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DERIVATIVES OF L-FUNCTIONS AT $s = 0$

(*after Stark, Tate, Bienenfeld and Lichtenbaum*)

T. Chinburg*

1. Introduction

Let K/k be a finite Galois extension of number fields with group $G = \mathrm{Gal}(K/k)$. Let V be a complex representation of G, and let $L(s, V)$ be its Artin L-function. Let $r(V)$ be the order of vanishing of $L(s, V)$ at $s = 0$. Stark's conjecture ([8], [9], [10]) concerns the number $c_V = \lim_{s \to 0} s^{-r(V)} L(s, V)$. In this paper we will give a new proof of a theorem of J. Tate ([11]) concerning c_V when the character of V assumes rational values. We will show how this result, Theorem 1 of §2, may be deduced from work of Bienenfeld and Lichtenbaum in [3] and [5].

With thanks to Tate, we describe in §2 the formulation of Stark's conjecture which led to Theorem 1. A result of Bienenfeld and Lichtenbaum is stated in §3. The regulators of Tate and of Bienenfeld–Lichtenbaum are compared in §4, leading to the completion of the proof of Theorem 1 in §5.

This work owes a great debt to H. Stark and to J. Tate. The author would like to thank Tate particularly for having initiated the comparison of his results to those of Bienenfeld and Lichtenbaum. Thanks also go to M. Bienenfeld and to S. Lichtenbaum for notes and conversations concerning their results.

2. Tate's form of Stark's conjecture

Let Y_K be the free abelian group with basis the set S of infinite places of K. Define X_K to be the kernel of the homomorphism $\phi: Y_K \to Z$ in-

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duced by $\phi(v) = 1$ for $v \in S$. Let E_K denote the unit group of K. If Λ is a ring containing the integers Z, and M is a G-module, let ΛM be the G-module $\Lambda \otimes_Z M$, where G acts trivially on Λ. The Dirichlet Unit Theorem shows that the homomorphism $\lambda: \mathbb{C}E_K \to \mathbb{C}X_K$ induced by $\lambda(e) = \sum_{v \in S} \log \|e\|_v v$ for $e \in E_K$ is a G-isomorphism. Therefore the rational vector spaces QE_K and QX_K are isomorphic.

Definition 1 (Tate): *Let V be a complex representation of G, and let $\bar{V}$ be the dual representation. Let $\varphi: X_K \to E_K$ be a G-homomorphism inducing an isomorphism $QX_K \simeq QE_K$. The regulator $R(V, \varphi)$ is the determinant of the homomorphism*

$$(\lambda \circ \varphi)_V : \mathrm{Hom}_G(\bar{V}, \mathbb{C}X_K) \to \mathrm{Hom}_G(\bar{V}, \mathbb{C}X_K)$$

induced by $\lambda \circ \varphi: \mathbb{C}X_K \to \mathbb{C}X_K$.

Conjecture 1 (Stark, à la Tate): *Let $A(V, \varphi) = R(V, \varphi)/c_V$. Then $A(V, \varphi)^\alpha = A(V^\alpha, \varphi)$ for all $\alpha \in \mathrm{Aut}(\mathbb{C}/Q)$.*

The functions c_V, $R(V, \varphi)$ and $A(V, \varphi)$ of V depend only on the isomorphism class of V. Therefore Conjecture 1 implies that $A(V, \varphi)$ is in the field $Q(\chi_V)$ generated by the values of the character χ_V of V. Furthermore, if V_1 and V_2 are two representations of G, we have $A(V_1 + V_2, \varphi) = A(V_1, \varphi)A(V_2, \varphi)$ because $c_{V_1 + V_2} = c_{V_1}c_{V_2}$ and $R(V_1 + V_2, \varphi) = R(V_1, \varphi)R(V_2, \varphi)$.

The author observed following consequence of this. Suppose that V is irreducible, and let $F = Q(\chi_V)$. Let W be a representation of G with character $\chi_W = \sum_{\alpha \in \mathrm{Gal}(F/Q)} \chi_V^\alpha$. Then Conjecture 1 implies that $A(V, \varphi) \in F$, and that $A(W, \varphi) = \prod_{\alpha \in \mathrm{Gal}(F/Q)} A(V, \varphi)^\alpha = N_{F/Q} A(V, \varphi)$. In particular, Conjecture 1 implies the

Norm Conjecture: *If V is irreducible, $F = Q(\chi_V)$ and W has character $\sum_{\alpha \in \mathrm{Gal}(F/Q)} \chi_V^\alpha$, then $A(W, \varphi) \in N_{F/Q}F$.*

The strongest result that has been shown concerning this conjecture is due to Tate:

Theorem 1 (Tate): *Let $I(F)$ denote the group of fractional ideals of F. Then the number $A(W, \varphi)$ is rational, and is positive if F is complex. The ideal $A(W, \varphi)Z$ generated by $A(W, \varphi)$ lies in $N_{F/Q}I(F)$.*

Corollary: *The norm conjecture is true for W if every generator of an ideal in $N_{F/Q}I(F)$ which is positive if F is complex lies in $N_{F/Q}F$.*

COROLLARY: *Conjecture* 1 *is true for representations with rational character.*

The proof to be given here of Theorem 1 consists of comparing a formula for a power of c_W due to Bienenfeld and Lichtenbaum with the above formulation of Stark's conjecture. There results a formula for a power of $A(W, \varphi)$ which with a group theoretic lemma is sufficient to complete the proof.

3. The Bienenfeld–Lichtenbaum theorem

Let M be a finitely generated $G = \mathrm{Gal}(K/k)$ module. Let us recall a theorem of Bienenfeld and Lichtenbaum ([3], [5]) concerning $c_{\mathbb{C}M}$.

Denote by O_k the ring of integers of k. Let $X = \mathrm{Spec}(O_k)$, and let X_∞ be the set of infinite places of k. The Artin–Verdier topology on $\tilde{X} = X \cup X_\infty$ is defined as follows. The connected objects $\mathcal{S}$ of the topology are the union of the generic point $\{0\}$ with the complement of a finite set of places in a finite extension of k, where each place of $\mathcal{S}$ is unramified over its restriction to k. The morphisms of the topology are induced by field homomorphisms over k of finite extensions of k. For further details, see [5], [3] or [2].

Let j and ψ be the inclusions

$$\mathrm{Spec}(k) \underset{j}{\hookrightarrow} X \underset{\psi}{\hookrightarrow} \tilde{X}$$

where $\mathrm{Spec}(k)$ and X are given the etale topology. By inflation, we may regard M as a $\mathrm{Gal}(\bar{k}/k)$ module, and hence as a sheaf on $\mathrm{Spec}(k)$. Let $\psi_!$ be the functor which takes a sheaf on X and extends it by zero to $\tilde{X}$ (cf. [2], p. 65). We will relate the cohomology of the sheaf $\psi_! j_* M$ to $c_{\mathbb{C}M}$. The next proposition appears in [5] and [3].

PROPOSITION 1 ([5], [3]): *The abelian groups* $H^1(\tilde{X}, \psi_! j_* M)$ *and* $\mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X})$ *are finitely generated of rank* $r(\mathbb{C}M)$.

Let X^0 be the set of closed points of X. For $x \in X^0 \cup X_\infty$ let $i_x : x \to \tilde{X}$ be inclusion. If T is a sheaf on $\mathrm{Spec}(k)$, define T_x to be the sheaf $(i_x)_*(i_x)^* \psi_* j_* T$ on $\tilde{X}$.

Let $\mathbf{Z}$ denote the constant sheaf with underlying group the integers Z. Define $G_{m,k} = G_{m,\mathrm{Spec}(k)}$. On $\tilde{X}$ there is an exact sequence of sheaves

$$0 \to \psi_! G_{m,X} \to \psi_* j_* G_{m,k} \to \coprod_{v \in X^0} i_v \mathbf{Z} \oplus \coprod_{w \in X_\infty} (G_{m,k})_w \to 0 \quad (1)$$

In [3] and [5], the first terms in the long exact cohomology sequence of this sequence are computed as follows:

$$0 \to 0 \to k^* \to \coprod_{v \in X^0} Z(v) \oplus \coprod_{w \in X_\infty} \tilde{k}_w^* \to H^1(\tilde{X}, \psi_! G_{m,X}) \to 0 \quad (2)$$

Here $H^0(\tilde{X}, i_v \mathbf{Z}) = Z(v)$ is a copy of the integers for $v \in X^0$. For $w \in X_\infty$, $H^0(\tilde{X}, (G_{m,k})_w) = \tilde{k}_w^*$ is the multiplicative group of the algebraic closure $\tilde{k}_w$ of k in the completion k_w of k at w.

Suppose $\tau = \bigoplus_{v \in X^0} n_v \oplus \bigoplus_{w \in X_\infty} \beta_w$ is an element of $\coprod_{v \in X^0} Z(v) \oplus \coprod_{w \in X_\infty} \tilde{k}_w^*$. We will write the image γ of τ in $H^1(\tilde{X}, \psi_! G_{m,X})$ as $\{n_v, \beta_w\}_{v,w}$. Define the absolute value of γ to be $\|\gamma\| = \prod_{v \in X^0} q_v^{-n_v} \prod_{w \in X_\infty} \|\beta_w\|_w$, where q_v is the order of the residue field of v, and where $\| \; \|_w$ is the normalized absolute value at w. From the sequence (2) and the product formula, the value of $\|\gamma\|$ does not depend on the choice of τ with image γ.

Let $\langle , \rangle$ denote the natural pairing

$$\langle , \rangle : H^1(\tilde{X}, \psi_! j_* M) \times \mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X}) \to H^1(\tilde{X}, \psi_! G_{m,X}).$$

Let A and B be torsion free subgroups of maximal rank $r(\not{C}M)$ in $H^1(\tilde{X}, \psi_! j_* M)$ and $\mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X})$, respectively. Let $\{a_i\}$ and $\{b_j\}$ be bases for A and B, respectively.

Definition 2 ([5], [3]): *The regulator* $R(M, A, B)$ *is* $|\det(\log\|\langle a_i, b_j \rangle\|)|$.

The value of $R(M, A, B)$ is independent of the choice of bases for A and B.

Theorem 2 (Bienenfeld–Lichtenbaum): *The number* $c_{\not{C}M} = \lim_{s \to 0} s^{-r(\not{C}M)} L(s, \not{C}M)$ *equals* $\theta(M, A, B) R(M, A, B)$, *where*

$$\theta(M, A, B) = \frac{\pm \# H^2(\tilde{X}, \psi_! j_* M)}{\#(H^1(\tilde{X}, \psi_! j_* M)/A) \#(\mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X})/B)}$$

and where $\# N$ *denotes the order of the finite abelian group* N.

4. A comparison of regulators

With the notations of the two previous sections, we will now relate Tate's regulators $R(\not{C}M, \varphi)$ to the regulators $R(M, A, B)$. For simplicity, let us suppose that M is torsion-free.

From [1, p. 69] we have the exact sequence

$$0 \to \psi_! j_* M \to \psi_* j_* M \to \bigoplus_{w \in X_\infty} M_w \to 0$$

of sheaves on $\tilde{X}$. The cohomology of this sequence yields

$$\begin{array}{ccccccc} 0 \to H^0(\tilde{X}, \psi_* j_* M) & \to & H^0(\tilde{X}, \bigoplus_{w \in X_\infty} M_w) & \to & H^1(\tilde{X}, \psi_! j_* M) & \to & H^1(\tilde{X}, \psi_* j_* M) \\ \| & & \| & & & & \\ M^G & & \bigoplus_{w \in X_\infty} M^{G(w)} & & & & \end{array} \tag{3}$$

where $G(w) \subseteq G$ is the decomposition group of $w \in X_\infty$. By the Leray spectral sequence of $\psi_* j_*$, the group $H^1(\tilde{X}, \psi_* j_* M)$ is contained in $H^1(\mathrm{Spec}(k), M)$. Because M is torsion-free, $H^1(\mathrm{Spec}(k), M) = H^1(G, M)$, and $H^1(G, M)$ is finite.

From the exact sequence of G-modules

$$0 \to X_K \to Y_K \to Z \to 0$$

and the fact that M is torsion-free, we have an exact sequence

$$0 \to M^G \to \mathrm{Hom}_G(Y_K, M) \to \mathrm{Hom}_G(X_K, M) \to H^1(G, M) \tag{4}$$

For $w \in X_\infty$, we will regard the algebraic closure $\bar{k}_w$ of k as a fixed subfield of the completion k_w of k at w. Let y_w be the place of Y_K above w that is determined by this embedding.

If $a = \bigoplus_{w \in X_\infty} a_w \in \bigoplus_{w \in X_\infty} M^{G(w)} = H^0(\tilde{X}, \bigoplus_{w \in X_\infty} M_w)$, let a^1 be the image of a in $H^1(\tilde{X}, \psi_! j_* M)$ under the connecting homomorphism of (3). Let a^Y the the unique element of $\mathrm{Hom}_G(Y_K, M)$ such that $a^Y(y_w) = a_w$ for $w \in X_\infty$. Let a^X be the image of a^Y in $\mathrm{Hom}_G(X_K, M)$ in the sequence (4).

LEMMA 1: *Suppose* $a \in H^0(\tilde{X}, \bigoplus_{w \in X_\infty} M_w)$ *and* $b \in \mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X})$. *Let* $\pi : \mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X}) \to \mathrm{Hom}_G(M, E_K)$ *be the isomorphism induced by restriction to stalks over the generic point of* $\tilde{X}$. *Let* Tr *be the trace homomorphism on* $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}X_K, \mathbb{C}X_K)$, *and define* $\mathrm{Tr}_G = (1/\#G)\,\mathrm{Tr}$. *Then* $\log \|\langle a^1, b \rangle\| = \mathrm{Tr}_G(\lambda \circ \pi(b) \circ a^X)$.

PROOF: The homomorphism $\pi(b) \circ a^X \in \mathrm{Hom}_G(X_K, E_K)$ is the restriction of $\pi(b) \circ a^Y \in \mathrm{Hom}_G(Y_K, E_K)$ to X_K. Since λ sends $\mathbb{C}E_K$ to $\mathbb{C}X_K$, one has $\mathrm{Tr}(\lambda \circ \pi(b) \circ a^X) = \mathrm{Tr}^Y(\lambda \circ \pi(b) \circ a^Y)$, where Tr^Y is the trace on $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C} Y_K, \mathbb{C} Y_K)$. The homomorphism $\pi(b) \circ a^Y \in \mathrm{Hom}_G(Y_K, E_K)$ sends σy_w to $\sigma\pi(b)(a_w)$ if $w \in X_\infty$ and $\sigma \in G$.

Therefore

$$\mathrm{Tr}(\lambda \circ \pi(b) \circ a^X) = \sum_{w \in X_\infty} \sum_{\sigma \bmod G(w)} \log \|\sigma\pi(b)(a_w)\|_{\sigma y_w} = $$
$$= \#G \sum_{w \in X_\infty} \log \|\pi(b)(a_w)\|_w.$$

We have a commutative diagram of pairings

$$\begin{array}{ccccc}
\bigoplus_{w \in X_\infty} M^{G(w)} & \times & \mathrm{Hom}_G(M, E_K) & \to & \bigoplus_{w \in X_\infty} \tilde{k}_w^* \\
\| & & \downarrow{\scriptstyle \pi^{-1}} & & \| \\
H^0(\tilde{X}, \bigoplus_{w \in X_\infty} M_w) & & & & H^0(\tilde{X}, \bigoplus_{w \in X_\infty} (G_{m,k})_w) \\
\downarrow & & & & \downarrow \\
\langle\ \rangle : H^1(\tilde{X}, \psi_! j_* M) & \times & \mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X}) & \to & H^1(\tilde{X}, \psi_! G_{m,X})
\end{array}$$

Here the homomorphisms $H^0(\tilde{X}, \bigoplus_{w \in X_\infty} M_w) \to H^1(\tilde{X}, \psi_! j_* M)$ and $H^0(\tilde{X}, \bigoplus_{w \in X_\infty} (G_{m,k})_w) \to H^1(\tilde{X}, \psi_! G_{m,X})$ result from the connecting homomorphisms of the sequences (3) and (2), respectively.

From this diagram, we see that $\langle a^1, b\rangle$ is the element of $H^1(\tilde{X}, \psi_! G_{m,X})$ given by $\{0_v, \pi(b)(a_w)\}_{v \in X^0, w \in X_\infty}$ in the notation introduced in §2, where 0_v is the integer zero. Therefore

$$\log \|\langle a^1, b\rangle\| = \sum_{w \in X_\infty} \log \|\pi(b)(a_w)\|_w =$$
$$= (1/\#G)\,\mathrm{Tr}(\lambda \circ \pi(b) \circ a^X) = \mathrm{Tr}_G(\lambda \circ \pi(b) \circ a^X).$$

To state the next proposition, we introduce some notations. Let $A(M)^1$ (resp. $A(M)^X$) be the additive group of $a^1 \in H^1(\tilde{X}, \psi_! j_* M)$ (resp. $a^X \in \mathrm{Hom}_G(X_K, M)$) for which $a \in H^0(\tilde{X}, \bigoplus_{w \in X_\infty} M_w)$. Let $i : \mathrm{Hom}_G(QX_K, QX_K) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C} X_K, \mathbb{C} X_K)$ be the homomorphism induced by tensoring with $\mathbb{C}$ and forgetting G. Let $[,]$ be the pairing of $\mathrm{Hom}_G(QX_K, QM)$ and $\mathrm{Hom}_G(QM, QX_K)$ defined by $[f, g] = \mathrm{Tr}_G(i(g \circ f))$ for $f \in \mathrm{Hom}_G(QX_K, QM)$ and $g \in \mathrm{Hom}_G(QM, QX_K)$. Let $A(M)^d$ be the submodule of $\mathrm{Hom}_G(M, QX_K)$ which is dual to $A(M)^X$ with respect to $[,]$.

Proposition 2: *Let $\varphi : X_K \to E_K$ be a G-homomorphism inducing an isomorphism $QX_K \simeq QE_K$. Then there exists a positive integer n such that $n\varphi$*

takes $A(M)^d$ *into a submodule* $B_0(n\varphi)$ *of* $\mathrm{Hom}_G(M, E_K)$. *Define* $B(n\varphi) = \pi^{-1}(B_0(n\varphi))$. *Then* $A(M)^1$ *and* $B(n\varphi)$ *are of maximal rank* $r(\mathbb{C}M)$ *in* $H^1(X, \psi_! j_* M)$ *and* $\mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X})$, *respectively, and the regulators* $R(\mathbb{C}M, n\varphi)$ *and* $R(M, A(M)^1, B(n\varphi))$ *are equal up to sign.*

PROOF: The homomorphism $a \to a^Y$ is an isomorphism between $H^0(\tilde{X}, \bigoplus_{w \in X_\infty} M_w)$ and $\mathrm{Hom}_G(Y_K, M)$. This induces an isomorphism $a^1 \to a^X$ between $A(M)^1$ and $A(M)^X$. By the remarks following the sequence (1), the index of $A(M)^1$ in $H^1(\tilde{X}, \psi_! j_* M)$ is finite. Therefore $A(M)^1$ and $A(M)^X$ have rank $r(\mathbb{C}M)$ by Proposition 1.

Because M is torsion-free, $A(M)^X$ is torsion-free. Let $\{a_j\}$ be a set of $r(\mathbb{C}M)$ elements of $H^0(\tilde{X}, \bigoplus_{w \in X_\infty} M_w)$ such that $\{a_j^X\}$ is a basis for $A(M)^X$. Then $\{a_j^1\}$ is a basis for $A(M)^1$. Let $\{a_j^d\}$ be the basis of $A(M)^d$ that is dual to $\{a_j^X\}$ with respect to the pairing [,].

If $n > 0$ is sufficiently large, $b_j = n\varphi \circ a_j^d$ is in $\mathrm{Hom}_G(M, E_K)$. Now $n\varphi$ induces an isomorphism $QX_K \simeq QE_K$, and the a_j^d generate the submodule $A(M)^d$ of rank $r(\mathbb{C}M)$ in $\mathrm{Hom}_G(M, QX_K)$. Hence the rank of the module $B_0(n\varphi)$ generated by the b_j is $r(\mathbb{C}M)$. Because $\pi: \mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X}) \to \mathrm{Hom}_G(M, E_K)$ is an isomorphism, the rank of $B(n\varphi) = \pi^{-1}(B_0(n\varphi))$ is also $r(\mathbb{C}M)$.

If $\lambda \circ b_j = \sum_i l_i a_i^d \in \mathrm{Hom}_G(\mathbb{C}M, \mathbb{C}X_K)$, we have that $l_i = \mathrm{Tr}_G(\lambda \circ b_j \circ a_i^X)$, since $\{a_i^d\}$ is dual to $\{a_i^X\}$. By Lemma 1, $l_i = \log\|\langle a_i^1, \pi^{-1}(b_j)\rangle\|$. Thus the matrix of the homomorphism $(\lambda \circ n\varphi)_{\overline{\mathbb{C}M}}: \mathrm{Hom}_G(\mathbb{C}M, \mathbb{C}X_K) \to \mathrm{Hom}_G(\mathbb{C}M, \mathbb{C}X_K)$ relative to the basis $\{a_j^d\}$ is $(\log\|\langle a_i^1, \pi^{-1}(b_j)\rangle\|)_{i,j}$. Since $\{a_i^1\}$ is a basis for $A(M)^1$, it follows that $|\det(\lambda \circ n\varphi)_{\overline{\mathbb{C}M}}| = R(M, A(M)^1, B(n\varphi))$. Now $\det(\lambda \circ n\varphi)_{\overline{\mathbb{C}M}}$ is real because each of the $\log\|\langle a_i^1, \pi^{-1}(b_j)\rangle\|$ are real. Hence $\det(\lambda \circ n\varphi)_{\mathbb{C}M} = \overline{R(\mathbb{C}M, n\varphi)} = R(\mathbb{C}M, n\varphi) = \pm R(M, A(M)^1, B(n\varphi))$.

COROLLARY: *Under the hypotheses of Proposition* 2, $A(\mathbb{C}M, n\varphi) = \pm\,\theta(M, A(M)^1, B(n\varphi))^{-1}$.

REMARK: Let $\mathscr{R} = \mathrm{Hom}_G(M, M) = \mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! j_* M)$. Then $H^1(\tilde{X}, \psi_! j_* M)$ and $A(M)^1$ are right $\mathscr{R}$ modules, and $\mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X})$ and $B(n\varphi)$ are left $\mathscr{R}$ modules.

5. Proof of Theorem 1

Let V be an irreducible representation of G and let $F = Q(\chi_V)$. For $\alpha \in \mathrm{Gal}(F/Q)$, let V^α be a representation of G with character χ_V^α. Let $W = \sum_{\alpha \in \mathrm{Gal}(F/Q)} V^\alpha$; then $A(W, \varphi) = \prod_{\alpha \in \mathrm{Gal}(F/Q)} A(V^\alpha, \varphi)$. Because complex conjugation is continuous, $\overline{A(V, \varphi)} = A(\bar{V}, \varphi)$. Therefore $A(W, \varphi)$ is real, and is positive if F is complex.

There is a smallest positive integer m such that mW is realizable by rational matrices. Let W' be a $Q[G]$ module such that $\mathbb{C}W' = mW$; then m is the Schur index of W'. Let $\mathscr{R}$ be a maximal order in the division ring $\mathrm{Hom}_G(W', W')$. Then there exists a torsion-free finitely generated $\mathscr{R}$ submodule M of W' such that $QM = W'$. By the corollary of Proposition 2, there is a positive integer n such that the number $A(W, n\varphi)^m = A(mW, n\varphi) = \pm\,\theta(M, A(M)^1, B(n\varphi))^{-1}$ is rational.

We have shown that $A(W, \varphi)$ is real and is positive if F is complex. We are to show that $A(W, \varphi)$ is rational and that the ideal $A(W, \varphi)Z$ lies in $N_{F/Q}I(F)$. It will suffice to show that $A(W, \varphi)^m$ is rational and that $A(W, \varphi)^m Z$ lies in $N_{F/Q}I(F)^m$. A simple calculation shows that $R(W, n\varphi) = n^r R(W, \varphi)$ and $A(W, n\varphi) = n^r A(W, \varphi)$, where $r = \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathbb{C}W, \mathbb{C}X_K)$. Since $[F:Q]$ divides r, we have that $n^r \in N_{F/Q}F$. Therefore $A(W, \varphi)^m = n^{-rm} A(W, n\varphi)^m = \pm\, n^{-rm}\theta(M, A(M)^1, B(n\varphi))^{-1}$ is rational, and to prove Theorem 1, it will suffice to show that the ideal generated by $\theta(M, A(M)^1, B(n\varphi))$ lies in $N_{F/Q}I(F)^m$.

The Bienenfeld–Lichtenbaum Theorem states that this is the ideal generated by

$$\frac{\#H^2(\tilde{X}, \psi_! j_* M)}{\#(H^1(\tilde{X}, \psi_! j_* M)/A(M)^1)\,\#(\mathrm{Hom}_{\tilde{X}}(\psi_! j_* M, \psi_! G_{m,X})/B(n\varphi))}$$

By the remark following the corollary to Proposition 3, each of the orders appearing in this expression is the order of a finite $\mathscr{R} = \mathrm{Hom}_G(M, M)$ module. The desired conclusion now results from the following lemma. For a proof of this lemma in the case of a simple $\mathscr{R}$ module N, from which the general case follows easily, see ([6], pp 214–216).

LEMMA 2: *Let N be a finite module for a maximal order $\mathscr{R}$ in a division ring D of dimension m^2 over F, where F is the center of D. Then the ideal $(\#N)Z$ lies in $N_{F/Q}I(F)^m$.*

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