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H. KISILEVSKY

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SOME NON-SEMI-SIMPLE IWASAWA MODULES

H. Kisilevsky

The purpose of this note is to show that the semisimplicity result of [2] may fail to be true in the cases not covered by the theorem in section 2. We base the examples on the idea of J.F. Jaulent [4] although our method in §1 is somewhat different. Theorem 1 of this note gives an alternate proof of Theorem 1 of [2] and Theorem 9 of [4]. We follow the notation in [2].

Let k/\mathbf{Q} be a totally complex abelian extension, and denote by $\Delta = \text{Gal}(k/\mathbf{Q})$. Let $J \in \Delta$ be the automorphism given by complex conjugation (under some fixed embedding of an algebraic closure of k into the complex numbers). Fix a prime p , such that $\delta^{p-1} = 1$ for all $\delta \in \Delta$. Let $\hat{\Delta} = \text{Hom}(\Delta, \mu_{p-1}) = \text{Hom}(\Delta, \mathbf{Z}_p^*)$ and denote by V the set of characters χ of Δ which are either odd or trivial, i.e. $V = \{\chi \in \hat{\Delta} \mid \chi(J) = -1 \text{ or } \chi = \chi_0\}$.

For each $\chi \in V$, there exists a (unique) \mathbf{Z}_p -extension (see [1]) K_χ/k , $\text{Gal}(K_\chi/k) = \Gamma_\chi$ such that K_χ/\mathbf{Q} is normal and $\text{Gal}(K_\chi/\mathbf{Q}) \cong \Gamma_\chi \cdot \Delta$ a semidirect product with

$$\delta\sigma\delta^{-1} = \sigma^{\chi(\delta)} \text{ for all } \sigma \in \Gamma_\chi, \delta \in \Delta.$$

Let L/K_χ be the maximal abelian unramified p -extension of K_χ so that $\text{Gal}(L/K_\chi) = X \simeq \varprojlim A_n$ (where A_n is the p -primary subgroup of the ideal class group of $k_n \subseteq K_\chi$, and the limit is taken as usual with respect to the norm maps).

Then, as usual, X is a noetherian torsion Δ -module, so we have

$$X/TX \sim {}_T X \sim {}_T X_0 \sim X_0/TX_0$$

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where ${}_T X = \{x \in X \mid Tx = 0\}$ and $X_0 = \{x \in X \mid T^k x = 0 \text{ some } k \geq 1\}$ and “ \sim ” here denotes pseudo-isomorphism.

Since Γ_χ acts trivially on X/TX and on ${}_T X$ we have a natural action of Δ on these groups and following [4], we study their Δ -decompositions.

If M is a Δ -module which is also a (pro) p -group then for $\phi \in \hat{\Delta}$ write

$$M_\phi = \{m \in M \mid \delta(m) = \phi(\delta) \cdot m \text{ for } \delta \in \Delta\}$$

and call this the ϕ -component of M .

Now $X_0 \sim A/T^{a_1} + \dots + A/T^{a_r}$ for integers $a_1, \dots, a_r \geq 1$. We say X_0 is semi-simple $\Leftrightarrow a_1 = a_2 = \dots = a_r = 1$ and in this case it is clear that

$${}_T X \sim X_0 \sim X/TX \text{ as } \Delta\text{-modules.}$$

In §1 we compute the Δ -decomposition of X/TX and in §2 we obtain some information of the Δ -decomposition of ${}_T X$.

§1. The Δ -structure of X/TX

Let L_0 be the subfield of L fixed by TX so that L_0 is the largest subfield of L abelian over k , and $\text{Gal}(L_0/k) \simeq X/TX$. Then L_0 is normal over \mathbf{Q} , and $L_0 \supseteq \Pi K_\phi$ (compositum taken over certain ϕ to be determined) where $[L_0 : \Pi K_\phi] < \infty$. (In fact the Galois group $\text{Gal}(L_0/\Pi K_\phi)$ is the torsion subgroup of X/TX and has certain interest c.f. [3].) To determine the characters ϕ for which $K_\phi \subseteq L_0$, we note $K_\phi \subseteq L_0 \Leftrightarrow K_\phi K_\chi / K_\chi$ is unramified at all primes over p and this is a condition which we shall determine locally.

Let \mathfrak{p} be a prime of k dividing p , and let $F = F_0$ be the completion of k at \mathfrak{p} . Let F_ϕ be the union of the completions of the finite layers of K_ϕ with respect to some consistent choice of primes over \mathfrak{p} . (In the notation of [2], $\mathfrak{p} = \mathfrak{p}_i$ some i , $F = F_{0,i}$, and $F_\phi = \bigcup_{n \geq 1} F_{n,i}$.) Then F_ϕ/F is a \mathbf{Z}_p -extension, infinitely ramified, such that F_ϕ/\mathbf{Q}_p is Galois, and $\text{Gal}(F_\phi/\mathbf{Q}_p) \simeq \mathbf{Z}_p \cdot D$ a semi-direct product where $D \subseteq \Delta$ is the decomposition group of p in $\text{Gal}(k/\mathbf{Q})$ and

$$\delta \sigma \delta^{-1} = \sigma^{\phi'(\delta)}$$

$$\text{for all } \sigma \in \mathbf{Z}_p = \text{Gal}(F_\phi/F) \text{ and } \delta \in V, \text{ and } \phi' = \phi|D.$$

We state the following two lemmas whose proofs we omit:

LEMMA 1: If M is the compositum of all \mathbf{Z}_p -extensions of F , and $G = \text{Gal}(M/F)$, then $G \simeq \mathbf{Z}_p^{|\mathcal{D}|+1}$ and we have the D -decomposition of G for all $\phi' \in \hat{D}$,

$$\begin{aligned} G_{\phi'} &\simeq \mathbf{Z}_p && \text{if } \phi' \neq \chi'_0 \\ &\simeq \mathbf{Z}_p + \mathbf{Z}_p && \text{if } \phi' = \chi'_0. \end{aligned}$$

LEMMA 2: $F_\phi F_\chi / F_\chi$ is unramified if and only if either (a) $F_\phi = F_\chi$ or (b) $F^{nr} \subseteq F_\phi F_\chi$, where F^{nr} is the unique non-ramified \mathbf{Z}_p -extension of F and is equal to $F \cdot \mathbf{Q}_p^{nr}$, the compositum of F with the non-ramified \mathbf{Z}_p -extension of \mathbf{Q}_p .

THEOREM 1: $K_\phi K_\chi / K_\chi$ is unramified if and only if

$$\phi \in V \text{ and } \phi|D = \chi|D.$$

PROOF: Suppose $K_\phi K_\chi / K_\chi$ is unramified so that for each \mathfrak{p} over p , we have $F_\phi F_\chi / F_\chi$ is unramified. Hence by Lemma 2, either (a) $F_\phi = F_\chi$ so that $\phi|D = \chi|D$ or (b) $F^{nr} \subseteq F_\phi F_\chi$. In this case, (b), we must have $\text{Gal}(F_\phi F_\chi / F)_{\chi'_0}$ is non-trivial since $\text{Gal}(F^{nr} / F)_{\chi'_0} \cong \mathbf{Z}_p$. Since both F_ϕ / F and F_χ / F are infinitely ramified it follows that only the χ'_0 component of $\text{Gal}(F_\phi F_\chi / F)$ is non-zero and so $\phi|D = \chi_0|D = \chi|D$. Hence in either case, $\phi \in V$ and $\phi|D = \chi|D$.

Conversely, suppose $\phi \in V$, and $\phi|D = \chi|D$. If $\chi|D \neq \chi_0|D$ then $F_\phi = F_\chi$ by Lemma 1, and so $K_\phi K_\chi / K_\chi$ is unramified at primes over \mathfrak{p} .

If $\phi|D = \chi|D = \chi_0|D$ then again by Lemma 1 either $F_\phi = F_\chi$; or $F^{nr} \subseteq F_\phi F_\chi$, so again $K_\phi K_\chi / K_\chi$ is unramified at primes over \mathfrak{p} . Since K_ϕ / k is unramified outside of primes over p , the conclusion of the theorem follows.

COROLLARY: $(X/TX)_\phi \sim \mathbf{Z}_p$ for $\phi \in V$, $\phi|D = \chi|D$, $\phi \neq \chi$, and ~ 0 otherwise.

REMARK: This corollary furnishes another proof of Theorem 1 in [2] and Theorem 9 of [4].

We also note if for any ϕ we have $F_\phi = F_\chi$, then it follows that $K_\phi K_\chi \subseteq L$ in the notation of [2] and for each ϕ , $(X'/TX')_\phi$ has non-zero \mathbf{Z}_p -rank. This gives many examples of \mathbf{Z}_p -extensions where X'/TX' and ${}_T X'$ are infinite.

§2. Δ -structure of ${}_T X$

Let $\Gamma = \Gamma_x = \text{Gal}(K_x/k)$, and so ${}_T X = \varprojlim A_n^\Gamma$. Since the limit is taken with respect to the norm maps $N_{m,n}$ and since $\delta N_{m,n} = N_{m,n} \delta$ for all $\delta \in \Delta$, it follows that

$$({}_T X)_\phi = \varprojlim (A_n^\Gamma)_\phi \text{ for } \phi \in \hat{\Delta}.$$

We consider the usual exact sequences

$$\begin{aligned} 1 &\rightarrow P_n \rightarrow I_n \rightarrow C_n \rightarrow 1 \\ 1 &\rightarrow E_n \rightarrow k_n^* \rightarrow P_n \rightarrow 1 \end{aligned}$$

where I_n, C_n, P_n, E_n are the ideal group, class group, group of principal ideals, and unit group of the n^{th} layer k_n of K_x respectively.

We obtain the exact sequence

$$1 \rightarrow P_n^\Gamma \rightarrow I_n^\Gamma \rightarrow C_n^\Gamma \xrightarrow{f} NP_n/P_n^{\gamma-1} \simeq E_0 \cap Nk_n^*/NE_n \rightarrow 1$$

where the map f is given below. Choose a fixed generator γ of Γ_x . Then for $x \in C_n^\Gamma$, $\gamma x = x$ and so $\frac{\gamma A}{A} = (\alpha) \in P_n$ for an ideal $A \in x$, define $f(x) = (\alpha) \bmod P_n^{\gamma-1}$. This is a group homomorphism which is not a Δ -map, (c.f. [4]), but satisfies

$$f: (A_n^\Gamma)_\phi \rightarrow ({}_N P_n/P_n^{\gamma-1})_{\phi x}.$$

Also the isomorphism is given by:

$$\begin{aligned} {}_N P_n/P_n^{\gamma-1} &\simeq E_0 \cap N(k_n^*)/N(E_n) \\ (\alpha) \bmod P_n^{\gamma-1} &\rightarrow N(\alpha) \bmod N(E_n) \end{aligned}$$

where N denotes the norm map $N_{n,0}$ from k_n to $k = k_0$. Hence we obtain the exact sequence

$$1 \rightarrow \frac{P_n \cap I_0}{I_0} \rightarrow P_n^\Gamma/P_0 \rightarrow I_n^\Gamma/I_0 \rightarrow C_n^\Gamma/j(C_0) \rightarrow \frac{E_0 \cap N(k_n^*)}{N(E_n)} \rightarrow 1 \quad (*)$$

where $j(C_0) \subseteq C_n^\Gamma$ is the subgroup generated by the ideals of $k = k_0$. We shall compute the ϕ -components of the groups $E_0 \cap N(k_n^*)/E_0^{\phi^n}$ and I_n^Γ/I_0 . Since the groups on either side of $C_n^\Gamma/j(C_0)$ are (at worst) quotients

of these, this will describe the set of ϕ -components of $A_n^\Gamma \sim C_n^\Gamma/j(C_0)$ which are possibly non-trivial. (As in [2], we use the notation $A_n \sim B_n$ for sequences of groups $\{A_n\}$ and $\{B_n\}$ to mean there are homomorphisms $\phi_n: A_n \rightarrow B_n$ whose kernels and cokernels have orders bounded independently of n .)

For each prime \mathfrak{p} of k dividing p , let $\mathfrak{p} = A(\mathfrak{p})^{e_{\mathfrak{p}}}$ in I_n where $e_{\mathfrak{p}} \sim p^n$ is the ramification index of \mathfrak{p} for k_n/k . Since Δ permutes the primes of k over (p) transitively it follows that

$$I_n^\Gamma/I_0 \simeq \bigoplus_{\mathfrak{p}|(p)} \langle A(\mathfrak{p}) \rangle / \langle \mathfrak{p} \rangle \sim \mathbf{Z}/p^n \mathbf{Z} [\Delta/D]$$

where $\langle A(\mathfrak{p}) \rangle, \langle \mathfrak{p} \rangle$ are the multiplicative subgroups of I_n^Γ generated by $A(\mathfrak{p})$ and \mathfrak{p} respectively.

$$\begin{aligned} \text{Hence it follows that } (I_n^\Gamma/I_0)_\phi &\sim \mathbf{Z}/p^n \mathbf{Z} \text{ if } \phi|D = \chi_0|D \\ &\sim 0 \text{ otherwise.} \end{aligned}$$

On the other hand by [2, Lemma 1] we have

$$\begin{aligned} (E_0 \cap N(k_n^*)/E_0 p^n)_{\phi_1} &\sim \mathbf{Z}/p^n \mathbf{Z} \text{ if } \phi_1 \text{ even, } \phi_1 \neq \chi_0 \text{ and } \phi_1|D \neq \chi|D \\ &\sim 0 \text{ otherwise.} \end{aligned}$$

Since $(A_n)_\phi^\Gamma \rightarrow (E_0 \cap N(k_n^*)/NE_n)_{\phi\chi}$, the possible ϕ -components of A_n^Γ which have non-trivial image in this group are among those ϕ ,

$$\phi(J) = \chi(J), \phi \neq \chi^{-1} \text{ and } \phi|D \neq \chi_0|D.$$

Hence the non-trivial ϕ -components of A_n^Γ are among

$$\{\phi|\phi|D = \chi_0|D\} \cup \{\phi|\phi(J) = \chi(J), \phi \neq \chi^{-1}, \phi|D \neq \chi_0|D\}.$$

This provides no restriction in the case that $D \subseteq \ker \chi$ when in fact X_0 is semisimple [2].

If $\chi|D \neq \chi_0|D$, then we see that the χ^{-1} component of A_n^Γ and that of ${}_\tau X$ must be pseudo-null.

§3. Examples

We now describe a set of characters χ so that for the \mathbf{Z}_p -extensions K_χ/k the groups X/TX and ${}_\tau X$ have *different* Δ -decompositions. This implies that the corresponding X_0 is *not* semi-simple.

By Corollary of §1, we see that $(X/TX)_{\chi^{-1}} \sim \mathbf{Z}_p$ if $\chi^{-1} \neq \chi$ and $\chi^{-1}|D = \chi|D$, i.e. if $\chi^2 \neq \chi_0$ and $\chi^2|D = \chi_0|D$. On the other hand §2 implies that $({}_\tau X)_{\chi^{-1}} \sim 0$ if $\chi|D \neq \chi_0|D$ so we have:

For any character χ , such that $\chi^2 \neq \chi_0$, $\chi|D \neq \chi_0|D$ and $\chi^2|D = \chi_0|D$ we have $({}_T X)_{\chi^{-1}} \sim 0$ and $(X/{}_T X)_{\chi^{-1}} \sim \mathbf{Z}_p$.

The examples of Jaulent [4] are of this type.

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Department of Mathematics
 Concordia University
 Sir George Williams Campus
 1455 De Maisonneuve Blvd. West
 Montreal
 Quebec H3G 1M8
 Canada