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GENERIC TORELLI FOR PROJECTIVE HYPERSURFACES

Ron Donagi *

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Introduction

The Torelli problem for a given family of varieties asks whether the period map is injective on that family, i.e., whether varieties of the family can be distinguished by means of their Hodge structures. The answer is affirmative for curves [1], abelian varieties, K3 surfaces [14] and cubic threefolds [6], and negative for several families of surfaces of general type [3,15].

In recent years, several variants have been studied, inquiring whether the period map is an immersion [13], whether its differential is injective on the deformation space [7], and whether the map is generically injective [4]. The corresponding problems are referred to as local, infinitesimal and generic Torelli, in comparison with the original "global" Torelli.

Our purpose in this work is to prove:

Generic Torelli for hypersurfaces. The period map for non-singular hypersurfaces of degree d in $\mathbb{P}^{n+1}$ is generically injective, except possibly for the following cases:

(0) $n = 2$, $d = 3$ (cubic surfaces).

(1) d divides $n + 2$.

(2) $d = 4$, $n = 4m$ or $d = 6$, $n = 6m + 1$ $(m \geqslant 1)$.

The result is false in case (0) and unknown in (1), (2). We prove the

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theorem in Section 6, where we also give some heuristic explanations as to why our proof should fail in the main exception, (1).

The proof is based on Griffiths' theory of infinitesimal variations of Hodge structures as developed in [4], [5], [8] and [10]. In a nutshell, the idea is this: let

$$p : M \to D$$

be a map of manifolds or varieties, defined almost everywhere. Let d be the rank of p at a generic point of M. The prolongation of p is the (almost everywhere defined) map

$$v : M \to G(d, TD),$$

where TD is the tangent bundle of D, $G(d, TD)$ is the Grassmannian bundle over D of d-dimensional subspaces of TD, and v is defined by

$$v(x) = (p(x), p_* T_x M).$$

The *Principle of Prolongation* is the obvious remark that p is generically injective if v is. Our plan is to apply this principle to the period map p. Its prolongation v is the "infinitesimal variation of Hodge structure".

The reason that we are able to recover a variety X from $v(X)$ but not directly from $p(X)$ is that $v(X)$ has in it more *algebraic* structure. The Hodge structure $p(X)$ consists of two types of data: an algebraic part giving a filtered vector space and a bilinear form satisfying various conditions, and a transcendental part giving a lattice in the vector space. Each part separately has no invariants: the invariants of a Hodge structure come from comparing the two parts. Thus the theory of Hodge structures is transcendental at heart.

While the infinitesimal variation $v(X)$ does not necessarily have more information in it than does $p(X)$ – the Torelli problem implies that it does *not* – it certainly does contain more *algebraic* structure. Thus the point of Griffiths' theory, and of our proof, becomes: try to handle the algebraic part of $v(X)$ efficiently. In the case of generic hypersurfaces we are able to show that this algebraic piece alone, without the lattice, determines the hypersurface.

The first three sections of this paper review the necessary background. The main body of the work shows that $v(X)$ determines the Jacobian ideal $J(X)$ of X. Thus we start in Section 1 by proving a variant of a well-known lemma, showing that $J(X)$ determines X up to projective automorphism.

Griffiths' method of converting the periods of a hypersurface into vector spaces of polynomials [7] is reviewed in Section 2. This is a basic tool, allowing us to translate the geometric question of injectivity into an algebraic question of recovering a ring from some partial data.

In Section 3 we review the only previously known generic Torelli result for hypersurfaces, that of [4] (for cubic hypersurfaces of dimension $3m$). Analysis of their method shows that they produce a vector space of polynomials and the Jacobian ideal in it, but the isomorphism of this vector space with the space of polynomials is not known except under some very special circumstances.

Thus in the remaining sections we concern ourselves with the problem of recovering the polynomial structure of a vector space from various non-linear bits of data. In Section 4, and again in Section 5, we are able to extend the Carlson-Griffiths method to numerous new cases. We show that in these cases the polynomial structure on our vector space W is determined by a certain family of quadrics on W^*, and that this family is determined by $v(X)$: in fact, $v(X)$ determines a bilinear map μ from $W \times W$ to another vector space (determined by $v(X)$), and our family consists of all quadrics of rank 4 on W^* whose image under μ is zero.

In Section 6 we adopt a somewhat different method. We consider only the map

$$B: H^1(\Theta_X) \times H^0(\Omega_X^n) \to H^1(\Omega_X^{n-1}),$$

i.e. the variation of the first piece of the Hodge structure. This ignores the lattice, the higher Hodge pieces, and the polarization. We associate to any bilinear map a new bilinear map which we call its symmetrizer. Iterative application to the map B above yields lower and lower pieces of the Jacobian ring R, allowing us eventually to recapture the polynomial structure, or rather to reduce its recapture to the special cases obtained in Sections 4 and 5 (Lemma 4.2, and the more complicated Proposition 5.3.).

We hope that the new techniques of Sections 4 and 5 and especially Section 6 should find their use in settling Torelli and related problems in cases other than hypersurfaces. Let us mention without proof that the arguments included here, with no alteration, suffice to prove generic Torelli for double covers of $\mathbb{P}^n$, and are quite likely to go over without major change to hypersurfaces in weighted projective spaces.

We assume throughout that the ground field is $\mathbb{C}$. Notice however that everything in Sections 3–6 is algebraic in nature and holds over any field with slight restrictions on the characteristic, i.e. the Jacobian ideal can always be recovered from the algebraic part of the infinitesimal variation of Hodge structure. The Jacobian ideal does not determine X in general (the proof in Section 1 is analytic), but we conjecture that it does if the characteristic does not divide the degree. Unfortunately, there seems to be no way of formulating the Torelli question itself other than over $\mathbb{C}$.

The indebtedness of this work to Griffiths' theories of the period map [7] and its infinitesimal variation [5] should be clear to any reader. I would like to thank Jim Carlson, Herb Clemens, Mark Green, Phil

Griffiths, and Loring Tu for stimulating conversations and encouragement. Special thanks go to Steve Zucker who helped me simplify considerably the proof of the crucial Proposition 6.2.

§1. Recovery of a function from its Jacobian ideal

Throughout this work V denotes the $(n+2)$-dimensional complex vector space

$$V := H^0(\mathbb{P}^{n+1}, \mathcal{O}(1)).$$

PROPOSITION 1.1: *If $f, g \in S^d V$ have the same Jacobian ideal then they are related by an invertible projective transformation.*

REMARK: In [4] a similar result is proven: One of the polynomials, say f, is required to be generic, and then the assertion is stronger: f and g must be proportional. The example of the Fermat hypersurfaces:

$$f = \sum x_i^d, \quad g = \sum a_i x_i^d \qquad (a_i \neq 0)$$

shows that, in a sense, both our result and that of [4] are best possible. A third proof, in [5], is more algebraic than ours: The authors require that no projective automorphism leave f invariant, and conclude that f and g (with the same Jacobian ideal) are projectively equivalent. The "analytic" lemma of Mather (below) is replaced by the existence of the moduli space of non-singular projective hypersurfaces of given degree and dimension.

PROOF OF 1.1: We adapt, and simplify, the argument from [12]. Start with:

LEMMA 1.2: *Let G be a Lie group acting on a manifold X, and let $U \subset X$ be a connected, locally closed submanifold satisfying:*

(a) For each $x \in U$,

$$T_x U \subset T_x(Gx),$$

where Gx is the orbit of G through x.
(b) dim $T_x(Gx)$ is independent of $x \in U$.
Then U is contained in an orbit of G.
(Proof: [11], lemma 3.1.)

In our application, $X = S^d V$ and $G = GL(V^*)$. For $f \in X$, we claim that $T_f(Gf)$, as a subspace of X, is $(J(f))^d$, the d^{th} piece of the Jacobian ideal of f, spanned by the functions

$$x_j \cdot \partial f / \partial x_i$$

where the x_i range over a basis of V. Indeed, G is generated by the 1-parameter subgroups

$$\{g_{ij}(t)\} = \{Id + t \cdot e_i \cdot x_j\}_{t\in\mathbf{C}}$$

where $\{e_i\}$ is the dual basis to $\{x_j\}$. This group acts on V^* by sending e_j to $e_j + te_i$, and fixing the other e_k, $k \neq j$. The Taylor expansion:

$$(g_{ij}(t))(f) = f + tx_j\frac{\partial f}{\partial x_i} + \dots$$

shows that the tangent vector to Gf at f along the subgroup g_{ij} is precisely $x_j\partial f/\partial x_i$, proving that $T_f(Gf) = (J(f))^d$.

Assume now that

$$(J(f))^d = (J(g))^d$$

and let $\overline{U} = \{f_t\}_{t\in\mathbf{C}}$ where $f_t = tf + (1-t)g$. Condition (a) of the lemma clearly holds for all $f_t \in \overline{U}$, since $f, g \in J(f)^d = J(g)^d$. Since:

$$T_{f_t}(Gf_t) = J(tf + (1-t)g)^d \subset J(f)^d + J(g)^d = J(f)^d = T_p(Gf),$$

we know, by semicontinuity, that there is a Zariski open subset $U \subset \overline{U}$ such that $f, g \in U$ and such that for $f_t \in U$,

$$T_{f_t}(Gf_t) = T_f(Gf)$$

so that condition (b) also holds, and we conclude that f, g represent projectively equivalent hypersurfaces. Q.E.D.

REMARK: As stated, the result is false in characteristic $p \neq 0$, since f and g may differ by a p^{th} power, all of whose partials vanish. The best possible conjecture would be: if f, g have the same Jacobian ideal, then there is an automorphism T of V such that all the partials of $Tf - g$ vanish. (I.e. $Tf - g$ would be a combination of p^{th} power monomials.)

§2. Hodge structures, residues and Jacobian rings

In this section we review some results, due mostly to Griffiths [7], concerning the explicit description of the Hodge structure of a non-singular hypersurface $X \subset \mathbb{P}^{n+1}$. We use the following notation: for a nonsingular hypersurface

$$X = \{f = 0\} \subset \mathbb{P}^{n+1}$$

We set:

$S = S^*V$: the homogeneous function ring of $\mathbb{P}^{n+1}$.

$J \subset S$: the homogeneous ideal generated by the partials of f.

$R = S/J$: We call this the Jacobian ring of f.

S^a, J^a, R^a: homogeneous pieces of degree a in S, J, R respectively.

We are interested in the Hodge groups $H^{i,n-i}$ of X, in the middle (= only interesting) dimension. More precisely, consider the Hodge filtration of the primitive cohomology:

$$H_0^n(X) = F^0 \supset F^1 \supset \ldots \supset F^n \supset F^{n+1} = (0)$$

defined by

$$F^a = \bigoplus_{i \geqslant a} H^{i,n-i}.$$

These subspaces vary holomorphically with X, and we ask for the successive quotients F^a/F^{a+1}. The basic result is:

THEOREM 2.1: *There are natural isomorphisms, depending holomorphically on* f:

$$\lambda_a : R^{t_a} \xrightarrow{\sim} F^a/F^{a+1}$$

where

$$t_a = (n - a + 1)d - (n + 2)$$

$$d = \deg(X), \qquad n = \dim(X).$$

The proof is in [7] and we only sketch it here. The residue map

$$\mathrm{Res} : H^{n+1}(\mathbb{P}^{n+1} \setminus X) \to H^n(X)$$

is defined as the adjoint of the tube map

$$H_n(X) \to H_{n+1}(\mathbb{P}^{n+1} \setminus X)$$

sending an n-cycle in X to the boundary of a normal disc tube around it in $\mathbb{P}^{n+1} \setminus X$. The image of Res is $H_0^n(X)$, the entire primitive cohomology. The cohomology of the affine $\mathbb{P}^{n+1} \setminus X$ can be computed using the algebraic deRham complex, in fact, using a bounded piece of it: any class in $H^{n+1}(\mathbb{P}^{n+1} \setminus X)$ can be represented by a meromorphic differential

$$\alpha = \frac{A \cdot \Omega}{f^{n+1}}$$

where

$$\Omega = \sum (-1)^i x_i \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_i} \wedge \ldots \wedge \mathrm{d}x_{n+2}$$

is the standard section of $\omega_{\mathbb{P}^{n+1}}(n+2)$, and A is a polynomial chosen so that $\deg(\alpha) = 0$, i.e.

$$\deg(A) = d \cdot (n+1) - (n+2) = t_n.$$

The key observation is that a class in $H^{n+1}(\mathbb{P}^{n+1} \setminus X)$ lands in $F^a(X)$ if and only if it can be represented by a meromorphic differential α with pole of order $\leqslant n - a + 1$, i.e., such that f^a divides A. This gives a map

$$\mathrm{Res}_a : S^{t_a} \twoheadrightarrow F^a$$

and the proof is concluded by checking that

$$\mathrm{Res}_a^{-1}(F^{a+1}) = J^{t_a}.$$

Next we consider the variation $v(X)$ of the Hodge structure of X. It consists of a linear map

$$v : H^1(\Theta_X) \to T_{p(X)}\mathfrak{F}$$

where $H^1(\Theta_X)$ is the tangent space to the deformation space of X (more precisely, its image under the Kodaira-Spencer isomorphism), $p(X)$ is the period of X, considered as a point of an appropriate flag manifold $\mathfrak{F}$. The tangent space to $\mathfrak{F}$ at the flag $F^0 \supset \ldots \supset F^n$ can be naturally identified with a subspace of the product of tangent spaces to the various Grassmannian quotients of $\mathfrak{F}$, i.e., with a subspace of

$$\bigoplus_a \mathrm{Hom}(F^a, F^0/F^a)$$

and by the infinitesimal period relations,

$$\mathrm{Im}(v) \subset \bigoplus_a \mathrm{Hom}(F^a/F^{a+1}, F^{a-1}/F^a).$$

In fact, we have the more precise version:

THEOREM 2.2 [7]: *The i^{th} piece v_i of the infinitesimal variation of the Hodge structure of a hypersurface can be identified with the homomorphism*

$$v_i : R^d \times R^{t_i} \to R^{t_i + d}$$

given by multiplication.

PROOF: This is quite easy: the identification of $H^1(\Theta_X)$ with R^d was explained in the proof of Proposition 1.1. Let $g \in S^d$, $A \in S^{t_i}$, and let $\bar{g}$, $\bar{A}$ be their images in $R^d \approx H^1(\Theta_X)$, $R^{t_i} \approx F^i/F^{i+1}$. To compute the variation of $\bar{A}$ in the direction $\bar{g}$, one simply differentiates

$$\frac{A \cdot \Omega}{(f+tg)^{n-i+1}}$$

with respect to t, then sets $t=0$. Up to a universal scalar, the answer is

$$\frac{g \cdot A \cdot \Omega}{f^{n-i+2}}$$

as claimed. Q.E.D.

The one remaining "algebraic" data in a Hodge structure, or its variation, is the cup product

$$F^a/F^{a+1} \otimes F^{n-a}/F^{n-a+1} \to H^{2n}(X).$$

Again, there is only one way this could fit with the Jacobian ring; the main computation in [4] proves that it does:

THEOREM 2.3 [4]: *Cup product*

$$F^a/F^{a+1} \times F^{n-a}/F^{n-a+1} \to H^{2n}(X)$$

can be naturally identified with the ring multiplication

$$R^{t_a} \times R^{t_{n-a}} \to R^{t_a+t_{n-a}} = R^{(d-2)(n+2)}.$$

In the rest of this section we describe the structure of the ring R. More generally, let

$$f_1, \dots, f_{n+2}$$

be $n+2$ homogeneous polynomials on $\mathbb{P}^{n+1}$ which have no common zero in $\mathbb{P}^{n+1}$. Let

$$I = (f_1, \dots, f_{n+2})$$

be the ideal they generate,

$$R = S/I$$

the quotient ring. In our application the f_i are the partial derivatives of f, all of degree $d-1$.

Our assumption on the f_i suffices to guarantee that they form a regular sequence ([9], p. 660). In particular, the Koszul complex for R:

$$0 \to S \otimes \Lambda^{n+2} V^* \to \dots \to S \otimes \Lambda^k V^* \to \dots S \otimes V^* \to S \to R \to 0$$

is exact ([9], p. 688).

COROLLARY 2.4: $\dim(R^a)$ *depends only on* a *and on the degrees* $d_i = \deg(f_i)$.

PROOF: The Koszul complex preserves the grading and computes the dimension of any graded piece of R as an alternating sum of dimensions of graded pieces of the free modules $S \otimes \Lambda^k V^*$. Q.E.D.

The main result concerning R is:

LOCAL DUALITY THEOREM 2.5: (1) R is an Artinian ring of top degree

$$\sigma = \sum (d_i - 1).$$

(2) R^σ is one dimensional.
(3) The pairing

$$R^a \times R^{\sigma - a} \to R^\sigma$$

is perfect, for any a.

For a proof, see [9], p. 659.

One application is used repeatedly in Torelli problems and deserves mention:

MACAULAY'S THEOREM 2.6: *If* $a + b \leqslant \sigma$, *the bilinear map*

$$R^a \times R^b \to R^{a+b}$$

is non-degenerate in each factor.

This is an immediate consequence of 2.5.

§3. The Carlson-Griffiths method

Let $X \subset \mathbb{P}^{n+1}$ be a generic, non-singular hypersurface of degree d, defined by $f = 0$, and $v(X)$ its infinitesimal variation of Hodge structure. We treat $v(X)$ as "known" and X as "unknown". To be precise, this means that we are given vector spaces W^{t_i} ($t_i = (i+1)d - (n+2)$, $0 \leqslant i \leqslant n$) and W^d, plus bilinear maps

$$v_i : W^d \times W^{t_i - d} \to W^{t_i}$$

and

$$Q_i : W^{t_i} \times W^{\sigma - t_i} \to \mathbb{C},$$

where $\sigma = (d-2)(n+2)$. We are also told that there exists a function f such that if $R = R(f)$ is the Jacobian ring of f, there exist vector space isomorphisms

$$\lambda_d : R^d \xrightarrow{\sim} W^d$$

$$\lambda_{t_i} : R^{t_i} \xrightarrow{\sim} W^{t_i}$$

which make the v_i, Q_i commute with multiplication in R. Needless to say, we are given neither f nor any of the λ's.

Let t be the smallest non-negative t_i:

$$t = t_{i_0} = (i_0 + 1)d - (n+2), \qquad 0 \leqslant t \leqslant d-1$$

where

$$i_0 = \left[\frac{n+1}{d}\right]$$

is the smallest i such that $F^{n-i} \neq (0)$. The main result of [4] is as follows:

THEOREM 3.1: *Assume that the remainder of n modulo d satisfies*

$$0 \leqslant \operatorname{rem}(n, d) \leqslant \frac{d-3}{2}$$

and that the isomorphism

$$\lambda_t : R^t \xrightarrow{\sim} W^t$$

can be recovered from $v(X)$. Then X can be recovered from $v(X)$, hence generic Torelli holds for X.

REMARK 3.2: The assumption on n is equivalent to:
(1) $t < d-1$, and
(2) $2t \geqslant d-1$.

By (1) we have:

$$R^t = S^t V$$

so the isomorphism λ_t is equivalent to providing W^t with "polynomial structure", i.e., an isomorphism:

$$S^t V \xrightarrow{\sim} W^t.$$

PROOF: Applying all the v_i in succession we get a multilinear map:

$$W^t \times \underbrace{W^d \times \ldots \times W^d}_{n-2i_0} \to W^{\sigma - t}.$$

Combining with the polarization, we have a multilinear map

$$W^t \times W^t \times W^d \times \ldots \times W^d \to W^t \times W^{\sigma - t} \to \mathbb{C}$$

hence a bilinear map

$$S^2 W^t \times S^{n-2i_0} W^d \to \mathbb{C}$$

or a linear map

$$\varphi : S^2 W^t \to S^{n-2i_0}(W^d)^*.$$

Composing on the left with the (known) map λ_t, and on the right with the (unknown) isomorphism λ_d, we get

$$\psi : S^2(S^t V) = S^2 R^t \to S^{n-2i_0}(R^d)^*.$$

At this stage, we cannot construct ψ from our data. However, $\ker(\psi)$ is known since it depends only on λ_t, φ. On the other hand, ψ is induced from multiplication in R, hence it factors through

$$\mu : S^2(S^t V) \to S^{2t} V$$

and by Macaulay's Theorem 2.6:

$$\psi^{-1}(0) = \mu^{-1}(J^{2t}).$$

Therefore, $\mu(\ker(\psi))$ gives us J^{2t} inside $S^{2t}V$. By Macaulay's theorem again, this determines J^{d-1} in $S^{d-1}V$ (we are assuming $2t \geqslant d-1$!) and by Proposition 1.1 this determines X. Q.E.D.

By the theorem and the remark following it, we are led to ask for which n, d can we recover the polynomial structure on W^t. The only case where this is clear is when $t = 1$, since then any isomorphism $V \xrightarrow{\sim} W$ will do. Unfortunately, the only way to satisfy conditions (1), (2) with $t = 1$ is to take $d = 3$, and n divisible by 3.

COROLLARY 3.3: [4] *Generic Torelli is true for cubic hypersurfaces of dimension* $n = 3m$.

§4. Quadrics of rank 4: the simplest case

Let V be a vector space, and let

$$\mu : S^2(S^t V) \to S^{2t} V$$

be the multiplication map. For any vector space U, let $\mathbb{P}(U)$ denote the projective space of hyperplanes in U, so that $U = H^0(\mathbb{P}(U), \mathcal{O}(1))$. Let

$$s : \mathbb{P}(V) \xrightarrow{\sim} S \subset \mathbb{P}(S^t V)$$

be the Veronese embedding, given by the complete linear system $|\mathcal{O}_{\mathbb{P}(V)}(t)|$.

Lemma 4.1: (1) $\ker(\mu)$ *is the system of quadrics in* $\mathbb{P}(S^t V)$ *containing the Veronese variety* S.
(2) S *is the base locus of* $\ker(\mu)$.
(3) $\ker(\mu)$ *is spanned by quadrics of* rank 4.

Proof: Choose a basis $(x_i)_{i \in I}$ for V. It induces bases (of "monomials") for $S^t V$, $S^2(S^t V)$, $S^{2t} V$. Any basis element of $S^2(S^t V)$ is taken by μ to a basis element of $S^{2t} V$, so the basis elements of $S^2(S^t V)$ can be grouped into equivalence classes indexed by the basis of $S^{2t} V$. Therefore $\ker(\mu)$ is generated by differences $q_1 - q_2$ where q_1, q_2 are basis elements of $S^2(S^t V)$ in the same class. Since each q_i is a quadric of rank 2, part (3) is proven.

Part (1) is a triviality: identifying $S^2(S^t V)$ with $H^0(\mathbb{P}(S^t V), \mathcal{O}(2))$ and $S^{2t} V$ with $H^0(\mathbb{P}(V), \mathcal{O}(2t))$, μ becomes the pullback:

$$s^* : H^0(\mathbb{P}(S^t V), \mathcal{O}(2)) \to H^0(S, \mathcal{O}_S(2)) \xrightarrow{\sim} H^0(\mathbb{P}(V), \mathcal{O}(2t)).$$

This proves that $S \subset$ base locus $(\ker(\mu))$. Let p be a point of the base locus, and single out a basis element x_0 of V. Using the quadrics $q_1 - q_2$ as above, one sees immediately that $p = s(p_0)$ where p_0 is the unique point of $\mathbb{P}(V)$ for which

$$(x_i(p_0))_{i \in I} = \left((x_0^{t-1} \cdot x_i)(p)\right)_{i \in I}. \quad \text{Q.E.D.}$$

Let W be another vector space, and

$$\lambda : S^t V \to W$$

an isomorphism which we treat as "unknown", to be recovered modulo an automorphism of V.

Lemma 4.2: *The polynomial structure* λ *on* W *is determined by the linear* *subspace* $\lambda_*(\ker \mu) \subset S^2W$, *and also by the image* $\lambda_*(S) \subset \mathbb{P}(W)$ *of the Veronese.*

PROOF: If $\lambda_*(\ker \mu)$ is given, we recover $T := \lambda_*(S)$ as its base locus. Choose any isomorphism

$$s : \mathbb{P}(V) \to T.$$

Then the isomorphism

$$W = H^0(\mathbb{P}(W), \mathcal{O}(1)) \xrightarrow{\sim} H^0(T, \mathcal{O}_T(1)) \xrightarrow{s^*} H^0(\mathbb{P}(V), \mathcal{O}(t)) = S^tV$$

is the inverse of our desired λ. Q.E.D.

THEOREM 4.3: *Generic Torelli holds for* $d = 2n + 3$, $n \geqslant 2$.

REMARK: Theorem 4.3 is a special case of Theorem 5.5. The only result of this section which is needed in the sequel is Lemma 4.2.

PROOF: The Carlson-Griffiths theorem 3.1 applies, since we have

$$\operatorname{rem}(n, d) = n = \frac{d-3}{2}, \qquad i_0 = 0, \qquad 2t = d - 1.$$

To determine the polynomial structure

$$\lambda_t : S^tV \xrightarrow{\sim} W^t$$

it suffices, by the previous lemma, to determine

$$(\lambda_t)_*(\ker(\mu)) \subset S^2W^t.$$

Using the map

$$\varphi : S^2W^t \to S^n(W^d)^*$$

constructed in the proof of Theorem 3.1, we propose the recipe:

$$(\lambda_t)_*(\ker(\mu)) = \operatorname{span}(\ker(\varphi) \cap \{\text{quadrics of rank } 4\}),$$

assuming that our hypersurface $X \subset \mathbb{P}(V)$ is "sufficiently general". To prove this, we use the "unknown" map λ_t to obtain an equivalent statement:

$$\ker(\mu) = \operatorname{span}(\ker(\psi) \cap \{\text{quadrics of rank } 4\}),$$

where

$$\psi: S^2R^t \to S^n(R^d)^*$$

is induced from the multiplication in R. The inclusion " $\subset$ " follows from part (3) of Lemma 4.1, which asserts that:

$$\ker(\mu) \subset \operatorname{span}(\ker(\mu) \cap \{\text{rank } 4\}).$$

Since

$$\mu(\ker(\psi)) = J^{2t} = J^{d-1},$$

the inclusion " $\supset$ " is equivalent to:

$$\mu(\{\text{quadrics of rank } 4\}) \cap \mathrm{J}^{\mathrm{d}-1} = \{0\}$$

for generic $f \in S^dV$. We see this by the following dimension count:

Linear functions on S^tV form a vector space of dimension $\binom{t+n+1}{n+1} = \binom{2t}{t}$. Any quadric of rank 4 can be written in the form

$$ab - cd$$

for some linear functions a, b, c, d, and no change is affected if a (respectively c) is multiplied by an arbitrary constant while b (respectively d) is divided by the same constant. Hence, the variety of quadrics of rank $\leqslant 4$ in S^tV, as well as its image under μ, have dimension no larger than

$$4 \cdot \binom{2t}{t} - 2.$$

Next we claim that for any $g \in S^{d-1}V$, $g \neq 0$, the family of $f \in S^dV$ such that

$$g \in J(f)$$

has dimension $\leqslant n + 2 + \binom{n+d}{n}$: This is clear since any element of $(J(f))^{d-1}$ is the derivative of f with respect to some derivation on V. The derivations form a vector space of dimension $n+2$, and the kernel of each is a vector space of dimension $\binom{n+d}{n}$.

Altogether, we have

$$\begin{aligned}
&\dim\{f\in S^dV \mid J(f)\cap\mu(\text{rank 4 quadrics})\supsetneqq\{0\}\}\\
&\quad\leqslant 4\cdot\binom{2t}{t}-2+n+2+\binom{n+d}{n}\\
&\quad=4\cdot\binom{2n+2}{n+1}+n+2-\binom{n+d}{n+1}+\binom{n+d+1}{n+1}-2\\
&\quad=\left(4\cdot\binom{2n+2}{n+1}+n+2-\binom{3n+3}{n+1}\right)+\binom{n+d+1}{n+1}-2\\
&\quad\leqslant\binom{n+d+1}{n+1}-2=\dim S^dV-2,
\end{aligned}$$

hence for generic $f\in S^dV$,

$$J(f)\cap\mu(\text{rank 4 quadrics})=\{0\}$$

as claimed, so our recipe does indeed produce $\lambda_*\ker(\mu)$. Q.E.D.

§5. Quadrics of rank 4: general case

The recipe for recovering $\ker(\mu)$ proposed in Section 4,

$$\ker(\mu)=\operatorname{span}(\ker(\psi)\cap\{\text{rank } 4\}),$$

is as simple as one may wish. Unfortunately, we do not know for which values of n, d it holds for the generic polynomial f. In this section we describe an alternative procedure, still based on [4], which works for "almost all" n in the Carlson-Griffiths range

$$0\leqslant\operatorname{rem}(n,d)\leqslant\frac{d-3}{2}.$$

Instead of $\ker(\mu)$, we search for the subvariety

$$\mathrm{LDiv}^k=\{a\in S^kV \mid \exists a'\in S^1V,\ a''\in S^{k-1}V \text{ such that } a=a'\cdot a''\}$$

of polynomials in S^kV which are divisible by a linear factor. Here k stands for any integer satisfying $k<d-1$, $2k\geqslant d-1$, and will eventually be identified with t.

LEMMA 5.1: *The subvariety $(\lambda_k)_*(\mathrm{LDiv}^k)\subset W^k$ determines, up to automorphism of V, the polynomial structure*

$$\lambda_k: S^kV\xrightarrow{\sim}W^k.$$

PROOF: LDiv^k determines the subset

$$A = \{a \in S^k V \mid a \text{ has } k \text{ distinct linear factors}\}$$

since $a \in \mathrm{LDiv}^k$ is in A if and only if the tangent cone to LDiv^k at a consists of k distinct linear subspaces of dimension equal to dim LDiv^k. The closure $\bar{A}$ in $S^k V$ of A is the image of the Segre map

$$\underbrace{V \times \ldots \times V}_{k \text{ times}} \to S^k V.$$

Finally, $\bar{A}$ determines the Veronese variety $S \subset \bar{A}$ as its smallest equisingular locus. Applying $(\lambda_k)_*$, we see that $(\lambda_k)_*(\mathrm{LDiv}^k)$ determines $(\lambda_k)_*(S) \subset W^k$, which determines λ_k by lemma 4.2. Q.E.D.

Let

$$\mathrm{Div}^k = \left\{ \begin{array}{l} a \in S^k V \mid \exists a' \in S^{k'} V,\ a'' \in S^{k''} V \text{ where} \\ \qquad 0 < k' \leqslant k'' = k - k' < k, \text{ such that } a = a' \cdot a'' \end{array} \right\}$$

be the locus of all divisible polynomials in $S^k V$, and for a fixed $a \in S^k V$ let

$$B(a) = \left\{ \begin{array}{l} b \in S^k V \mid \exists c, d \in S^k V \text{ such that} \\ \qquad a \otimes b - c \otimes d \in \ker(\mu) \setminus \{0\} \end{array} \right\}.$$

LEMMA 5.2: (1) $\mathrm{Div}^k = \{a \in S^k V \mid B(a) \neq \phi\}$

(2) $\mathrm{LDiv}^k = \{a \in S^k V \mid \dim(B(a)) \geqslant \dim(\mathrm{LDiv}^k) = n + 1 + \binom{n+k}{n+1}\}$.

PROOF: (1) $a \otimes b - c \otimes d$ is a non-zero element of $\ker(\mu)$ for some c, d if and only if the polynomial $a \cdot b$ has a second factorization into polynomials of degree k. This happens, for some b, if and only if a itself is divisible.

(2) $a \cdot b$ can be refactored if and only if a, b can be factored in the form:

$$a = a' \cdot a'' \qquad b = b' \cdot b''$$

where $0 < k' = \deg(a') = \deg(b') < k$, and $a' \neq b'$, $a'' \neq b''$. (The refactorization is then $(a' \cdot b'') \cdot (a'' \cdot b')$.) Hence, for a given $a \in S^k V$, $B(a)$ is an open dense subset of

$$\{b \in \mathrm{Div}^k \mid a \text{ and } b \text{ have factors of the same degree } k'\}$$

so we have

$$\dim(B(a)) = \max\{\dim \mathrm{Div}^k_{k'} \mid a \text{ has a factor of degree } k'\}$$

where $\mathrm{Div}^k_{k'}$ is the irreducible component of Div^k consisting of polynomials b with a factor of degree k'. Observe that

$$c(k') := \dim(S^{k'}V) = \binom{k'+n+1}{n+1}$$

is a strictly convex function of k' (since the second difference:

$$\begin{aligned} c(k'-1)+c(k'+1)-2c(k') &= \binom{k'+n+1}{n} - \binom{k'+n}{n} \\ &= \binom{k'+n}{n-1} \end{aligned}$$

is positive), therefore the function

$$d(k') := \dim(\mathrm{Div}^k_{k'}) = c(k') + c(k-k') - 1$$

takes its maximum (on the interval $1 \leqslant k' \leqslant k-1$) at the endpoints, $k'=1$ or $k'=k-1$. Thus:

$$\dim(B(a)) = \max\{d(k') \mid a \text{ has a factor of degree } k'\}$$

is $\geqslant d(1) = \dim \mathrm{LDiv}^k$ if and only if a has a linear factor. Q.E.D.

PROPOSITION 5.3: *The kernel of the multiplication map*

$$\alpha : S^2W^k \to W^{2k}$$

determines the polynomial structure on W^k *if* $k < d-1$, $2k \geqslant d-1$ *and*

$$\dim(\mathrm{LDiv}^k) - 3 > \dim(J^{2k}).$$

Here $\alpha = (\lambda_{2k})_* \circ \mu \circ S^2(\lambda_k)_*{}^{-1}$.

REMARK: Explicitly, the inequality is

$$\binom{n+k}{n+1} + n - 2 > (n+2)\binom{n+2k-d+2}{n+1}.$$

Note that $\dim(J^{2k})$ equals the right hand term of the inequality since an element of J^{2k} can be written uniquely as a linear combination of the $n+2$ basic elements of J^{d-1} with arbitrary polynomials of degree $2k-(d-1)$ for coefficients.

We remark also that the main use we make of the Proposition is in the proof of Theorem 6.4, where we need only the case $d=2k$, $n>1$. In this case the messy inequality simplifies, miraculously, to:

$$k \geqslant 4.$$

PROOF: Our recipe this time is as follows: For $a \in W^k$, let

$$B_\alpha(a) = \left\{ \begin{matrix} b \in W^k \mid \exists c, d \in W^k \text{ such that} \\ \qquad a \otimes b - c \otimes d \in \ker(\alpha) \setminus \{0\} \end{matrix} \right\}$$

and let

$$D = \{ a \in W^k \mid \dim B_\alpha(a) \geqslant \dim(\mathrm{LDiv}^k) \}.$$

Then, we claim, D has a unique irreducible component whose dimension is $\geqslant \dim(\mathrm{LDiv}^k)$, and this component, which is thus determined by $\ker(\alpha)$, is $(\lambda_k)_*(\mathrm{LDiv}^k)$. By lemma 5.1 we recover the polynomial structure on W^k as claimed. Our claim follows immediately from the proof of lemma 5.2, where we saw that LDiv^k is the only maximal-dimensional irreducible component of Div^k, and from the following inclusions:

LEMMA 5.4: $(\lambda_k)_*(\mathrm{LDiv}^k) \subset D \subset (\lambda_k)_*(\mathrm{Div}^k)$.

PROOF: Pulling back to S^kV via

$$\lambda_k : S^kV = R^k \xrightarrow{\sim} W^k,$$

the lemma is equivalent to:

$$\mathrm{LDiv}^k \subset (\lambda_k)^{-1}(D) \subset \mathrm{Div}^k.$$

Let

$$\nu : S^2(S^kV) \to S^{2k}V/J^{2k} = R^{2k}$$

be the multiplication map (corresponding via λ_k, λ_{2k} to α) and for $a \in S^kV$ let

$$B_\nu(a) = \left\{ \begin{matrix} b \in S^kV \mid \exists c, d \in S^kV \text{ such that} \\ \qquad a \otimes b - c \otimes d \in \ker(\nu) \setminus \{0\} \end{matrix} \right\}.$$

Since

$$(\lambda_k)^{-1}(D) = \{ a \in S^kV \mid \dim(B_\nu(a)) \geqslant \dim(\mathrm{LDiv}^k) \}$$

and for all $a \in S^k V$ there is the trivial inclusion

$$B(a) \subset B_\nu(a),$$

the inclusion

$$\mathrm{LDiv}^k \subset (\lambda_k)^{-1}(D)$$

follows from lemma 5.2(2).

From now on, assume that $a \in (\lambda_k)^{-1}(D)$, i.e., that

$$\dim(B_\nu(a)) \geqslant \dim(\mathrm{LDiv}^k).$$

Let

$$C(a) = \{a \otimes b - c \otimes d \in \ker(\nu) \setminus \{0\} \mid b, c, d \in S^k V\}$$
$$= \{Q \in \ker(\nu) \setminus \{0\} \mid \mathrm{rank}(Q) \leqslant 4 \text{ and } \mathrm{Sing}(Q) \subset \{a = 0\}\}.$$

There is a correspondence $\sigma : C(a) \to B_\nu(a)$ defined by:

$$\sigma(Q) \ni b \Leftrightarrow \exists c, d \in S^k V \text{ such that } Q = a \otimes b - c \otimes d.$$

It satisfies:

(1) $\dim(\sigma(Q)) \leqslant 3$ for any $Q \in C(a)$.

(Proof: if $b \in \sigma(Q)$ then the hyperplane $\{b = 0\}$ must be tangent to the quadric Q at some non-singular point of Q, i.e., the hyperplane $\{b = 0\}$ (which determines b up to scalar) must lie in the dual variety of Q, which is a quadric surface.)

(2) σ is surjective, by the definition of $B_\nu(a)$.

Combining (1) and (2) we find:

$$\begin{aligned}\dim(C(a)) &\geqslant \dim(B_\nu(a)) - 3 & \\ &\geqslant \dim(\mathrm{LDiv}^k) - 3 & \text{since } a \in (\lambda_k)^{-1}(D) \\ &> \dim(J^{2k}) & \text{by hypothesis.}\end{aligned}$$

Consider now the restriction μ_0 of the multiplication map

$$\mu : \ker(\nu) \to J^{2k}$$

to $C(a)$. By the inequality just established, there is a non-zero $Q \in \mu_0^{-1}(0)$. If $Q = a \otimes b - c \otimes d$ then $b \in B(a)$, so $a \in \mathrm{Div}^k$, concluding the proof of the lemma and of the proposition. Q.E.D.

Combining Proposition 5.3 (with $k = t$) with the Carlson-Griffiths theorem 3.1, we obtain:

THEOREM 5.5: *If the remainder of n modulo d satisfies*

$$0 \leqslant \operatorname{rem}(n, d) \leqslant \frac{d-3}{2}$$

and if n satisfies

$$\binom{n+t}{n+1} + n - 2 > (n+2)\binom{n+s}{n+1}$$

where

$$t = \left(\left[\frac{n+1}{d}\right] + 1\right)d - (n+2)$$

and

$$s = 2t - d + 2,$$

then a generic n-dimensional hypersurface X of degree d is determined by its infinitesimal variation of Hodge structure $v(X)$, hence the generic Torelli theorem holds for X.

It remains to check for which n, d the inequality of the theorem holds. For $n = 1$ it is easy to check that it never holds, so we might as well assume $n \geqslant 2$. We obtain the sufficient condition:

$$\binom{n+t}{n+1} > (n+2)\binom{n+s}{n+1}$$

or, upon division:

$$\frac{t\cdot(t+1)\cdot\ldots\cdot(t+n)}{s\cdot(s+1)\cdot\ldots\cdot(s+n)} > n+2.$$

We now fix d, s, t, i.e. we let n vary in a fixed congruence class modulo d. For $n \geqslant t - s - 1$ the left-hand side equals

$$\frac{(s+1+n)\cdot\ldots\cdot(t+n)}{s\cdot(s+1)\cdot\ldots\cdot(t-1)}$$

and therefore is given by a polynomial in n, all of whose coefficients are positive, of degree

$$t - s = d - 2 - t = \operatorname{rem}(n, d).$$

If we require $t - s \geqslant 2$ then for n sufficiently large ($n \geqslant s^2 - s - 1$ will do, for instance) the required inequality will hold.

COROLLARY 5.6: *Fix d and choose a remainder*

$$2 \leqslant r \leqslant \frac{d-3}{2}.$$

Generic Torelli is true for all but a finite number of n which are congruent to r modulo d.

The results are less impressive when we list the good d's for fixed n:

$n=2$:	$d=7$ or 8	
$n=3$:	$d=3$ or	$9 \leqslant d \leqslant 12$
$n=4$:		$11 \leqslant d \leqslant 17$
$n=5$:		$13 \leqslant d \leqslant 22$
$n=6$:	$d=3, 5$ or	$15 \leqslant d \leqslant 28$
etc.		

§6. The polynomial structure via symmetrizers

In this section we abandon the map

$$\psi: S^2R^t \to S^{n-2i_0}(R^d)^*$$

of [4], and consider instead the bilinear map

$$B_{t,d}: R^t \times R^d \to R^{d+t}.$$

DEFINITION 6.1: Given a bilinear map

$$B: U \times V \to W$$

we define its symmetrizer to be the vector space

$$T = \{P \in \operatorname{Hom}(U, V) \mid \forall l, m \in U, \quad B(l, P(m)) = B(m, P(l))\}$$

together with the natural bilinear map

$$B_-: T \times U \to V$$

defined by

$$B_-(P, l) = P(l).$$

Our main observation concerns the symmetrizer of the multiplication map

$$B_{t,d}: R^t \times R^d \to R^{t+d}.$$

We claim that it can be identified with the "previous" multiplication

$$B_{d-t,t}: R^{d-t} \times R^t \to R^d,$$

that is, that any $P: R^t \to R^d$ satisfying the symmetry condition 6.1 is given by multiplication with a uniquely determined element $p \in R^{d-t}$. More generally, we have:

PROPOSITION 6.2: *Let R be the Jacobian ring of a generic polynomial f of degree d in $n+2$ variables, where d, n satisfy*

$$(d-2)(n-1) \geqslant 3.$$

Then for $a \leqslant d-1$, $b \leqslant d$, the symmetrizer of the multiplication map

$$B_{a,b}: R^a \times R^b \to R^{a+b}$$

is the multiplication map

$$B_{b-a,a}: R^{b-a} \times R^a \to R^b.$$

REMARK: We do not know whether the proposition is true for all f or only for generic f.

Since the proof of this proposition seems less interesting than its application, we defer the proof to the end of this section. Our use of the proposition is based on the following intrinsic version:

LEMMA 6.3: *For $i = a, b, a+b$, let W^i be a vector space and let*

$$\lambda_i: R^i \to W^i$$

be an isomorphism of vector spaces, where R^i is the i^{th} graded piece of the Jacobian ring R of a polynomial f (of degree d, in $n+2$ variables). Assume that the diagram

$$\begin{array}{cccccc} B_{a,b}: & R^a \times & R^b & \to & R^{a+b} \\ & \downarrow \lambda_a & \downarrow \lambda_b & & \downarrow \lambda_{a+b} \\ B\ \ : & W^a \times & W^b & \to & W^{a+b} \end{array}$$

commutes, where $B: W^a \times W^b \to W^{a+b}$ is a given bilinear form. Let

$$B_{-}: W^{b-a} \times W^a \to W^b$$

be the symmetrizer of B, where W^{b-a} is the vector space denoted T in Definition 6.1. There is then a unique homomorphism

$$\lambda_{b-a}: R^{b-a} \to W^{b-a}$$

which makes the following diagram commute:

$$\begin{array}{llclcl} B_{b-a,a}: & R^{b-a} & \times & R^{a} & \to & R^{b} \\ & \downarrow \lambda_{b-a} & & \downarrow \lambda_{a} & & \downarrow \lambda_{b} \\ B_{-} \quad : & W^{b-a} & \times & W^{a} & \to & W^{b}. \end{array}$$

Further, λ_{b-a} is injective if $a \geqslant 0$, $b \leqslant (d-2)(n+2)$, and λ_{b-a} is an isomorphism if f is generic and if a, b, d, n satisfy the numerical restrictions of Proposition 6.2: $a \leqslant d-1$, $b \leqslant d$, and

$$(d-2)(n-1) \geqslant 3.$$

PROOF: Clearly, λ_{b-a} must satisfy

$$(\lambda_{b-a}(p))(u) = \lambda_b\big(p \cdot \lambda_a^{-1}(u)\big)$$

for any $p \in R^{b-a}$ and $u \in W^a$, and this defines λ_{b-a} uniquely and unambiguously. If $\lambda_{b-a}(p) = 0$ then for all $m \in R^a$,

$$p \cdot m = 0 \in R^b$$

so by Macaulay's theorem 2.6 we have $p = 0$ (assuming $a \geqslant 0$ and $b \leqslant (d-2)(n+2)$). Finally, if a, b, d, n satisfy the hypotheses of Proposition 6.2 then $B_{b-a,a}$ is the symmetrizer of $B_{a,b}$, so λ_{b-a} is an isomorphism by functoriality of the symmetrizer construction. Q.E.D.

We can now prove the main result of this paper.

THEOREM 6.4: *Generic Torelli for hypersurfaces.*

Assume that n, d fall into none of the following cases:

(0) $n = 2$, $d = 3$. (*The cubic surface*)

(1) *d divides $n+2$.*

(2) $d = 4$, $n \equiv 0(4)$, *or* $d = 6$, $n \equiv 1(6)$.

Then a generic n-dimensional hypersurface X of degree d can be recovered from its infinitesimal variation of Hodge structure $v(X)$. In particular, the period map for such hypersurfaces is generically injective.

Remarks on the exceptions

In case (0) the theorem is false, as is local Torelli, since cubic surfaces depend on four parameters while their Hodge structure is trivial, hence has no moduli.

The exceptions in case (2) seem to be due to a technical weakness of our method, rather than to any genuine difference in behavior. However, we feel that such an intrinsic difference might exist for type (1). The difficulty is easy to document in the two special cases $n = 2$, $d = 4$ (quartic K3 surfaces) and $n = 4$, $d = 3$ (cubic fourfolds). These special cases share two properties: the period domain D is symmetric, vaguely suggesting an arithmetic flavor; and the period map is étale, showing on the one hand that generic and global Torelli are equivalent in these cases, and on the other than the infinitesimal variation $v(X)$ contains no information beyond the bare Hodge structure $H(X)$, so our techniques are totally useless here.

Both of these properties have (weaker) analogues for the general case of type (1). The period domain D has a symmetric quotient D_0, parametrizing partial Hodge filtrations

$$F^0 \supset F^1 \supset F^n \supset (0).$$

The partial period map, to a subvariety of D_0, is again étale. In other words, the infinitesimal variation of the first non-zero piece of the Hodge filtration is an isomorphism, hence contains no information beyond $H(X)$. (This is the only piece used in our proof.)

We do not know, of course, whether this special behavior is significant. It might well be, for instance, that the variation of the second piece, given by the map

$$S^2R^d \to R^{2d},$$

is sufficient to recover X in all cases except for the quartic surface and the cubic fourfold.

PROOF OF THEOREM 6.4: Recall first the cases in which Torelli (even the global version) is already known:

$d \leqslant 2$: hyperplanes and quadrics have no moduli.

$d = 3$: the result is trivial for plane cubic curves, false (and excluded) for surfaces, and known for threefolds [6].

$d = 4$: the case of quartic surfaces is a special case of the global Torelli for K3 surfaces [14].

$n = 1$, $d \geqslant 4$: the result follows from the Torelli theorem for curves plus the uniqueness of a non-singular g_n^2, proved in [2].

Excluding these cases amounts precisely to the assumption

$$(d-2)(n-1) \geqslant 3$$

which we make from now on.

The infinitesimal variation of the first piece, $H^{n-i_0,i_0}(X)$, of the cohomology of X is given to us as a bilinear form

$$B_{t,d}: W^t \times W^d \to W^{t+d}.$$

We know that isomorphisms $\lambda_i : R^i \to W^i$ (for $i = t, d, t+d$) exist which make the diagram

$$\begin{array}{ccccc} R^t \times & R^d & \to & R^{t+d} \\ \downarrow \lambda_t & \downarrow \lambda_d & & \downarrow \lambda_{t+d} \\ W^t \times & W^d & \to & W^{t+d} \end{array}$$

commute. By Lemma 6.3, we get a new diagram

$$\begin{array}{ccccc} R^{d-t} \times & R^t & \to & R^d \\ \downarrow \lambda_{d-t} & \downarrow \lambda_t & & \downarrow \lambda_d \\ W^{d-t} \times & W^t & \to & W^d \end{array}$$

where the vertical maps are isomorphisms and the bottom row consists entirely of data determined by $v(X)$.

If $d - t > t$ we exchange t, $d - t$, and in any case we can iterate this construction. We obtain a sequence of diagrams

$$\begin{array}{ccccc} R^{a_i} \times & R^{b_i} & \to & R^{a_i + b_i} \\ \downarrow \lambda_{a_i} & \downarrow \lambda_{b_i} & & \downarrow \lambda_{a_i + b_i} \\ W^{a_i} \times & W^{b_i} & \to & W^{a_i + b_i} \end{array}$$

in which the λ's are isomorphisms, the bottom rows are successive symmetrizers of $B_{t,d}$ and hence are determined by $v(X)$, and where the triples $(a_i, b_i, a_i + b_i)$ follow the Euclidean algorithm for t, d. In the last diagram we have $a_i = b_i = k$, where k is the greatest common divisor

$$k = g.c.d.\ (t, d) = g.c.d.\ (d, n+2).$$

Excluding the cases where $d = 2$ or where d divides $n + 2$, we have $k < d - 1$. We claim that the polynomial structure on W^k can be determined from the map $W^k \times W^k \to W^{2k}$: if $2k < d - 1$ then $W^{2k} \approx S^{2k}$ so we can use Lemma 4.2. Otherwise, we use Proposition 5.3. The required identity becomes

$$\binom{n+k}{n+1} + n - 2 > (n+2)\binom{n+2}{n+1} = (n+2)^2,$$

which is easily seen to hold for $k \geqslant 4$ (i.e., $d \geqslant 8$) and $n \geqslant 2$. We conclude that the isomorphism $\lambda_k : S^k = R^k \to W^k$ can be determined, modulo an automorphism of the underlying vector space $V = S^1$.

Since the "polynomial structure" maps

$$\nu_\alpha : S^\alpha \to W^\alpha$$

for $\alpha = a$ and $\alpha = b$ determine ν_{a+b}, we can start with $\nu_k = \lambda_k$ and recover, inductively, the maps $\nu_{a_i+b_i}$, beginning with ν_{2k} and ending with ν_d. The kernel

$$J^d = \ker(\nu_d)$$

determines J^{d-1} by Macaulay's theorem, and this determines the isomorphism class of X by Proposition 1.1. Q.E.D.

We still need to prove Proposition 6.2. We replace it by a slightly stronger version:

PROPOSITION 6.5: *The symmetrizer of $B_{a,b}$ is $B_{b-a,a}$ whenever $a + b \leqslant (d-2)(n+1)$ and $2a + b \leqslant (d-2)(n+2)+1$. (This contains 6.2 as a special case.)*

PROOF OF PROPOSITION 6.5: Consider the homomorphism

$$\alpha : R^{b-a} \to T$$

(T is the symmetrizer of $B_{a,b}$) defined by

$$\alpha(p)(l) = p \cdot l \in R^b$$

for $p \in R^{b-a}$, $l \in R^a$. It is clear that for $p \in R^{b-a}$, $P = \alpha(p)$ satisfies the symmetry condition, hence $P \in T$. Moreover, α is injective by Macaulay's theorem.

As we saw in 2.4, the dimensions of R^a, R^b, R^{a+b}, R^{b-a} are independent of f for non-singular f. We think of each R^i as a vector bundle over the parameter space ($\mathbb{P}(S^d) \setminus$ Discriminant) of non-singular hypersurfaces. T becomes a subvariety of the vector bundle

$$\mathrm{Hom}(R^a, R^b),$$

defined by fiber-linear equations and containing the vector sub-bundle $\alpha(R^{b-a})$. By semicontinuity of $\dim(T)$, the equality $\alpha(R^{b-a}) = T$ for one non-singular f implies the same equality for generic f. We prove the proposition for a particular f, namely the Fermat hypersurface $f = \sum_{i=1}^{n+2} x_i^d$. In this case J is generated by the monomials x_i^{d-1}.

We claim that if $a + b \leqslant (d-2)(n+1)$ then

$$\left((x^I) : R^a\right)^b = (x^I)^b,$$

where I is any multi-index of degree a. (The superscript "b" indicates b^{th}

graded piece of the respective ideals.) It is clear that both ideals (x^I), $((x^I): R^a)$ are generated by monomials, so it suffices to prove that for a multi-index K of degree b,

$$x^K \cdot R^a \subset (x^I) \Leftrightarrow K \geqslant I.$$

The implication $\Leftarrow$ is trivial, so assume $x^K \cdot R^a \subset (x^I)$, while $K_i < I_i$ for some i, $1 \leqslant i \leqslant n+2$. Let L be the multi-index defined by:

$$L_i = I_i - K_i \dot{-} 1 \geqslant 0,$$

$$L_j = (d-2) - K_j, \qquad j \neq i.$$

We have

$$x^K \cdot x^L = \left(\prod_{j \neq i} x_j^{d-2}\right) \cdot x_i^{I_i - 1} \notin J + (x^I)$$

so our assumption $x^K \cdot R^a \subset (x^I)$ implies

$$a > \deg(L) = (d-2)(n+1) - b + I_i - 1$$

or

$$a + b > (d-2)(n+1),$$

contradiction. This proves our claim.

From the symmetry condition for $P \in T$:

$$x^I \cdot P(x^J) = x^J \cdot P(x^I) \in R^{a+b}$$

for I, J multi-indices of degree a, we conclude

$$P(x^I) \in \left((x^I) : R^a\right)^b = (x^I)^b$$

so there is an element

$$p_I \in R^{b-a}$$

such that

$$P(x^I) = p_I \cdot x^I \in R^b.$$

We need to prove that p_I is independent of I. Write

$$p_I = \sum p_{I,H} x^H,$$

the summation extending over (non-negative) multi-indices H of degree

$b-a$. Note that p_I can be modified by anything in $\text{Ann}(x^I)$ without altering P. If we let M denote the "maximal" multi-index $(d-2, \ldots, d-2)$, we have

$$\text{Ann}(x^I) = \left(x^H \mid I + H \not\leqslant M\right),$$

so we are reduced to proving the following statement:

If H, I, J are multi-indices of degree $b-a$, a, a respectively, and $H + I \leqslant M$, $H + J \leqslant M$, then $p_{I,H} = p_{J,H}$.

Observe that if, in addition, $H + I + J \leqslant M$, then the desired statement follows from the symmetry condition:

$$x^I \cdot P(x^J) = x^J \cdot P(x^I),$$

by comparing coefficients of the non-zero monomial x^{H+I+J}. The general case of our statement then follows from:

LEMMA 6.6: *Let H, I, J be multi-indices of degrees $b-a$, a, a respectively, satisfying*

$$H + I \leqslant M, \qquad H + J \leqslant M.$$

Then either

$$H + I + J \leqslant M$$

of there is a multi-index K of degree a satisfying

$$H + I + K \leqslant M, \qquad H + J + K \leqslant M.$$

PROOF: Let $L = \max(I, J)$. The condition on K is

$$K \leqslant M - H - L,$$

and the assumption implies $M - H - L \geqslant 0$. Now if I, J are disjoint (i.e., $L = I + J$) then we have

$$H + I + J \leqslant M,$$

while otherwise $\deg(L) < \deg(I) + \deg(J)$, hence:

$$\begin{aligned}
\deg(M - H - L) &\geqslant \deg(M) - \deg(H) - \deg(I) - \deg(J) + 1 \\
&= (d-2)(n+2) - (b-a) - 2a + 1 \\
&= (d-2)(n+2) + 1 - (a+b) \\
&\geqslant a,
\end{aligned}$$

so it is possible to choose K of degree a satisfying

$$K \leqslant M - H - L$$

as required. Q.E.D.

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