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REPRESENTATION-FINITE SELFINJECTIVE ALGEBRAS OF CLASS D_n

Christine Riedtmann

1. Introduction

In this paper, we complete the classification of finite-dimensional, selfinjective, representation-finite algebras over an algebraically closed field k. If such an algebra Λ is connected, we can associate with it a Dynkin-graph $\Delta = A_n$, D_n, E_6, E_7, or E_8, the tree class of Λ ([5], 2). The classification has been carried out in [5] for algebras of tree class A_n and in [2] for algebras of tree class E_6, E_7, and E_8 as well as for a class of algebras of tree class D_n. We gave an explicit description of the Auslander–Reiten quivers for algebras of tree class D_n in [6]. Here we will determine how many non-isomorphic basic algebras of tree class D_n give rise to a given Auslander–Reiten quiver. Throughout the article, we assume the field k to be algebraically closed.

Let Δ be one of the Dynkin-graphs A_n, D_n, E_6, E_7, or E_8, and let $\mathbb{Z}\Delta$ be the corresponding translation-quiver. We associate with a subset $\mathscr{C}$ of vertices of $\mathbb{Z}\Delta$ a translation-quiver $(\mathbb{Z}\Delta)_{\mathscr{C}}$ in the following way. The underlying quiver of $(\mathbb{Z}\Delta)_{\mathscr{C}}$ is obtained by adding to $\mathbb{Z}\Delta$ a vertex c^* and the two arrows $c \to c^*$ and $c^* \to \tau^{-1}c$ for every c in $\mathscr{C}$. We take the translation of $(\mathbb{Z}\Delta)_{\mathscr{C}}$ to be the translation of $\mathbb{Z}\Delta$ on the common vertices and to be undefined on the vertices c^*. A set $\mathscr{C}$ is called a *configuration* of $(\mathbb{Z}\Delta)_{\mathscr{C}}$ is a representable translation-quiver [2]; i.e., if $(\mathbb{Z}\Delta)_{\mathscr{C}}$ satisfies the conditions listed in [1], 2.8. If Δ ranges over all Dynkin-graphs, $\mathscr{C}$ over all configurations of $\mathbb{Z}\Delta$, and Π over all non-trivial admissible automorphism groups of $(\mathbb{Z}\Delta)_{\mathscr{C}}$, the residue quivers $(\mathbb{Z}\Delta)_{\mathscr{C}}/\Pi$ provide a complete list of Auslander–Reiten quivers of finite-dimensional, basic, connected k-algebras which are representation-finite and selfinjective, but not equal to k ([2], 1.3). Two translation-quivers $(\mathbb{Z}\Delta)_{\mathscr{C}}/\Pi$ and $(\mathbb{Z}\Delta')_{\mathscr{C}'}/\Pi'$ are isomorphic if and only if there is an isomorphism

$f: \mathbb{Z}\Delta \to \mathbb{Z}\Delta'$ such that $\mathscr{C}' = f\mathscr{C}$ and $\Pi' = f\Pi f^{-1}$. In particular, Δ' equals Δ.

In case $\Delta = A_n$, E_6, E_7, or E_8, any two basic algebras with a given Auslander–Reiten quiver $(\mathbb{Z}\Delta)_{\mathscr{C}}/\Pi$ are isomorphic. Our main result is the following:

THEOREM: *Let $\mathscr{C}$ be a configuration of $\mathbb{Z}D_n$, and let $\Pi \neq \{1\}$ be an admissible automorphism group of $(\mathbb{Z}D_n)_{\mathscr{C}}$.*

(a) *In case* $\operatorname{char} k = 2$ *and* $n = 3m$ *for some integer m, and if in addition $\mathscr{C}$ is $\tau^{(2m-1)\mathbb{Z}}$-stable and $\Pi = \tau^{(2m-1)\mathbb{Z}}$, there are exactly two isomorphism classes of basic k-algebras with Auslander–Reiten quiver $(\mathbb{Z}D_n)_{\mathscr{C}}/\Pi$.*

(b) *In all other cases, any two basic k-algebras with Auslander–Reiten quiver $(\mathbb{Z}D_n)_{\mathscr{C}}/\Pi$ are isomorphic.*

By $\tau^{(2m-1)\mathbb{Z}}$ we denote the infinite cyclic group generated by τ^{2m-1}. Notice that an algebra with Auslander–Reiten quiver $(\mathbb{Z}\Delta)_{\mathscr{C}}/\Pi$ is necessarily connected, selfinjective, and representation-finite.

Let Λ be a basic k-algebra with Auslander–Reiten quiver Γ_Λ, and let $\operatorname{ind}\Lambda$ be the full subcategory of the category $\operatorname{mod}\Lambda$ of finitely generated Λ-modules whose objects are specific representatives of the isomorphism classes of indecomposable modules. Then Λ is called *standard* if $\operatorname{ind}\Lambda$ is isomorphic to the mesh-category $k(\Gamma_\Lambda)$ ([1], 5.1). Part (a) of our theorem provides a large family of non-standard algebras. In fact, we obtain one for each isomorphism class of $\tau^{(2m-1)\mathbb{Z}}$-stable configurations of $\mathbb{Z}D_{3m}$, or equivalently for each configuration of $\mathbb{Z}A_{m-1}$ ([6], 6). For all such non-standard algebras Λ, we will describe $\operatorname{ind}\Lambda$ by its quiver and relations.

Let us explain for which cases the theorem was proved in [2]. An admissible automorphism group of $(\mathbb{Z}D_n)_{\mathscr{C}}$ is given by an admissible automorphism group of $\mathbb{Z}D_n$ stabilizing $\mathscr{C}$. The admissible automorphism groups Π of $\mathbb{Z}D_n$ were described in [4], 4.2: if Π is non-trivial, it is generated by $\tau^r\psi$ for some $r \geq 1$, where ψ is an automorphism of $\mathbb{Z}D_n$ with a fixed point. In [2], 1, we gave a proof for part (b) of the theorem in case Π is generated by $\tau^r\psi$ with $r \geq 2n-3$. We now describe the configurations $\mathscr{C}$ of $\mathbb{Z}D_n$ which admit an automorphism $\tau^r\psi$ with $1 \leq r < 2n-3$. Representatives of the two isomorphism classes of configurations of $\mathbb{Z}D_4$ are displayed in [2], 7.6, and they clearly do not admit such an automorphism. Let ϕ be the automorphism of $\mathbb{Z}D_n$ which exchanges $(p, n-1)$ and (p, n) for each p and fixes all other vertices, where we use the coordinates introduced in [5], 1.3 for the vertices of $\mathbb{Z}D_n$. The set of vertices (p, q) with $q \geq n-1$ of a ϕ-stable configuration $\mathscr{C}$ consists of the $\tau^{(2n-3)\mathbb{Z}}$-orbits of $(i, n-1)$ and (i, n) for some integer i

([2], 1.6 or [6], 4), and thus $2n-3$ divides r for any automorphism $\tau^r\psi$ stabilizing $\mathscr{C}$. Let $\mathscr{C}$ be a ϕ-unstable configuration of $\mathbb{Z}D_n$ for $n \geq 5$, and assume $\tau^r\psi$ stabilizes $\mathscr{C}$, where $1 \leq r < 2n-3$. The set of vertices (p,q) in $\mathscr{C}$ with $q \geq n-1$ consists of three $\tau^{(2n-3)\mathbb{Z}}$-orbits ([2], 1.6 or [6], 4). Therefore, $2n-3$ and hence n must be divisible by 3, say $n = 3m$, and either $r = 2m-1$ or $r = 2(2m-1)$. Since τ^{2n-3} stabilizes $\mathscr{C}$, ψ^3 does as well, and thus ψ is the identity. To summarize, we have to prove the theorem for basic algebras Λ with Auslander–Reiten quiver $\Gamma_\Lambda = (\mathbb{Z}D_{3m})_{\mathscr{C}}/\Pi$, where $\mathscr{C}$ is a $\tau^{(2m-1)\mathbb{Z}}$-stable configuration of $\mathbb{Z}D_{3m}$ and either $\Pi = \tau^{(2m-1)\mathbb{Z}}$ or $\Pi = \tau^{2(2m-1)\mathbb{Z}}$.

Let Λ be such an algebra, and let $\pi:(\mathbb{Z}D_{3m})_{\mathscr{C}} \to \Gamma_\Lambda$ be the canonical map. In case $\Pi = \tau^{2(2m-1)\mathbb{Z}}$, we prove the theorem by constructing a Π-invariant well-behaved functor $F:k((\mathbb{Z}D_{3m})_{\mathscr{C}}) \to \operatorname{ind}\Lambda$; i.e., a k-linear functor F with $Fx = \pi x$ for every vertex x of $(\mathbb{Z}D_{3m})_{\mathscr{C}}$, such that $F\bar{\alpha}:\pi x \to \pi y$ is an irreducible morphism in $\operatorname{ind}\Lambda$ for the canonical image $\bar{\alpha}$ in $k((\mathbb{Z}D_{3m})_{\mathscr{C}})$ of every arrow $\alpha: x \to y$ in $(\mathbb{Z}D_{3m})_{\mathscr{C}}$, and such that $F(\overline{g\alpha}) = F\bar{\alpha}$ for each g in Π ([5], 2.5). Such a functor F induces a well-behaved functor

$$H: k(\Gamma_\Lambda) \to \operatorname{ind}\Lambda,$$

which is an isomorphism ([5], 2.5). The construction of F goes along the lines of the corresponding construction in the case A_n ([5], 4). In particular, we need some information about morphisms in $k((\mathbb{Z}D_{3m})_{\mathscr{C}})$, which we collect in chapter 2. In fact, we provide a k-basis for $k((\mathbb{Z}D_n)_{\mathscr{C}})(x,y)$ for any two vertices x and y, where $\mathscr{C}$ is a ϕ-unstable configuration of $\mathbb{Z}D_n$, for $n \geq 5$.

In the remaining case $\Pi = \tau^{(2m-1)\mathbb{Z}}$, we define an ideal J in the path-category $k\Delta$, where $\Delta = (\mathbb{Z}D_{3m})_{\mathscr{C}}/\tau^{(2m-1)\mathbb{Z}}$, and we show that $\operatorname{ind}\Lambda$ is isomorphic either to the mesh-category $k(\Delta)$ or to $k\Delta/J$, for every algebra Λ with Auslander–Reiten quiver Δ. In case $\operatorname{char} k \neq 2$, we construct an isomorphism to $k(\Delta)$, which completes the proof of part (b) of the theorem. As for part (a), it suffices to show that $k\Delta/J$ is isomorphic to $\operatorname{ind}\Lambda'$ for some Λ' and that $k\Delta/J$ and $k(\Delta)$ are not isomorphic if $\operatorname{char} k = 2$. It is possible to check the second fact directly by showing that some huge system of linear equations has no solution. However, we will take a different approach, describing Λ' and the standard algebra Λ with Auslander–Reiten quiver Δ by quivers and relations (see also [7]) and proving that Λ and Λ' are not isomorphic. Moreover, we will show that $\operatorname{ind}\Lambda'$ has only even-fold coverings. More precisely, the map $(\mathbb{Z}D_{3m})_{\mathscr{C}}/\tau^{2(2m-1)\mathbb{Z}} \to \Delta$, which is a covering of translation-quivers for all s, gives rise to a covering functor $k((\mathbb{Z}D_{3m})_{\mathscr{C}})/\tau^{s(2m-1)\mathbb{Z}} \to \operatorname{ind}\Lambda'$ if and only if s is even.

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2. Morphisms in $k((\mathbb{Z}D_n)_{\mathscr{C}})$

Let $\mathscr{C}$ be a ϕ-unstable configuration of $\mathbb{Z}D_n$, for $n \geq 5$. By Γ we denote the translation-quiver $(\mathbb{Z}D_n)_{\mathscr{C}}$. Our aim is to construct a k-basis for $k(\Gamma)(x, y)$ for any two objects x and y of the mesh-category $k(\Gamma)$.

2.1 A vertex (p, q) of $\mathbb{Z}D_n$ or $(p, q)^*$ of Γ with $(p, q) \in \mathscr{C}$ is called *low* if $q \leq n - 2$ and *high* otherwise. For any two vertices x and y of $\mathbb{Z}D_n$, we let $\delta(x, y)$ be the maximal number of high projective vertices on any path in Γ from x or $\phi(x)$ to y or $\phi(y)$. Notice that $\delta(x, z) = \delta(x, y) + \delta(y, z)$, provided there are any paths in Γ from x to y and from y to z, and also that $\delta((p, q),\ (p', q')) = \delta((p, n-1),\ (p' + \min(q', n-1) + 1 - n, n-1))$. Define a high vertex (p, q) of $\mathbb{Z}D_n$ to be *$\mathscr{C}$-congruent* if the high vertex (i, j) in $\mathscr{C}$ with minimal $i \geq p$ satisfies $i + j \equiv p + q$ modulo 2, and call (p, q) *$\mathscr{C}$-incongruent* otherwise.

Let h_p, h'_p, and l_p be the three paths from $(p, n-2)$ to $(p+1, n-2)$ in $\mathbb{Z}D_n$, where h_p and h'_p contain the $\mathscr{C}$-congruent and $\mathscr{C}$-incongruent high vertex with first coordinate p, respectively, and l_p goes through $(p+1, n-3)$, for any integer p. We call h_p and h'_p the *$\mathscr{C}$-congruent* and *$\mathscr{C}$-incongruent crenel path* starting at $(p, n-2)$. Define a path w in Γ to be *stable* if all vertices in w lie in $\mathbb{Z}D_n$. Call w *low* if it is stable and contains no crenel path, and *$\mathscr{C}$-congruent* if it is stable and contains no $\mathscr{C}$-incongruent crenel path. Notice that a low path may start or stop in a high vertex and a $\mathscr{C}$-congruent path in a high $\mathscr{C}$-incongruent vertex. We say that a path f is *free* (with respect to $\mathscr{C}$) if f is low and if no low vertex (p, q) of f satisfies $2p + q = 2i + j + 1$ and $q < j$ for any low projective vertex $(i, j)^*$ of Γ. Note that $2p + \min(q, n-1)$ is constant on "vertical lines" of $\mathbb{Z}D_n$. Fig. 1 shows a low path which is not free.

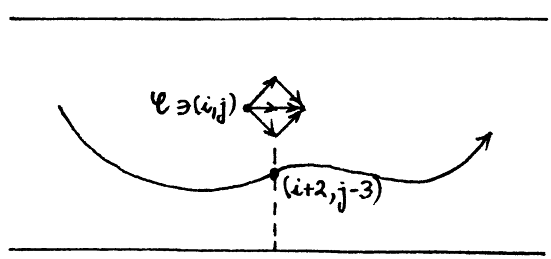


Fig. 1

DEFINITION: A path $w: x \to y$ in Γ is *$\mathscr{C}$-forbidden* if w is $\mathscr{C}$-congruent and satisfies at least one of the following conditions:

(i) w contains a free subpath $f: x' \to y'$, where x' and y' are high, one $\mathscr{C}$-congruent and one $\mathscr{C}$-incongruent, and $\delta(x', y') = 0$.

(ii) w contains a proper free subpath $f: x' \to y'$, where $x' \neq y'$ are high $\mathscr{C}$-congruent and $\delta(x', y') = 0$.

(iii) w is free, x and y are $\mathscr{C}$-incongruent, and $\delta(x, y) = 1$.

(iv) w contains a proper free subpath $f: x' \to y'$, where x' and y' are high, one $\mathscr{C}$-congruent and one $\mathscr{C}$-incongruent, and $\delta(x', y') = 1$.

(v) w contains a subpath $h_{p'} f h_p$, where f is free and

$$\delta((p, n-2), (p'+1, n-2)) = 1.$$

A subpath v of w is a proper subpath of $v \neq w$.

We call w *$\mathscr{C}$-admissible* if it is $\mathscr{C}$-congruent and not $\mathscr{C}$-forbidden. Clearly, any subpath of a $\mathscr{C}$-admissible path is again $\mathscr{C}$-admissible.

LEMMA: (a) *If $w_2 h_p w_1: x \to y$ is $\mathscr{C}$-admissible, then $w_2 l_p w_1$ is, too.*

(b) *If $f h_p w$ is $\mathscr{C}$-admissible for some free path $f: (p+1, n-2) \to y$, then $\alpha f l_p w$ is $\mathscr{C}$-admissible for any arrow $\alpha: y \to z$ for which $\alpha f l_p w$ is $\mathscr{C}$-congruent.*

PROOF: (a) Let (p, q) be the high $\mathscr{C}$-congruent vertex of $\mathbb{Z}D_n$ with first coordinate p. Inspection of the five possible cases shows that, if $w_2 l_p w_1$ is $\mathscr{C}$-forbidden, then either the subpath from x to (p, q) or the one from (p, q) to y of $w_2 h_p w_1$ is $\mathscr{C}$-forbidden as well.

(b) Assume $v = \alpha f l_p w$ is $\mathscr{C}$-forbidden. Since $f l_p w$ is $\mathscr{C}$-admissible, any $\mathscr{C}$-forbidden subpath of v contains $\alpha f l_p$, and hence we may assume all proper subpaths of v to be $\mathscr{C}$-admissible. Again we look at all possibilities separately, and it turns out that, whenever v is $\mathscr{C}$-forbidden, $h_p w$ is $\mathscr{C}$-forbidden, too. We treat the first case as an example; i.e., we let $v = \alpha f l_p f': x \to z$, where f' is free, x and z are high, one $\mathscr{C}$-congruent and one $\mathscr{C}$-incongruent, and $\delta(x, z) = 0$. Then $h_p f'$ is $\mathscr{C}$-forbidden of type (ii) if x is $\mathscr{C}$-congruent and of type (i) if x is $\mathscr{C}$-incongruent.

2.2 DEFINITION: Two paths w and w' are *$\mathscr{C}$-neighbors* if $w = w_2 v w_1$ and $w' = w_2 v' w_1$, where the set $\{v, v'\}$ consists either of the two paths $\beta\alpha$ and $\delta\gamma$ from (p, q) to $(p+1, q)$ for some $(p, q) \notin \mathscr{C}$ with $1 < q < n-2$ or of the two paths $h_{p+1} l_p$ and $l_{p+1} h_p$ for some integer p for which $(p, n-1) \notin \mathscr{C}$ and $(p, n) \notin \mathscr{C}$ (see Fig. 2). Call w and w' *$\mathscr{C}$-homotopic* if they are linked by a sequence $w = w_0, w_1, \ldots, w_r = w'$ of successive $\mathscr{C}$-neighbors.

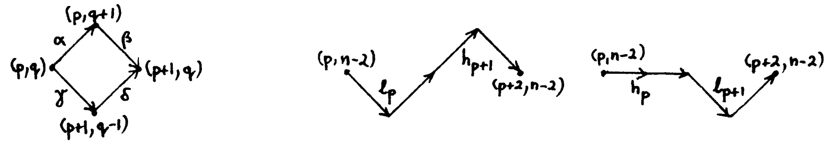


Fig. 2

Note that a $\mathscr{C}$-neighbor of a $\mathscr{C}$-admissible path is $\mathscr{C}$-admissible. We call a $\mathscr{C}$-admissible path w *$\mathscr{C}$-marginal* if w is $\mathscr{C}$-homotopic to some w' containing $(p,1) \to (p,2) \to (p+1,1)$ for a p such that $(p,1) \notin \mathscr{C}$. Call w *$\mathscr{C}$-essential* if it is $\mathscr{C}$-admissible, but not $\mathscr{C}$-marginal. Compare [5], 4.2.

We say that the low projective vertex $(i,j)^*$ lies between the low paths w and w' from x to y if w contains a vertex (p,q) and w' a vertex (p',q') with $2p+q = 2i+j+1 = 2p'+q'$ and either $q<j<q'$ or $q'<j<q$ (compare [5], 5.5).

LEMMA: (a) *Two low paths w and w' are $\mathscr{C}$-homotopic if and only if no low projective vertex lies between w and w'.*

(b) *A low path w is $\mathscr{C}$-homotopic to some free path if and only if w is free.*

PROOF: For (a), we refer to [5], 5.5, and (b) follows from (a) and the definition of free paths.

2.3 With any arrow α of Γ, we associate its *sign* $s(\alpha)$: we set $s(\alpha) = 1$, unless α is a stable arrow of the form $(p,q) \to (p,q+1)$ with $q < n-2$, in which case we set $s(\alpha) = (-1)^{n-q}$. For a path $w = \alpha_r \ldots \alpha_1$, we let $s(w) = s(\alpha_r) \ldots s(\alpha_1)$. We obtain a functor from the path category of Γ onto the mesh-category $k(\Gamma)$ by sending any path w to $\tilde{w} = s(w)\bar{w}$, where $\bar{w}$ denotes the canonical image of w in $k(\Gamma)$. Its kernel I_s is the ideal generated by the elements

$$\theta_z = \sum s(\alpha(\sigma\alpha))\alpha(\sigma\alpha),$$

where z is a stable vertex of Γ, the sum is taken over all arrows $\alpha: z' \to z$, and $\sigma\alpha$ is the arrow $\tau z \to z'$. We call I_s the ideal of *modified mesh-relations.*

LEMMA: *If $f:(p,n-2) \to (p',n-2)$ is free, then $f - w$ lies in I_s, where $w = l_{p'-1} \ldots l_p$.*

Proof: Since f is free, w must be free, too, and hence w and f are $\mathscr{C}$-homotopic by Lemma 2.2. Clearly, differences of low $\mathscr{C}$-neighbors, and hence of low $\mathscr{C}$-homotopic paths, lie in I_s.

2.4 **Proposition**: *For any two stable vertices x and y of Γ, we have*

$$k(\Gamma)(x, y) = \oplus k\tilde{w},$$

where w runs through a set of representatives of the $\mathscr{C}$-homotopy classes of $\mathscr{C}$-essential paths from x to y.

Remark: This proposition yields a basis for $k(\Gamma)(x, y)$ in case x or y or both are projective, too. In fact, if e.g. $y = (p, q)^*$ for some $(p, q) \in \mathscr{C}$ and ι is the arrow $(p, q) \to (p, q)^*$, composition with $\tilde{\iota}$ induces a bijection

$$k(\Gamma)(x, (p, q)) \to k(\Gamma)(x, (p, q)^*)$$

for any $x \neq (p, q)^*$ ([1], 2.6).

Proof: Let W be the vector space freely generated by all paths from x to y in Γ. Let $C \subset S \subset W$ be the subspaces spanned by the $\mathscr{C}$-congruent and the stable paths, respectively, and let A_i be the subspace spanned by the $\mathscr{C}$-congruent paths $\alpha_r \ldots \alpha_1$ for which $\alpha_i \ldots \alpha_1$ is $\mathscr{C}$-admissible. If r is the common length of all paths in W, we have

$$C = A_1 \supset A_2 \supset \ldots \supset A_r = A,$$

where A is spanned by the $\mathscr{C}$-admissible paths. We will define a string of projections

$$W \xrightarrow{\pi_0} S \xrightarrow{\pi_1} C = A_1 \xrightarrow{\pi_2} A_2 \to \ldots \to A_{r-1} \xrightarrow{\pi_r} A_r$$

such that the kernel of each π_i lies in $I_s(x, y)$. In addition, we will show that the image of $I_s(x, y)$ under $\pi = \pi_r \ldots \pi_0$ is the subspace of A spanned by the $\mathscr{C}$-marginal paths and the differences of $\mathscr{C}$-neighbors. This will imply our proposition.

In order to define $\pi_0: W \to S$, we notice that any path w in W can be written as

$$w = w_m \kappa_m \iota_m w_{m-1} \ldots w_1 \kappa_1 \iota_1 w_0,$$

where w_i is stable and ι_i and κ_i are arrows with projective head and tail, respectively, for any i. We set

$$\pi_0 w = (-1)^m w_m(\sum s(\alpha_m(\sigma\alpha_m))\alpha_m(\sigma\alpha_m))w_{m-1}\dots w_1(\sum s(\alpha_1(\sigma\alpha_1))\alpha_1(\sigma\alpha_1))w_0,$$

where for each i the α_i range over all stable arrows whose head is the head of κ_i. By induction on m, the vector $w - \pi_0 w$ lies in I_s, and the kernel of π_0 is spanned by such vectors.

Let w be a stable path and write

$$w = w_m h'_{p_m} w_{m-1}\dots w_1 h'_{p_0} w_0,$$

where w_i is $\mathscr{C}$-congruent for any i. Setting

$$\pi_1 w = w_m(l_{p_m} - h_{p_m})w_{m-1}\dots w_1(l_{p_1} - h_{p_1})w_0,$$

we obtain a vector in C. By definition, $s(h_p) = s(h'_p) = -s(l_p) = 1$ for any p, so that $h_p + h'_p - l_p$ lies in I_s, provided that $(p, n-2) \notin \mathscr{C}$. But we know from [6], 6 that the second coordinate of any low point of a ϕ-unstable configuration $\mathscr{C}$ is strictly less than $n-2$. As before, we conclude that the kernel of π_1 lies in I_s.

Let us define $\pi_i : A_{i-1} \to A_i$, for $i = 2, \dots, r$. Let $w = \alpha_r \dots \alpha_1$ be a path in A_{i-1}. If $w \in A_i$, we set $\pi_i w = w$. Otherwise, the path $v = \alpha_i \dots \alpha_1 : x \to z$ is $\mathscr{C}$-forbidden, whereas $\alpha_{i-1}\dots\alpha_1$ is not. Thus v contains a unique $\mathscr{C}$-forbidden subpath of minimal length, which includes α_i. In each of the possible cases listed in 2.1, we define a linear combination ψv of $\mathscr{C}$-admissible paths from x to z, and we show that $v - \psi v$ lies in I_s. We set $\pi_i w = \alpha_r \dots \alpha_{i+1}(\psi v)$.

(i) Assume v contains a free subpath $f : x' \to z$, where $x' = (p, q)$ and $z = (p', q')$ are high, one $\mathscr{C}$-congruent and one $\mathscr{C}$-incongruent, with $\delta(x', z) = 0$. Set $\psi v = 0$. In order to see that v lies in I_s, it suffices by Lemma 2.3 to show that $\beta l_{p'-1}\dots l_{p+1}\alpha$ does, where $\alpha : (p, q) \to (p+1, n-2)$ and $\beta : (p', n-2) \to (p', q')$ are arrows. Assume first $p' = p+1$. The condition $\delta((p, q), (p+1, q')) = 0$ implies that neither $(p, n-1)$ nor (p, n) belongs to $\mathscr{C}$. Since one of the vertices (p, q), $(p+1, q')$ is $\mathscr{C}$-congruent and one $\mathscr{C}$-incongruent, we see that $p + q \not\equiv p + 1 + q'$ modulo 2, so that $q' = q$. Clearly, the path $\beta\alpha : (p, q) \to (p+1, n-2) \to (p+1, q)$ lies in I_s. In case $p' = p + t + 1$ for some $t > 0$, we write

$$\beta l_{p+t}\dots l_{p+1}\alpha = \beta(l_{p+t} - h_{p+t} - h'_{p+t})l_{p+t-1}\dots l_{p+1}\alpha + \\ + \beta h_{p+t} l_{p+t-1}\dots l_{p+1}\alpha + \beta h'_{p+t} l_{p+t-1}\dots l_{p+1}\alpha.$$

The first summand lies in I_s by definition, the second and third by induction on t.

(ii) If v contains a proper free subpath from x' to y', where $x' \neq y'$ are high $\mathscr{C}$-congruent and $\delta(x', y') = 0$, two cases are possible (see Fig. 3). In case $x = x'$, $y' = (p, q)$, $z = (p+1, n-2)$, and $v = h_p f$ for some free path f, we set $\psi v = l_p f$, which is $\mathscr{C}$-admissible. By (i), the path $h'_p f$ lies in I_s, so that

$$v - \psi v = (h_p + h'_p - l_p)f - h'_p f$$

does as well. In the second case, we have $z = y' = (p', q')$, $x' = (p, q)$, and $v = \beta f h_p v_1$ for some free path $f:(p+1, n-2) \to (p', n-2)$. We set $\psi v = \beta f l_p v_1$, which is $\mathscr{C}$-admissible by Lemma 2.1(b). As in the first case, $v - \psi v$ lies in I_s.

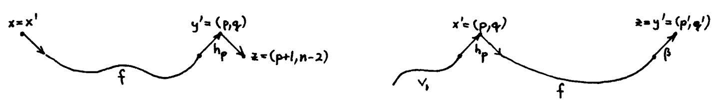


Fig. 3

(iii) In case v is free, x and z are high $\mathscr{C}$-incongruent and $\delta(x, z) = 1$, we must have $v = w$, and we set $\pi_r w = 0$. In order to see that w lies in I_s, it suffices to prove that $u = \beta l_{p'-1} \dots l_{p+1} \alpha : (p, q) \to (p', q')$ does, provided that (p, q) and (p', q') are high $\mathscr{C}$-incongruent and there is exactly one high point $(i, j) \in \mathscr{C}$ with $p \leq i < p'$. In case $p = i = p' - 1$, we have $(p, q) = (i, q) \notin \mathscr{C}$ and $q' = q$, since the high point (i', j') in $\mathscr{C}$ with minimal $i' \geq p' = i + 1$ satisfies $p' + q' \not\equiv i' + j' \not\equiv i + j \not\equiv i + q$ modulo 2. Indeed, consecutive high points (i, j) and (i', j') of a ϕ-unstable configuration $\mathscr{C}$ satisfy $i + j \not\equiv i' + j'$ modulo 2 ([6], 4). Clearly $\beta\alpha : (p, q) \to (p+1, n-2) \to (p+1, q)$ lies in I_s. Let $p' = p + t + 1$ for some $t > 0$, and assume $i + 1 < p'$. Then

$$u = \beta(l_{p+t} - h_{p+t} - h'_{p+t}) l_{p+t-1} \dots l_{p+1} \alpha + \beta h_{p+t} l_{p+t-1} \dots l_{p+1} \alpha + \beta h'_{p+t} l_{p+t-1} \dots l_{p+1} \alpha$$

lies in I_s, by induction on t and since βh_{p+t} does by (i). In case $p' = i + 1$, we obtain

$$u = \beta l_{p+t} \dots l_{p+2}(l_{p+1} - h_{p+1} - h'_{p+1})\alpha + \beta l_{p+t} \dots l_{p+2} h_{p+1} \alpha + \beta l_{p+t} \dots l_{p+2} h'_{p+1} \alpha.$$

(iv) Assume v contains a proper free subpath from x' to y', where x' and y' are high, one $\mathscr{C}$-congruent and one $\mathscr{C}$-incongruent, and $\delta(x', y')$

$= 1$. In case $x = x'$ is $\mathscr{C}$-incongruent, $y' = (p, q)$, $z = (p + 1, n - 2)$, and $v = h_p f$ for some free path f, we set $\psi v = l_p f$, and in case $z = y' = (p', q')$ is $\mathscr{C}$-incongruent, $x' = (p, q)$, and $v = w = \beta f h_p v_1$ for some free path $f:(p + 1, n - 2) \to (p', n - 2)$, we set $\psi v = \beta f l_p v_1$ (Fig. 3). In both cases, ψv is $\mathscr{C}$-admissible by Lemma 2.1, and using (iii) it is easy to check that $v - \psi v$ lies in I_s.

(v) In case $v = h_{p'} f h_p v_1$, where f is free and $\delta((p, n - 2), (p' + 1, n - 2)) = 1$, we set $\psi v = h_{p'} f l_p v_1 + l_{p'} f h_p v_1 - l_{p'} f l_p v_1$. The first one of these paths is $\mathscr{C}$-admissible by Lemma 2.1(b), the second one because $f h_p v_1$ is, and the third one by Lemma 2.1(a). Moreover, we have

$$v - \psi v = h'_{p'} f h'_p v_1,$$

which belongs to I_s by (iii).

It remains to be seen that $\pi I_s(x, y)$ is the subspace of A spanned by the $\mathscr{C}$-marginal paths and the differences of $\mathscr{C}$-neighbors. Clearly, $\mathscr{C}$-marginal paths as well as differences of $\mathscr{C}$-neighbors lie in I_s, since

$$l_{p+1} h_p - h_{p+1} l_p = h_{p+1}(h_p + h'_p - l_p) \\ - h_{p+1} h'_p - (h_{p+1} + h'_{p+1} - l_{p+1}) h_p + h'_{p+1} h_p$$

does, whenever $(p, n - 1) \notin \mathscr{C}$ and $(p, n) \notin \mathscr{C}$.

As $I_s(x, y)$ is spanned by the vectors

$$\mu = w_2 \sum s(\alpha(\sigma\alpha))\alpha(\sigma\alpha) w_1,$$

where w_1 and w_2 are paths from x to τz and from z to y for some stable z, respectively, and where the sum is taken over all arrows α with head z, it suffices to write $\pi\mu$ as a linear combination of $\mathscr{C}$-marginal paths and differences of $\mathscr{C}$-neighbors. We may assume that τz does not lie in $\mathscr{C}$, since otherwise $\pi_0 \mu = 0$, and that μ lies in S. Similarly, we have $\pi_1 \mu = 0$ if the second coordinate of z is $n - 2$. The proof in case z is high is straightforward, the main problems being the large number of possible cases and the bookkeeping. In most cases, $\pi\mu$ turns out to be zero. As an example, we treat one of the harder cases, and we skip the rest.

Assume $z = (p + 1, q) \neq y$ is high $\mathscr{C}$-incongruent and $\tau z = (p, q) \neq x$ is $\mathscr{C}$-congruent. Then μ has the form

$$\mu = v_2(\sigma^{-1}\alpha)\alpha(\sigma\alpha)(\sigma^2\alpha) v_1 = v_2 h'_{p+1} h_p v_1,$$

and we may assume that

$$\pi_1 \mu = v_2(l_{p+1} - h_{p+1}) h_p v_1.$$

Let i be the length of v_1. Then $\pi_{i+1}\ldots\pi_1\mu$ is either zero or a linear combination of vectors of the form

$$v = v_2(l_{p+1} - h_{p+1})h_p v_3.$$

Let us assume that $v_3 = fh_{p'}v_4$, where $f:(p'+1, n-2) \to (p, n-2)$ is free and $\delta((p', n-2), (p+1, n-2)) = 1$; i.e., we suppose $h_p v_3$ to be $\mathscr{C}$-forbidden of type v). We obtain

$$v = v_2(l_{p+1} - h_{p+1})h_p f h_{p'} v_4,$$
$$v_1 = \pi_{i+2} v = v_2(l_{p+1} - h_{p+1})(h_p f l_{p'} + l_p f h_{p'} - l_p f l_{p'})v_4.$$

By our assumptions, neither $(p, n-1)$ nor (p, n) lies in $\mathscr{C}$, so that $\delta((p, n-1), (p+1, n-1)) = 0$ and $\delta((p', n-1), (p+1, n-1)) = 1$. Hence $v_2 h_{p+1} h_p f l_{p'} v_4$ is the only path occurring in v_1 which does not lie in A_{i+3}. We obtain

$$v_2 = \pi_{i+3} v_1 = v_2(l_{p+1}h_p - h_{p+1}l_p)fl_{p'}v_4 + v_2(l_{p+1} - h_{p+1})l_p f(h_{p'} - l_{p'})v_4,$$
$$\rho = \pi_{i+4} v_2 = v_2(l_{p+1}h_p - h_{p+1}l_p)fl_{p'}v_4.$$

Suppose $\rho = v_2(l_{p+1}h_p - h_{p+1}l_p)v_5$ belongs to A_j, but not to A_{j+1} for some j with $i+4 \leq j < r$, and let $v_2 = v_7 v_6$, where the length of v_6 is $j - i - 3$. In case v_6 itself is $\mathscr{C}$-forbidden, we clearly have

$$\pi_{j+1}\rho = v_7 v_6'(l_{p+1}h_p - h_{p+1}l_p)v_5 \text{ or } \pi_{j+1}\rho = 0.$$

Otherwise,

$$v_6 l_{p+1} h_p \text{ and } v_6 h_{p+1} l_p$$

are $\mathscr{C}$-forbidden of the same type, since $\delta((p, n-1), (p+1, n-1)) = 0$. Unless they are $\mathscr{C}$-forbidden of type (v), we have $\pi_{j+1}\rho = 0$, since π_{j+1} either annihilates both summands separately, or

$$\pi_{j+1}(v_7 v_6 l_{p+1} h_p v_5) = v_7 v_6 l_{p+1} l_p v_5 = \pi_{j+1}(v_7 v_6 h_{p+1} l_p v_5).$$

In the remaining case, there is a free path $f:(p+2, n-2) \to (p', n-2)$, where $\delta((p+1, n-2), (p'+1, n-2)) = 1$, such that $v_6 = h_{p'}f$. Then

$$\pi_{j+1}\rho = v_7(h_{p'}fl_{p+1}l_p + l_{p'}fl_{p+1}h_p - l_{p'}fl_{p+1}l_p - h_{p'}fl_{p+1}l_p - l_{p'}fh_{p+1}l_p + l_{p'}fl_{p+1}l_p)v_5 = v_7 l_{p'} f(l_{p+1}h_p - h_{p+1}l_p)v_5,$$

so that by induction we may assume ρ lies in A, and hence it is the difference of two $\mathscr{C}$-neighbors.

Finally, if $\tau z = (p, q)$ does not lie in $\mathscr{C}$ and $q \leq n-3$, $\pi\mu$ is a linear combination of vectors of the form

$$v_2 \sum s(\alpha(\sigma\alpha))\alpha(\sigma\alpha)v_1,$$

each of which is either the difference of two $\mathscr{C}$-neighbors or $\mathscr{C}$-marginal.

2.5 In the remainder of this chapter, we derive the auxiliary results needed in the proof of the theorem. From now on, we assume that $\mathscr{C}$ contains the vertex $(0, n-1)$. This condition can always be fulfilled by replacing $\mathscr{C}$ by an isomorphic configuration. We recall the following description of $\mathscr{C}$ from [6], 6. The set of high vertices of $\mathscr{C}$ consists of the $\tau^{(2n-3)\mathbb{Z}}$-orbits of

$$(0, n-1),\ \phi^{n_1+n_3}(n_1+n_3+1, n-1), \text{ and } \phi^{n-1+n_1}(n-1+n_1, n-1)$$

for some natural numbers (including zero) n_1, n_2, and n_3 with $n_1 + n_2 + n_3 = n-3$. There are configurations $\mathscr{D}_1$, $\mathscr{D}_2$, and $\mathscr{D}_3$ of $\mathbb{Z}A_{n_1}$, $\mathbb{Z}A_{n_2}$, and $\mathbb{Z}A_{n_3}$, respectively, such that the set of low vertices of $\mathscr{C}$ is the disjoint union of the sets

$$\tau^{1-n}\psi_{n_1}\mathscr{D}_1,\ \tau^{-(n+n_1+n_3)}\psi_{n_2}\mathscr{D}_2, \text{ and } \tau^{-(2n-2+n_1)}\psi_{n_3}\mathscr{D}_3.$$

For any natural number $m \leq n-2$, the injection

$$\psi_m : (\mathbb{Z}A_m)_0 \to (\mathbb{Z}D_n)_0$$

from the vertex set of $\mathbb{Z}A_m$ to the vertex set of $\mathbb{Z}D_n$ is defined by

$$\psi_m(p, q) = \begin{cases} (p, q) & \text{if } 0 \leq p < p+q \leq m \\ (p+q+n-2-m, m+1-q) & \text{if } p < m < p+q \end{cases}$$

and by requiring that $\psi_m \tau^m = \tau^{2n-3}\psi_m$, where τ denotes the translation of $\mathbb{Z}A_m$ on the left-hand side and $\mathbb{Z}D_n$ on the right-hand side. Notice that, for any $m < n-2$, ψ_m factors through ψ_{m+1}. In fact, we have $\psi_m = \psi_{m+1}\,\omega_m$, where the injection

$$\omega_m : (\mathbb{Z}A_m)_0 \to (\mathbb{Z}A_{m+1})_0$$

is given by

$$\omega_m(p, q) = \begin{cases} (p, q) & \text{if } 0 \leq p < p+q \leq m \\ (p, q+1) & \text{if } p < m < p+q \end{cases}$$

and by the rule $\omega_m \tau^m = \tau^{m+1} \omega_m$ (see Fig. 4).

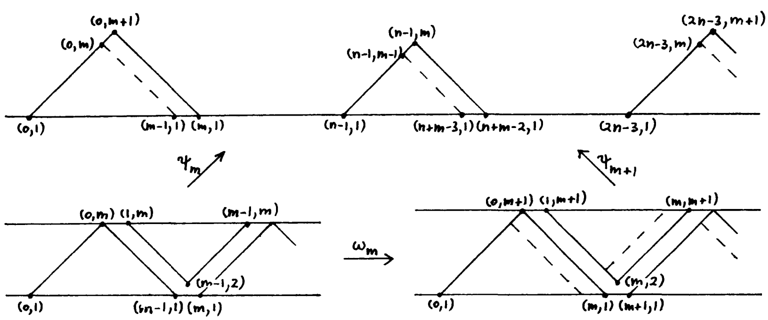


Fig. 4

Lemma: *A set $\mathscr{D}$ in $(\mathbb{Z}A_m)_0$ is a configuration of $\mathbb{Z}A_m$ if and only if*

$$\mathscr{D}^+ = \omega_m \mathscr{D} \cup \tau^{(m+1)\mathbb{Z}}(m, 1)$$

is a configuration of $\mathbb{Z}A_{m+1}$.

Proof: We use the characterization of configurations of $\mathbb{Z}A_m$ and $\mathbb{Z}A_{m+1}$ in terms of rectangles ([5], 2.6). By $R_s(x)$ we denote the rectangle of $\mathbb{Z}A_s$ starting at x, for $s = m, m+1$. The following facts are easy to verify, and they clearly imply the lemma:

$$\omega_m^{-1} R_{m+1}(\omega_m(p, q)) = R_m(p, q) \text{ for any } (p, q) \text{ in } (\mathbb{Z}A_m)_0,$$

$$\omega_m^{-1} R_{m+1}(t(m+1) - 1, q) = R_m(tm, q - 1) \text{ for } q \geq 2 \text{ and } t \in \mathbb{Z},$$

$$\mathrm{R}_{m+1}(\omega_m(\mathrm{p}, \mathrm{q})) \cap \tau^{(m+1)\mathbb{Z}}(m, 1) = \emptyset \text{ for any } (p, q) \text{ in } (\mathbb{Z}A_m)_0.$$

2.6 Set

$$\chi_1 = \tau^{1-n} \psi_{n_1+1} : (\mathbb{Z}A_{n_1+1})_0 \to (\mathbb{Z}D_n)_0,$$

$$\chi_2 = \tau^{-(n+n_1+n_3)} \psi_{n_2+1} : (\mathbb{Z}A_{n_2+1})_0 \to (\mathbb{Z}D_n)_0,$$

$$\chi_3 = \tau^{-(2n-2+n_1)} \psi_{n_3+1} : (\mathbb{Z}A_{n_3+1}) \to (\mathbb{Z}D_n)_0.$$

Fig. 5 shows the images of χ_1, χ_2, and χ_3. In chapter 5, we will show that χ_k can be extended to a k-linear functor

$$\chi_k : k((\mathbb{Z}A_{n_k+1})_{\mathscr{D}_k^+}) \to k((\mathbb{Z}D_n)_{\mathscr{C}})$$

for $k = 1$, 2, and 3. This will enable us to describe the full subcategory of projective objects in $k((\mathbb{Z}D_n)_{\mathscr{C}})$ in terms of the full subcategories of projectives in $k((\mathbb{Z}A_{n_k+1})_{\mathscr{D}_k^+})$.

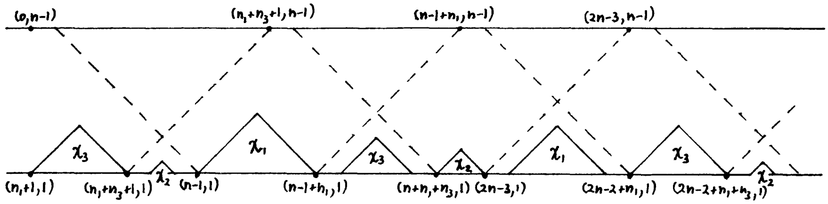


Fig. 5

Lemma: *Any $\mathscr{C}$-essential path in $\Gamma = (\mathbb{Z}D_n)_{\mathscr{C}}$ from $(n-1, 1)$ to $(n + n_1 - q, q)$ and from $(n-1, q')$ to $(n + n_1 - 1, 1)$ is free.*

Remark: The same statement holds for $\mathscr{C}$-essential paths in Γ from $(n_1 + n_3 + n, 1)$ to $(2n - 2 - q, q)$, from $(n_1 + n_3 + n, q')$ to $(2n - 3, 1)$, from $(2n - 2 + n_1, 1)$ to $(2n - 1 + n_1 + n_3 - q, q)$, and from $(2n - 2 + n_1, q')$ to $(2n - 2 + n_1 + n_3, 1)$.

Proof: Clearly, χ_1 extends to an isomorphism from the full subquiver Δ of $(\mathbb{Z}A_{n_1+1})_{\mathscr{D}_1^+}$ given by the vertices x for which there are paths $(0, 1) \to x \to (n_1, 1)$ in $(\mathbb{Z}A_{n_1+1})_{\mathscr{D}_1^+}$ to the full subquiver Δ' of $(\mathbb{Z}D_n)_{\mathscr{C}}$ given by the vertices x' for which there are paths $(n-1, 1) \to x' \to (n - 1 + n_1, 1)$ in $(\mathbb{Z}D_n)_{\mathscr{C}}$. The stable vertices of Δ and Δ' are the (p, q) and $\chi_1(p, q)$ with $0 \leq p < p + q \leq n_1 + 1$, respectively. Notice that χ_1 induces a bijection between $\mathscr{D}_1^+$-homotopy classes of stable paths from x to y in Δ and $\mathscr{C}$-homotopy classes of stable paths from $\chi_1 x$ to $\chi_1 y$ in Δ', under which $\mathscr{D}_1^+$-essential paths correspond to $\mathscr{C}$-essential paths ([5], 4.2).

Since $(-1, 1)$ lies in $\mathscr{D}_1^+$ by construction, any $\mathscr{D}_1^+$-essential path $\tau^{-1}(-1, 1) = (0, 1) \to (n_1 + 1 - q, q)$ is $\mathscr{D}_1^+$-homotopic to a subpath of the "α-path" $(0, 1) \to (0, n_1 + 1) \to (n_1, 1)$ (see [5], 5). Thus any $\mathscr{C}$-essential path $w: (n-1, 1) \to (n + n_1 - q, q)$ is $\mathscr{C}$-homotopic to $(n-1, 1) \to (n-1, n_1 + 1) \to (n + n_1 - q, q)$, which is free, since all low vertices of $\mathscr{C}$ lie in the image of χ_1, χ_2, or χ_3. Since $\mathscr{C}$-neighbors of free paths are free, w is free as well. The proof in the other case is analogous.

2.7 Let $\mathscr{C}$ be a configuration of $\mathbb{Z}D_n$ as in 2.5, and assume $n = 3m$, $n_1 = n_2 = n_3 = m - 1$ (see Fig. 6). We will need the following proposition only in case $\mathscr{C}$ is $\tau^{(2m-1)\mathbb{Z}}$-stable. However, this assumption does not simplify the proof. Set $\Gamma = (\mathbb{Z}D_n)_{\mathscr{C}}$.

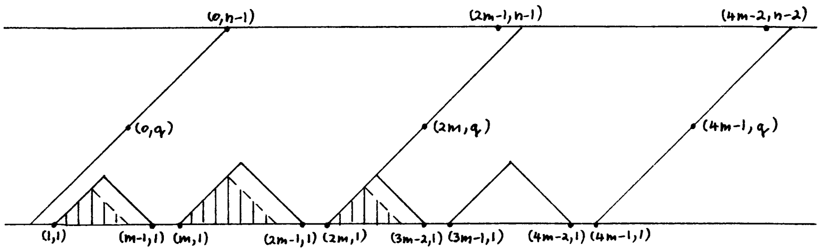


Fig. 6

PROPOSITION: (a) *If* $2 \leq q \leq n-2$, *any* $\mathscr{C}$*-essential path in* Γ *from* $(0,q)$ *to* $(2m,q)$ *or* $(4m-1,q)$ *starting with the arrow* $(0,q) \to (1,q-1)$ *is* $\mathscr{C}$*-homotopic to a path starting with* $(0,q) \to (1,q-1) \to (1,q)$.

(b) *If* $q \geq n-1$, *there is no* $\mathscr{C}$*-essential path from* $(0,q)$ *to* $(4m-1,q)$.

(c) *Any* $\mathscr{C}$*-admissible path from* $(0,n)$ *to* $(2m,n)$ *is* $\mathscr{C}$*-homotopic to* $\beta l_{2m-1} \ldots l_1 \alpha$, *where* α *and* β *are the arrows* $\alpha:(0,n) \to (1,n-2)$ *and* $\beta:(2m,n-2) \to (2m,n)$.

(d) *Any* $\mathscr{C}$*-admissible path from* $(0,n-1)$ *to* $(2m,n-1)$ *is* $\mathscr{C}$*-homotopic to either* $\delta l_{2m-1} \ldots l_1 \gamma$ *or* $\delta l_{2m-1} \ldots l_2 h_1 \gamma$, *where* γ *and* δ *are the arrows* $\gamma:(0,n-1) \to (1,n-2)$ *and* $\delta:(2m,n-2) \to (2m,n-1)$.

PROOF: Notice that by 2.5 the set of high points of $\mathscr{C}$ is the $\tau^{(2m-1)\mathbb{Z}}$-orbit of $(0,n-1)$.

(a) Assume our assertion is wrong for some $\mathscr{C}$-essential path w: $(0,q) \to (x,q)$ starting with $(0,q) \to (1,q-1)$, where $x = 2m$ or $x = 4m - 1$. Then there is a low point $(i,j) \in \mathscr{C}$ with $i + j = q$ and $2 \leq j \leq q$, such that w contains the only path w_1 from $(0,q)$ to $(i+1,j-1)$ (see Fig. 7).

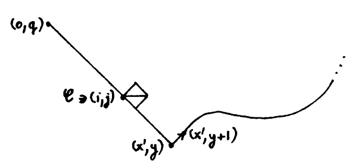


Fig. 7

Indeed, if such an $(i,j) \in \mathscr{C}$ does not exist, the subpath $(1,q-1) \to (x',y) \to (x',y+1)$ is $\mathscr{C}$-homotopic to $(1,q-1) \to (1,q) \to (x',y+1)$, and we are done (2.2). Notice that any low path from $(i+1,j-1)$ to (x,q) is $\mathscr{C}$-homotopic to a path containing $(x-1,1) \to (x-1,2) \to (x,1)$, which is $\mathscr{C}$-marginal, since neither $(2m-1,1)$ nor $(4m-2,1)$ lie in $\mathscr{C}$. Therefore, w has the form $w = w_3 h_p w_2 w_1$, where w_2 is low, but w_3 need not be. Clearly, we have $p \geq i+1$. If $x = 2m$, there is no path in Γ from $(p+1, n-2)$ to $(x, q) = (2m, q)$: since the second coordinate j of $(i,j) \in \mathscr{C}$ is less than m, we have $p+n-1 \geq i+n = q-j+n > q+2m$. This proves (a) in case $x = 2m$.

If $(x, q) = (4m-1, q)$, we distinguish three cases, depending on the position of (i,j) (compare Fig. 6).

(i) $1 \leq i < i+j \leq m-1$: We must have $p \leq m-1$, since otherwise w_2 contains $(m-1, 1) \to (m-1, 2) \to (m, 1)$, up to $\mathscr{C}$-homotopy. A similar argument, using $(4m-2, 1) \notin \mathscr{C}$, shows that w_3 cannot be low. Hence $w_3 = w_5 h_{p'} w_4$ for some low path $w_4 : (p+1, n-2) \to (p', n-2)$, which must not be free, since $0 \leq \delta((p+1, n-2), (p', n-2)) \leq 1$. This implies that $3m \leq p'$. But there is no path in Γ from $(p'+1, n-2)$ to $(4m-1, q)$, since $q = i+j \leq m-1$ forces $p'+n-1 \geq 6m-1 > 4m-1+q$.

(ii) $m \leq i < i+j \leq 2m-1$: That w_2 is $\mathscr{C}$-essential implies $p \leq 2m-1$. Then any low path $(p+1, n-2) \to (p', n-2)$ is free, provided that $p' \leq 4m-1$, and therefore w_3 must be low and free. Up to $\mathscr{C}$-homotopy, we may choose $w_3 = w_4 l_{2m-1} l_{2m-2} \ldots l_{p+1}$, where w_4 is a free path from $(2m, n-2)$ to $(4m-1, q)$. Here we use that $4m-1+q = 4m-1+i+$ $+j > 5m-1$. Then $w_3 h_p$ is $\mathscr{C}$-homotopic to $w_4 h_{2m-1} l_{2m-2} \ldots l_{p+1} l_p$. Hence we can choose $p = 2m-1$, and we can choose w_1 to contain $(2m-1, 1)$, up to $\mathscr{C}$-homotopy. By Lemma 2.6, w_1 is $\mathscr{C}$-homotopic to the path $(0, q) \to (m, q-m) \to (m, m) \to (2m-1, 1)$, which contradicts our assumption.

(iii) $2m \leq i < i+j \leq 3m-2$: We must have $p \leq 3m-2$, since otherwise w_1 is $\mathscr{C}$-marginal. Then w_3 is free, and we may assume $w_3 = w_4 l_{3m-1} \ldots l_{p+1}$, since $4m-1+q > 6m-1$. As before, $w_3 h_p$ is $\mathscr{C}$-homotopic to $w_4 h_{3m-1} l_{3m-2} \ldots l_p$, which is a contradiction.

(b) Assume there is a $\mathscr{C}$-essential path $w : (0, q) \to (4m-1, q)$ for $q \geq n$ -1. If $q = n$, both $(0, n)$ and $(4m-1, n)$ are $\mathscr{C}$-incongruent. For any high $\mathscr{C}$-congruent vertex (p, q') with $1 \leq p \leq 4m-2$, either $\delta((0, q), (p, q')) = 1$ or $\delta((p, q'), (4m-1, q)) = 1$, so that w must be low, which is impossible. In case $q = n-1$, w has the form $w_2 h_p w_1$, where $p \leq 3m-2$ and w_1 is low, and thus free.

(i) $p \leq 2m-1$: We may assume $p = 1$. Then w_2 cannot be low; i.e., $w_2 = w_4 h_{p'} w_3$ for some p' with $2m \leq p' \leq 4m-2$ and some low path w_3, which must not be free. Thus w_3 contains a vertex $(3m-1, y)$ with $y \leq m-1$. Since w_4 is free, we can choose $p' = 4m-2$, and we may assume that w_3 contains $(4m-2, 1)$. By Lemma 2.6, w_3 is free, which is a contradiction.

(ii) $2m \leq p$: Since w_2 is free, we can "push the crenel to the right" and violate the condition $p \leq 3m-2$.

(c) and (d) follow from the definition and Lemma 2.3, since in these cases all low paths are free.

3. Proof of part (b) of the theorem

Let Λ be a basic algebra with Auslander–Reiten quiver $\Gamma_\Lambda = (\mathbb{Z}D_n)_{\mathscr{C}}/\tau^{r\mathbb{Z}}$, where $n = 3m$ for some $m > 1$, $\mathscr{C}$ is stable under $\tau^{(2m-1)\mathbb{Z}}$, and $r = 2m - 1$ or $r = 2(2m - 1)$. We choose $\mathscr{C}$ to contain $(0, n-1)$, and we let $\pi : \Gamma \to \Gamma_\Lambda$ be the canonical map. As explained in the introduction, we have to construct a $\tau^{r\mathbb{Z}}$-invariant well-behaved functor $k(\Gamma) \to \operatorname{ind} \Lambda$, provided that either $\operatorname{char} k \neq 2$ or $r \neq 2m - 1$. It suffices to find a k-linear functor

$$F : k\Gamma \to \operatorname{ind} \Lambda$$

from the path-category $k\Gamma$ of Γ to $\operatorname{ind} \Lambda$ such that $Fx = \pi x$ for all vertices x, $F\alpha \in \operatorname{Hom}_\Lambda(\pi x, \pi y)$ is irreducible for all arrows $\alpha : x \to y$, $F(\tau^r \alpha) = F\alpha$, and $F\theta_z = 0$ for all stable vertices z, where

$$\theta_z = \sum s(\alpha(\sigma\alpha))\alpha(\sigma\alpha)$$

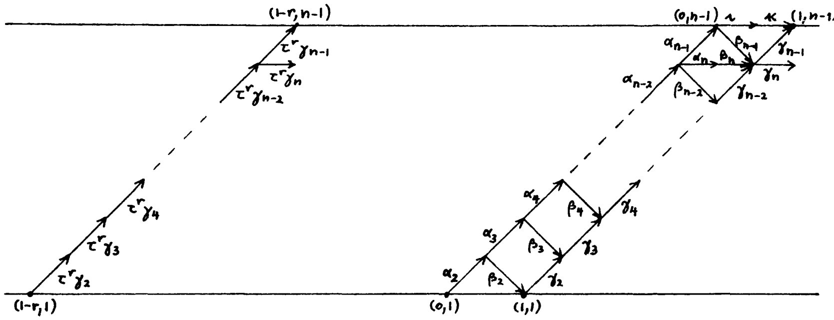
is the modified mesh-relation arising from the mesh of Γ which stops at z. Then sending $\tilde{w}$ to Fw, for any path w in Γ, yields our desired $\tau^{r\mathbb{Z}}$-invariant well-behaved functor.

3.1 In a first step, we construct the irreducible $F\alpha$ so that $F(\tau^r\alpha) = F\alpha$ and so that $F\theta_z = 0$ for all z which do not belong to $\tau^{r\mathbb{Z}}(1, n-1)$ or $\tau^{r\mathbb{Z}}(1, n)$. We make no assumption on $\operatorname{char} k$ or r yet. Start from any well-behaved functor $F_0 : k(\Gamma) \to \operatorname{ind} \Lambda$. Such a functor exists, since $\pi : \Gamma \to \Gamma_\Lambda$ is the universal covering, and F_0 is a covering functor; i.e., for any two vertices x and y of Γ, F_0 induces isomorphisms

$$\bigoplus_{\pi z = \pi y} k(\Gamma)(x, z) \to \operatorname{Hom}_\Lambda(\pi x, \pi y),$$

$$\bigoplus_{\pi z = \pi x} k(\Gamma)(z, y) \to \operatorname{Hom}_\Lambda(\pi x, \pi y)$$

(see [4], 2 and [1], 3.1). Set $F\alpha = F_0\tilde{\alpha}$ for any arrow $\alpha : x \to y$ of Γ for which the stable vertices in $\{x, y\}$ lie in the set $\{(p, q) : 1 - r \leq p \leq 0\}$, and set $F(\tau^r\gamma_q) = F\gamma_q$, for $q = 2, \ldots, n$, $F\beta_2 = F_0\tilde{\beta}_2$ (see Fig. 8).



Fig, 8

By induction on q, we define $F\beta_q$ in such a way that

$$F\beta_q F\alpha_q - F\gamma_{q-1} F\beta_{q-1} = 0$$

for $q = 3, \ldots, n-2$. The construction is analogous to the one used in [5], 1.6 and 4; it is based on Proposition 2.7(a). As an example, we show how to find $F\beta_{n-1}$ and $F\beta_n$ so that

$$F\beta_{n-1} F\alpha_{n-1} + F\beta_n F\alpha_n - F\gamma_{n-2} F\beta_{n-2} = 0.$$

Choose an Auslander–Reiten sequence

$$\pi(0, n-2) \xrightarrow{[F\alpha_{n-1} F\alpha_n F\beta_{n-2}]^{\mathrm{T}}} \pi(0, n-1) \oplus \pi(0, n) \oplus \pi(1, n-3)$$

$$\xrightarrow{[\underline{\beta}\ \underline{\beta}'\ \underline{\gamma}]} \pi(1, n-2)$$

in mod Λ. There exists a $\lambda \in k$ such that

$$\mu = \lambda \underline{\gamma} - F\gamma_{n-2} \in \mathscr{R}^2(\pi(1, n-3), \pi(1, n-2)),$$

where $\mathscr{R}$ denotes the radical of ind Λ. Since F_0 is a covering functor, we can write

$$\mu F\beta_{n-2} = \sum \lambda_w F_0 \tilde{w},$$

where λ_w is a scalar and the w's are $\mathscr{C}$-essential paths in Γ from $(0, n-2)$ to $(sr+1, n-2)$ with $s \geq 1$. Notice that $(sr+1, n-2)$ must be either $(2m, n-2)$ or $(4m-1, n-2)$, since the length of any $\mathscr{C}$-essential path in Γ is at most $2(2n-3)$ ([2], 1.2). Suppose one of the paths w has the form

$w'\beta_{n-2}$. By Proposition 2.7(a), we may assume $w = v\gamma_{n-2}\beta_{n-2} = vl_0$. Since $\tilde{l}_0 = \tilde{h}_0 + \tilde{h}'_0$, we see that we can write

$$\mu F\beta_{n-2} = \mu_1 F_0\tilde{\alpha}_{n-1} + \mu_2 F_0\tilde{\alpha}_n = \mu_1 F\alpha_{n-1} + \mu_2 F\alpha_n$$

for some $\mu_1 \in \mathscr{R}^2(\pi(0, n-1), \pi(1, n-2))$ and $\mu_2 \in \mathscr{R}^2(\pi(0,n), \pi(1, n-2))$. We set

$$F\beta_{n-1} = -\lambda\underline{\beta} - \mu_1 \text{ and } F\beta_n = -\lambda\underline{\beta'} - \mu_2,$$

which are irreducible. By construction,

$$F\theta_{(1,n-2)} = F\beta_{n-1}F\alpha_{n-1} + F\beta_n F\alpha_n - F\gamma_{n-2}F\beta_{n-2} = 0.$$

Finally, we find a irreducible morphism $F\kappa \in \operatorname{Hom}_\Lambda(\pi(0, n-1)^*, \pi(1, n-1))$ such that

$$F\kappa F\iota + F\gamma_{n-1}F\beta_{n-1} \in \mathscr{R}^{2r+2}(\pi(0, n-1), \pi(1, n-1)),$$

and we extend F first to all arrows of Γ by periodicity, requiring that $F(\tau^r\alpha) = F\alpha$, and then to a k-linear functor $F: k\Gamma \to \operatorname{ind}\Lambda$.

3.2 Let $r = 2(2m-1)$. Write

$$F\gamma_{n-1}F\beta_{n-1} + F\kappa F\iota = \sum \lambda_w F_0\tilde{w},$$
$$F\gamma_n F\beta_n = \sum \mu_v F_0\tilde{v},$$

where $\lambda_w, \mu_v \in k$, the $w: (0, n-1) \to (2(2m-1)s + 1, n-1)$ are $\mathscr{C}$-essential with $s \geq 1$, and the $v: (0,n) \to (2(2m-1)t + 1, n)$ are $\mathscr{C}$-essential with $t \geq 0$. There are no such paths for $t = 0$, $t \geq 2$, or $s \geq 2$, since the length of a $\mathscr{C}$-essential path is at most $2(2n-3)$. By Proposition 2.7(b), there is none for $s = 1$, $t = 1$ either, so that $F\theta_{(1,n-1)} = F\theta_{(1,n)} = 0$. *This completes the proof of the theorem in case* $r = 2(2m-1)$.

3.3 From now on, we let $r = 2m - 1$. By Proposition 2.7(b), (c), (d), we obtain

$$F\gamma_{n-1}F\beta_{n-1} + F\kappa F\iota = \lambda' F_0(\tilde{\gamma}'_{n-1}\tilde{l}_{2m-1}\cdots\tilde{l}_1\tilde{\beta}_{n-1}) + \mu' F_0(\tilde{\gamma}'_{n-1}\tilde{l}_{2m-1}\cdots\tilde{l}_2\tilde{h}_1\tilde{\beta}_{n-1})$$

$$F\gamma_n F\beta_n = v' F_0(\tilde{\gamma}'_n\tilde{l}_{2m-1}\cdots\tilde{l}_1\tilde{\beta}_n),$$

where λ', μ', ν' are scalars and $\gamma'_{n-1} = \tau^{-(2m-1)}\gamma_{n-1}$, $\gamma'_n = \tau^{-(2m-1)}\gamma_n$. Since for any arrow α, $F\alpha$ and $F_0\tilde{\alpha}$ differ only by a non-zero scalar modulo $\mathscr{R}^2$, and since

$$\mathscr{R}^{8m-2}(\pi(0, n-1), \pi(1, n-1)) = 0 = \mathscr{R}^{8m-2}(\pi(0, n), \pi(1, n)),$$

we obtain

$$(*)\begin{cases} F\gamma_{n-1}F\beta_{n-1} + F\kappa F\iota = \lambda F(\gamma'_{n-1}l_{2m-1}\dots l_1\beta_{n-1}) \\ \qquad\qquad\qquad\qquad\qquad + \mu F(\gamma'_{n-1}l_{2m-1}\dots l_2h_1\beta_{n-1}) \\ F\gamma_nF\beta_n = \nu F(\gamma'_n l_{2m-1}\dots l_1\beta_n) \end{cases}$$

for some $\lambda, \mu, \nu \in k$.

Let J be the ideal in $k\Gamma_\Lambda$ generated by the images $\pi\theta_z$ under $\pi: k\Gamma \to k\Gamma_\Lambda$ of all modified mesh-relations with $z \notin \tau^{(2m-1)\mathbb{Z}}(1, n-1)$ along with

$$\pi(\gamma_{n-1}\beta_{n-1}) + \pi(\kappa\iota) - \pi(\gamma'_{n-1}l_{2m-1}\dots l_1\beta_{n-1}).$$

Notice that the associated graded category ([1], 5.1) of $k\Gamma_\Lambda/J$ is the mesh-category $k(\Gamma_\Lambda)$. In particular, we have

$$\begin{aligned}\dim_k k\Gamma_\Lambda/J(\pi x, \pi y) &= \dim_k k(\Gamma_\Lambda)(\pi x, \pi y) \\ &= \sum_{\pi z = \pi y} \dim_k k(\Gamma)(x, z) = \dim_k \operatorname{Hom}_\Lambda(\pi x, \pi y),\end{aligned}$$

for any x and y in Γ.

Proposition: *The category* ind Λ *is isomorphic to either* $k(\Gamma_\Lambda)$ *or* $k\Gamma_\Lambda/J$.

Proof: It is enough to show that we can choose $\mu = \nu = 0$ and either $\lambda = 0$ or $\lambda = 1$ in (*). Indeed, then the full k-linear functor $k\Gamma_\Lambda \to$ ind Λ

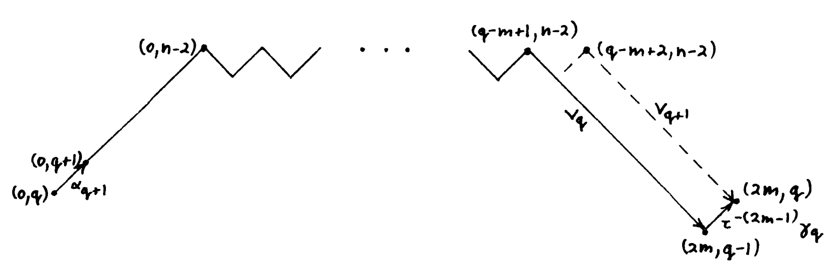


Fig. 9

induced by F factors through either $k(\Gamma_\Lambda)$ or $k\Gamma_\Lambda/J$. By the dimension formulas above, we obtain an isomorphism from $k(\Gamma_\Lambda)$ or $k\Gamma_\Lambda/J$ to ind Λ.

Let us get rid of μ and ν. For any q with $2m+1 \leq q \leq n-2$, we let $v_q: (0,q) \to (2m, q-1)$ be the path composed from the only path $(0,q) \to (0,n-2)$, the path $l_{q-m} \dots l_0 : (0,n-2) \to (q-m+1, n-2)$, and the only path $(q-m+1, n-2) \to (2m, q-1)$ (see Fig. 9).

Set $v = l_{2m-1} \dots l_1 : (1, n-2) \to (2m, n-2)$, and define

$$F'\beta_q = \begin{cases} F\beta_q - \nu F v_q & \text{if } 2m+1 \leq q \leq n-2, \\ F\beta_q & \text{if } 2 \leq q \leq 2m, \end{cases}$$

$$F'\beta_{n-1} = F\beta_{n-1} - \nu F(v\beta_{n-1}),$$

$$F'\beta_n = F\beta_n - \nu F(v\beta_n),$$

$$F'\kappa = F\kappa + \mu F(\gamma'_{n-1} l_{2m-1} \dots l_2(\sigma^{-1}\gamma_{n-1})\kappa),$$

(see Fig. 8).

In order to check that

$$F'\beta_{q+1}F\alpha_{q+1} - F\gamma_q F'\beta_q = 0$$

for $q = 2, \dots, n-3$, we have to show that

$$F(v_{q+1}\alpha_{q+1}) = F(\tau^{-(2m-1)}\gamma_q v_q), \text{ for } q = 2m+1, \dots, n-3,$$

and that

$$F(v_{2m+1}\alpha_{2m+1}) = 0.$$

Since $F\theta_z = 0$ for all low vertices z, the value of F is constant on $\mathscr{C}$-homotopy classes of low paths. Clearly, $v_{q+1}\alpha_{q+1}$ and $\tau^{-(2m-1)}\gamma_q v_q$ are $\mathscr{C}$-homotopic, for $q = 2m+1, \dots, n-3$ (see Fig. 9), and $v_{2m+1}\alpha_{2m+1}$ is $\mathscr{C}$-homotopic to $(0, 2m) \to (2m-1, 1) \to (2m-1, 2) \to (2m, 1) \to (2m, 2m)$, which is $\mathscr{C}$-marginal (Fig. 6). A direct computation yields:

$$F'\beta_{n-1}F\alpha_{n-1} + F'\beta_n F\alpha_n - F\gamma_{n-2}F'\beta_{n-2} = 0,$$

$$F\gamma_n F'\beta_n = 0,$$

$$F\gamma_{n-1}F'\beta_{n-1} + F'\kappa F\iota = (\lambda - \nu)F(\gamma'_{n-1} v\beta_{n-1}),$$

where for the last equation we use $\mathscr{R}^{8m-2}(\pi(0, n-1), \pi(1, n-1)) = 0$ again.

It follows that we may assume $\mu = \nu = 0$ in (*). If $\lambda = 0$, we are done. Otherwise, choose $\lambda' \in k$ with $\lambda'^{2(2m-1)} = \lambda$ and replace $F\alpha$ by $F'\alpha = \lambda' F\alpha$ for all arrows α. Then we still have $F'\theta_z = 0$ for all $z \notin \tau^{(2m-1)\mathbb{Z}}(1, n-1)$. However,

$$F'\gamma_{n-1}F'\beta_{n-1} + F'\kappa F'\iota = F'(\gamma'_{n-1}v\beta_{n-1}).$$

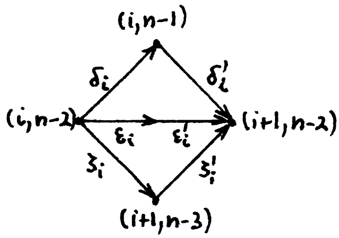
To summarize, we find a $\tau^{(2m-1)\mathbb{Z}}$-invariant k-linear functor F: $k\Gamma \to \operatorname{ind}\Lambda$ such that $Fx = \pi x$ for all x, $F\alpha$ is irreducible for all α, $F\theta_z = 0$ for all $z \notin \tau^{(2m-1)\mathbb{Z}}(1, n-1)$, and either $F\theta_{(1,n-1)} = 0$ or $F\theta_{(1,n-1)} = F(\gamma'_{n-1}v\beta_{n-1})$. This finishes the proof of our proposition.

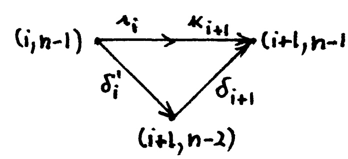
3.4 Assume that char $k \neq 2$. Suppose F does not induce a well-behaved $\tau^{(2m-1)\mathbb{Z}}$-invariant functor $k(\Gamma) \to \operatorname{ind}\Lambda$; i.e., $F\theta_{(1,n-1)} = F(\gamma'_{n-1}v\beta_{n-1})$. Notice that F *vanishes on all vectors in the ideal I_s of modified mesh-relations which are linear combinations of stable paths.* Our next step is to construct a $\tau^{(2m-1)\mathbb{Z}}$-invariant k-linear functor $F_1 : k\Gamma \to \operatorname{ind}\Lambda$ such that $F_1 x = \pi x$ for all x, $F_1\alpha - F\alpha \in \mathscr{R}^{4m-1}$ for all α, and

$$F_1\theta_z \in \mathscr{R}^{8m-2}(\pi\tau z, \pi z)$$

for all stable vertices z. In the following sections, we will modify F_1 further in order to obtain a $\tau^{(2m-1)\mathbb{Z}}$-invariant well-behaved functor.

We name the arrows in the meshes of Γ stopping at $(i+1, n-2)$, for $i \in \mathbb{Z}$, or $(i+1, n-1)$, for $i = s(2m-1)$ and $s \in \mathbb{Z}$, as follows:





Set $v_i = l_{i+2m-2} \ldots l_i$ and $w_i = l_{i+2m-2} \ldots l_{i+1} h_i$ for each $i \in \mathbb{Z}$. For $1 \leq i \leq 2m-1$, we define:

$$F_1\delta_i = \begin{cases} F\delta_i - \tfrac{1}{2}F(\delta_{i+2m-1}v_i) + \tfrac{1}{2}F(\delta_{i+2m-1}w_i) & \text{if } i \text{ is odd,} \\ F\delta_i + \tfrac{1}{2}F(\delta_{i+2m-1}v_i) & \text{if } i \text{ is even,} \end{cases}$$

$$F_1\delta'_i = F\delta'_i + (-1)^i \tfrac{1}{2}F(v_{i+1}\delta'_i),$$

$$F_1\varepsilon_i = \begin{cases} F\varepsilon_i & \text{if } i \text{ is odd,} \\ F\varepsilon_i + \frac{1}{2}(\varepsilon_{i+2m-1}w_i) & \text{if } i \text{ is even,} \end{cases}$$

$$F_1\varepsilon_i' = F\varepsilon_i',$$

$$F_1\zeta_i = F\zeta_i + (-1)^i\tfrac{1}{2}F(\zeta_{i+2m-1}v_i),$$

$$F_1\zeta_i' = F\zeta_i'.$$

We set

$$F_1\kappa_1 = F\kappa_1 + \tfrac{1}{2}F(\delta_{2m}l_{2m-1}\ldots l_2\delta_1'\kappa_1),$$

$$F_1\iota_{2m-1} = F\iota_{2m-1}.$$

We extend F_1 to all arrows δ_i, δ_i', ε_i, ε_i', ζ_i, ζ_i'; $\iota_{s(2m-1)}$, $\kappa_{s(2m-1)+1}$ by $\tau^{(2m-1)\mathbb{Z}}$-periodicity. We have to check that

$$F_1\theta_{(i+1,q)} \in \mathscr{R}^{8m-2}(\pi(i,q), \pi(i+1,q))$$

for all (i,q) with $1 \leq i \leq 2m-1$ and $q \geq n-2$. Notice that we need not take products of "correction terms" in $\mathscr{R}^{4m-1}$ into account.

The case $(i,q) = (2m-1, n-1)$ and all combinations $q = n-2, n-1, n$ and i even or odd for (i,q) have to be treated separately. Observe that, for $1 \leq i \leq 2m-1$, $(i, n-1)$ is $\mathscr{C}$-congruent if and only if i is odd. This implies that, for $1 \leq i \leq 2m-2$,

$$F(h_{i+1}\delta_i') = 0 \text{ and hence } F(w_{i+1}\delta_i') = 0 \text{ if } i \text{ is even,}$$

$$F(h_{i+1}\varepsilon_i') = 0 \text{ and hence } F(w_{i+1}\varepsilon_i') = 0 \text{ if } i \text{ is odd.}$$

If we combine these two equations with the facts that F is $\tau^{(2m-1)\mathbb{Z}}$-invariant, that $F\theta_z = 0$ if $z \notin \tau^{(2m-1)\mathbb{Z}}(1, n-1)$, and that $F\theta_{(1,n-1)} = F(\delta_{2m}v_1\delta_0')$, a straightforward computation shows that $F_1\theta_{(i+1,q)} \in \mathscr{R}^{8m-2}$ for all high vertices (i,q) with $1 \leq i \leq 2m-1$.

Let i be even and $1 \leq i \leq 2m-1$. Then

$$\begin{aligned} F_1\theta_{(i+1,n-2)} &= F_1(\delta_i'\delta_i + \varepsilon_i'\varepsilon_i - \zeta_i'\zeta_i) \\ &\equiv \tfrac{1}{2}F(\delta_{i+2m-1}'\delta_{i+2m-1}v_i + v_{i+1}\delta_i'\delta_i \\ &\quad + \varepsilon_{i+2m-1}'\varepsilon_{i+2m-1}w_i - \zeta_{i+2m-1}'\zeta_{i+2m-1}v_i) \end{aligned}$$

modulo $\mathscr{R}^{8m-2}$.

Since i is even, we have $\delta_i'\delta_i = h_i'$, $\delta_{i+2m-1}'\delta_{i+2m-1} = h_{i+2m-1}'$, and

$\varepsilon'_{i+2m-1}\varepsilon_{i+2m-1} = h_{i+2m-1}$. We may replace

h'_{i+2m-1} by $-h_{i+2m-1} + l_{i+2m-1}$ in the first summand and

h'_i by $-h_i + l_i$ in the second summand.

The third summand is $\mathscr{C}$-forbidden of type (v), since $\delta((i, n-2), (i+2m, n-2)) = 1$, so that we may replace it by

$$v_{i+1}h_i + h_{i+2m-1}v_i - v_{i+1}l_i$$

(2.4). We obtain

$$F_1\theta_{(i+1,n-2)} \equiv \tfrac{1}{2}F(-h_{i+2m-1}v_i + l_{i+2m-1}v_i - v_{i+1}h_i + v_{i+1}l_i + v_{i+1}h_i + h_{i+2m-1}v_i - v_{i+1}l_i - l_{i+2m-1}v_i) \equiv 0 \text{ modulo } \mathscr{R}^{8m-2}.$$

If i is odd, we have

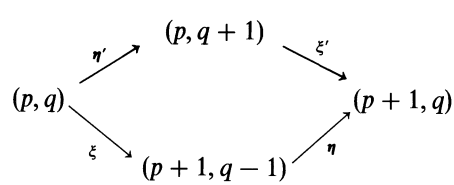
$$F_1\theta_{(i+1,n-2)} \equiv \tfrac{1}{2}F(-\delta'_{i+2m-1}\delta_{i+2m-1}v_i + \delta'_{i+2m-1}\delta_{i+2m-1}w_i - v_{i+1}\delta'_i\delta_i + \zeta'_{i+2m-1}\zeta_{i+2m-1}v_i) \equiv \tfrac{1}{2}F(-h_{i+2m-1}v_i + h_{i+2m-1}v_i + v_{i+1}h_i - l_{i+2m-1}v_i - v_{i+1}h_i + l_{i+2m-1}v_i) \equiv 0 \text{ modulo } \mathscr{R}^{8m-2},$$

because now $\delta'_i\delta_i = h_i$ and $\delta'_{i+2m-1}\delta_{i+2m-1} = h_{i+2m-1}$.

Let us define F_1 on the remaining arrows of Γ. For $\xi:(i, q) \to (i+1, q-1)$ with $1 \leq i \leq 2m-1$ and $m+1 \leq q \leq n-3$, we set

$$F_1\xi = F\xi + (-1)^i F v_\xi,$$

where $v_\xi:(i, q) \to (i+2m, q-1)$ is the path composed from the only path $(i, q) \to (i, n-2)$, the path $l_{i+q-m}\ldots l_i:(i, n-2) \to (i+q-m+1, n-2)$, and the only path $(i+q-m+1, n-2) \to (i+2m, q-1)$ (compare Fig. 9). We extend this definition to the $\tau^{(2m-1)\mathbb{Z}}$-orbit of such a ξ by $\tau^{(2m-1)\mathbb{Z}}$-periodicity, and we set $F_1\alpha = F\alpha$ for all remaining arrows of Γ. Consider a mesh



with $m \leq q \leq n-3$. If $q \geq m+1$, $v_{\xi'}\eta'$ is $\mathscr{C}$-homotopic to $\tau^{-(2m-1)}\eta v_{\xi}$ (Fig. 9), because the second coordinates of all low points of $\mathscr{C}$ are less than m. We claim that $v_{\xi'}\eta'$ is $\mathscr{C}$-marginal for $q = m$. Modulo $\tau^{(2m-1)\mathbb{Z}}$, we may assume $2 \leq p+m \leq 2m$ (see Fig. 6). If $p \leq 0$, $v_{\xi'}\eta'$ is $\mathscr{C}$-homotopic to the $\mathscr{C}$-marginal path $(p,m) \to (1,p+m-1) \to (1,m-1) \to (m-1,1) \to (m-1,2) \to (m,1) \to (m,p+2m) \to (p+2m,m)$ (see Fig. 10). If $p \geq 1$, $v_{\xi'}\eta'$ is $\mathscr{C}$-homotopic to $(p,m) \to (m,p) \to (m,m) \to (2m-1,1) \to (2m-1,2) \to (2m,1) \to (2m,p+m) \to (p+2m,m)$.

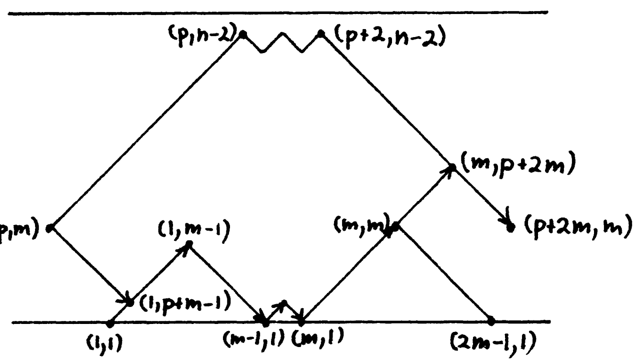


Fig. 10

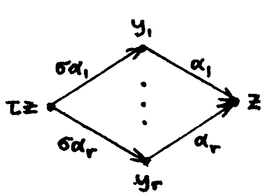
We conclude that $F_1\theta_z \in \mathscr{R}^{8m-2}$ for all stable z.

3.5 We construct a k-linear functor $F_2: k\Gamma \to \operatorname{ind} \Lambda$ such that

$$F_2\alpha - F_1\alpha \in \mathscr{R}^{8m-3}(\pi x, \pi y),$$

for every arrow $\alpha: x \to y$ of Γ, and such that $F_2\theta_z = 0$ for all stable z. Compare [4], 2.2 and [1], 3.1.

Let $\kappa: \Gamma_0 \to \mathbb{Z}$ be given by $\kappa(p,q) = 2p + \min(q, n-1)$ for stable vertices and $\kappa(i,j)^* = \kappa(i,j) + 1$ for $(i,j) \in \mathscr{C}$. We set $F_2\alpha = F_1\alpha$ for all arrows $\alpha: x \to y$ with $\kappa(x) = 0$ and for all $\alpha: (i,j) \to (i,j)^*$ with $\kappa(i,j) \geq 0$. Let z be stable with $\kappa(z) = s \geq 2$, and assume $F_2\alpha$ is defined for all arrows stopping at some y with $1 \leq \kappa(y) < s$, in such a way that $F\theta_y = 0$ if y is stable. Consider the mesh



of Γ, and observe that $\kappa(y_i) = \kappa(z) - 1$, so that $F_2(\sigma\alpha_i)$ is defined. We have

$$\sum s(\alpha_i(\sigma\alpha_i))F_1\alpha_iF_2(\sigma\alpha_i) \\ = F_1\theta_z + \sum_i s(\alpha_i(\sigma\alpha_i))F_1\alpha_i(F_2(\sigma\alpha_i) - F_1(\sigma\alpha_i)) \in \mathscr{R}^{8m-2}(\pi\tau z, \pi z).$$

We find $F_2\alpha_i$ such that $F_2\theta_z = 0$ by Lemma 3.7. In order to define $F_2\alpha$ for arrows $\alpha: x \to y$ with $\kappa(x) < 0$, we use the dual arguments.

3.6 The functor F_2 has all the desired properties, but it need not be $\tau^{(2m-1)\mathbb{Z}}$-invariant. However, it satisfies

$$F_2(\tau^{2m-1}\alpha) - F_2\alpha \in \mathscr{R}^{8m-3}(\pi x, \pi y)$$

for every arrow $\alpha: x \to y$. Sending w to $F_2\tilde{w}$ yields a well-behaved functor $F_2: k(\Gamma) \to \operatorname{ind} \Lambda$. We will now define a k-linear $\tau^{(2m-1)\mathbb{Z}}$-invariant functor $F_3: k\Gamma \to \operatorname{ind} \Lambda$ having all the desired properties.

We set $F_3\alpha = F_2\alpha$ for all arrows $\alpha: x \to y$ in Γ for which the stable vertices in $\{x, y\}$ lie in $\{(p, q): 2 - 2m \leq p \leq 0\}$, and we set $F_3\gamma_q = F_3(\tau^{2m-1}\gamma_q) = F_2(\tau^{2m-1}\gamma_q)$, for $q = 2, \ldots, n$, $F_3\beta_2 = F_2\beta_2$, and $F_3\kappa = F_2\kappa$ (see Fig. 8). By induction on q, we define $F_3\beta_q$ in such a way that

$$F_3\beta_q - F_2\beta_q \in \mathscr{R}^{8m-3}(\pi(0, q), \pi(1, q-1)),$$

for $q = 3, \ldots, n$, and that $F_3\theta_{(1,q)} = 0$, for $q = 2, \ldots, n-2$. Assume $F_3\beta_3, \ldots, F_3\beta_{q-1}$ are already defined for some $q \leq n-2$. Then

$$\mu = F_2\beta_qF_3\alpha_q - F_3\gamma_{q-1}F_3\beta_{q-1} \in \mathscr{R}^{8m-2}(\pi(0, q-1), \pi(1, q-1)),$$

and we can write

$$\mu = \sum \lambda_w F_2\tilde{w},$$

where $\lambda_w \in k$ and the $w: (0, q-1) \to (1 + (2m-1)s, q-1)$ are $\mathscr{C}$-essential of length $\geq 8m - 2$. Hence $s = 2$, and we may assume that all the $w: (0, q-1) \to (4m-1, q-1)$ begin with α_q, by Proposition 2.7(a). We obtain

$$\mu = vF_2\tilde{\alpha}_q = vF_3\alpha_q$$

for some $v \in \mathscr{R}^{8m-3}(\pi(0, q), \pi(1, q-1))$, and we set $F_3\beta_q = F_2\beta_q - v$. In the same way, we define $F_3\beta_{n-1}$ and $F_3\beta_n$. By construction,

$$F_3\theta_{(1,n-1)} \in \mathscr{R}^{8m-2}(\pi(0, n-1), \pi(1, n-1))$$

and

$$F_3\theta_{(1,n)} \in \mathscr{R}^{8m-2}(\pi(0,n), \pi(1,n)),$$

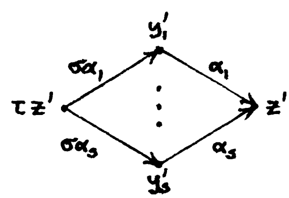
which are zero by Proposition 2.7(b). We extend F_3 by $\tau^{(2m-1)\mathbb{Z}}$-periodicity.

This completes the proof of part (b) of the theorem.

3.7 Let A be a basic, connected, representation-finite k-algebra, let $\operatorname{ind} A$ be a category of specific representatives of the indecomposables, $\mathscr{R}$ its radical, and Γ_A its quiver, the Auslander–Reiten quiver of A.

LEMMA: *Let z be a non-projective vertex of Γ_A and $\alpha_i : y_i \to z$, for $i = 1, \ldots, s$, the arrows with head z. Given irreducible morphisms $f_i : \tau z \to y_i$ and $g_i : y_i \to z$ such that $\sum g_i f_i \in \mathscr{R}^{c+1}(\tau z, z)$, for some $c \geq 2$, there are morphisms $g'_i \in \operatorname{Hom}_A(y_i, z)$ with $g'_i - g_i \in \mathscr{R}^c(y_i, z)$ such that $\sum g'_i f_i = 0$.*

PROOF: Let $\pi : \tilde{\Gamma}_A \to \Gamma_A$ be the universal cover of Γ_A ([1], 1.3), and choose $z' \in \pi^{-1} z$. Consider the mesh



of $\tilde{\Gamma}_A$, where $\pi y'_i = y_i$. Choose $\kappa : \tilde{\Gamma}_A \to \mathbb{Z}A_2$ such that $\kappa(\tau z') = 0$ ([1], 1.6). There exists a well-behaved functor $F : k(\tilde{\Gamma}_A) \to \operatorname{ind} A$ with $F(\overline{\sigma\alpha_i}) = f_i$, where $\overline{\sigma\alpha_i}$ is the canonical image of $\sigma\alpha_i$ in $k(\tilde{\Gamma}_A)$. Since F is a covering functor, we can write

$$\sum_i g_i f_i = \sum_w \lambda_w F\bar{w},$$

where $\lambda_w \in k$ and w ranges over paths from $\tau z'$ to some $x' \in \pi^{-1} z$. We may assume that the length of any w is not less than $c + 1$. Every w has the form $v(\sigma\alpha_i)$, for some i, so that we obtain

$$\sum_i g_i f_i = \sum_i \mu_i F(\overline{\sigma\alpha_i}) = \sum_i \mu_i f_i$$

for some $\mu_i \in \mathscr{R}^c(y_i, z)$. Choose $g'_i = g_i - \mu_i$.

4. Proof of part (a) of the theorem

Let $\mathscr{C}$ be a $\tau^{(2m-1)\mathbb{Z}}$-stable configuration of $\mathbb{Z}D_{3m}$ containing $(0, n-1)$, where $n = 3m$. Let $\Gamma = (\mathbb{Z}D_{3m})_{\mathscr{C}}$, and let $\pi: \Gamma \to \Delta = \Gamma/\tau^{(2m-1)\mathbb{Z}}$ be the canonical map.

4.1 In 3.3, we defined an ideal J in the path-category $k\Delta$, and we showed that, for any algebra Λ with Auslander–Reiten quiver Δ, the category ind Λ is isomorphic to either $k\Delta/J$ or the mesh-category $k(\Delta)$. The following proposition implies that there actually exists an algebra Λ with ind $\Lambda \simeq k\Delta/J$, or, in the terminology of [1], that $k\Delta/J$ is an Auslander-category. Indeed, $k(\Gamma)$ has this property by definition, and it is preserved under covering functors ([1], 3.5).

Proposition: *There exists a* $\tau^{2(2m-1)\mathbb{Z}}$*-invariant covering functor* $F: k(\Gamma) \to k\Delta/J$.

Proof: Let $G: k\Gamma \to k\Delta/J$ be the composition of $\pi: k\Gamma \to k\Delta$ with the canonical functor $k\Delta \to k\Delta/J$. By definition, $G\theta_z = 0$ for all modified mesh-relations θ_z with $x \notin \tau^{(2m-1)\mathbb{Z}}(1, n-1)$. Therefore, G vanishes on all vectors in I_s which are linear combinations of stable paths.

In order to define F, we use the notations introduced in 3.4. We set

$$F\kappa_1 = G\kappa_1 + G(\delta_{2m}l_{2m-1}\dots l_2\delta'_1\kappa_1),$$
$$F\delta_1 = G\delta_1 - G(\delta_{2m}v_1) + G(\delta_{2m}w_1),$$
$$F\zeta'_1 = G\zeta'_1 - G(v_2\zeta'_1) + G(w_2\zeta'_1),$$
$$F\zeta_i = G\zeta_i - G(\zeta_{i+2m-1}w_i), \text{ for } i = 2,\dots,2m-1,$$
$$F\zeta'_i = G\zeta'_i + G(w_{i+1}\zeta'_i) + G(w_{i+2m}v_{i+1}\zeta'_i), \text{ for } i = 2,\dots,2m-2,$$
$$F\delta'_{2m-1} = G\delta'_{2m-1} - G(v_{2m}\delta'_{2m-1}).$$

We extend this definition by $\tau^{2(2m-1)\mathbb{Z}}$-periodicity to all arrows in the $\tau^{2(2m-1)\mathbb{Z}}$-orbits of the ones for which F is already defined, and we let F coincide with G on the remaining δ_i, δ'_i, ε_i, ε'_i, ζ_i, ζ'_i; $l_{s(2m-1)}$, $\kappa_{s(2m-1)+1}$. In Fig. 11, the arrows on which F differs from G are drawn full, the other ones broken.

By definition $F\theta_{(i+1,q)} = G\theta_{(i+1,q)}$, which is zero, for all (i, q) with $i = 0, 1, \dots, 2(2m-1)-1$ and $q \geq n-2$ except $(0, n-1)$, $(2m-1, n-1)$, and $(i, n-2)$ with $i = 1, \dots, 2m-1$. Straightforward computations yield

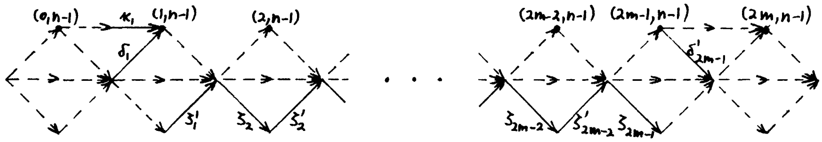


Fig. 11

$F\theta_{(i+1,q)} = 0$ in these cases, too, given that G vanishes on all stable paths whose length exceeds $2(2n-3)$ as well as on the following vectors:

$$\delta'_{2m}\delta_{2m}w_1 - \delta'_{2m}\delta_{2m}v_1 - v_2\delta'_1\delta_1 + v_2\zeta'_1\zeta_1,$$

$$\zeta'_{i+2m-1}\zeta_{i+2m-1}w_i - w_{i+1}\zeta'_i\zeta_i, \text{ for } i = 1,\dots,2m-2,$$

$$v_{i+2m-1}v_i, \text{ for any } i,$$

$$w_{i+2m}\zeta'_{i+2m-1}\zeta_{i+2m-1}w_i - w_{i+2m}\zeta'_{i+2m-1}\zeta_{i+2m-1}v_i, \quad \text{for } i = 1,\dots,2m-2.$$

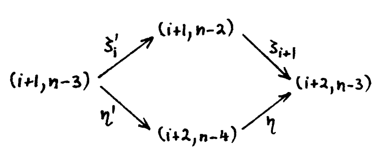
The first one of these vectors is $v - \pi'v \in I_s$, where $v = h_{2m}w_1$ and where π' is the projection of 2.4. That the second one lies in I_s follows from the fact that $h_{i+1}l_i$ and $l_{i+1}h_i$ are $\mathscr{C}$-neighbors if i is not a multiple of $2m-1$. For the third one, we use the following lemma. As a consequence, $v_{i+2m}l_{i+2m-1}w_i$ and $v_{i+2m}l_{i+2m-1}v_i$ lie in I_s for all i, and hence

$$\begin{aligned} &w_{i+2m}l_{i+2m-1}w_i - w_{i+2m}l_{i+2m-1}v_i \\ &\quad = v - \pi'v + v_{i+2m}l_{i+2m-1}w_i - v_{i+2m}l_{i+2m-1}v_i \end{aligned}$$

does as well, for $i = 1,\dots,2m-2$, where $v = w_{i+2m}l_{i+2m-1}w_i$. Remember also that

$$G(\kappa_1\iota_0 + \delta_1\delta'_0) = G(\delta_{2m}v_1\delta'_0),$$

and that any $\mathscr{C}$-admissible path from $(0, n-1)$ to $(4m-1, n-1)$ is $\mathscr{C}$-marginal (2.7). Consider a mesh



.

For $i = 2,\dots,2m-2$, we have

$$F(\zeta_{i+1}\zeta'_i) - G(\eta\eta') = G(-\zeta_{i+4m-1}w_{i+2m}w_{i+1}\zeta'_i + \zeta_{i+4m-1}w_{i+2m}v_{i+1}\zeta'_i),$$

which is zero, since

$$w_{i+2m}w_{i+1} - w_{i+2m}v_{i+1}$$
$$= w_{i+2m}w_{i+1} - \pi'(w_{i+2m}w_{i+1}) + v_{i+2m}w_{i+1} - v_{i+2m}v_{i+1}$$

lies in I_s by 2.4 and the following lemma. For $i = 1$, we obtain

$$F(\zeta_2\zeta_1') - G(\eta\eta') = -G(\zeta_{2m+1}v_2\zeta_1').$$

We set

$$F\xi = G\xi - Gu_\xi,$$

for all arrows $\xi:(2,q)\to(3,q-1)$ with $2m-1\leq q\leq n-3$, where u_ξ is the path composed from $(2,q)\to(2,n-2)$, $l_{q-m+2}\ldots l_2:(2,n-2)\to(q-m+3,n-2)$, and the path $(q-m+3,n-2)\to(2m+2,q-1)$ (compare Fig. 9). We let $F\alpha = F\xi$ for all arrows α in the $\tau^{2(2m-1)\mathbb{Z}}$-orbit of such a ξ, and $F\alpha = G\alpha$ for all remaining arrows of Γ. It is easy to check that $F\theta_z = 0$ for all stable z. Notice that the path

$$(2,2m-2)\to(2,n-2)\xrightarrow{l_2}(3,n-2)\ldots$$
$$\ldots\xrightarrow{l_{m+1}}(m+2,n-2)\to(2m+2,2m-2)$$

is $\mathscr{C}$-marginal (Fig. 6, compare 3.4).

Therefore, F induces a k-linear functor $F:k(\Gamma)\to k\Delta/J$. For any two vertices x and y of Γ, the two maps

$$\bigoplus_{\pi z=\pi y} k(\Gamma)(x,z)\to k\Delta/J(\pi x,\pi y)$$

$$\bigoplus_{\pi z=\pi x} k(\Gamma)(z,y)\to k\Delta/J(\pi x,\pi y)$$

given by F are surjective. Comparing dimensions (3.3), we see that they are bijective, and hence F is a covering functor.

LEMMA: *For any $p\in\mathbb{Z}$, $l_{p+4m-4}\ldots l_p:(p,n-2)\to(p+4m-3,n-2)$ is $\mathscr{C}$-marginal.*

PROOF: Modulo $\tau^{(2m-1)\mathbb{Z}}$, we may assume $2\leq p+n-2\leq 2m$ (see Fig. 6). If $p+n-2\leq m$, the subpath $l_{m-1}\ldots l_p$ is $\mathscr{C}$-homotopic to $(p,n-2)\to(1,p+n-3)\to(1,m-1)\to(m-1,1)\to(m-1,2)\to(m,1)\to(m,n-2)$, which is $\mathscr{C}$-marginal. In case $m+1\leq p+n-2$, the subpath

$l_{2m-1} \dots l_p$ is $\mathscr{C}$-homotopic to the $\mathscr{C}$-marginal path $(p, n-2) \to (m, p+n-2-m) \to (m, m) \to (2m-1, 1) \to (2m-1, 2) \to (2m, 1) \to (2m, n-2)$.

4.2 Let Λ' be the full subcategory of $k\Delta/J$ whose objects are the projective vertices of Δ. We claim that $k\Delta/J$ is isomorphic to $\operatorname{ind} \Lambda'$ and that Δ is the Auslander–Reiten quiver of Λ'. Recall from [1], 2.4 that an object x of a locally finite-dimensional category M is top-torsionfree if there exists a non-zero morphism $\mu \in M(x, y)$ for some y such that $\mu v = 0$ for each non-invertible morphism v with range x. The top-torsionfree objects of $k(\Gamma)$ are precisely the projective vertices of Γ ([1], 2). Let $F: k(\Gamma) \to k\Delta/J$ be the covering functor constructed in 4.1. A vertex x of Γ is top-torsionfree in $k(\Gamma)$ or projective in Γ if and only if $Fx = \pi x$ is top-torsionfree in $k\Delta/J$ or projective in Δ, respectively. Thus the top-torsionfree objects of $k\Delta/J$ are precisely the projective vertices of Δ, and hence $\operatorname{ind} \Lambda'$ is isomorphic to $k\Delta/J$ ([1], 2.4). Therefore, the underlying quivers of Δ and the Auslander–Reiten quiver $\Gamma_{\Lambda'}$ of Λ' are isomorphic, and it suffices to show that the Auslander–Reiten translation τ_A on $\Gamma_{\Lambda'}$ coincides with the translation τ of Δ. For each non-projective vertex x of Γ, the simple representation k_x of $k(\Gamma)$ has a minimal projective resolution

$$0 \to k(\Gamma)(?, \tau x) \to \bigoplus k(\Gamma)(?, y_i) \to k(\Gamma)(?, x) \to k_x \to 0,$$

where y_i ranges over the tails of the arrows with head x ([1], 2.6). Since F is a covering functor, we obtain a minimal projective resolution

$$0 \to k\Delta/J(?, \pi\tau x) \to \bigoplus k\Delta/J(?, \pi y_i) \to k\Delta/J(?, \pi x) \to k_{\pi x} \to 0$$

for the simple representation $k_{\pi x}$ of $k\Delta/J$, which implies that $\tau = \tau_A$ for all vertices of Δ ([1], 2 and 3).

In chapter 3 we showed that, in case $\operatorname{char} k \neq 2$, Λ' is isomorphic to the standard category Λ with Auslander–Reiten quiver Δ; i.e., the full subcategory of $k(\Delta)$ whose objects are the projective vertices of Δ. In order to complete the proof of the theorem, it is enough to show that, in case $\operatorname{char} k = 2$, $k(\Delta)$ and $k\Delta/J$ or equivalently Λ and Λ' are not isomorphic. This is a consequence of the following proposition if we set $s = 1$.

4.3 Assume $\operatorname{char} k = 2$.

PROPOSITION: *There exists a covering functor*

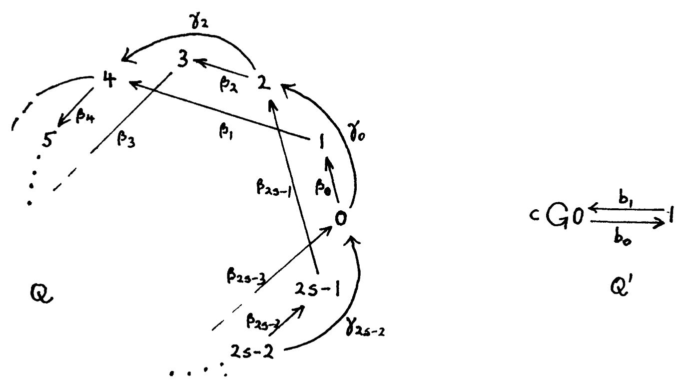
$$H: k(\Gamma/\tau^{s(2m-1)\mathbb{Z}}) \to k\Delta/J$$

if and only if s is even.

This proposition expresses that a covering $\Gamma_A \to \Gamma_B$ between the Auslander–Reiten quivers of two representation-finite categories A and B need not be induced by a covering functor from ind A to ind B.

PROOF: By 4.1, there exists such a covering functor for $s = 2$ and hence for all even numbers s. Conversely, assume that there is such a covering functor, or, equivalently, that there exists a $\tau^{s(2m-1)\mathbb{Z}}$-invariant covering functor $H': k(\Gamma) \to k\Delta/J$ for some s. Then H' maps projective vertices of Γ to projective vertices of Δ, and, if x is not projective, we have $H'(\tau x) = \tau H'(x)$. Thus the covering $\Gamma \to \Delta$ of translation-quivers induced by H' ([1], 3.3) coincides with π.

Let $(n-1, q)$, with $q \leq m-1$, be the unique point of $\mathscr{C}$ with first coordinate $n-1$ (Fig. 5). Let $\tilde{A}$ be the full subcategory of $k(\Gamma)$ whose objects are the projective vertices $(t(2m-1), n-1)^*$ and $(n-1+t(2m-1), q)^*$ of Γ, for $t \in \mathbb{Z}$, and let A' be the full subcategory of $k\Delta/J$ whose objects are the projective vertices $\pi(0, n-1)^*$ and $\pi(n-1, q)^*$ of Δ. Then H' induces a $\tau^{s(2m-1)\mathbb{Z}}$-invariant covering functor $G': \tilde{A} \to A'$. Using the description of $\tilde{A}$ and A' by quivers and relations (chapter 5), we obtain a covering functor $G: kQ/I \to kQ'/I'$, where Q and Q' are the following quivers:



The ideal I is generated by

$$\gamma_{2i+2}\gamma_{2i} + \beta_{2i+1}\beta_{2i} \text{ and } \beta_{2i+4}\beta_{2i+1},$$

for $i = 0, \ldots, s-1$, where we set $\gamma_{2s} = \gamma_0$, $\beta_{2s} = \beta_0$, and $\beta_{2s+2} = \beta_2$. The ideal I' is generated by

$$c^2 + b_1 b_0,\ b_0 b_1 + b_0 c b_1, \text{ and } c^4.$$

Observe that

$$c^2b_1 \equiv b_1b_0b_1 \equiv b_1b_0cb_1 \equiv c^3b_1 \equiv c^4b_1 \equiv 0 \text{ modulo } I',$$

and similarly $b_0c^2 \in I'$. Thus the residue classes of c, c^2, c^3; b_0, b_0c and b_1, cb_1 modulo I' form k-bases for the vector spaces of non-invertible morphisms in $kQ'/I'(0,0)$; $kQ'/I'(0,1)$, and $kQ'/I'(1,0)$, respectively. Therefore, we can write

$$G\gamma_{2i} = \lambda_{2i,1}c + \lambda_{2i,2}c^2 + \lambda_{2i,3}c^3,$$
$$G\beta_{2i} = \mu_{2i,1}b_0 + \mu_{2i,2}b_0c,$$
$$G\beta_{2i+1} = \mu_{2i+1,1}b_1 + \mu_{2i+1,2}cb_1$$

for some scalars $\lambda_{2i,1} \neq 0$, $\lambda_{2i,2}$, $\lambda_{2i,3}$, $\mu_{j,1} \neq 0$, and $\mu_{j,2}$. Since G maps I into I', we obtain the following relations:

$$\lambda_{2i+2,1}\lambda_{2i,1} = \mu_{2i+1,1}\mu_{2i,1},$$
$$\lambda_{2i+2,1}\lambda_{2i,2} + \lambda_{2i+2,2}\lambda_{2i,1} = \mu_{2i+1,1}\mu_{2i,2} + \mu_{2i+1,2}\mu_{2i,1},$$
$$\mu_{2i+4,1}\mu_{2i+1,1} + \mu_{2i+4,1}\mu_{2i+1,2} + \mu_{2i+4,2}\mu_{2i+1,1} = 0,$$

for $i = 0, \ldots, s-1$. This implies that

$$0 = 2\sum_{i=0}^{s-1} \frac{\lambda_{2i,2}}{\lambda_{2i,1}} = \sum_{i=0}^{s-1}\left(\frac{\lambda_{2i,2}}{\lambda_{2i,1}} + \frac{\lambda_{2i+2,2}}{\lambda_{2i+2,1}}\right) = \sum_{i=0}^{s-1}\left(\frac{\mu_{2i,2}}{\mu_{2i,1}} + \frac{\mu_{2i+1,2}}{\mu_{2i+1,1}}\right)$$
$$= \sum_{i=0}^{s-1}\left(\frac{\mu_{2i+4,2}}{\mu_{2i+4,1}} + \frac{\mu_{2i+1,2}}{\mu_{2i+1,1}}\right) = \sum_{i=0}^{s-1} 1 = s \cdot 1_k.$$

Hence s is even.

5. Quivers and relations

5.1 Let $\mathscr{C}$ be a ϕ-unstable configuration of $\mathbb{Z}D_n$ containing $(0, n-1)$ for $n \geq 5$. Our goal in this chapter is to describe the full subcategory $\tilde{\Lambda} = \tilde{\Lambda}_{\mathscr{C}}$ of $k((\mathbb{Z}D_n)_{\mathscr{C}})$ whose objects are the projective vertices of $(\mathbb{Z}D_n)_{\mathscr{C}}$ by quiver and relations ([1], 2.1). We use the notations n_1, n_2, n_3, $\mathscr{D}_1^+$, $\mathscr{D}_2^+$, $\mathscr{D}_3^+$, χ_1, χ_2, χ_3 introduced in 2. First we extend

$$\chi_k : (\mathbb{Z}A_{n_k+1})_0 \to (\mathbb{Z}D_n)_0$$

to a k-linear functor

$$\chi_k : k((\mathbb{Z}A_{n_k+1})_{\mathscr{D}_k^+}) \to k((\mathbb{Z}D_n)_{\mathscr{C}})$$

for $k = 1, 2, 3$. We carry the construction out for $k = 1$; χ_2 and χ_3 are defined in an analogous way.

First we extend χ_1 to a k-linear functor $\chi_1 : k\mathbb{Z}A_{n_1+1} \to k\mathbb{Z}D_n$ between the path categories associated with $\mathbb{Z}A_{n_1+1}$ and $\mathbb{Z}D_n$. We send an arrow $\alpha:(p,q) \to (p,q+1)$ with $q \leq n_1$ and $p+q \equiv 0$ modulo n_1+1 to the only path from $\chi_1(p,q)$ to $\chi_1(p,q+1)$ containing a $\mathscr{C}$-congruent crenel path, and we do the same for an arrow $\alpha:(p,q) \to (p+1,q-1)$ with $q \geq 2$ and $p+q \equiv -1$ modulo n_1+1. Fig. 12 exemplifies this definition. For all other arrows $\alpha: x \to y$, there exists an arrow $\beta: \chi_1 x \to \chi_1 y$, and we set $\chi_1\alpha = \beta$. On paths, χ_1 is defined by composition.

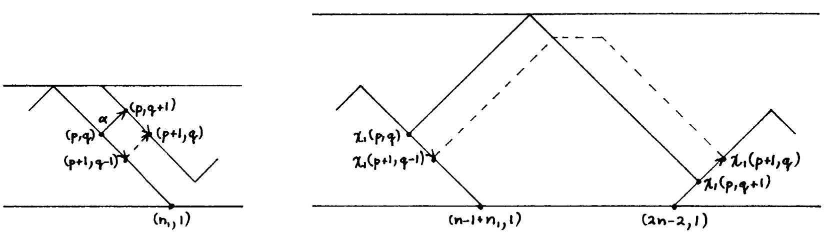


Fig. 12

Next we extend χ_1 to a k-linear functor

$$\chi_1 : k\Gamma_1 \to k\Gamma,$$

where $\Gamma_1 = (\mathbb{Z}A_{n_1+1})_{\mathscr{D}_1^+}$ and $\Gamma = (\mathbb{Z}D_n)_{\mathscr{C}}$. If $(i,j) \in \mathscr{D}_1^+$ lies in $\omega_{n_1}\mathscr{D}_1$, χ_1 maps the mesh of $\mathbb{Z}A_{n_1+1}$ starting at (i,j) bijectively onto the mesh of $\mathbb{Z}D_n$ starting at $\chi_1(i,j) \in \mathscr{C}$, so that we can send $(i,j)^*$ to $(\chi_1(i,j))^*$ and the arrows with head and tail $(i,j)^*$ to the arrows with head and tail $\chi_1(i,j)^*$, respectively. Let

$$(p,1) \xrightarrow{\iota} (p,1)^* \xrightarrow{\kappa} (p+1,1)$$

belong to a mesh of Γ_1 starting at some point in $\tau^{(n_1+1)\mathbb{Z}}(n_1,1)$ and set $\chi_1(p,1) = (p',1)$. Note that $\chi_1(p+1,1) = (p'+2n-3-n_1,1)$, and that p' is the first coordinate of a high point (p',j) of $\mathscr{C}$ (Fig. 5). Let

$$(p',j) \xrightarrow{\iota'} (p',j)^* \xrightarrow{\kappa'} (p'+1,j)$$

be part of the mesh of Γ starting at (p',j). We set

$$\chi_1(p,1)^* = (p',j)^*,$$
$$\chi_1\iota = \iota' w_1,$$
$$\chi_1\kappa = w_2 h_{p'+n-n_1-1} l_{p'+n-n_1-2} \dots l_{p'+2}\alpha\kappa',$$

where w_1 and w_2 are the only paths in Γ from $(p',1)$ to (p',j) and from $(p'+n-n_1, n-2)$ to $(p'+2n-3-n_1, 1)$, respectively, and $\alpha:(p'+1, j) \to (p'+2, n-2)$ is an arrow (see Fig. 13).

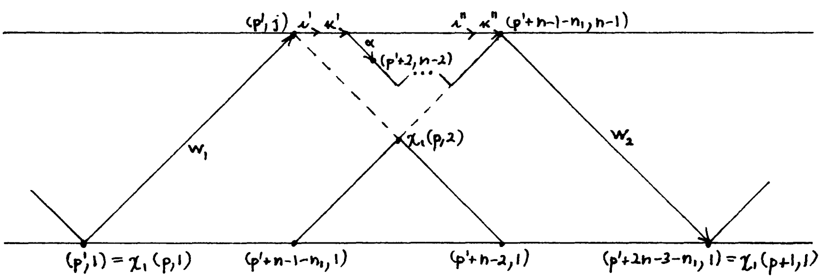


Fig. 13

We define the sign $s'(\alpha)$ of a stable arrow α of Γ_1 to be $+1$, unless α has the form $\alpha:(i,j) \to (i,j+1)$, in which case $s'(\alpha) = (-1)^j$ (compare [5], 4.2). We set $s'(\kappa) = 1$ for all arrows κ with projective tail, and we require $s'(\tau^{n_1+1}\iota) = s'(\iota)$ if ι is an arrow with projective head. For $\iota:(i,j) \to (i,j)^*$ with $0 \le i \le n_1$, we set

$$s'(\iota) = \begin{cases} (-1)^n & \text{if } i+j < n_1+1, \\ -1 & \text{if } i+j = n_1+1, \\ (-1)^{n+n_1+1} & \text{if } i+j > n_1+1. \end{cases}$$

Let $\tilde{w} = s'(w)\bar{w}$, where $s'(w) = s'(\alpha_r)\dots s'(\alpha_1)$ for $w = \alpha_r \dots \alpha_1$ and where $\bar{w}$ is the canonical image of w in $k(\Gamma_1)$. The kernel of the functor $k\Gamma_1 \to k(\Gamma_1)$ obtained by sending w to $\tilde{w}$ is the ideal J of $k\Gamma_1$ generated by the modified mesh-relations

$$\theta_z = \sum s'(\alpha(\sigma\alpha))\alpha(\sigma\alpha),$$

where z is a stable vertex and α ranges over all arrows with head z. By [5], 4.2, J is generated by the θ_z for $\tau z \in \mathscr{D}_1^+$, differences of $\mathscr{D}_1^+$-neighbors of length 2, and $\mathscr{D}_1^+$-marginal paths of length 2. We defined the sign functions s' and s (2.3) in such a way that $\chi_1\theta_z$ lies in I_s for all z with $\tau z \in \mathscr{D}_1^+$. In addition, χ_1 maps $\mathscr{D}_1^+$-neighbors of length 2 to $\mathscr{C}$-admissible $\mathscr{C}$-homotopic paths and $\mathscr{D}_1^+$-marginal paths of length 2 to $\mathscr{C}$-admissible $\mathscr{C}$-marginal paths in Γ (see Fig. 12). Hence we obtain an induced functor $\chi_1 : k(\Gamma_1) \to (\Gamma)$.

REMARK: This functor χ_1 is actually fully faithful. However, we will not prove this, since we only need the weaker statement of Corollary 5.2.

5.2 LEMMA: *Let $w:(x,y) \to (p,q)$ be $\mathscr{C}$-essential.*

(a) *If $n-1 \le x < x+y \le n+n_1$, then*
$n-1 \le p \le n+n_1-1$ *or*
$2n-2 \le p+\min(q,n-1)$ *and* $p \le 2n-3+n_1$ *or*
$3n-3 \le p+\min(q,n-1) \le 3n-4+n_1$.

(b) *If $1 \le x < x+y \le n_1+1$, then*
$1 \le p \le n_1$ *or*
$n \le p+\min(q,n-1) \le n+n_1-1$ *or*
$n \le p \le n+n_1-2$ *or*
$2n-1 \le p+\min(q,n-1) \le 2n-3+n_1$.

See Fig. 14. Analogous results hold for $\mathscr{C}$-essential paths starting in the images of χ_2 and χ_3: Replace $\mathscr{C}$ by $\tau^{n_1+n_3+1}\phi^{n_1+n_3}\mathscr{C}$ and $\tau^{n-1+n_1}\phi^{n-1+n_1}\mathscr{C}$, respectively.

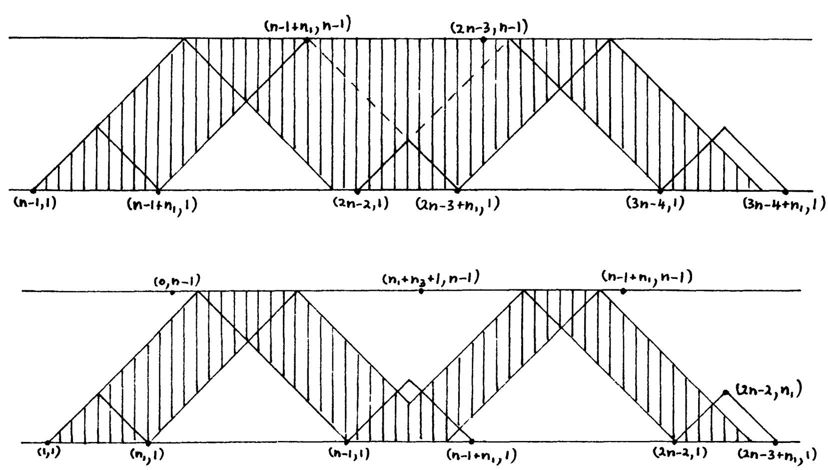


Fig. 14

PROOF: We only prove (b). The proof of (a) uses the same methods, and it is somewhat simpler.

If w is low, we have $1 \leq p \leq n_1$, since any path containing a vertex (n_1+1, j) is $\mathscr{C}$-homotopic to a path containing $(n_1, 1) \to (n_1, 2) \to (n_1+1, 1)$, and $(n_1, 1) \notin \mathscr{C}$. Next suppose $w = w_2 h_{p_1} w_1$, where both w_1 and w_2 are low. We see that $1 \leq p_1 \leq n_1$, and $p \leq n + n_1 - 1$ holds for any low $\mathscr{C}$-essential path $(p_1+1, n-2) \to (p, q)$. We are done if $n \leq p + \min(q, n-1) \leq n + n_1 - 1$. Hence we can assume $n + n_1 \leq p + \min(q, n-1)$ and $n_1 + 1 \leq p$, since $\delta((p_1, n-1), (n_1+1, n-1)) = 0$. We claim that w_2 cannot be free. If it were, w_2 would be $\mathscr{C}$-homotopic to $w_2' l_{n_1+1} \dots l_{p_1+1}$ and w to $w_2' h_{n_1+1} l_{n_1} \dots l_{p_1} w_1$, which is $\mathscr{C}$-marginal. Since any path $(p_1+1, n-2) \to (n-1, q)$ is free, we obtain $n \leq p \leq n + n_1 - 1$, and we only have to exclude the possibility $p = n + n_1 - 1$. But any low path $(p_1+1, n-2) \to (n+n_1-1, q)$ is $\mathscr{C}$-homotopic to a path containing a $\mathscr{C}$-essential subpath $(n-1, p_1) \to (n+n_1-1, 1)$, which is free by Lemma 2.6. Finally, let $w = w_3 h_{p_2} w_2 h_{p_1} w_1$, where w_1, w_2, and w_3 are low. Examining the subpath $w_2 h_{p_1} w_1$, we obtain $1 \leq p_1 \leq n_1$ and either $1 \leq p_2 \leq n_1$ or $n \leq p_2 \leq n + n_1 - 2$. The first possibility yields a $\mathscr{C}$-forbidden path $h_{p_2} w_2 h_{p_1}$, so that $n \leq p_2 \leq n + n_1 - 2$. For any $\mathscr{C}$-essential low path $w_3 : (p_2+1, n-2) \to (p, q)$, we have $2n - 1 \leq p + \min(q, n-1)$ and $p \leq 2n - 3 + n_1$, and it suffices to exclude the possibility $p + \min(q, n-1) = 2n - 2 + n_1$. As before, w_3 must not be free. Hence we may assume that $q \leq n_1$. By [1], 2.8, there is a path $v : (p, q) \to (i, j)^*$ for some $(i, j) \in \mathscr{C}$ such that vw does not lie in I_s. Since $2n - 3 < p$, $(i, j) \neq (2n-3, n-1)$, and thus $2i + \min(j, n-1) \geq 2(2n - 2 + n_1) + 1$; i.e., (i, j) lies "to the right" of the "vertical line" through $(2n - 2 + n_1, 1)$ Since the length of any $\mathscr{C}$-essential path does not exceed $2(2n-3)$, we obtain on the other hand that $2i + \min(j, n-1) \leq 2x + y + 2(2n-3) \leq 2n_1 + 1 + 2(2n-3)$, which is impossible. Clearly, $w_3 h_{p_2} w_2 h_{p_1} w_1$ cannot stop at a high vertex, and hence w has at most two crenels.

Set $\Gamma_k = (\mathbb{Z}A_{n_k+1})_{\mathscr{D}_k^+}$, for $k = 1, 2, 3$.

COROLLARY: *For any two stable vertices z and z' of Γ_k, χ_k induces a surjection*

$$k(\Gamma_k)(z, z') \to k(\Gamma)(\chi_k(z), \chi_k(z')).$$

PROOF: We give a proof for $k = 1$. It is enough to show that any $\mathscr{C}$-essential path $w : (x, y) \to (p, q)$ is $\mathscr{C}$-homotopic to a path $\chi_1 v$ for some $v : z \to z'$, where $(x, y) = \chi_1(z)$ and $(p, q) = \chi_1(z')$. Translating z and z' by $\tau^{s(n_1+1)}$ and (x, y), (p, q), and w by $\tau^{s(2n-3)}$ for a suitable s, we may assume that either $n - 1 \leq x < x + y \leq n + n_1$ or $1 \leq x < x + y \leq n_1 + 1$.

Clearly $w = \chi_1 v$ if (p, q) lies in the same "connected component" of the image of χ_1 as (x, y), that is, if (p, q) satisfies the same inequalities. Therefore it suffices to consider $\mathscr{C}$-essential paths $w:(x, y) \to (p, q)$ for which (x, y) and (p, q) are the only vertices in the image of χ_1.

Assume $x + y = n + n_1$, $y \leq n_1 + 1$ and $p = 2n - 2$, $q \leq n_1$ (Fig. 14), and let $w = w_2 h_{p_1} w_1$. Then $n - 1 \leq p_1 \leq n + n_1 - 1$, and we may exclude $p_1 = n - 1$, since otherwise w_2 is $\mathscr{C}$-marginal. Replace w_1 by the path $w'_1:(x, y) \to (p_1, n + n_1 - p_1) \to (p_1, n - 2)$ and w_2 by $w'_2:(p_1 + 1, n - 2) \to (2n - 2, p_1 + 1 - n) \to (p, q)$. The path $w' = w'_2 h_{p_1} w'_1$ is $\mathscr{C}$-homotopic to w, and $w' = \chi_1 v$, where v is the path $(n_1 + 1 - y, y) \to (1 + p_1 - n, n + n_1 - p_1) \to (1 + p_1 - n, n + n_1 - p_1 + 1) \to (q, n_1 + 2 - q)$ in Γ_1.

In case $x + y = n_1 + 1$, $y \leq n_1$ and $p = n - 1$, $q \leq n_1 + 1$, the argument is analogous. The last possibility is that $x + y = n + n_1$, $y \leq n_1 + 1$ and $p = 3n - 4$, $q \leq n_1 + 1$ and that $w = w_3 h_{p_2} w_2 h_{p_1} w_1$, where $n \leq p_1 \leq n + n_1 - 1$ and $2n - 2 \leq p_2 \leq 2n - 3 + n_1$. Then w_2 is $\mathscr{C}$-homotopic to $(p_1 + 1, n - 2) \to (2n - 2, p_1 + 1 - n) \to (2n - 2, n_1) \to (p_2, 2n - 2 + n_1 - p_2) \to (p_2, n - 2)$, which reduces the problem to the cases already treated.

5.3 Lemma: *Let $w:(1, n - 1) \to (p, q)$ be $\mathscr{C}$-essential. Then we have either*

$$n \leq p + \min(q, n - 1) \text{ and } p \leq n - 1 + n_1 \text{ or}$$
$$n + n_1 + n_3 + 1 \leq p + \min(q, n - 1) \text{ and } p \leq 2n - 3.$$

See Fig. 15. Again, analogous results hold for $\mathscr{C}$-essential paths starting in $\tau^{-1}(i, j)$, where (i, j) is any high point of $\mathscr{C}$.

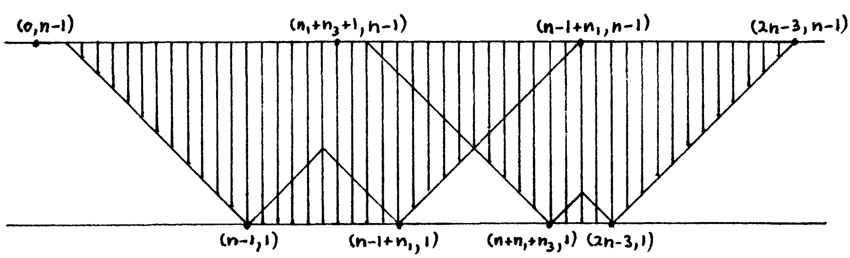


Fig. 15

Proof: If w is low, we must have $n \leq p + \min(q, n - 1)$ and $p \leq n - 1 + n_1$. Assume $w = w_2 h_{p_1} w_1$, where w_1 and w_2 are low. We claim that w_1 is free. If not, it is $\mathscr{C}$-homotopic to a path $(1, n - 1) \to (n - 1, 1) \to (p_1, n + n_1 - p_1) \to (p_1, n - 2)$, which is free by Lemma 2.6, a contradiction.

Since w_1 is $\mathscr{C}$-admissible, we see that $n_1 + n_3 + 2 \leq p_1$, and we may assume $w_1 = l_{p_1 - 1} \dots l_2 \alpha$, where α is the arrow $(1, n-1) \to (2, n-2)$. Then $h_{p_1} w_1$ is $\mathscr{C}$-homotopic to $l_{p_1} \dots l_{n_1+n_3+3} h_{n_1+n_3+2} l_{n_1+n_3+1} \dots l_2 \alpha$, so that we may assume $p_1 = n_1 + n_3 + 2$. For any low $\mathscr{C}$-essential path $w_2 : (n_1 + n_3 + 3, n-2) \to (p, q)$, we have $n + n_1 + n_3 + 1 \leq p + \min(q, n - 1)$ and $p \leq 2n - 3$. Finally, assume $w = w_3 h_{p_2} w_2 h_{p_1} w_1$ for some low paths w_1, w_2, and w_3, where $p_1 = n_1 + n_3 + 2$. As before, w_2 must be free, and since $p_2 \leq 2n - 3$, w is $\mathscr{C}$-forbidden.

5.4 We recall from [5] the description of the full subcategory $\tilde{\Lambda}_k$ of $k(\Gamma_k)$ whose objects are the projective vertices of Γ_k, for $k = 1, 2, 3$. For each integer i, there is exactly one point $(i, \beta_k i - i)$ in $\mathscr{D}_k^+$ with first coordinate i, and the map $i \to \beta_k i$ is a permutation of $\mathbb{Z}$. Since $\mathscr{D}_k^+$ is $\tau^{(n_k+1)\mathbb{Z}}$-stable, $\beta_k(i + n_k + 1) = \beta_k i + n_k + 1$ for all i. Let α_k be the permutation given by $i \to \alpha_k i = \beta_k^{-1} i + n_k + 2$. For each $i \in \mathbb{Z}$, choose $a_k i$ and $b_k i$ such that

$$\alpha_k^{a_k i}(i) = i + n_k + 1 = \beta_k^{b_k i}(i).$$

We let $\tilde{Q}_k$ be the quiver with vertex set $\mathbb{Z}$ containing an arrow $\alpha : i \to \alpha_k i$ and $\beta : i \to \beta_k i$ for each i. By $\tilde{I}_k$ we denote the ideal of $k\tilde{Q}_k$ generated by all paths of the form

$$\alpha\beta \text{ and } \beta\alpha$$

along with the vectors

$$\alpha^{a_k i} - \beta^{b_k i},$$

for each i, where $\alpha^{a_k i}$ and $\beta^{b_k i}$ are the paths from i to $i + n_k + 1$ composed from $a_k i$ α-arrows and $b_k i$ β-arrows respectively.

Let $d_k(i)$ be the vertex $(\alpha_k i - n_k - 2, n_k + 2 - \alpha_k i + i)$ of $\mathscr{D}_k^+$, which is the only point (p, q) of $\mathscr{D}_k^+$ with $p + q = i$. By $U_k(i, \alpha)$ we denote the "α-path" in Γ_k from $\tau^{-1} d_k(i)$ to $d_k(i + n_k + 1)$ ([5], 5.6). For an arrow $\alpha : i \to \alpha_k i$, we let

$$\mathrm{u}_k(\alpha) : d_k(i)^* \to d_k(\alpha_k i)^*$$

be the path composed from the arrow $d_k(i)^* \to \tau^{-1} d_k(i)$, the subpath

$$\begin{aligned}\tau^{-1} d_k(i) = (\alpha_k i - n_k - 1, n_k + 2 - \alpha_k i + i) &\to (\alpha_k i - n_k - 1, n_k + 1) \\ &\to (\alpha_k^2 i - n_k - 2, n_k + 2 - \alpha_k^2 i + \alpha_k i) = d_k(\alpha_k i)\end{aligned}$$

of $U_k(i,\alpha)$, and the arrow $d_k(\alpha_k i) \to d_k(\alpha_k i)^*$. By $U_k(i,\beta)$ we denote the "β-path" from $\tau^{-1}d_k(i)$ to $d_k(i+n_k+1)$, and we let $u_k(\beta): d_k(i)^* \to d_k(\beta_k i)^*$ be defined in an analogous way, using the subpath from $\tau^{-1}d_k(i)$ to $d_k(\beta_k i)$ of $U_k(i,\beta)$, for each arrow $\beta: i \to \beta_k i$.

There exist non-zero scalars $\lambda_k(i,\alpha)$ and $\lambda_k(i,\beta)$, such that sending the vertex i to $d_k(i)^*$ and the arrows $\alpha: i \to \alpha_k i$ and $\beta: i \to \beta_k i$ to $\lambda_k(i,\alpha)\tilde{u}_k(\alpha)$ and $\lambda_k(i,\beta)\tilde{u}_k(\beta)$, respectively, we obtain an isomorphism from $k\tilde{Q}_k/\tilde{I}_k$ to $\tilde{\Lambda}_k$. In fact, the non-zero scalars can be chosen to be ± 1. The quiver of $\tilde{\Lambda}_k$ is obtained from $\tilde{Q}_k$ be deleting the arrows from i to $i+n_k+1$, except in case $n_k = 0$, where only one of the two arrows $\alpha, \beta: i \to i+1$ may be deleted.

Notice that $\alpha_k 0 = n_k + 1$, since $\mathscr{D}_k^+$ contains $(-1,1)$ by definition. For i in the $\beta_k^{\mathbb{Z}}$-orbit of 0, but $i \not\equiv 0$ modulo n_k+1, we let $c_k i < b_k i$ be such that

$$\beta_k^{c_k i}(i) \equiv 0 \text{ modulo } n_k + 1.$$

5.5 Now we can describe the full subcategory $\tilde{\Lambda}$ of projective objects of $k(\Gamma)$ by quiver and relations. First we define a quiver $\tilde{Q} = \tilde{Q}(\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)$. We start from the disjoint union K of $\tilde{Q}_1$, $\tilde{Q}_2$, and $\tilde{Q}_3$, and we denote its vertices by pairs $[k,i]$, for $k = 1,2,3$ and $i \in \mathbb{Z}$. We delete the arrows

$$\alpha: [k, s(n_k+1)] \to [k, (s+1)(n_k+1)]$$

$$\beta: [k, s(n_k+1)] \to [k, s(n_k+1) + \beta_k 0]$$

in K for all $s \in \mathbb{Z}$. We add the following arrows:

$$[1, s(n_1+1)] \xrightarrow{\gamma} [2, s(n_2+1)] \xrightarrow{\beta} [1, s(n_1+1) + \beta_1 0],$$

$$[2, s(n_2+1)] \xrightarrow{\gamma} [3, s(n_3+1)] \xrightarrow{\beta} [2, s(n_2+1) + \beta_2 0],$$

$$[3, s(n_3+1)] \xrightarrow{\gamma} [1, (s+1)(n_1+1)] \xrightarrow{\beta} [3, s(n_3+1) + \beta_3 0],$$

for all $s \in \mathbb{Z}$. This is $\tilde{Q}$.

We let $\tilde{I}$ be the ideal of $k\tilde{Q}$ generated by the paths

$$\begin{cases} \alpha\beta \text{ and } \beta\alpha \\ [1, s(n_1+1) + \beta_1^{-1} 0] \xrightarrow{\beta} [1, s(n_1+1)] \xrightarrow{\beta} [3, (s-1)(n_3+1) + \beta_3 0], \\ [2, s(n_2+1) + \beta_2^{-1} 0] \xrightarrow{\beta} [2, s(n_2+1)] \xrightarrow{\beta} [1, s(n_1+1) + \beta_1 0], \\ [3, s(n_3+1) + \beta_3^{-1} 0] \xrightarrow{\beta} [3, s(n_3+1)] \xrightarrow{\beta} [2, s(n_2+1) + \beta_2 0], \end{cases}$$

along with the differences of paths $[k,i] \to [k, i+n_k+1]$

$$\begin{cases} \alpha^{a_k i} - \beta^{b_k i} \text{ if } i \notin \beta_k^{\mathbb{Z}} 0, \\ \alpha^{a_k i} - \beta^{b_k i - c_k i} \gamma \beta^{c_k i} \text{ if } i \in \beta_k^{\mathbb{Z}} 0, \text{ but } i \not\equiv 0 \text{ modulo } n_k + 1, \end{cases}$$

and finally the differences

$$\begin{cases} \gamma^2 - \beta^{b_3 0} : [1, s(n_1+1)] \to [3, s(n_3+1)], \\ \gamma^2 - \beta^{b_1 0} : [2, s(n_2+1)] \to [1, (s+1)(n_1+1)], \\ \gamma^2 - \beta^{b_2 0} : [3, s(n_3+1)] \to [2, (s+1)(n_2+1)], \end{cases}$$

for all $s \in \mathbb{Z}$.

Fig. 16 shows Γ and Γ_k, portions of the quivers of $\tilde{\Lambda}$ and $\tilde{\Lambda}_k$, and the quivers $Q = \tilde{Q}/\tau^{(2n-3)\mathbb{Z}}$ and $Q_k = \tilde{Q}_k/\tau^{(n_k+1)\mathbb{Z}}$, where $k = 1, 2, 3$, for a configuration $\mathscr{C}$ of $\mathbb{Z}D_{10}$ with $n_1 = 0$, $n_2 = 3$, $n_3 = 4$. The α- and γ-arrows are drawn full, the β-arrows broken.

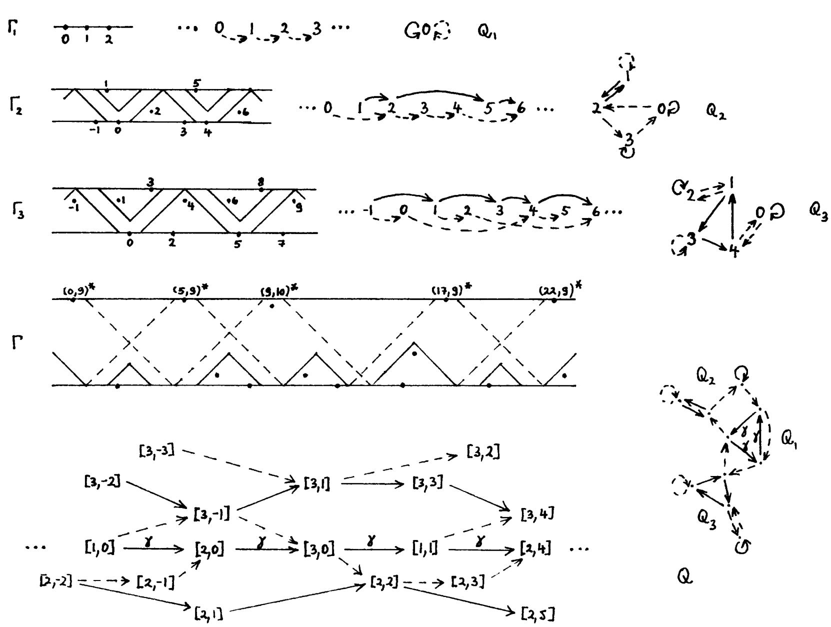


Fig. 16

Proposition: *The category $k\tilde{Q}/\tilde{I}$ is isomorphic to $\tilde{\Lambda}$.*

PROOF: We identify the vertices of $\tilde{Q}$ with the objects of $\tilde{\Lambda}$, sending $[k, i]$ to $\psi[k, i] = \chi_k d_k(i)^*$. Note that

$$\psi[k, i + n_k + 1] = \tau^{-(2n-3)}\psi[k, i]$$

and that

$$\psi[1, 0] = (\phi^{n+n_1-1}(2 - n + n_1, n - 1))^*,$$
$$\psi[2, 0] = (0, n - 1)^*,$$
$$\psi[3, 0] = (\phi^{n_1+n_3}(n_1 + n_3 + 1, n - 1))^*,$$

(see 5.1). For each arrow $\delta: [k, i] \to [k', i']$ of $\tilde{Q}$, we define a path $v(\delta): \psi[k, i] \to \psi[k', i']$ in Γ. For an arrow $\alpha: [k, i] \to [k, \alpha_k i]$ or $\beta: [k, i] \to [k, \beta_k i]$ with $i \not\equiv 0$ modulo $n_k + 1$, we set

$$v(\alpha) = \chi_k u_k(\alpha) \text{ and } v(\beta) = \chi_k u_k(\beta).$$

For an arrow $\gamma: [k, s(n_k + 1)] \to [j, t(n_j + 1)]$, the vertices $\psi[k, s(n_k + 1)] = (p_1, q_1)^*$ and $\psi[j, t(n_j + 1)] = (p_2, q_2)^*$ are consecutive high projective vertices of Γ, and we set

$$v(\gamma) = \begin{cases} \iota_2 \varepsilon_2 l_{p_2-1} \ldots l_{p_1+2} \delta_1 \kappa_1 & \text{if } p_2 > p_1 + 1 \\ \iota_2 \kappa_1 & \text{if } p_2 = p_1 + 1 \end{cases}$$

with

$$(p_1, q_1)^* \xrightarrow{\kappa_1} (p_1 + 1, q_1) \xrightarrow{\delta_1} (p_1 + 2, n - 2) \text{ and}$$

$$(p_2, n - 2) \xrightarrow{\varepsilon_2} (p_2, q_2) \xrightarrow{\iota_2} (p_2, q_2)^* \text{ (Fig. 17).}$$

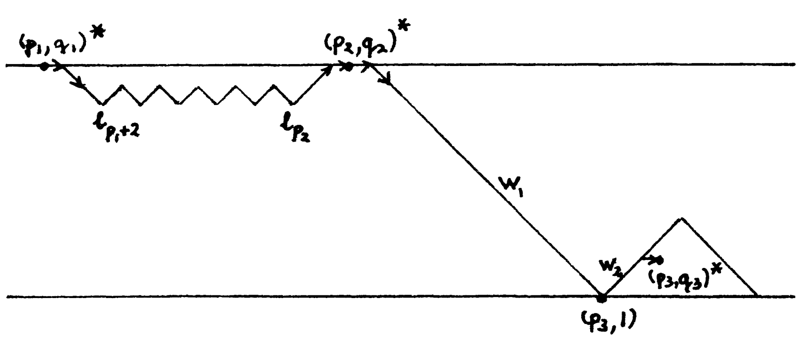


Fig. 17

For the arrow $\beta:[j, t(n_j+1)] \to [k, s(n_k+1)+\beta_k 0]$, the vertex $\psi[k, s(n_k+1)+\beta_k 0] = (p_3, q_3)^*$ satisfies $p_3 = p_2 + n - 1$; it is high if and only if $n_k = 0$. We set

$$v(\beta) = \iota_3 w_2 w_1 \delta_2 \kappa_2$$

with

$$(p_2, q_2)^* \xrightarrow{\kappa_2} (p_2+1, q_2) \xrightarrow{\delta_2} (p_2+2, n-2) \text{ and}$$

$$(p_3, q_3) \xrightarrow{\iota_3} (p_3, q_3)^*,$$

where w_1 and w_2 are the only paths $w_1:(p_2+2, n-2) \to (p_2+n-1, 1) = (p_3, 1)$ and $w_2:(p_3, 1) \to (p_3, q_3)$ (Fig. 17).

We claim that

$$\tilde{v}(\beta)\tilde{v}(\gamma) = -\chi_k \tilde{u}_k(\beta),$$

where on the left-hand side $\gamma:[k, s(n_k+1)] \to [j, t(n_j+1)]$ and $\beta:[j, t(n_j+1)] \to [k, s(n_k+1)+\beta_k 0]$ are arrows of $\tilde{Q}$ and on the right-hand side $\beta: s(n_k+1) \to s(n_k+1)+\beta_k 0$ is an arrow of $\tilde{Q}_k$. Indeed modulo vectors in I_s, we have

$$\delta_2 \kappa_2 \iota_2 \varepsilon_2 = -h_{p_2+1} h_{p_2} = -l_{p_2+1} h_{p_2} - h_{p_2+1} l_{p_2} + l_{p_2+1} l_{p_2},$$

and $w_1 l_{p_2+1}$ is $\mathscr{C}$-marginal (Fig. 17). In case $p_2 > p_1 + 1$, we see that

$$\tilde{v}(\beta)\tilde{v}(\gamma) = -\tilde{\iota}_3 \tilde{w}_2 \tilde{w}_1 \tilde{h}_{p_2+1} \tilde{l}_{p_2} \ldots \tilde{l}_{p_1+2} \tilde{\delta}_1 \tilde{\kappa}_1 = -\chi_k \tilde{u}_k(\beta)$$

(5.1, Fig. 13). In case $p_2 = p_1 + 1$, we replace $\kappa_2 \iota_2$ by $-(\sigma\varepsilon_2)(\sigma^2\varepsilon_2)$.

In 5.3, we saw that any $\mathscr{C}$-essential path in Γ from $(1, n-1)$ to $(2n-3, n-1)$ is $\mathscr{C}$-homotopic to

$$w = \delta_4 l_{2n-3} \ldots l_{n_1+n_3+3} h_{n_1+n_3+2} l_{n_1+n_3+1} \ldots l_2 \varepsilon_1$$

or equivalently to

$$w' = \delta_4 l_{2n-3} \ldots l_{n+n_1} h_{n+n_1-1} l_{n+n_1-2} \ldots l_2 \varepsilon_1$$

with $\varepsilon_1:(1, n-1) \to (2, n-2)$ and $\delta_4:(2n-3, n-2) \to (2n-3, n-1)$. On the other hand, we know by [2], 1.2 that

$$k(\Gamma)((0,n-1)^*,(2n-3,n-1)^*) \neq 0,$$

and hence w and w' are $\mathscr{C}$-essential. It is easy to see that the subpath $v:(1,n-1) \to (2n-3,n-1)$ of $v(\gamma_3)v(\gamma_2)v(\gamma_1)$ satisfies $\pi' v = w$, where π' is the projection to the space of $\mathscr{C}$-admissible paths defined in 2.4, and where γ_1, γ_2, γ_3 are the arrows

$$[2,0] \xrightarrow{\gamma_1} [3,0] \xrightarrow{\gamma_2} [1,n_1+1] \xrightarrow{\gamma_3} [2,n_2+1].$$

The subpath $\delta_3 l_{n-2+n_1} \ldots l_2 \varepsilon_1 : (1,n-1) \to \phi^{n+n_1-1}(n+n_1,n-1)$ of w is $\mathscr{C}$-homotopic to the path $\delta_3 w_3 w_2 w_1 \varepsilon_1$ with $w_1:(2,n-2) \to (n-1,1)$, $w_2:(n-1,1) \to (n-1,n_1+1) \to (n+n_1-1,1)$ and $w_3:(n+n_1-1,1) \to (n+n_1-1,n-2)$. The path w_3 is the image under χ_1 of the α-path $U_1(0,\alpha):(0,1) \to (n_1,1)$ in Γ_1, and hence it is $\mathscr{C}$-homotopic to $\chi_1 U_1(0,\beta)$. We see that

$$\tilde{v}(\gamma_2)\tilde{v}(\gamma_1) = \pm \tilde{v}(\beta_{b_1 0}) \ldots \tilde{v}(\beta_1),$$

where $\beta_1 : [2,0] \to [1,\beta_1 0]$, $\beta_r : [1,\beta_1^{r-1} 0] \to [1,\beta^r 0]$, for $r = 2,\ldots,b_1 0$. In the same way, we obtain

$$\tilde{v}(\gamma_3)\tilde{v}(\gamma_2) = \pm \tilde{v}(\beta_{b_2 0}) \ldots \tilde{v}(\beta_1),$$

where $\beta_1 : [3,0] \to [2,\beta_2 0]$, $\beta_r : [2,\beta_2^{r-1} 0] \to [2,\beta_2^r 0]$, for $r = 2,\ldots,b_2 0$. On the other hand, any low $\mathscr{C}$-essential path from $(1,n-1)$ to a low point of $\mathscr{C}$ factors through $w_1\varepsilon_1$ (5.3), and by 5.2 it has the form $\chi_1(v')w_1\varepsilon_1$, where $v':(1,1) \to d_1(i) \in \mathscr{D}_1^+$ is $\mathscr{D}_1^+$-essential. Then we know that $i = \beta_1^b 0$ for some $b < b_1 0$ by [5], 5.7. To summarize, the paths $\delta_r \ldots \delta_1$ in $\tilde{Q}$ starting at $[2,0]$ which give rise to non-zero morphisms $\tilde{v}(\delta_r) \ldots \tilde{v}(\delta_1)$ in $\tilde{\Lambda}$ are precisely the paths

$$\gamma^r \text{ for } r \leq 3,\ \beta^b \text{ for } b \leq b_1 0,\ \gamma\beta^{b_1 0}, \text{ and } \beta^b\gamma \text{ for } b \leq b_2 0.$$

Because by [2], 1.2,

$$k(\Gamma)(\psi[2,0],\psi[k,i]) \neq 0$$

if and only if

$$k(\Gamma)(\psi[k,i],\psi[2,n_2+1]) \neq 0,$$

we obtain that the paths $\delta_r \ldots \delta_1$ of $\tilde{Q}$ stopping at $[2, n_2 + 1]$ which give rise to non-zero morphisms $\tilde{v}(\delta_r) \ldots \tilde{v}(\delta_1)$ are precisely the

$$\gamma^r \text{ for } r \leq 3,\ \beta^b \text{ for } b \leq b_2 0,\ \beta^{b_2 0}\gamma, \text{ and } \gamma\beta^b \text{ for } b \leq b_1 0.$$

Of course, we obtain analogous descriptions for all paths $\delta_r \ldots \delta_1$ starting or stopping at any vertex $[k, s(n_k + 1)]$ with $\tilde{v}(\delta_r) \ldots \tilde{v}(\delta_1) \neq 0$.

Let $[k, i]$ be a vertex of $\tilde{Q}$ with $i \not\equiv 0 \bmod n_k + 1$. There exists a $\mathscr{C}$-essential path $w: \tau^{-1}\chi_k d_k(i) \to \chi_k d_k(i + n_k + 1)$ in Γ, and, by 5.2, w is $\mathscr{C}$-homotopic to $\chi_k v$ for some $v: \tau^{-1} d_k(i) \to d_k(i + n_k + 1)$. Any such v is $\mathscr{D}_k^+$-homotopic to both the α-path $U_k(i, \alpha)$ and the β-path $U_k(i, \beta)$ ([5], 5.7). Let $\alpha_{a_k i} \ldots \alpha_2 \alpha_1$ and $\beta_{b_k i} \ldots \beta_2 \beta_1$ be paths from i to $i + n_k + 1$ in $\tilde{Q}_k$. Then

$$\tilde{u}(\alpha_{a_k i}) \ldots \tilde{u}(\alpha_1) = \pm \tilde{\iota} \tilde{U}_k(i, \alpha) \tilde{\kappa},$$
$$\tilde{u}(\beta_{b_k i}) \ldots \tilde{u}(\beta_1) = \pm \tilde{\iota} \tilde{U}_k(i, \beta) \tilde{\kappa},$$

where $\kappa: d_k(i)^* \to \tau^{-1} d_k$ and $\iota: d_k(i + n_k + 1) \to d_k(i + n_k + 1)^*$. Therefore we see that the following paths $\delta_r \ldots \delta_1$ of $\tilde{Q}$ starting at $[k, i]$ give rise to non-zero morphisms $\tilde{v}(\delta_r) \ldots \tilde{v}(\delta_1)$ in $\tilde{\Lambda}$:

$$\begin{cases} \alpha^a \text{ for } a \leq a_k i, \\ \beta^b \text{ for } b \leq b_k i, \text{ if } i \notin \beta_k^{\mathbb{Z}} 0, \\ \beta^b \text{ for } b \leq c_k i \text{ and } \beta^b \gamma \beta^{c_k i} \text{ for } b \leq b_k i - c_k i, \text{ if } i \in \beta_k^{\mathbb{Z}} 0. \end{cases}$$

On the other hand, let $w: \tau^{-1}\chi_k d_k(i) \to \chi_{k'} d_{k'}(i')$ be a $\mathscr{C}$-essential path. We may assume that $i' \not\equiv 0$ modulo $n_{k'} + 1$. Then $k' = k$ by 5.2, and w is $\mathscr{C}$-homotopic to some $\chi_k v$. Thus $i' = \beta_k^b(i)$ for $b \leq b_k i$ or $i' = \alpha_k^a(i)$ for $a \leq a_k i$, and the paths $\delta_r \ldots \delta_1$ listed above are the only ones with $\tilde{v}(\delta_r) \ldots \tilde{v}(\delta_1) \neq 0$.

By definition, $\tilde{I} \subset k\tilde{Q}$ is the ideal generated by the differences of paths yielding non-zero morphisms in $\tilde{\Lambda}$ along with the paths yielding zero. We conclude that $k\tilde{Q}/\tilde{I}$ is isomorphic to $\tilde{\Lambda}$ ([2], 5). In fact, for each arrow δ of $\tilde{Q}$ we can choose $\lambda_\delta = \pm 1$ such that the functor $\psi: k\tilde{Q} \to \tilde{\Lambda}$ induced by sending δ to $\psi\delta = \lambda_\delta \tilde{v}(\delta)$ induces the above isomorphism.

REMARK: The quiver $Q_k = \tilde{Q}_k / \tau^{(n_k + 1)\mathbb{Z}}$ is an oriented Brauer-quiver with $n_k + 1$ vertices containing an α-loop in $\tau^{(n_k + 1)\mathbb{Z}} 0$, for $k = 1, 2, 3$ ([3], [5], 6.2). Denote the Brauer-quiver obtained by changing the orientation of Q_k by P_k. Then $\tilde{\Lambda}/\tau^{(2n-3)\mathbb{Z}}$ is isomorphic to the category defined by the quiver and the relations describing the three-cornered algebra $D(P_3 P_2 P_1)$ ([2], 7.2).

5.6 Let $\mathscr{C}$ be a configuration of $\mathbb{Z}D_n$ for which all numbers n_1, n_2, and n_3 are positive, and let $\tilde{A}$ be the full subcategory of $k(\Gamma)$ whose objects are the high projective vertices of Γ together with the $(i,j)^*$ for which i is congruent to $n-1$, $n+n_1+n_3$, or $2n-2+n_1$ modulo $2n-3$ (compare 4.3). The category $\tilde{A}$ is isomorphic to the full subcategory of $k\tilde{Q}/\tilde{I}$ whose objects are the $[k, s(n_k+1)]$ and $[k, s(n_k+1)+\beta_k 0]$, for $k=1,2,3$ and $s \in \mathbb{Z}$. Write $i \in \mathbb{Z}$ as $i = 6s_i + t_i$ with $0 \leq t_i < 6$, and identify $\mathbb{Z}$ with the objects of $\tilde{A}$ by sending i to

$$\begin{cases} [1, s_i(n_1+1)], [2, s_i(n_2+1)], [3, s_i(n_3+1)] \\ \qquad \text{for } t_i = 0, 2, 4, \text{respectively}, \\ [3, (s_i-1)(n_3+1)+\beta_3 0], [1, s_i(n_1+1)+\beta_1 0], \\ \qquad [2, s_i(n_2+1)+\beta_2 0] \text{ for } t_i = 1, 3, 5, \text{ respectively.} \end{cases}$$

We obtain that $\tilde{A}$ is isomorphic to $k\tilde{K}/\tilde{J}$, where $\tilde{K}$ is the quiver with vertex set $\mathbb{Z}$ which contains the arrows

$$\gamma_{2i}: 2i \to 2i+2, \ \beta_{2i}: 2i \to 2i+1, \text{ and } \beta_{2i+1}: 2i+1 \to 2i+4,$$

for each $i \in \mathbb{Z}$, and where $\tilde{J}$ is the ideal of $k\tilde{K}$ generated by

$$\gamma_{2i+2}\gamma_{2i} - \beta_{2i+1}\beta_{2i} \text{ and } \beta_{2i+4}\beta_{2i+1}$$

for all i.

5.7 Let $\mathscr{C}$ be a $\tau^{(2m-1)\mathbb{Z}}$-stable configuration of $\mathbb{Z}D_{3m}$ containing $(0, n-1)$, where $n = 3m$, and let $\pi: \Gamma \to \Delta = \Gamma/\tau^{(2m-1)\mathbb{Z}}$ be the universal covering. Our aim is to describe the standard category Λ, and if char $k = 2$, the non-standard category Λ' with Auslander–Reiten quiver Δ by quivers and relations.

The three numbers n_1, n_2, and n_3 associated with $\mathscr{C}$ are all equal to $m-1$, and the three configurations $\mathscr{D}_1^+$, $\mathscr{D}_2^+$, and $\mathscr{D}_3^+$ of $\mathbb{Z}A_m$ coincide (2.5). By α and β we denote the permutations $\alpha = \alpha_1$ and $\beta = \beta_1$ of $\mathbb{Z}$, and we set $ai = a_1 i$, $bi = b_1 i$, and $ci = c_1 i$, for each $i \in \mathbb{Z}$ (5.4). The automorphism τ^m of Γ_1 induces an automorphism τ^m of $\tilde{Q}_1$, which is given by $\tau^m i = i - m$. We let Q_1 be the residue quiver $\tilde{Q}_1/\tau^{m\mathbb{Z}}$. We identify the vertex $\tau^{m\mathbb{Z}} i$ of Q_1 with the residue class $\bar{i}$ of i modulo m, and we set $\tau^{m\mathbb{Z}}\alpha = \bar{\alpha}$ and $\tau^{m\mathbb{Z}}\beta = \bar{\beta}$ for the arrows. The quiver Q_1 is an oriented Brauer-quiver with m vertices ([3], 1.4, [5], 3.4). Since $\overline{\alpha 0} = \bar{0}$, Q_1 contains an $\bar{\alpha}$-loop in $\bar{0}$.

The automorphism τ^{2m-1} of Γ induces an automorphism τ^{2m-1} of $\tilde{Q}$, which takes

$[3,i]$ to $[2,i]$, $[2,i]$ to $[1,i]$, and $[1,i]$ to $[3,i-m]$.

The residue quiver $Q = \tilde{Q}/\tau^{(2m-1)\mathbb{Z}}$ is obtained from Q_1 by replacing the loop $\bar{\alpha}:\bar{0} \to \bar{0}$ by the loop $\bar{\gamma}:\bar{0} \to \bar{0}$ (5.5). We let $\pi:\tilde{Q} \to Q$ be the natural map. Fig. 18 shows Q for a configuration $\mathscr{C}$ of $\mathbb{Z}D_{24}$.

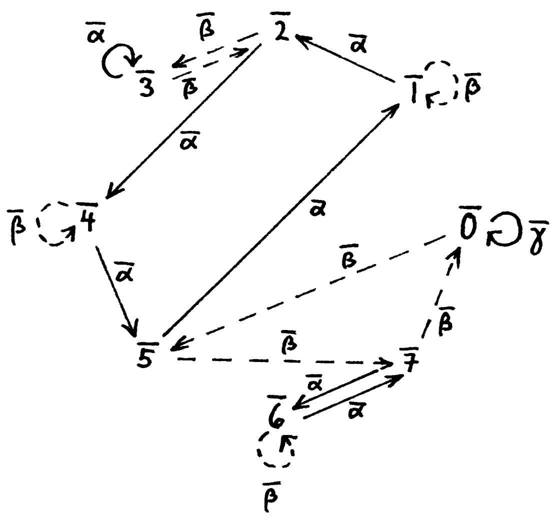


Fig. 18

Let I and I' be the ideals of kQ generated by

$$\bar{\beta}^2 : \overline{\beta^{-1}0} \to \overline{\beta 0} \text{ and } \bar{\beta}^2 + \bar{\beta}\bar{\gamma}\bar{\beta} : \overline{\beta^{-1}0} \to \overline{\beta 0},\ \bar{\gamma}^4 : \bar{0} \to \bar{0},$$

respectively, along with

$$\begin{cases} \text{all paths } \bar{\alpha}\bar{\beta} \text{ and } \bar{\beta}\bar{\alpha}, \\ \bar{\alpha}^{ai} - \bar{\beta}^{bi} : \bar{i} \to \bar{i} \text{ if } i \notin \beta^{\mathbb{Z}}0, \\ \bar{\alpha}^{ai} - \bar{\beta}^{bi-ci}\bar{\gamma}\bar{\beta}^{ci} : \bar{i} \to \bar{i} \text{ if } \bar{i} \neq \bar{0},\ i \in \beta^{\mathbb{Z}}0, \\ \bar{\gamma}^2 - \bar{\beta}^{b0} : \bar{0} \to \bar{0}. \end{cases}$$

Proposition: (a) *The category Λ is isomorphic to kQ/I.*

(b) *The category Λ' is isomorphic to kQ/I'.*

REMARKS: (i) The standard and non-standard algebras

$$\oplus\, \Lambda(x,y) \quad\text{and}\quad \oplus\, \Lambda'(x,y)$$

with Auslander–Reiten quiver Δ are given by the quiver Q and the relations I and I', respectively; the summations range over all objects x and y of Λ and Λ'.

(ii) As a consequence of (b), we obtain the description of the full subcategory of $k\Delta/J$ whose objects are $\pi(0,n-1)^*$ and $\pi(n-1,\beta 0)^*$, or equivalently the full subcategory of Λ' whose objects are $\bar{0}$ and $\overline{\beta 0}$, by quiver and relations used in 4.3.

PROOF: (a) By [2], 5.3, Λ is isomorphic the residue category of kQ modulo the image of $\tilde{I}$ under $\pi: k\tilde{Q} \to kQ$, which is I (5.5).

(b) Let $\operatorname{char} k = 2$. Then the functor $\psi: k\tilde{Q} \to \tilde{\Lambda}$ defined in 5.5 is given by $\psi(\delta) = \tilde{v}(\delta)$ for all arrows δ; in other words, all scalars λ_δ equal $+1$. We will define a functor $\psi': kQ/I' \to \Lambda'$ and a covering functor $F': k\tilde{Q}/\tilde{I} \to kQ/I'$ so that the following diagram commutes

$$\begin{array}{ccc} k\tilde{Q}/\tilde{I} & \xrightarrow{\psi} & \tilde{\Lambda} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ kQ/I' & \xrightarrow{\psi'} & \Lambda' \end{array}$$

where $F: \tilde{\Lambda} \to \Lambda'$ is induced by the covering functor $F: k(\Gamma) \to k\Delta/J$ defined in 4.1. Remember that Λ' is the full subcategory of $k\Delta/J$ whose objects are the projective vertices of Δ. Then ψ' is a covering functor, and hence an isomorphism, because it is bijective on the objects.

First we define F'. We set $F'[k,i] = \bar{i}$ and

$F'\alpha = \bar{\alpha}$ for all arrows α,

$F'\beta = \bar{\beta} + \overline{\beta\gamma}$ if β lies in the $\tau^{2(2m-1)\mathbb{Z}}$-orbit of

$$[2,0] \xrightarrow{\beta} [1,\beta 0],$$

$F'\beta = \bar{\beta} + \overline{\gamma\beta}$ if β lies in the $\tau^{2(2m-1)\mathbb{Z}}$-orbit of

$$[3,\beta^{-1}0] \xrightarrow{\beta} [3,0],$$

$F'\beta = \bar{\beta}$ for all other arrows β,

$F'\gamma = \bar{\gamma} + \overline{\gamma^2}$ if γ lies in the $\tau^{2(2m-1)\mathbb{Z}}$-orbit of

$$[2,0] \xrightarrow{\gamma} [3,0],$$

$F'(\gamma) = \bar{\gamma}$ for all other arrows γ.

It is easy to check that F' maps $\tilde{I}$ into I'.

Next we show that F' is a covering functor; i.e., that for any two vertices x and y of $\tilde{Q}$, F' induces bijections

$$\bigoplus_{\pi z = \pi y} k\tilde{Q}/\tilde{I}(x, z) \to kQ/I'(\pi x, \pi y) \leftarrow \bigoplus_{\pi z = \pi x} k\tilde{Q}/\tilde{I}(z, y).$$

We will prove that the first map is an isomorphism. Notice that

$$\bar{\gamma}^2\bar{\beta} : \overline{\beta^{-1}0} \to \bar{0} \text{ and } \bar{\beta}\bar{\gamma}^2 : \bar{0} \to \overline{\beta 0}$$

lie in I'; indeed,

$$\bar{\gamma}^2\bar{\beta} \equiv \bar{\beta}^{b0-1}\bar{\beta}\bar{\gamma}\bar{\beta} \equiv \bar{\gamma}^3\bar{\beta} \equiv \bar{\gamma}^4\bar{\beta} \equiv 0 \text{ modulo } I'.$$

If $\bar{i} \neq \bar{j}$ and $i \notin \beta^{\mathbb{Z}}0$, $j \notin \beta^{\mathbb{Z}}0$, there is at most one path from $\bar{i}$ to $\bar{j}$ which does not lie in I'; if there is one, or equivalently if $j \in \alpha^{\mathbb{Z}}i$ or $j \in \beta^{\mathbb{Z}}i$, we choose its residue class modulo I' as a basis for $kQ/I'(\bar{i}, \bar{j})$. If $\bar{i} \neq 0$, we choose the trivial path at $\bar{i}$ and $\bar{\alpha}^{ai}$ as a basis for $kQ/I'(\bar{i}, \bar{i})$. In the remaining cases, we choose the residue classes of the following paths as a basis of $kQ/I'(\bar{i}, \bar{j})$:

$1_{\bar{0}}$, $\bar{\gamma}$, $\bar{\gamma}^2$, $\bar{\gamma}^3$ for $\bar{i} = \bar{j} = \bar{0}$,

$\bar{\beta}^{ci}$, $\bar{\gamma}\bar{\beta}^{ci}$ for $\bar{j} = \bar{0}$, $\bar{i} \neq \bar{0}$,

$\bar{\beta}^{bj-cj}$, $\bar{\beta}^{bj-cj}\bar{\gamma}$ for $\bar{i} = \bar{0}$, $\bar{j} \neq \bar{0}$,

$\bar{\beta}^{b}$ for $\bar{j} = \overline{\beta^{b}i}$ with $0 < b < ci$, $\bar{i} \neq \bar{0}$,

$\bar{\beta}^{b}\bar{\gamma}\bar{\beta}^{ci}$ for $\bar{j} = \overline{\beta^{b}i}$ with $ci < b < bi$, $\bar{i} \neq \bar{0}$.

If $k\tilde{Q}/\tilde{I}([k,i],[k',j]) \neq 0$, we choose the only path from $[k,i]$ to $[k',j]$ in $\tilde{Q}$ which does not lie in $\tilde{I}$ as a basis. With respect to these bases, the map

$$F' : \bigoplus_{s \in \mathbb{Z}} k\tilde{Q}/\tilde{I}([k,i], \tau^{s(2m-1)}[k,j]) \to kQ/I'(\bar{i}, \bar{j})$$

of (*) is given by the identity matrix if $i \not\equiv 0$ and $j \not\equiv 0$ modulo m or if $[k,i]$ lies in the $\tau^{2(2m-1)\mathbb{Z}}$-orbit of $[3,0]$ and $j \not\equiv 0$ modulo m or if $[k,j]$ lies in the $\tau^{2(2m-1)\mathbb{Z}}$-orbit of $[1,m]$ and $i \not\equiv 0$ modulo m. It is given by

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ if $[k,i] \in \tau^{2(2m-1)\mathbb{Z}}[2,0]$ and $j \not\equiv 0$ modulo m or if $[k,j] \in \tau^{2(2m-1)\mathbb{Z}}[3,0]$ and $i \not\equiv 0$ modulo m,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{if } [k, i] \in \tau^{2(2m-1)\mathbb{Z}}[2,0] \text{ and } j \equiv 0 \text{ modulo m,}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{if } [k, i] \in \tau^{2(2m-1)\mathbb{Z}}[3,0] \text{ and } i \equiv 0 \text{ modulo } m.$$

Since all these matrices, as well as the ones obtained from the second map in (*), are non-singular, F' is a covering functor.

Define $\psi': kQ \to \Lambda' \subset k\Delta/J$ to be the functor induced by $\psi'\bar{\imath} = \pi\psi[1, i]$ and $\psi'\bar{\delta} = Gv(\delta)$ for all arrows $\bar{\delta}$ of Q, where δ is an arrow of $\tilde{Q}$ with $\pi\delta = \bar{\delta}$ and where $G: k\Gamma \to k\Delta/J$ is composed from $\pi: k\Gamma \to k\Delta$ and the natural functor $k\Delta \to k\Delta/J$ (4.1). Remember that $G\theta_z = 0$ for all (modified) mesh-relations θ_z with $z \notin \tau^{(2m-1)\mathbb{Z}}\mathbb{1} \phi\, n-1)$. Therefore $G(\sum \lambda_i v_i) = 0$ if $\sum \lambda_i v_i \in I_s$ and if none of the paths v_i contains a subpath

$$(s(2m-1), n-1) \xrightarrow{\iota_s} (s(2m-1), n-1)^* \xrightarrow{\kappa_s} (s(2m-1)+1, n-1).$$

Hence ψ' vanishes on all generators of I' for which no summand factors through $\bar{0}$. If $\bar{\delta}_t \ldots \bar{\delta}_1$ is a path in Q which does factor through $\bar{0}$, we choose $\delta_t \ldots \delta_1$ in $\tilde{Q}$ with $\pi(\delta_t \ldots \delta_1) = \bar{\delta}_t \ldots \bar{\delta}_1$ and we write

$$v(\delta_t) \ldots v(\delta_1) = w_r \kappa_{s_r} \iota_{s_r} w_{r-1} \ldots w_1 \kappa_{s_1} \iota_{s_1} w_0,$$

where no w_j factors through a $(s(2m-1), n-1)^*$. Then

$$\begin{aligned} \psi'(\bar{\delta}_t \ldots \bar{\delta}_1) &= Gw_r G(\varepsilon_{s_r}\varepsilon'_{s_r} + \varepsilon_{s_r+1} v_{s_r} \varepsilon'_{s_r}) Gw_{r-1} \ldots \\ &\quad \ldots Gw_1 G(\varepsilon_{s_1}\varepsilon'_{s_1} + \varepsilon_{s_1+1} v_{s_1} \varepsilon'_{s_1}) Gw_0 \\ &= G(w_r \varepsilon_{s_r} \varepsilon'_{s_r} w_{r-1} \ldots w_1 \varepsilon_{s_1} \varepsilon'_{s_1} w_0) + \sum Gu_j, \end{aligned}$$

where $(s(2m-1), n-1) \xrightarrow{\varepsilon'_s} (s(2m-1), n-1)^* \xrightarrow{\varepsilon_s} (s(2m-1)+1,$ $n-1)$ and $v_s = l_{(s+1)(2m-1)-1} \ldots l_{s(2m-1)+1}$. Notice that each u_j is strictly longer than $v(\delta_t) \ldots v(\delta_1)$. In particular, ψ' vanishes on $\bar{\alpha}^{ai} + \bar{\beta}^{bi-ci}\bar{\gamma}\bar{\beta}^{ci}$ for $i \in \beta^{\mathbb{Z}}0$, $\bar{\imath} \neq 0$, and on $\bar{\gamma}^4$, $\bar{\beta}\bar{\gamma}^2$, and $\bar{\gamma}^2\bar{\beta}$, since in these cases all u_j lie in I_s (5.2, 5.3). We see that

$$\begin{aligned} \psi'\bar{\gamma}^2 &= G(\iota_2(\sigma\varepsilon'_2) l_{4m-3} \ldots l_{2m+1} h_{2m} h_{2m-1} l_{2m-2} \ldots \\ &\quad \ldots l_2(\bar{\sigma}^1 \varepsilon_0)\kappa_0) + Gu = \psi'\bar{\beta}^{b0}, \end{aligned}$$

since G vanishes on

$$u = \iota_3(\sigma\varepsilon_3')l_{2n-4}\dots l_{4m}h_{4m-1}l_{4m-2}\dots l_{2m}h_{2m-1}l_{2m-2}\dots l_2(\sigma^{-1}\varepsilon_0)\kappa_0$$

(5.3). Similarly, we obtain

$$\psi'\bar{\beta}^2 = \psi'\bar{\beta}\bar{\gamma}\bar{\beta},$$

for $\bar{\beta}^2 : \overline{\beta^{-1}0} \to \overline{\beta 0}$. Hence ψ' induces a functor $\psi' : kQ/I' \to \Lambda'$.

As for the commutativity, it suffices to show that $F\psi(\delta) = \psi' F'(\delta)$ for all arrows δ of $\tilde{Q}$. By definition of F (4.1), we have $Fv = Gv + \sum Gu_j$ for any path $v : x \to y$ in Γ, where $u_j : x \to \tau^{-s_j(2m-1)}y$ for $s_j > 0$. This implies that

$$F\psi(\delta) = Fv(\delta) = Gv(\delta) = \psi' F'(\delta),$$

whenever $F'\delta = \bar{\delta}$. For arrows $\delta : [k, i] \to [k, j]$ with $i \not\equiv 0 \not\equiv j$ modulo m, this follows from the fact that any path in Γ from $\psi[k, i]$ to $\tau^{-s(2m-1)}\psi[k, j]$ lies in I_s for $s > 0$. For the other arrows with $F'\delta = \bar{\delta}$, it is a direct consequence of the definition of F. It suffices to prove that

$$Fv(\beta) = Gv(\beta) + Gv(\beta)v(\gamma) \text{ for } \beta : [2, 0] \to [1, \beta 0],$$

$$Fv(\beta) = Gv(\beta) + Gv(\gamma)v(\beta) \text{ for } \beta : [3, \beta^{-1}0] \to [3, 0],$$

$$Fv(\gamma) = Gv(\gamma) + Gv(\gamma)v(\gamma) \text{ for } \gamma : [2, 0] \to [3, 0].$$

Using the notations of 4.1, we obtain in the first case $v(\beta) = w\zeta_2\delta_1'\kappa_1$ and

$$Fv(\beta) = Gv(\beta) + G(w\zeta_2\kappa_1)G(\delta_{2m}l_{2m-1}\dots l_2\delta_1'\kappa_1) + GwG(\zeta_{2m+1}w_2)G(\delta_1'\kappa_1).$$

The third summand vanishes, since $(\tau^{-(2m-1)}w)\zeta_{2m-1}w_2\delta_1'\kappa_1$ lies in I_s, and the second summand equals $Gv(\beta)v(\gamma)$. Notice that any path from $(0, n-1)^*$ to $\tau^{-s(2m-1)}\psi[1, \beta 0]$ with $s \geq 2$ lies in I_s as well. The argument in the second case is analogous. In the third case, we have

$$v(\gamma) = \iota_{2m-1}\delta_{2m-1}\zeta_{2m-2}'\zeta_{2m-2}\dots\zeta_2'\zeta_2\delta_1'\kappa_1,$$

and a computation yields

$$Fv(\gamma) = Gv(\gamma) + (1 + 2(2m-3))Gu_1 + 2(2m-3)(2m-2)Gu_2,$$

where

$$u_1 = \iota_{4m-2}\delta_{4m-2}l_{4m-3}\dots l_2\delta'_1\kappa_1,$$

$$u_2 = \iota_{2n-3}\delta_{2n-3}l_{2n-4}\dots l_{2m+1}h_{2m}l_{2m-1}\dots l_2\delta'_1\kappa_1.$$

This ends the proof, since char $k = 2$ and $Gv(\gamma)v(\gamma) = Gu_1$.

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