_i(t)), v_i \rangle + \langle (\text{grad } g)(\mu_i(t)), v_i \rangle.$$

The scalar products on the right hand side are cosines of certain angles, they will be estimated using comparison theory. Consider the triangle Δ formed by $\mu_1(t)$, $\mu_2(t)$, and x and denote by ω_i the angle at $\mu_i(t)$. Observe that Δ is isosceles since $f(\mu_1(t)) = f(\mu_2(t))$. Now Proposition 1.3 (ii) and hyperbolic trigonometry imply $\cos \omega_i \geqslant \tanh(c(t)/2)$, hence

$$\langle (\text{grad } f)(\mu_i(t)), v_i \rangle \geqslant \tanh(c(t)/2).$$

Similarly we get

$$\langle (\text{grad } g)(\mu_i(t)), v_i \rangle \geqslant \tanh(c(t)/2)$$

by considering the triangle formed by $\mu_1(t)$, $\mu_2(t)$, and y.

Finally we obtain

$$\langle (\text{grad } e)(\mu_i(t)), v_i \rangle \geqslant \tanh(c(t)/2). \tag{2}$$

Recalling that $(\text{grad } e)(\mu_i(t)) = \|(\text{grad } e)(\mu_i(t))\|^2 \dot{\mu}_i(t)$ with $\|\text{grad } e\| \leqslant 1$, and observing that $\tanh(c(t)/2)$ is positive, we get from (1) and (2)

$$c'(t) \geqslant 2\tanh(c(t)/2).$$

We set $a(t) = \sinh(c(t)/2)$. Then $a(1) = \sinh(d(m_1, m_2)/2)$ and $a'(t) \geqslant a(t)$. By integration this implies

$$a(t) \geqslant a(1)\mathrm{e}^{t-1}$$

for $t \geqslant 1$. Using $2a(t) \leqslant \mathrm{e}^{c(t)/2}$ and $2a(1) \geqslant d(m_1, m_2)$ we get

$$\mathrm{e}^{c(t)/2} \geqslant d(m_1, m_2)\mathrm{e}^{t-1},$$

and hence our assertion.

After these preparations we are able to prove our main result.

THEOREM 3.8: *The map* $\varphi\colon E_1 \to M'(\infty)$ *is a homeomorphism satisfying* $h \circ \varphi = h$.

PROOF: Since h is continuous the property $h \circ \varphi_t = h$ carries over to φ. Local uniform convergence of the sequence $\{\varphi_t\}$ ensures that φ is continuous. The injectivity of φ follows from the preceding two lemmata exactly as in Im Hof [5]. Now we will prove that φ is surjective. We choose a point $z \in M'(\infty)$ and a sequence $\{r_i\} \subset M$ converging to z. For a suitable $c > 0$ we have $z \in \overline{M}(c)$ and $\{r_i\} \subset \overline{M}(c)$. Let $\mu_i\colon (0, \infty) \to M$ be the maximal integral curve of Z satisfying $\mu_i(t_i) = r_i$ for $t_i = e(r_i)$, and consider the sequence of points $m_i = \mu_i(1) \in E_1(c)$. Since $E_1(c)$ is compact we may assume that $\{m_i\}$ converges to a point $m \in E_1(c)$. We claim $\varphi(m) = z$. Obviously $\varphi(m) \in M'(\infty)$, so it is sufficient to estimate horospherical distances on $F = f^{-1}(-c)$. From the triangle inequality we get

$$d_F(\varphi(m), z) \leqslant d_F(\varphi(m), \varphi(m_i)) + d_F\big(\varphi(m_i), \varphi_{t_i}(m_i)\big) + d_F(r_i, z).$$

We fix $\epsilon > 0$ and assert that each of the distances on the right hand side is smaller than ϵ as soon as i is sufficiently large. For the first distance we use the convergence of $\{m_i\}$ to m and the continuity of φ, for the second we use the convergence of $\{t_i\}$ to ∞ and the uniform convergence of $\{\varphi_{t_i}\}$ to φ on the set $E_1(c)$, and for the third we use the convergence of $\{r_i\}$ to z.

Observing that $E_1(c)$ is compact and $M'(\infty)$ Hausdorff, we conclude that φ restricted to $E_1(c)$ is a homeomorphism onto its image. Together with the fact that the global map $\varphi\colon E_1 \to M'(\infty)$ is bijective, this implies $\varphi\colon E_1 \to M'(\infty)$ is a homeomorphism.

Our final aim is to determine the set of infinite points of a single slice H_s. A priori there are different ways to define such a set. We choose the definition $H_s(\infty) = \{z \in M(\infty);\ h(z) = s\}$. Alternatively, we might form the closure cl H_s in $\overline{M}$ with respect to the ball topology and intersect it with $M(\infty)$. However, we have

LEMMA 3.9: $H_s(\infty) = \mathrm{cl}\ H_s \cap M(\infty)$.

PROOF: If $z \in H_s(\infty)$, then $z = \lim_{t\to\infty}\mu_m(t)$ for $m = \varphi^{-1}(z)$, hence $\mu_m(t) \in H_s$. This implies $z \in \mathrm{cl}\ H_s$. If $z \in \mathrm{cl}\ H_s \cap M(\infty)$, then by the continuity of h on $\overline{M}$ we have $h(z) = s$, hence $z \in H_s(\infty)$.

As a consequence of Theorem 3.8 we now get

COROLLARY 3.10: *$H_s(\infty)$is homeomorphic to S^{n-2}.*

PROOF: The homeomorphism φ: $E_1 \to M'(\infty)$ of Theorem 3.8 satisfies $h \circ \varphi = h$, so it restricts to a homeomorphism between $E_1 \cap H_s$ and $H_s(\infty)$. In the proof of Theorem 2.4 we have observed that $E_1 \cap H_s$ is diffeomorphic to S^{n-2}. This concludes the proof.

Coda

Having studied the families of ellipsoids and hyperboloids with respect to two finite or two infinite foci, we are left with the case of one finite and one infinite focus. Again there are two families of hypersurfaces, the level sets of the sum and those of the difference of a distance and a Busemann function.

In euclidean space, where horospheres coincide with hyperplanes, we obtain in this way two families of paraboloids situated symmetrically with respect to the finite focus. In manifolds of strictly negative curvature both types of hypersurfaces are still submanifolds diffeomorphic to euclidean spaces, but they behave differently at infinity. Those hypersurfaces which correspond to ellipsoids have a single point at infinity, the infinite focus, the ones corresponding to hyperboloids have a sphere at infinity, as in the cases described earlier.

The proofs of these facts are left to the reader.

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