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ON THE CLASSICAL CHARACTERISTIC LINEAR SERIES OF PLANE CURVES WITH NODES AND CUSPIDAL POINTS: TWO EXAMPLES OF BENIAMINO SEGRE *

Allen Tannenbaum

Introduction

B. Segre in [8] studies families of plane curves with singularities. (See also Zariski [12] for an account of some of Segre's work on curves with nodes and cusps.) For example if we consider a maximal irreducible algebraic (i.e. "complete") family of plane curves of degree d with precisely δ nodes and κ cusps (by "cusp" we mean a singularity analytically isomorphic to y^2+x^3), then Segre shows that the expected ("virtual") dimension of such a family should be $d(d+3)/2-\delta-2\kappa$ (that is each node imposes one independent condition, and each cusp two independent conditions). If a certain first cohomology group vanishes (see Section 1 below for the precise definition), then it follows from general deformation theory that this family will be non-singular of dimension precisely $d(d+3)/2-\delta-2\kappa$. If this cohomology group is not zero, then the deformations may be obstructed ("the characteristic series is incomplete"), and in point of fact J. Wahl in [11] constructs explicitly such an example of an obstructed deformation space. It may also occur, that the cohomology group is not zero, but that the deformations are nevertheless unobstructed. In other words, that the complete algebraic family is non-singular, but of dimension strictly greater than the expected dimension of $d(d+3)/2-\delta-2\kappa$. Such an example is constructed by B. Segre in [8] which we will discuss in Section 3 below.

Now in studying such deformations of plane curves with singularities (the deformations we are referring to are formally locally trivial, i.e. they don't formally change the singularities; see Section 1), the first thing to do is to identify the "characteristic linear series" which is basically the linear system cut out on a general curve of the family by nearby curves in the family. In the case of plane curves with nodes and cusps we will show

* This work was partially done while the author was a guest at the Sonderforschungsbereich of the University of Bonn.

that the classical Italian analysis of this works by giving an explicit modern expression for the characteristic linear series (see the details in Section 1 below). However for the formally locally trivial families of deformations of plane curves with higher order singularities, we will show that the classical techniques (as evidenced by Segre [8]) do not give a correct description of the characteristic linear series. (See also Segre [7], Severi [9], and Zariski [12] for an account of these classical works as well as an extensive bibliography.)

Indeed in [8], Segre has a second example of a complete family of plane curves of degree $6d$ with precisely $6d^2-2$ cusps and one cuspidal triple point analytically isomorphic to y^3+x^4. Segre claims that this family is obstructed by computing the characteristic linear series and showing that the dimension is greater than the actual dimension of the family. However because of the erroneous classical description of the characteristic linear series, Segre's claim about the obstructedness of his family is false, and we will show in point of fact that his family is unobstructed by using the correct characteristic linear series (see Section 3).

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1. Some deformation theory

We will sketch here some of the relevant deformation theory from Horikawa [4], Tannenbaum [10], and Wahl [11] which we will need in the sequel. All our schemes will be defined over k a fixed algebraically closed field of characteristic 0. We begin with the following definitions from [11]:

DEFINITIONS (1.1): (i) Let V be a non-singular variety, $D \subset V$ an effective Cartier divisor, N_D the normal sheaf of D in V. Using the Lichtenbaum-Schlessinger cotangent complex, define

$$N'_D := \ker\left(N_D \to T^1(D/k, \mathcal{O}_D)\right).$$

(ii) Define a functor from the category of finite local artinian k-algebras to the category of sets by

$H'(A) :=$ {subschemes of $V x_{\mathrm{Spec}\,k}$ Spec A flat over A inducing D on V which are locally trivial deformations of D in the Zariski topology}.

Then H' is pro-representable, $H'(k[\epsilon]/\epsilon^2) = H^0(D, N'_D)$ and smoothness is obstructed by elements in $H^1(D, N'_D)$.

(iii) Let T be an algebraic k-scheme and $X \to T$ a flat morphism of finite type. Then $X \to T$ is a *formally locally trivial family of deformations* if for every closed point $t \in T$ and every $n > 0$, in the diagram

$$\begin{array}{ccccc} X & \longleftarrow & X_n & \longleftarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ T & \longleftarrow & \operatorname{Spec} \mathcal{O}_{T,t}/m_t^n & \longleftarrow & \operatorname{Spec} k \end{array}$$

X_n is a locally trivial deformation of X_1 in the Zariski topology.

(iv) Define a functor $J_{\delta,\kappa,d}$ on the category of algebraic k-schemes by

$J_{\delta,\kappa,d}(S) := \{$relative effective Cartier divisors $C \hookrightarrow \mathbb{P}^2 \times S$ which as flat families of curves over S are formally locally trivial at all $s \in S$, and whose geometric fibers are plane curves of degree d with precisely δ nodes and κ cusps and no other singularities$\}$.

Then in [11], pages 560–563 it is proven that $J_{\delta,\kappa,d}$ is representable by an algebraic k-scheme $V_{\delta,\kappa,d}$.

REMARKS (1.2): (i) Note that if $D \in V_{\delta,\kappa,d}$ (by this we mean that there is some closed point in $V_{\delta,\kappa,d}$ corresponding to D), then the Zariski tangent space of $V_{\delta,\kappa,d}$ at D may be identified with $H^0(D, N_D')$, and if $H^1(D, N_D') = 0$, D is a smooth point of $V_{\delta,\kappa,d}$ ((1.1)(ii)).

(ii) In order to compute the cohomology groups of N_D' we will first relate it to a certain line bundle. We therefore make the following definition (see also Arbarello-Cornalba [1]):

DEFINITION (1.3): Let S be a non-singular irreducible projective surface and $D \subset S$ a reduced curve. Let $\pi' : \tilde{D} \to D$ be the normalization morphism, and if i: $D \hookrightarrow S$ is the inclusion map of D into S, set $\pi = i \circ \pi'$: $\tilde{D} \to S$. Let R be the ramification divisor of π, i.e. the divisor $\tilde{D}$ defined by the zeros of the differential $d\pi$. Then we define a sheaf N_π' by the exact sequence

$$0 \to T_{\tilde{D}}(R) \to \pi^* T_S \to N_\pi' \to 0 \tag{1}$$

where for a scheme X, T_X denotes the tangent sheaf (the dual of the sheaf of Kähler differentials).

REMARK (1.4): Note it is clear that N_π' is locally free, i.e. it is a line bundle. If we define N_π by

$$0 \to T_{\tilde{D}} \to \pi^* T_S \to N_\pi \to 0$$

then N'_π fits into an exact sequence

$$0 \to Q_\pi \to N_\pi \to N'_\pi \to 0$$

(see [1], pages 23–24 for details).

We also note that $H^0(\tilde{D}, N_\pi)$ is isomorphic to the set of infinitesimal deformations of the pair $(\tilde{D}, \pi)$ ([4]).

We can now state the following result:

LEMMA (1.5): *Let $D \subset S$, π, π', N'_π be as in* (1.3). *Then* (*a*) $\pi'_* N'_\pi \cong N_D \otimes \pi'_* J0_{\tilde{D}}$ *where N_D is the normal bundle of D in S, and J is the Jacobian ideal of D* (*the first Fitting ideal of the sheaf of Kähler differentials*). *In particular, there exists an exact sequence*

$$0 \to N'_D \to \pi'_* N'_\pi \to T \to 0$$

where T is a torsion sheaf, and

$$h^0(N'_D) \leqslant h^0(N'_\pi) \text{ and } h^1(N'_D) \geqslant h^1(N'_\pi).$$

(b) If D only has nodes and (ordinary) cusps as singularities, then $\pi'_* N'_\pi \cong N'_D$. In particular, $H^i(\tilde{D}, N'_\pi) \cong H^i(D, N'_D)$ for $i = 0,1$.

PROOF: We prove (a) and (b) together. First note that by Piene [6], page 261 we have that $\omega_{\tilde{D}} \cong C\mathcal{O}_{\tilde{D}} \otimes \pi'^*\omega_D$ where $\omega_{\tilde{D}}$, ω_D are the dualizing sheaves on $\tilde{D}$, D respectively, and C is the conductor ideal (i.e. $C := \mathrm{Ann}_{\mathcal{O}_D}(\pi'_*\mathcal{O}_{\tilde{D}}/\mathcal{O}_D)$). Then taking the highest exterior powers of the exact sequence (1) of (1.3) we get that

$$\begin{aligned} N'_\pi &\cong \pi^*\mathcal{O}_S(-K_S) \otimes \omega_{\tilde{D}}(-R) \\ &\cong \pi'^*(i^*\mathcal{O}_S(-K_S) \otimes \omega_D) \otimes C\mathcal{O}_{\tilde{D}}(-R) \\ &\cong \pi'^*N_D \otimes C\mathcal{O}_{\tilde{D}}(-R) \text{ (adjunction formula).} \end{aligned}$$

But again from [6], page 261, we have that $C\mathcal{O}_{\tilde{D}}(-R) = J\mathcal{O}_{\tilde{D}}$. Hence applying the projection formula, we get that

$$\pi'_* N'_\pi \cong N_D \otimes \pi'_* J\mathcal{O}_{\tilde{D}}$$

Next from the exact sequence

$$0 \to N'_D \to N_D \to T^1(D/k, \mathcal{O}_D) \to 0$$

it is clear that $N'_D \cong J \cdot N_D$. From this the existence of the exact sequence asserted in (a) is immediate, and from this exact sequence and the fact that π' is affine, we get the relations about the cohomology groups stated in (a).

Finally if D only has nodes and cusps, a simple calculation ([6], page 268) shows that

$$J \cdot N_D \cong N_D \otimes \pi'_* J\mathcal{O}_{\tilde{D}}$$

from which (b) follows. Q.E.D.

REMARKS (1.6): (i) (1.5) above is a generalization of Proposition (1.4) in [10], page 110.

(ii) The proof of (1.5) shows that

$$N'_\pi \cong \pi^*\mathcal{O}_S(-K_S) \otimes \mathcal{O}_{\tilde{D}}(K_{\tilde{D}} - R)$$

where $K_{\tilde{D}}$ is the canonical divisor on $\tilde{D}$. In particular if we let $g = g(\tilde{D})$ be the genus of $\tilde{D}$, then by Riemann-Roch

$$h^0(N'_\pi) - h^1(N'_\pi) = g - 1 - K_S \cdot D - \deg R. \tag{2}$$

By Serre duality we have that

$$h^1(N'_\pi) = h^0(\pi^*\mathcal{O}_S(K_S) \otimes \mathcal{O}_{\tilde{D}}(R)).$$

Let $\tilde{D} = \tilde{D}_1 \cup \ldots \cup \tilde{D}_s$ where the $\tilde{D}_i$ are the irreducible components of $\tilde{D}$, $i = 1, \ldots, s$. (We only assumed that D is reduced; it may be reducible.) If on each irreducible component of $\tilde{D}$, $\tilde{D}_i$, we have that

$$\deg(\pi^*\mathcal{O}_S(K_S) \otimes \mathcal{O}_{\tilde{D}}(R))|_{\tilde{D}_i} = K_S \cdot D_i + \deg \mathcal{O}_{\tilde{D}}(R)|_{\tilde{D}_i} < 0 \tag{3}$$

then $h^1(N'_\pi) = 0$.

Note that our above arguments are valid for an arbitrary reduced curve D on an arbitrary non-singular projective surface S. We specialize now to the case $S = \mathbb{P}^2$, and we suppose that D is a plane curve of degree d with precisely δ nodes and κ cusps as its only singularities. By (1.5) $h^i(N'_\pi) = h^i(N'_D)$. Then $h^0(N'_D) - h^1(N'_D)$ is the "virtual dimension" in the sense of B. Segre [2] of the irreducible component containing D of the parameter space ((1.2) of plane curves of degree d with δ nodes and κ cusps. In this case (2) gives the well-known classical formula

$$\begin{aligned} h^0(N'_D) - h^1(N'_D) &= g - 1 + 3d - \kappa \\ &= d(d+3)/2 - \delta - 2\kappa \end{aligned}$$

(note that $K_{\mathbb{P}^2} \sim -3l$ where l is a line and " $\sim$ " denotes "is linearly equivalent to").

Finally the sufficient condition (3) for $H^1(N'_D) \cong H^1(N'_\pi)$ to vanish reduces in this case to the famous classical criterion that for D irreducible, if $3d > \kappa$, then the characteristic series is complete.

2. The dependence of cusps

We are now ready to construct the example of Segre [8] of an unobstructed deformation space of plane curves with cusps but with dimension strictly greater than the virtual dimension. We set some notation, that we will use throughout this section.

Let d be an integer $\geqslant 1$. Let f be a general homogeneous form of degree $2d$ in the variables x, y, z (the homogeneous coordinates of $\mathbb{P}^2$), and let g be a general homogeneous form of degree $3d$. Let $Y_1 \subset \mathbb{P}^2$ be the curve defined by f, and let $Y_2 \subset \mathbb{P}^2$ be the curve defined by g. Since f and g have been chosen generally, Y_1 and Y_2 will intersect transversally in $6d^2$ distinct points $p_1,\dots,p_{6d^2}$. Let $Y \subset \mathbb{P}^2$ be the curve defined by the equation $f^3 + g^2 = 0$. Then Y has degree $6d$, and cusps precisely at the points $p_1,\dots,p_{6d^2}$ but no other singular points. Let π': $\tilde{Y} \to Y$ be the normalization of Y, and $\pi := \pi' \circ i$ where i: $Y \hookrightarrow \mathbb{P}^2$ is the inclusion map, and let N'_π be defined as in (1.3). Then we have the following key result:

THEOREM (2.1): $h^1(N'_\pi) = (d-1)(d-2)/2.$

PROOF: Blow up the surface $\mathbb{P}^2$ at $p_1,\dots,p_{6d^2}$ and call the resulting surface S'. Let $e'_1,\dots,e'_{6d^2}$ be the corresponding exceptional divisors. Let q_i be the point of intersection of the proper transform of Y on S' with the exceptional divisor e'_i for each $i = 1,\dots,6d^2$. Now blow up S' at the points $q_1,\dots,q_{6d^2}$ and call the resulting surface S, and the corresponding exceptional divisors $e_1,\dots,e_{6d^2}$. Let P_i denote the total transform of p_i on S for $i = 1,\dots,6d^2$. Then note that $P_i \cdot P_j = e_i \cdot e_j = -\delta_{ij}$ (δ_{ij} is the Kronecker delta), and $P_i \cdot e_j = 0$ for all $i, j = 1,\dots,6d^2$. Then letting " $\sim$ " denote "is linearly equivalent to", if $\tilde{Y}$ is the proper transform of Y on S (which is of course the normalization), then $\tilde{Y} \sim 6dl - \Sigma_i(2P_i + e_i)$. Finally let $\bar{l}$ denote a divisor on $\tilde{Y}$ corresponding to $\mathcal{O}_S(l) \otimes \mathcal{O}_{\tilde{Y}}$. Then from (1.6)(ii)

$$\begin{aligned} N'_\pi &\cong \pi^*\mathcal{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}) \otimes \mathcal{O}_{\tilde{Y}}(K_{\tilde{Y}} - R) \\ &\cong \mathcal{O}_{\tilde{Y}}(3\bar{l} - K_{\tilde{Y}} - R). \end{aligned}$$

We now break up the proof into several steps:

Step (1). Let $\bar{Y}$ be any effective divisor linearly equivalent to $6dl - \Sigma_i(P_i + e_i)$. Then

$$h^1(\mathcal{O}_{\tilde{Y}}(3\bar{l} + K_{\tilde{Y}} - R)) = h^1(\mathcal{O}_S(\bar{Y})).$$

Proof: In what follows we slightly abuse notation by letting $C \cdot D$ denote the intersection number as well as a divisor on C corresponding to $\mathcal{O}_S(D) \otimes \mathcal{O}_C$ for $C \subset S$ an effective divisor. Then we will first show that $3l + K_{\tilde{Y}} - R \sim \tilde{Y} \cdot \bar{Y}$. But $K_{\tilde{Y}} \sim \tilde{Y} \cdot \tilde{Y} + \tilde{Y} \cdot K_S$ and $K_S \sim -3l + \Sigma_i(P_i + e_i)$. A simple computation then reduces us to showing that

$$R \sim \tilde{Y} \cdot \Big(-\sum_i P_i + \sum_i (P_i + e_i) \Big)$$

$$= \tilde{Y} \cdot \Big(\sum_i e_i \Big).$$

But this follows immediately from the fact that each point of the ramification divisor has multiplicity 1, and that the intersection number $e_i \cdot \tilde{Y} = e_i \cdot (-2P_i - e_i) = 1$ for each $i = 1, \dots, 6d^2$.

Next consider the exact sequence

$$0 \to \mathcal{O}_S(-\tilde{Y} + \bar{Y}) \to \mathcal{O}_S(\bar{Y}) \to \mathcal{O}_{\tilde{Y}}(\tilde{Y} \cdot \bar{Y}) \to 0. \tag{*}$$

Then using the above computation and the long exact cohomology sequence applied to (*), we'll be done with the proof of Step (1) once we have shown that $h^1(\mathcal{O}_S(-\tilde{Y} + \bar{Y})) = h^2(\mathcal{O}_S(-\tilde{Y} + \bar{Y})) = 0$. But note that $-\tilde{Y} + \bar{Y} \sim \Sigma_i P_i =: A$. We use now the long exact cohomology sequence associated to $0 \to \mathcal{O}_S \to \mathcal{O}_S(A) \to \mathcal{O}_A(A^2) \to 0$. Since $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$, we immediately have that $h^2(\mathcal{O}_S(A)) = 0$, and we must prove that $h^1(\mathcal{O}_S(A)) = h^1(\mathcal{O}_A(A^2)) = 0$. But recall that each P_i is simply the total transform on S of $p_i \in \mathbb{P}^2$ $(i = 1, \dots, 6d^2)$ and as such is a reduced reducible curve of arithmetic genus 0 with two irreducible non-singular rational components intersecting in one point. Moreover $P_i \cdot P_j = -\delta_{ij}$. Therefore in order to show that $h^1(\mathcal{O}_A(A^2)) = 0$, it is enough to compute that $h^1(\mathcal{O}_{P_i}(P_i^2)) = 0$, and this is trivial.

Step (2). Let $\tilde{Y}_2$ be an effective divisor on S linearly equivalent to $3dl - \Sigma_i(P_i + e_i)$ (recall that $Y_2 \subset \mathbb{P}^2$ was defined by the form g of degree $3d$). Then we claim that $h^1(\mathcal{O}_S(\bar{Y})) = h^1(\mathcal{O}_{\tilde{Y}_2}(\tilde{Y}_2 \cdot \bar{Y}))$.

Proof: Consider the exact sequence

$$0 \to \mathcal{O}_S\big(-\tilde{Y}_2 + \bar{Y}\big) \to \mathcal{O}_S(\bar{Y}) \to \mathcal{O}_{\tilde{Y}_2}\big(\tilde{Y}_2 \cdot \bar{Y}\big) \to 0. \tag{**}$$

Applying the long exact cohomology sequence to (**) we must show that $h^1(\mathcal{O}_S(-\tilde{Y}_2 + \bar{Y})) = h^2(\mathcal{O}_S(-\tilde{Y}_2 + \bar{Y})) = 0$. But note that $-\tilde{Y}_2 + \bar{Y} \sim 3dl$. Then the vanishing of these groups follows easily from the long exact cohomology sequence associated to

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(3dl) \to \mathcal{O}_{3dl}\big((3dl)^2\big) \to 0$$

since $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$, and $h^1(\mathcal{O}_S(3dl)) = h^1(\mathcal{O}_{3dl}((3dl)^2))$ may easily be computed to be 0.

Step (3). $\mathcal{O}_{\tilde{Y}_2}(\tilde{Y}_2 \cdot \bar{Y}) \cong \mathcal{O}_{\tilde{Y}_2}(\tilde{Y}_2 \cdot 2dl)$.

PROOF: Note that $\bar{Y} - 2dl \sim 4dl - \Sigma(P_i + e_i)$. But we have that (the intersection number) $\tilde{Y}_2 \cdot (\bar{Y} - 2dl) = 12d^2 - 12d^2 = 0$ since $(P_i + e_i)^2 = -2$ for $i = 1,\ldots,6d^2$. Now $\tilde{Y}_2$ and $\bar{Y} - 2dl$ may be taken (in their respective linear systems) to be effective divisors without any common components which implies that $\mathcal{O}_{\tilde{Y}_2}(\tilde{Y}_2 \cdot (\bar{Y} - 2dl)) \cong \mathcal{O}_{\tilde{Y}_2}$.

Step (4).

$$h^1\left(\mathcal{O}_{\tilde{Y}_2}\left(\tilde{Y}_2 \cdot 2dl\right)\right) = h^0\left(\mathcal{O}_{\mathbf{P}^2}(d-3)\right)$$
$$= (d-1)(d-2)/2.$$

PROOF: Clearly $h^0(\mathcal{O}_{\mathbf{P}^2}(d-3)) = h^0(\mathcal{O}_S(d-3)l))$. We have the sequence

$$0 \to \mathcal{O}_S\left(-\tilde{Y}_2 + 2dl\right) \to \mathcal{O}_S(2dl) \to \mathcal{O}_{\tilde{Y}_2}\left(\tilde{Y}_2 \cdot 2dl\right) \to 0 \qquad (***)$$

and a simple argument as in Step (2) shows that $h^1(\mathcal{O}_S(2dl)) = h^2(\mathcal{O}_S(2dl)) = 0$. Therefore applying the long exact cohomology sequence to (***) and Serre duality, we see that

$$h^1\left(\mathcal{O}_{\tilde{Y}_2}\left(\tilde{Y}_2 \cdot 2dl\right)\right) = h^2\left(\mathcal{O}_S\left(-\tilde{Y}_2 + 2dl\right)\right)$$
$$= h^0\left(\mathcal{O}_S((d-3)l)\right).$$

This completes the proof of Step (4) and using the previous 3 steps obviously completes the proof of the theorem. Q.E.D.

COROLLARY (2.2): *$h^0(N'_\pi) = \frac{1}{2}(13d + 2)(d + 1)$.*

PROOF: This follows immediately from the fact that $h^0(N'_\pi) - h^1(N'_\pi) = 6d(6d+3)/2 - 12d^2$ (see (1.6)(ii)) and from (2.1) above. Q.E.D.

COROLLARY (2.3): *Let V_{6d^2} be the irreducible component containing Y of the parameter scheme of plane curves of degree $6d$ with precisely $6d^2$ cusps and no other singular points* ((1.1)). *Then dim $V_{6d^2} = \frac{1}{2}(13d+2)(d+1)$, and V_{6d^2} is non-singular at Y.*

PROOF: From (2.2) we see that dim $V_{6d^2} \leqslant \frac{1}{2}(13d+2)(d+1)$. Clearly the dimension of the family of plane curves $\{C | C$ is the plane curve defined by $\tilde{f}^3 + \tilde{g}^2 = 0$ where $\tilde{f}$ is a homogeneous form in x, y, z of degree $2d$, and $\tilde{g}$ is a homogeneous form in x, y, z of degree $3d\}$ is

$$\frac{2d(2d+3)}{2}+\frac{3d(3d+3)}{2}+1=\tfrac{1}{2}(13d+2)(d+1).$$

From this (2.3) follows immediately. Q.E.D.

COROLLARY (2.4): *If* $d>2$, *then dim* $V_{6d^2}>6d(6d+3)/2-12d^2$ (*the virtual dimension*).

PROOF: In this case from (2.1), $h^1(N'_\pi)>0$. Q.E.D.

3. The classical characteristic linear series

As we mentioned in the Introduction, in his paper [8], page 35, Segre gives a second example which he claims defines an obstructed family of integral plane singular curves with ordinary cusps and one higher cuspidal point. We will show in point of fact that Segre's family is *not* obstructed and will show that his error (and apparently the classical Italian geometers' error [7,8,9]) is a sometimes incorrect identification of the characteristic linear series for such kinds of deformations. Segre associates his characteristic linear series to $H^0(N'_\pi)$ instead of $H^0(N'_D)$ (see also Wahl [11], Mumford [5], and Mumford's appendix to Chapter V of Zariski [12] for discussions of the characteristic linear series for curves). In the case of formally locally trivial deformations of plane curves whose only singularities are nodes and cusps, we have shown in (1.6) that $H^i(N'_\pi)\cong H^i(N'_D)$, $i=0,1$, so in this case there is no problem. But for the formally locally trivial deformations of higher order singularities one no longer has such an identification.

We now completely analyze the Segre example from a modern point of view. Let g be a homogeneous form in x, y, z of degree $3d$ such that the curve $Y_2\subset\mathbb{P}^2$ defined by g has a node at some point p and no other singularities. We will assume (as is done in [8], page 35) that $d>2$. Let f be a homogeneous form in x, y, z of degree $2d$ such that the curve $Y_1\subset\mathbb{P}^2$ defined by f is non-singular and passes through p so that the curve $Y_1\cup Y_2$ has an ordinary triple point (i.e. with distinct tangents) at p. Let Y be defined by $f^3+g^2=0$. Then Y has $6d^2-2$ ordinary cusps (defined by the points of intersection of Y_1 and Y_2 outside of p) and a cuspidal triple point analytically isomorphic to y^3+x^4 at p. (To see this, note that if we represent Y_2 locally around p as $x^2+y^2=0$, and Y_1 as $y=0$, we get that locally around p, Y is isomorphic to $(x^2+y^2)^2+y^3=0$. One can now either make a rather long messy calculation that $(x^2+y^2)^2+y^3$ is analytically equivalent to y^3+x^4, or as mercifully pointed out to us by G. Barthel and C. Gibson one can use the results of Arnold [2], page 33.) Define N'_π as before. Then:

THEOREM (3.1): $h^1(N'_\pi)=(d-1)(d-2)/2$.

PROOF: The proof is almost identical to that of (2.1) so we just sketch the steps. Blow up $\mathbb{P}^2$ at $p_0 := p, p_1, \ldots, p_{6d^2-2}$ and denote the resulting surface by S''. Let e_i'' be the corresponding exceptional divisors ($i = 0, \ldots, 6d^2-2$), and let q_i be the point of intersection of e_i'' with the proper transform of Y on S'' for each i. Next blow up S'' at the points $q_0, q_1, \ldots, q_{6d^2-2}$ and denote the resulting surface by S' and the exceptional curve corresponding to q_0 by e'. Blow up S'' at the point of intersection of the proper transform of Y on S'' with e', and call this (final) blown up surface S and the resulting exceptional vivisor e. Let P_i be the total transform of p_i on S, and similarly Q_i the total transform of q_i on S for each $i = 0, \ldots, 6d^2-2$. Note that

$$P_i \cdot P_j = Q_i \cdot Q_j = -\delta_{ij},$$

$$P_i \cdot Q_j = P_i \cdot e = Q_i \cdot e = 0$$

for all $i, j = 0, \ldots, 6d^2-2$. The proper transform of Y on S, $\tilde{Y}$ (which is of course the normalization), is linearly equivalent to the divisor

$$6dl - (3P_0 + Q_0 + e) - \sum_{i=1}^{6d^2-2} (2P_i + Q_i)$$

where l is the total transform of a line in $\mathbb{P}^2$. Let $\bar{l}$ denote a divisor on $\tilde{Y}$ corresponding to $\mathcal{O}_S(l) \otimes \mathcal{O}_{\tilde{Y}}$. Then just as in (2.1) we have:

Step (1). Let $\overline{Y}$ be any effective divisor linearly equivalent to $6dl - (2P_0 + Q_0 + e) - \Sigma_{i=1}^{6d^2-2}(P_i + Q_i)$. Then $h^1(\mathcal{O}_{\tilde{Y}}(3\bar{l} + K_{\tilde{Y}} - R)) = h^1(\mathcal{O}_S(\overline{Y}))$.

PROOF: Again we show that $3\bar{l} + K_{\tilde{Y}} - R \sim \tilde{Y} \cdot \overline{Y}$. We note that $K_S \sim -3l + \Sigma_{i=0}^{6d^2-2}(P_i + Q_i) + e$, and then using the adjunction formula, we are reduced to showing that

$$R \sim \tilde{Y} \cdot \left(\sum_{i=0}^{6d^2-2} Q_i + e \right).$$

Note that the point in R whose image is p has multiplicity 2, while the other ramification points have multiplicity 1. But we have that (the intersection numbers) $\tilde{Y} \cdot (Q_0 + e) = -(Q_0 + e)^2 = 2$, and $\tilde{Y} \cdot Q_i = 1$ for $i = 1, \ldots, 6d^2-2$.

Next following the argument of Step (1) of (2.1) we see that if we set $-\tilde{Y} + \overline{Y} \sim \Sigma_{i=0}^{6d^2-2} P_i =: A$, we must show that $h^1(\mathcal{O}_A(A^2)) = 0$. But again as in (2.1) this is trivial (note in this case P_0 is a reduced, reducible curve of arithmetic genus 0 with three irreducible non-singular rational components, while $P_1, \ldots, P_{6d^2-2}$ have the same description as in (2.1)).

Step (2). Let Y' be an effective divisor on S linearly equivalent to $3dl - \sum_{i=0}^{6d^2-2}(P_i + Q_i) - e$. Then we claim that $h^1(\mathcal{O}_S(\bar{Y})) = h^1(\mathcal{O}_{Y'}(Y' \cdot \bar{Y}))$.

PROOF: Note that $-Y' + \bar{Y} \sim 3dl - P_0 =: C$. Note that we may assume that C is an irreducible smooth curve (it is linearly equivalent to the proper transform of a curve of degree $3d$ in $\mathbb{P}^2$ passing through P_0), and following Step (2) of (2.1) we must show that $h^1(\mathcal{O}_C(C^2)) = 0$. But this is immediate.

Step (3). $\mathcal{O}_{Y'}(Y' \cdot \bar{Y}) \cong \mathcal{O}_{Y'}(Y' \cdot 2dl)$.

PROOF: We have that $\bar{Y} - 2dl \sim 4dl - (2P_0 + Q_0 + e) - \sum_{i=1}^{6d^2-2}(P_i + Q_i)$ and it is easily seen that $\bar{Y} - 2dl$ is linearly equivalent to an effective divisor with no common components of Y'. Then the fact that $(\bar{Y} - 2dl) \cdot Y' = 0$ completes the proof.

Step (4). $h^1(\mathcal{O}_{Y'}(Y' \cdot 2dl)) = (d-1)(d-2)/2$.

PROOF: Identical to that of Step (4) of (2.1). This completes the proof of (3.1). Q.E.D.

COROLLARY (3.2): (*i*) $h^0(N'_\pi) - h^1(N'_\pi) = 6d^2 + 9d - 1$.
(*ii*) $h^0(N'_\pi) = (13d^2 + 15d)/2$.

PROOF: (i) is immediate from (1.6)(ii), equation (2), and (ii) follows from (i) and (3.1). Q.E.D.

Now on this basis Segre ([8], page 35) concludes that he has an example of an obstructed deformation of deficiency $\bar{\omega} = 1$ (the "deficiency" is the difference between the dimension of the Zariski tangent space and actual dimension of the space). Indeed his argument runs as follows: Consider the algebraic family of curves ($d > 2$ as always in this section)

$V := \{C \mid C$ is a curve defined by $\tilde{f}^3 + \tilde{g}^2 = 0$ where $\tilde{g}$ is a form of degree $3d$ such that the curve $\{\tilde{g} = 0\}$ has a node and $\tilde{f}$ is a form of degree $2d$ such that $\{\tilde{f} = 0\}$ passes through this node$\}$.

On an open subset of V we get a family of plane curves with $6d^2 - 2$ cusps and one triple point of type $y^3 + x^4$. The dimension of V can easily be computed to be $\frac{1}{2}(13d^2 + 15d - 2)$. Segre claims V is a complete family and indeed we will prove this (see (3.3) and (3.5) below). Then since $h^0(N'_\pi) > \frac{1}{2}(13d^2 + 15d - 2) = \dim V$, Segre concludes the family is obstructed (and $h^0(N'_\pi) - \dim V = \bar{\omega} = 1$).

We will now show that this family is unobstructed by analyzing the correct characteristic linear series. We first prove the following:

THEOREM (3.3): *Let $r \in \tilde{Y}$ be such that $\pi'(r) = p$ (the triple point of Y). Then $h^1(N'_\pi(-4r)) = h^1(N'_\pi) = (d-1)(d-2)/2$.*

PROOF: The same method used in (2.1) and (3.1) works again. To see this we outline the four steps one last time using the same notation as in the proof of (3.1).

Step (1). Let $\bar{\bar{Y}}$ be any effective divisor linearly equivalent to $6dl - 4P_0 - \sum_{i=1}^{6d^2-2}(P_i + Q_i)$ (which is linearly equivalent to $\bar{Y} - 2P_0 + Q_0 + e$ in the notation of Step (1) of (3.1)). Then

$$h^1\left(\mathcal{O}_{\tilde{Y}}(3\bar{l} + K_{\tilde{Y}} - R - 4r)\right) = h^1\left(\mathcal{O}_S(\bar{\bar{Y}})\right).$$

PROOF: As usual we show that $3\bar{l} + K_{\tilde{Y}} - R - 4r \sim \tilde{Y} \cdot \bar{\bar{Y}}$. But $3\bar{l} + K_{\tilde{Y}} - R \sim \tilde{Y} \cdot \bar{Y}$ (from (3.1)) and since $\bar{\bar{Y}} \sim \bar{Y} - 2P_0 + Q_0 + e$ we need only show that $\tilde{Y} \cdot (-2P_0 + Q_0 + e) \sim -4r$ (on $\tilde{Y}$) and this is immediate.

Following the argument of Step (1) of (2.1) we must now show that $h^1(\mathcal{O}_S(-\tilde{Y} + \bar{\bar{Y}})) = h^2(\mathcal{O}_S(-\tilde{Y} + \bar{\bar{Y}})) = 0$. Since $-\tilde{Y} + \bar{\bar{Y}} \sim -P_0 + Q_0 + e$ this is quite obvious. Indeed using the exact sequence

$$0 \to \mathcal{O}_S(-P_0 + Q_0 + e) \to \mathcal{O}_S(Q_0 + e) \to \mathcal{O}_{P_0}(Q_0 + e) \to 0$$

and the facts that $h^1(\mathcal{O}_{P_0}(Q_0 + e)) = h^2(\mathcal{O}_S(Q_0 + e)) = 0$ we get that $h^2(\mathcal{O}_S(-P_0 + Q_0 + e)) = 0$. Moreover, it is easy to see that $h^1(\mathcal{O}_S(Q_0 + e)) = h^0(\mathcal{O}_S(-P_0 + Q_0 + e)) = 0$, and $h^0(\mathcal{O}_S(Q_0 + e)) = h^0(\mathcal{O}_{P_0}(Q_0 + e)) = 1$. Therefore from the exact sequence

$$\begin{array}{ccc} 0 \to H^0(\mathcal{O}_S(Q_0 + e)) & \to & H^0(\mathcal{O}_{P_0}(Q_0 + e)) \\ \cong & & \cong \\ k & & k \end{array}$$

$$\to H^1(\mathcal{O}_S(-P_0 + Q_0 + e)) \to 0$$

we have that $h^1(\mathcal{O}_S(-P_0 + Q_0 + e)) = 0$.

Step (2). $h^1(\mathcal{O}_S(\bar{\bar{Y}})) = h^1(\mathcal{O}_{Y'}(Y' \cdot \bar{\bar{Y}}))$ where Y' is as in Step (2) of (3.1).

PROOF: $-Y' + \bar{\bar{Y}} \sim 3dl - 3P_0 + Q_0 + e$ which is clearly linearly equivalent to a reduced effective divisor C'. As before we must show that $h^1(\mathcal{O}_{C'}((C')^2)) = 0$, and this is obvious.

Step (3). $\mathcal{O}_{Y'}(Y' \cdot \bar{\bar{Y}}) \cong \mathcal{O}_{Y'}(Y' \cdot 2dl)$.

PROOF: $\bar{\bar{Y}} - 2dl \sim 4dl - 4P_0 - \sum_{i=1}^{6d^2-2}(P_i + Q_i)$.

From the representation

$$4dl - 4P_0 - \sum_{i=1}^{6d^2-2}(P_i + Q_i) = (2dl - 4P_0) + \left(2dl - \sum_{i=1}^{6d^2-2}(P_i + Q_i)\right)$$

and the assumption $d > 2$, we get that $\overline{\overline{Y}} - 2dl$ is linearly equivalent to an effective divisor with no common components of Y'. Since $Y' \cdot \overline{\overline{Y}} - 2dl = 0$, we are done.

Using Step (4) of (1.3) we can complete the proof of the theorem.

Q.E.D.

COROLLARY (3.4): $h^0(N'_Y) = \dim V = \frac{1}{2}(13d^2 + 15d - 2)$. *In particular, V is non-singular at Y (i.e. the deformations are unobstructed at Y) and the family V is complete.*

PROOF: Recall the exact sequence

$$0 \to N'_Y \to \pi'_* N'_\pi \to T \to 0 \tag{1}$$

from (1.5)(a), where T is a torsion sheaf supported at p. We claim that $\dim T_p = 1$. Indeed we have that $N'_Y \cong JN_Y$, and $\pi'_* N'_\pi \cong (\pi'_* J\mathcal{O}_{\tilde{Y}})N_Y$. Now Piene [6], page 268, computes that $\hat{J}_p \cong (t^8, t^9)k[[t^3, t^4]]$, and $\widehat{(\pi'_* J\mathcal{O}_{\tilde{Y}})}_p \cong t^8 k[[t]]$. Note then that $\hat{J}_P$ consists of all formal power series of the form $\{a_0 t^8 + a_1 t^9 + \Sigma_{i \geqslant 0} b_i t^{i+11}\}$, i.e. these power series are missing the term t^{10}, while $\widehat{(\pi'_* J\mathcal{O}_{\tilde{Y}})}_p$ consists of all formal power series of the form $\{\Sigma_{i \geqslant 0} b_i t^{i+8}\}$. From this it is immediate that $\dim T_p = 1$. Moreover from this description of J, it is clear that $\pi'_* N'_\pi(-4r)$ is naturally contained in N'_Y (recall that $\pi'(r) = p$), and so we have an exact sequence

$$0 \to \pi'_* N'_\pi(-4r) \to N'_Y \to \overline{T} \to 0 \tag{2}$$

for some torsion sheaf $\overline{T}$. Applying the long exact cohomology sequence to (2), we conclude that $h^1(N'_\pi(-4r)) \geqslant h^1(N'_Y)$, and therefore from (3.3), $h'(N'_\pi) \geqslant h^1(N'_Y)$. But from the exact sequence (1) we have that $h^0(N'_Y) \geqslant h^1(N'_\pi)$, and so we conclude that $h^1(N'_Y) = h^1(N'_\pi)$. Thus from (1) we get an exact sequence

$$0 \to H^0(N'_Y) \to H^0(N'_\pi) \to k \to 0$$

from which we see that

$$\begin{aligned} h^0(N'_Y) &= \dim h^0(N'_\pi) - 1 \\ &= (13d^2 + 15d - 2)/2. \end{aligned}$$

From this we conclude that $h^0(N'_Y) = \dim V$, so that V is a complete family of curves non-singular at Y, and thus the characteristic linear series is complete.

Q.E.D.

REMARKS (3.5): (i) There is still the question of proving the existence of a universal family of plane curves of δ nodes, κ cusps, and λ cuspidal points of type $y^3 + x^4$ as was done for nodes and cusps in [11]. We don't doubt that the methods of [11] would work in this case as well but actually here this is unnecessary. Indeed let us use the notation of (1.1). So again we have a functor H' from the category of artinian k-algebras to sets defined by

$$H'(A) := \{\text{subschemes of } \mathbb{P}^2 x_{\operatorname{Spec} k} \operatorname{Spec} A \text{ flat over } A \text{ inducing } Y \text{ on } \mathbb{P}^2 \text{ which are locally trivial deformations of } Y \text{ in the Zariski topology}\}.$$

Then H' is pro-representable, say by the complete ring R. But by the above R is smooth, i.e. is a complete power series ring and hence we can apply the criterion of Artin [3] to conclude that R is algebraisable, so that there exists a "universal" algebraic k-scheme X parametrizing formally locally trivial deformations of Y such that $\hat{\mathcal{O}}_{X,x} \cong R$ where $x \in X$ corresponds to Y. Moreover it is clear that X must be isomorphic to V in a neighborhood of Y, and this is enough for our purposes.

(ii) Note that we have proven the existence of an exact sequence

$$0 \to H^0(N'_Y) \to H^0(N'_\pi) \to k \to 0.$$

Moreover one has an exact sequence (see (1.4))

$$0 \to H^0(Q_\pi) \to H^0(N_\pi) \to H^0(N'_\pi) \to 0$$

and since these are vector spaces, we may identify $H^0(N'_\pi)$ as a subvector space of $H^0(N_\pi)$. Now sections of $H^0(N_\pi)$ correspond to infinitesimal deformations of the pair $(\tilde{Y}, \pi)$. The argument above then shows that there exist infinitesimal deformations of $(\tilde{Y}, \pi)$ corresponding to sections of $H^0(N'_\pi)$ which do not induce formally locally trivial deformations of Y. But of course in no way does this imply that such deformations of Y are obstructed.

References

[1] E. ARBARELLO and M. CORNALBA: Su una congettura di Petri. *Comment. Math. Helvetici 56* (1981) 1–38.

[2] V.I. ARNOLD: Normal forms of functions in neighborhoods of degenerate critical points. *Russian Math. Surveys 29* (1974) 10–50.

[3] M. ARTIN: Algebraization of formal moduli. *Global Analysis* (papers in honor of K. Kodaira; edited by D. Spencer and S. Iyanaga), University of Tokyo Press, 21–71 (1969).

[4] E. HORIKAWA: On deformations of holomorphic maps I. *J. Math. Soc. Japan 25* (1978) 372–396.

[5] D. MUMFORD: *Lectures on Curves on an Algebraic Surface*. Princeton University Press, Princeton, N.J. (1966).

[6] R. PIENE: Polar classes of singular varieties. *Ann. Scient. Ec. Norm. Sup. Ser. 4 Vol. II* (1978) 247–276.

[7] B. SEGRE: Dei sistemi lineari tangenti ad un qualunque sistema di forme. *Atti Acad. naz. Lincei Rendiconti serie 5 Vol. 33* (1924) 182–185.

[8] B. SEGRE: Esistenza e dimensione di sistemi continui di curve piane algebriche con dati caraterri. *Atti Acad. naz. Lincei Rendiconti serie 6 Vol. 10* (1929) 31–38.

[9] F. SEVERI: Osservazioni sui sistemi continui di curve appartenenti ad una superficie algebrica. *Atti Acad. Sci. Torino Vol. 39* (1904) 376–392.

[10] A. TANNENBAUM. Families of algebraic curves with nodes. *Comp. Math. 41* (1980) 107–126.

[11] J. WAHL: Deformations of plane curves with nodes and cusps. *Amer. J. Math. 96* (1974) 529–577.

[12] O. ZARISKI: *Algebraic Surfaces*. 2nd supplemented edition, Springer, Berlin/New York (1971).

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