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D-DIMENSIONS OF ALGEBRAIC SURFACES AND NUMERICALLY EFFECTIVE DIVISORS

Fumio Sakai

Introduction

Let X be a non-singular projective surface over an algebraically closed field k, char$(k) = 0$. Given a divisor D on X, we denote by $\kappa(D, X)$ the D-dimension of X ([5]). The purpose of this note is to study the relation between $\kappa(D, X)$ and numerical properties of D.

We say that D is *pseudo effective* (resp. *numerically effective*) if $DH \geq 0$ for all ample divisors H on X (resp. if $DC \geq 0$ for all curves C on X). If D is pseudo effective, there exists a unique Zariski decomposition: $D = P + N$ where the P is a numerically effective $\mathbb{Q}$-divisor (Zariski [15], Fujita [4]). Since $\kappa(D, X) = \kappa(P, X)$, it suffices to consider P. We define the *numerical type* of D as follows:

D	P	Numerical Type
not pseudo effective		(a)
pseudo effective	$P \approx 0$	(b)
	$P^2 = 0$, $P \not\approx 0$	(c)
	$P^2 > 0$	(d)

Here $\approx$ denotes the numerical equivalence. If D is of type(a), obviously $\kappa(D, X) = -\infty$. If D is of type(b), we have $\kappa(D, X) = 0$ if and only if P is a torsion (i.e., $\exists n > 0, nP \sim 0$), and otherwise we have $\kappa(D, X) = -\infty$. In §1 we shall see that $\kappa(D, X) = 2$ if and only if D is of type(d). For D of type(c), the situation is complicated. Suppose that there exists an effective divisor $F \in |mP|$ for a positive integer m such that mP is integral. If we write $F = \sum n_i E_i$ with irreducible components E_i, we find $FE_i = 0$

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for all i. §2 and §3 are devoted to consider curves having this numerical property. To determine $\kappa(F, X)$ is equivalent to answer the following question raised by Mumford ([10], p. 336): when does nF lie in a pencil for some n (in other words $\kappa(F, X) = 1$)? Of course a necessary condition is that the normal sheaf of F is a torsion. Our main observation is that if $KF \leq 0$, we get $\kappa(F, X) = 1$ except in the following few cases (after suitably contracting exceptional curves of the first kind): (i) X is a rational surface with $\kappa^{-1}(X) = 0$, $F = nC$ where the C is an indecomposable curve of canonical type satisfying $C \sim -K$, (ii-1) X is an elliptic ruled surface, $F = nC$ where the C is a section, (ii-2) X is an elliptic ruled surface with $\kappa^{-1}(X) = 0$, $F = nC + n'C'$ where the C and the C' are disjoint sections satisfying $C + C' \sim -K$. Now we state the results concerning $\kappa(D, X)$ for D of type(c) (See §4): in case $PK < 0$, we get $\kappa(D, X) = 1$ which gives a ruled fibration on X, in case $PK = 0$, we have $\kappa(D, X) = -\infty$ only if $\chi(\mathcal{O}_X) \leq 0$ (essentially due to Fujita [4]), we have $\kappa(D, X) = 0$ only on a rational surface or on an elliptic ruled surface, otherwise we get $\kappa(D, X) = 1$ which gives an elliptic fibration on X, in case $PK > 0$, we can only say that $\kappa(D, X) \leq 1$.

As for the canonical divisor K, it is known that the Kodaira dimension $\kappa(X)$ is completely determined by the numerical type of K ([10]). In the second half of §4, we shall prove that $\kappa(K + \Delta, X)$ is also determined by the numerical type of $K + \Delta$ when Δ is a reduced curve. In [12] we shall see the same result for a not necessarily reduced curve Δ under the hypothesis $\omega_\Delta \cong \mathcal{O}_\Delta$. For further applications, see [13], [14].

NOTATIONS:

X a non-singular projective surface
K a canonical divisor of X
$\kappa(X)$ the Kodaira dimension of X $(= \kappa(K, X))$
$\kappa^{-1}(X)$ the anti-Kodaira dimension of X $(= \kappa(-K, X))$
$p_g(X)$ the geometric genus of X
$q(X)$ the irregularity of X
$\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$
An effective divisor C on X is regarded as a curve(1-dimensional scheme) with the structure sheaf $\mathcal{O}_C \cong \mathcal{O}/\mathcal{O}(-C)$
$\omega_C \cong \mathcal{O}(K + C) \otimes \mathcal{O}_C$ the dualizing sheaf of C
$\mathcal{N}_C \cong \mathcal{O}(C) \otimes \mathcal{O}_C$ the normal sheaf of C
The cohomology sequence of

$$0 \to \mathcal{O}(K) \to \mathcal{O}(K + C) \to \omega_C \to 0$$

yields the inequality

$$(*)\quad \dim H^0(X, \mathcal{O}(K + C)) \geq \dim H^0(C, \omega_C) - 1 + \chi(\mathcal{O}_X).$$

§1. Preliminaries

(A) Q-*divisor*. Let $\mathrm{Div}(X)$ be the group of divisors on X. A Q-divisor is an element of $\mathrm{Div}(X, \mathbf{Q}) = \mathrm{Div}(X) \otimes \mathbf{Q}$. We write a Q-divisor as $D = \sum \alpha_i D_i$ where the D_i are reduced irreducible curves and the α_i are rational numbers. If all $\alpha_i \geq 0$, we say that D is *effective*, written $D \geq 0$. Two Q-divisors D and D' are *linearly equivalent*, written $D \sim D'$, if $D - D'$ is a principal divisor of a non-zero rational function. We denote by $L(D)$ the vector space of non-zero rational functions f such that $(f) + D \geq 0$. Let $\mathcal{O}(D)$ be the sheaf given by $U \to L(D_U)$ where D_U is the restriction of D to an open set U. There is a one to one correspondence between $L(D)$ and $H^0(X, \mathcal{O}(D))$. To a Q-divisor $D = \sum \alpha_i D_i$, we associate two divisors $[D] = \sum [\alpha_i] D_i$ and $\{D\} = \sum \{\alpha_i\} D_i$, where $[\alpha]$ is the greatest integer smaller than or equal to α and $\{\alpha\}$ is the least integer greater than or equal to α. Since $[-\alpha] = -\{\alpha\}$, we have $[-D] = -\{D\}$. Since the principal divisor (f) of a non-zero rational function f is integral, we have $L(D) = L([D])$ and $\mathcal{O}(D) = \mathcal{O}([D])$. The complete linear system $|D| = \{D' \geq 0 \mid D' \sim D\}$ is in one to one correspondence with the projective space $(L(D) - \{0\})/k^*$. In fact, if $D' \sim D$, there is a rational function f such that $D' = (f) + D$ hence $f \in L(D)$. We refer to Demazure [3], for further discussions on Q-divisors. We write as $h^i(D) = \dim H^i(X, \mathcal{O}(D))$. In case $h^0(D) > 0$, we define a rational map Φ_D by

$$\Phi_D : X \ni z \to (\psi_0(z) : \ldots : \psi_N(z)) \in \mathbf{P}^N,$$

where $\psi_0, \ldots, \psi_N$ is a basis of $H^0(X, \mathcal{O}(D))$ and $N = h^0(D) - 1$.

(B) *D-dimension.* For a Q-divisor D, the D-dimension $\kappa(D, X)$ of X takes one of the values $-\infty, 0, 1, 2$ and has the following three equivalent interpretations ([5]):

(i) there exist a positive integer m_0, positive numbers σ, τ so that for large m,

$$\sigma m^{\kappa(D,X)} \leq h^0(m m_0 D) \leq \tau m^{\kappa(D,X)},$$

$$\text{(ii)}\quad \kappa(D, X) = \begin{cases} \max\limits_{m>0} \dim\{\Phi_{mD}(X)\} \\ -\infty \text{ if } h^0(mD) = 0 \text{ for all } m > 0, \end{cases}$$

$$\text{(iii)}\ \kappa(D,X) = \begin{cases} \text{tr.deg.} \bigoplus_{m \geq 0} H^0(X, \mathcal{O}(mD)) - 1 \\ -\infty \text{ if } h^0(mD) = 0 \text{ for all } m > 0. \end{cases}$$

REMARK: We note that $\kappa(D,X) = 0$ if and only if $h^0(mD) \leq 1$ for all $m > 0$ and $h^0(mD) = 1$ for some $m > 0$. For any positive rational number α, $\kappa(D,X) = \kappa(\alpha D, X)$.

(C) *Numerical Equivalence.* The properties "pseudo effective" and "numerically effective" are determined by the numerical equivalence class. A Q-divisor D is *numerically equivalent to zero*, written $D \approx 0$, if $DZ = 0$ for all $Z \in \mathrm{Div}(X)$. Two Q-divisors D and D' are numerically equivalent, written $D \approx D'$, if $D - D'$ is numerically equivalent to zero. We write

$$\mathrm{N}(X, \dot{\mathrm{Q}}) = \mathrm{Div}(X, \mathrm{Q})/\approx\ \cong \{\mathrm{Div}(X)/\approx\} \otimes \mathrm{Q}.$$

Let $\mathscr{C}_E$ be the convex cone generated by effective Q-divisors. The dual cone $\mathscr{C}_P$ is nothing but the cone generated by numerically effective Q-divisors. The double dual cone of $\mathscr{C}_E$ coincides with the closure $\overline{\mathscr{C}_E}$. A Q-divisor D is pseudo effective if its numerical equivalence class belongs to $\overline{\mathscr{C}_E}$. This is the original definition (Fujita [4]). We see this.

LEMMA 1: *A Q-divisor D is pseudo effective if and only if $DP \geq 0$ for all numerically effective divisors P on X.*

PROOF: Assume that $DH \geq 0$ for all ample divisors H. Let P be a numerically effective divisor and H an ample divisor. For all $n \geq 0$, $H + nP$ is ample by Nakai's criterion. Therefore $D(H + nP) \geq 0$. This is however possible only if $DP \geq 0$. The other implication is trivial, because an ample divisor is numerically effective. Q.E.D.

Given a sequence of pseudo effective Q-divisors $D^{(n)} = \sum \alpha_i^{(n)} D_i$. If $\alpha_i^{(n)} \to \alpha_i \in \mathrm{Q}$ as $n \to \infty$ for all i, then clearly the limit $D = \sum \alpha_i D_i$ is again pseudo effective.

LEMMA 2: *If a Q-divisor D is not pseudo effective, then for any Q-divisor Z, $h^0(nD + Z) = 0$ for all large integers n.*

PROOF (cf. [4]): Suppose otherwise that $h^0(n_k D + Z) > 0$ for a sequence of integers $n_k \to \infty$. Then $D + (1/n_k)Z$ is pseudo effective and the limit is D. Q.E.D.

REMARK: We note that $\mathscr{C}_P \subset \overline{\mathscr{C}_E}$. So if P is numerically effective, then $P^2 \geq 0$. If both D and $-D$ are pseudo effective, then $D \approx 0$ ([4]).

PROPOSITION 1: *Let P be a numerically effective* Q*-divisor on a surface X. Then $\kappa(P, X) = 2$ if and only if $P^2 > 0$.*

PROOF: By the Riemann–Roch theorem, we have

$$h^0(mP) + h^2(mP) = h^1(mP) + \tfrac{1}{2}m^2P^2 - \tfrac{1}{2}mPK + \chi(\mathcal{O}_X)$$

for positive integers m such that mP is integral. If $P \not\approx 0$, then $h^2(mP) = h^2(K - mP) = 0$ for all large m (Lemma 2). Thus the condition $P^2 > 0$ implies that $\kappa(P, X) = 2$. Conversely, suppose that $\kappa(P, X) = 2$. Assume $P^2 = 0$. Take any positive integer m such that mP is integral and $h^0(mP) \geq 2$. We write as $|mP| = |M| + Z$ where the Z is the fixed part. Then $0 = (mP)^2 = M^2 + MZ + mPZ$. Since $M^2 \geq 0$, $MZ \geq 0$ and $PZ \geq 0$, we get $M^2 = 0$. The image $\Phi_{mP}(X) = \Phi_M(X)$ is thus always a curve, which contradicts the hypothesis $\kappa(P, X) = 2$. Q.E.D.

(D) *Zariski Decomposition.* As stated in introduction, a pseudo effective Q-divisor D has a *Zariski decomposition*: $D = P + N$ where

(i) the N is an effective Q-divisor and either $N = 0$ or the intersection matrix of the irreducible components of N is negative definite,

(ii) the P is a numerically effective Q-divisor and the intersection of P with each irreducible component of N is zero.

We see some properties of P and N. First we recall

LEMMA ([15], [4]): *Let $E_1, \ldots, E_k$ be irreducible curves such that the intersection matrix (E_iE_j) is negative definite. Given a* Q*-divisor $Z = \sum \alpha_i E_i$ and an effective (resp. pseudo effective)* Q*-divisor D satisfying $(D - Z)E_i \leq 0$ for all i, then $D - Z$ is effective (resp. pseudo effective).*

LEMMA 3: *Let $D = P + N$ be the Zariski decomposition of a pseudo effective* Q*-divisor D. Then $|D| = |P| + N$.*

PROOF: If $D' \in |D|$, then $D' - N \sim P$. So $(D' - N)E = 0$ for all irreducible components E of N. By the above lemma, $D' - N$ is effective. Q.E.D.

COROLLARY: *For $m > 0$, $h^0(mD) = h^0(mP)$ and hence $\kappa(D, X) = \kappa(P, X)$.*

Corollary: *If furthermore D is integral, then $|D| = |[P]| + \{N\}$.*

Remark: Let D' be another pseudo effective $\mathbf{Q}$-divisor with a Zariski decomposition $D' = P' + N'$. If $D \approxeq D'$, then $N = N'$. If $D \sim D'$, then $P \sim P'$.

(E) *Redundant Exceptional Curves.* Given a $\mathbf{Q}$-divisor F on X, an exceptional curve of the first kind E with $FE = 0$ is said to be *F-redundant.*

Proposition 2: *Let F be a $\mathbf{Q}$-divisor on a surface X. Then there is a birational morphism $\mu: X \to X_0$ onto a surface X_0 and a $\mathbf{Q}$-divisor F_0 such that* (i) $F = \mu^*F_0$, (ii) *X_0 contains no F_0-redundant exceptional curves,* (iii) $KF = K_0F_0$ *where the K_0 denotes a canonical divisor of X_0. Furthermore F is (numerically) effective if and only if F_0 is (numerically) effective.*

Proof: If E is an F-redundant exceptional curve, let $\pi: X \to X'$ be the contraction of E. Since $FE = 0$, there is a $\mathbf{Q}$-divisor F' such that $F = \pi^*F'$. Since $K \sim \pi^*K' + E$, $KF = (\pi^*K' + E)(\pi^*F') = K'F'$. By successive such contractions, we arrive at the desired surface X_0 and a $\mathbf{Q}$-divisor F_0. Q.E.D.

§2. Curves of fibre type

Definition: A curve $C = \sum n_iE_i$ on a surface X is said to be of (numerically) *fibre type* if $CE_i = 0$ for all i. We say that C is *indecomposable* if C is connected and g.c.d. $(n_i) = 1$.

First we consider indecomposable curves of fibre type. We recall the following

Lemma 4 (cf. [10], see also [1], [2]): *Let $C = \sum n_iE_i$ be an indecomposable curve of fibre type on X. Then*

(i) $h^0(\mathcal{O}_C) = 1$,

(ii) *if $\mathcal{L}$ is an invertible sheaf on C such that* $\deg(\mathcal{L} \otimes \mathcal{O}_{E_i}) = 0$ *for all i, then $H^0(C, \mathcal{L}) \neq 0$ if and only if $\mathcal{L} \cong \mathcal{O}_C$,*

(iii) *if $Z = \sum m_iE_i$ satisfies $Z^2 = 0$, then $Z = \alpha C$, $\alpha \in \mathbf{Q}$,*

(iv) *if Z is an effective divisor such that $ZE_i = 0$ for all i, then $Z = nC + Z'$ where $n \geq 0$ and the Z' is an effective divisor disjoint from C.*

Corollary: *The arithmetic genus $p_a(C)$ of an indecomposable curve of* *fibre type is given by*

$$p_a(C) = h^0(\omega_C) = \tfrac{1}{2}KC + 1.$$

LEMMA 5: *Suppose that X has a fibration $\Phi: X \to B$ onto a curve B with connected fibres. If an indecomposable curve C is contained in a fibre of Φ, then some multiple of C is actually a fibre of Φ and $\kappa(C, X) = 1$.*

PROOF: Immediate from Lemma 4, (iii).

As a divisor, a curve of fibre type C is numerically effective. According to Proposition 1, we have either $\kappa(C, X) = 0$ or $\kappa(C, X) = 1$.

LEMMA 6: *Let C be an indecomposable curve of fibre type with $\kappa(C, X) = 1$. Then for a large integer m, Φ_{mC} gives a fibration onto a curve with connected fibres. Some multiple of C is a fibre of this fibration. If f denotes a general fibre, then $p_a(f) = 0, 1, \geq 2$ according as $p_a(C) = 0, 1, \geq 2$.*

PROOF: The first part is a general result ([5, Theorem 5]). By Lemma 5, there exists a positive integer n such that nC becomes a fibre. Therefore $p_a(f) = \frac{1}{2}Kf + 1 = \dfrac{n}{2}KC + 1$, which proves the last assertion. Q.E.D.

COROLLARY: *If there is another indecomposable curve of fibre type C' disjoint from C, then $\kappa(C', X) = 1$ and $p_a(C') = 0, 1, \geq 2$ according as $p_a(C) = 0, 1, \geq 2$.*

LEMMA 7: *Let C be an indecomposable curve of fibre type with $p_a(C) = = 0$. Then $\kappa(C, X) = 1$. Furthermore if C' is another indecomposable curve of fibre type disjoint from C, then also $p_a(C') = 0$.*

PROOF: By using the Riemann–Roch theorem, we obtain $\kappa(C, X) = 1$. The second assertion follows from the above corollary. Q.E.D.

To see the case in which $\kappa(C, X) = 0$, we begin with

LEMMA 8: *Let C be an indecomposable curve of fibre type with $\kappa(C, X) = 0$. Then $\kappa(Z, X) \leq 0$ for any divisor Z satisfying $ZC = 0$.*

PROOF: Suppose that there is a divisor Z such that $ZC = 0$ and $\kappa(Z, X) \geq 1$. Let $Z = P + N$ be the Zariski decomposition. It follows that $0 \leq PC = ZC - NC \leq 0$, hence $PC = 0$. We note that $\kappa(Z, X) = \kappa(P, X)$. If $\kappa(P, X) = 2$, then $P^2 > 0$, by Proposition 1. Then the Hodge index theorem implies that either $C \approxeq 0$ or $C^2 < 0$, a contradiction. If $\kappa(P, X) = 1$, C would be contained in a fibre of the fibration given by Φ_{mP} for a large m such that mP is integral. It follows from Lemma 5 that $\kappa(C, X) = 1$, which is a contradiction. Q.E.D.

LEMMA 9: *Let C be an indecomposable curve of fibre type with $p_a(C) = 1$. Then the case $\kappa(C, X) = 0$ occurs only if X is either a rational, or an elliptic ruled surface.*

PROOF: Suppose $\kappa(C, X) = 0$. Since $p_a(C) = 1$, $KC = 0$. By Lemma 8, we must have $\kappa(X) \leq 0$. The case $\kappa(X) = 0$ with $p_g(X) > 0$ is excluded by

LEMMA (Mumford [10]): *Let C be an indecomposable curve of fibre type with $p_a(C) = 1$. If $p_g(X) > 0$, then $\kappa(C, X) = 1$.*

Other surfaces with $\kappa(X) = 0$ are Enriques surfaces and hyperelliptic surfaces. In these cases, X has an étale covering $\pi : \tilde{X} \to X$ with $p_g(\tilde{X}) = 1$. Applying the above lemma to an indecomposable component of $\tilde{C} = \pi^* C$, we obtain $\kappa(C, X) = \kappa(\tilde{C}, \tilde{X}) = 1$. Finally if X were a ruled surface of genus ≥ 2, then since $p_a(C) = 1$, C must be contained in a fibre of the ruled fibration of X, which would contradict Lemma 5. Thus it remains the two possibilities as required. Q.E.D.

Now we consider a curve of fibre type F which need not be indecomposable. We decompose as

$$F = \sum_{i=1}^{r} n_i C_i$$

where the C_i are mutually disjoint indecomposable curves of fibre type.

LEMMA 10: (i) *$KF < 0$ if and only if $p_a(C_i) = 0$ for all i,*
(ii) *$KF = 0$ if and only if $p_a(C_i) = 1$ for all i.*

PROOF: If $p_a(C_i) = 0$ for at least one i, from Lemma 7, we see that $p_a(C_i) = 0$ for all i.

PROPOSITION 3: *If a surface X has a curve of fibre type F with $KF < 0$, then $\kappa(F, X) = 1$ and X is a ruled surface.*

PROOF: This follows from Lemma 7.

PROPOSITION 4: *If a surface X has a curve of fibre type F with $KF = 0$, then either*

(i) $\kappa(F, X) = 0$, *X is a rational surface and $r = 1$,*

(ii) $\kappa(F, X) = 0$, *X is an elliptic ruled surface and $r \leq 2$, or*

(iii) $\kappa(F, X) = 1$, *X is an elliptic surface.*

Here the r denotes the number of distinct indecomposable components of F.

PROOF: In view of Lemma 10, the assertions follow from Lemma 6 and Lemma 9 except the restrictions on r. Suppose $\kappa(F, X) = 0$. In our situation, clearly $h^0(\omega_F) \geq r$. Using the inequality (*), we get

$$h^0(K + F) \geq r - 1 + \chi(\mathcal{O}_X).$$

On the other hand, by Lemma 8, we have $\kappa(K + F, X) \leq 0$ and hence $h^0(K + F) \leq 1$. So we must have $2 \geq r + \chi(\mathcal{O}_X)$. If X is rational, since $\chi(\mathcal{O}_X) = 1$, we get $r = 1$, and if X is elliptic ruled, since $\chi(\mathcal{O}_X) = 0$, we get $r \leq 2$. Q.E.D.

EXAMPLE 1: Take a non-singular curve C_0 of genus $g \geq 1$ with $C_0^2 = n \geq 0$ on a surface X_0. Choose distinct points $x_1, \ldots, x_n$ on C_0 so that the divisor $\delta = C_0 | C_0 - \sum x_i$ is not a torsion on C_0. This is possible, for instance if $n \geq 2g + 1$ (cf. [15], p. 562). Let $\pi : X \to X_0$ be the blowing up of X_0 at $x_1, \ldots, x_n$ and we define C to be the strict transform of C_0 by π. Then $C^2 = 0$ and the normal sheaf $\mathcal{N}_C$ is isomorphic to the pull back of $\mathcal{O}(\delta)$. Hence $\mathcal{N}_C$ is not a torsion. Via the exact sequence

$$0 \to H^0(X, \mathcal{O}((m - 1)C)) \to H^0(X, \mathcal{O}(mC)) \to H^0(C, \mathcal{N}_C^{\otimes m}),$$

we get $h^0(mC) = 1$ for all $m > 0$. Thus we obtain an example of a non-singular curve C of genus g with $C^2 = \kappa(C, X) = 0$.

REMARK: In [10], the results in Lemma 4 and in Mumford's lemma (in the proof of Lemma 9) are stated for an indecomposable curve of canonical type. But the proofs also work in our form (cf. §3).

§3. Curves of canonical type

In this section we consider curves of fibre type on a rational surface and on an elliptic ruled surface. Given a curve of fibre type F on a surface X, by using Proposition 2, we get a surface X_0 and a curve F_0 with a birational morphism $\mu: X \to X_0$ such that $F = \mu^* F_0$ and X_0 contains no F_0-redundant exceptional curves. It is then easy to see that F_0 is again of fibre type. As the dimension $\kappa(F, X)$ is concerned, from the first, we may assume that X contains no F-redundant exceptional curves. In case F is of fibre type, an exceptional curve of the first kind E is F-redundant, either if E is a component of F, or if E is disjoint from F. So the hypothesis that X has no F-redundant exceptional curves means that

(i) F contains no exceptional curves of the first kind,

(ii) any exceptional curve if the first kind (if exists) meets F.

The condition (i) is a minimality condition of F. It is easy to see the following

REMARK: Let C be an indecomposable curve of fibre type with $p_a(C) = 0$. The following two conditions are equivalent:

(i) C contains no exceptional curves of the first kind,

(ii) C is a non-singular rational curve.

DEFINITION (Mumford [10]): A curve $C = \sum n_i E_i$ is said to be of *canonical type* if $CE_i = KE_i = 0$ for all i. As before, C is *indecomposable* if C is connected and if g.c.d. $(n_i) = 1$.

LEMMA 11: *Let C be an indecomposable curve of fibre type with $p_a(C) = 1$. The following three conditions are equivalent*:

(i) *C contains no exceptional curves of the first kind,*

(ii) *C is an indecomposable curve of canonical type,*

(iii) $\omega_C \cong \mathcal{O}_C$.

PROOF: The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are clear. Lemma 4 proves (ii) $\Rightarrow$ (iii). We have only to prove (i) $\Rightarrow$ (ii). This is trivial, if C is irreducible. If not, we let $C = \sum n_i E_i$ be the irreducible decomposition. Then $E_i^2 < 0$ for all i, because C is connected and $CE_i = 0$ for all i. If $KE_i < 0$, E_i would be an exceptional curve of the first kind. Hence $KE_i \geq 0$ for all i. Since $p_a(C) = 1$, $KC = 0$, hence $KE_i = 0$ for all i. Q.E.D.

The dual graph of an indecomposable curve of canonical type C is classified in the theory of elliptic surfaces. If C is irreducible, C is either a non-singular elliptic curve, or a rational curve having a node or a cusp. If C is reducible, all irreducible components are non-singular rational curves with self-intersection -2.

PROPOSITION 5: *Let X be a rational surface. If C is an indecomposable curve of canonical type with $\kappa(C,X) = 0$ on X, meeting all exceptional curves of the first kind, then $C \sim -K$. Furthermore, the normal sheaf $\mathcal{N}_C$ is not a torsion.*

PROOF: According to the inequality (*), we have $h^0(K + C) = 1$. If $K^2 > 0$, then by the Riemann–Roch theorem, $\kappa(K + C, X) = 2$, which contradicts Lemma 8. So $K^2 \leq 0$. Since K is not pseudo effective (cf. [10]), by Lemma 2, we can find a positive integer m such that $h^0(iK + C) = 1$ for $i = 1, \ldots, m$, but $h^0((m + 1)K + C) = 0$. Take an effective divisor $Z \in |mK + C|$. First we see that $K^2 = 0$. This is trivial, if $Z = 0$. Assume $Z \neq 0$. Then Z is disjoint from C. In fact, since $ZC = 0$, by Lemma 4, (iv), we can write as $Z = nC + Z'$ where $n \geq 0$ and Z' is disjoint from C. If $n > 0$, then $mK \sim (n - 1)C + Z$, which is absurd, because $\kappa(X) = -\infty$. Since $h^0(K + Z) = 0$ and $h^1(\mathcal{O}_X) = 0$, we infer that $h^0(\omega_Z) = 0$. Hence every irreducible component of Z is a non-singular rational curve. Let E be an irreducible component of Z. If $E^2 \geq 0$, then $\kappa(E, X) \geq 1$, which is not the case, because of Lemma 8. Therefore $E^2 < 0$. If $K^2 < 0$, then $KZ < 0$ and hence there is at least one irreducible component E with $KE < 0$. In this case, this E would be an exceptional curve of the first kind, which does not meet C, a contradiction. Thus $K^2 = 0$. By the Riemann-Roch theorem, we get $h^0(-K) > 0$. Take an effective divisor $G \in |-K|$. Then $mG + Z \sim C$. Since $\kappa(C, X) = 0$, the equality holds: $mG + Z = C$. This is possible only if $Z = 0$, $G = C$ and $m = 1$. Hence $C \sim -K$. Now we see that $\mathcal{N}_C$ is not a torsion. The cohomology sequence of

$$0 \to \mathcal{O}((m - 1)C) \to \mathcal{O}(mC) \to \mathcal{N}_C^{\otimes m} \to 0$$

yields $h^1(mC) \geq h^1(\mathcal{N}_C^{\otimes m})$ for $m \geq 2$. Using the Riemann–Roch theorem, we then get $h^0(mC) \geq h^1(\mathcal{N}_C^{\otimes m}) + 1$. If $\mathcal{N}_C$ is a torsion, taking m multiple of the order of $\mathcal{N}_C$, we have $\mathcal{N}_C^{\otimes m} \cong \mathcal{O}_C$, hence $h^1(\mathcal{N}_C^{\otimes m}) = h^1(\mathcal{O}_C) = 1$. So we would have $h^0(mC) \geq 2$, which is a contradiction. Q.E.D.

COROLLARY: *Let X and C be the same as above. Then there is a birational morphism (composite of 9 points blowing ups) $\pi: X \to \mathsf{P}^2$ and the image of C is a cubic curve.*

PROOF: It is known that a relatively minimal model of X is among P^2, F_e $(e \geq 0, \neq 1)$ where the F_e is the rational rule surface $\mathsf{P}(\mathcal{O} \oplus \mathcal{O}(-e))$ over P^1. If E is a non-singular rational curve on X, then $E^2 \geq -2$, because $E^2 = -EK - 2 = CE - 2$. Noting that on F_e there is a section with self-intersection $-e$, we must have $e = 0$ or 2 in order that F_e is a relatively minimal model of X. For both cases, we can easily find a birational morphism π of X onto P^2. Since $K^2 = 0, \pi$ consists of blowing ups at 9 points. We also see that $\pi_* C \sim -K_{\mathsf{P}^2}$. Q.E.D.

We turn our attention to an elliptic ruled surface. If C is an indecomposable curve of canonical type on an elliptic ruled surface X, then clearly C is a non-singular elliptic curve. In what follows, we mean by an elliptic curve a non-singular one.

PROPOSITION 6: *Let X be an elliptic ruled surface. If C is an elliptic curve of fibre type (i.e., $C^2 = 0$) with $\kappa(C, X) = 0$ on X, then C is a section of the ruled fibration of X.*

PROOF: By Lemma 8, $\kappa(K + C, X) \leq 0$. If $\kappa(K + C, X) = 0$, then according to the classification of such C in [11, Theorem (2.7)], X has an elliptic fibration such that C is in a fibre. It follows from Lemma 5 that $\kappa(C, X) = 1$. Thus $\kappa(K + C, X) = -\infty$, from which we infer that C is a section (cf. Lemma 12 below). Q.E.D.

Let C be a section in an elliptic ruled surface X. Suppose that $C^2 = 0$ and that C meets all exceptional curves of the first kind. We see easily that such X and C can be constructed as follows. Take a relatively minimal elliptic ruled surface X_0 and a section C_0 with $C_0^2 = n \geq 0$. Inductively, choose a point x_i in C_i, and let X_{i+1} be the blowing up of X_i at x_i and let C_{i+1} be the strict transform of C_i. Put $X = X_n$ and $C = C_n$.

If the normal sheaf $\mathcal{N}_C$ is not a torsion, we know that $\kappa(C, X) = 0$ (cf. Example 1). However, if $\mathcal{N}_C$ is a torsion, the question when $\kappa(C, X) = 0$ seems to be open (cf. Introduction).

EXAMPLE 2: On an elliptic curve B, we have a non-trivial extension

$$0 \to \mathcal{O}_B \to \mathcal{E} \to \mathcal{O}_B \to 0.$$

Let X be the ruled surface $\mathsf{P}(\mathcal{E})$. There is a unique section $C \in |\mathcal{O}_{\mathsf{P}(\mathcal{E})}(1)|$. We know that $K \sim -2C$ and that $\mathcal{N}_C \cong \mathcal{O}_C$. By a result of Atiyah, $h^0(S^m(\mathcal{E})) = 1$ and hence $h^0(mC) = 1$ for all $m > 0$. So $\kappa(C, X) = 0$.

REMARK: When X is relatively minimal, this is the only case in which $\mathcal{N}_C$ is a torsion and $\kappa(C, X) = 0$. (For a proof, see [12]).

We proceed to the case of two elliptic curves (cf. Proposition 4).

PROPOSITION 7: *Let X be an elliptic ruled surface. Suppose that there exist two mutually disjoint elliptic curves of fibre type C and C' with $\kappa(C, X) = \kappa(C', X) = 0$ on X, and that every exceptional curve of the first kind meets at least one of C and C'. Then $C + C' \sim -K$ and X is a geometrically ruled surface of the form $\mathrm{P}(\mathcal{O} \oplus \mathcal{O}(\delta))$ where the δ is a non-torsion divisor of degree zero on the base curve. Neither $\mathcal{N}_C$ nor $\mathcal{N}_{C'}$ is a torsion.*

PROOF: It follows from Proposition 6 that both C and C' are sections of the ruled fibration of X. By Lemma 8, we have $\kappa(C + C', X) = 0$, $\kappa(K + C + C', X) \leq 0$. On the other hand $h^0(K + C + C') \geq 1$, by the inequality (*). So $\kappa(K + C + C', X) = 0$. Take an effective divisor $Z \in |K + C + C'|$. If f denotes a fibre of the ruled fibration, $Zf = 0$, because $Kf = -2$ and $Cf = C'f = 1$. Therefore Z must be contained in fibres. So each irreducible component of Z is a non-singular rational curve. Since $ZC = ZC' = 0$, Z is disjoint from C and C'. We know that $K^2 \leq 0$. If $K^2 < 0$, there would appear an exceptional curve of the first kind in Z, which does not meet $C \cup C'$ (cf. the proof of Proposition 5). So $K^2 = 0$. In this case, by the Riemann–Roch theorem, the equality $h^0(2(K + C + C')) = h^1(2(K + C + C'))$ holds. Using this and the cohomology sequence of

$$0 \to \mathcal{O}(2K + C + C') \to \mathcal{O}(2(K + C + C')) \to \mathcal{O}_C \oplus \mathcal{O}_{C'} \to 0,$$

we obtain $h^0(2(K + C + C')) \geq 2$ unless $h^0(-(K + C + C')) \geq 1$. We must have $h^0(-(K + C + C')) \geq 1$, because $\kappa(K + C + C', X) = 0$. Since $h^0(K + C + C') \geq 1$, we infer that $K + C + C' \sim 0$ as required. By the fact $K^2 = 0$, we conclude that X is relatively minimal. Furthermore the existence of two disjoint sections implies that X is defined to be $\mathrm{P}(\mathcal{O} \oplus \mathcal{O}(\delta))$ with a divisor δ on the base curve. By a calculation, it is easy to see that $\deg(\delta) = 0$ and δ cannot be a torsion and that neither $\mathcal{N}_C$ nor $\mathcal{N}_{C'}$ is a torsion. Q.E.D.

We summarize the results in Propositions 3, 4, 5, 6 and 7.

THEOREM 1: *Let F be a curve of fibre type with $KF = 0$ on a surface X. Suppose that X contains no F-redundant exceptional curves. Then we have $\kappa(F, X) = 1$ except in the following cases:*

(i) *X is a rational surface with $\kappa^{-1}(X) = 0$ and $F = nC$ where the C is an indecomposable curve of canonical type satisfying $C \sim -K$,* (ii-1) *X is an elliptic ruled surface and $F = nC$ where the C is a section of the ruled fibration of X, meeting all exceptional curves of the first kind in X,*

(ii-2) *X is an elliptic ruled surface defined as $\mathsf{P}(\mathcal{O} \oplus \mathcal{O}(\delta))$ where the δ is a non-torsion divisor of degree zero on the base curve $(\kappa^{-1}(X) = 0)$ and $F = nC + n'C'$ where the C and the C' are disjoint sections of the ruled fibration of X satisfying $C + C' \sim -K$. Here the n and the n' denote positive integers.*

Furthermore the case $\kappa(F, X) = 1$ occurs if and only if X has a minimal elliptic fibration and F is a sum of fibres with positive rational coefficients.

PROOF: It remains to check the case $\kappa(F, X) = 1$. By Proposition 4, there is an elliptic fibration $\Phi: X \to B$. We decompose F into indecomposable curves of fibre type: $F = \sum n_i C_i$. By Lemma 5, some multiple $m_i C_i$ becomes a fibre f_i over some point x_i in B. Thus we can write as: $F = \sum (n_i/m_i) f_i$. If E were an exceptional curve of the first kind contained in a fibre, clearly $FE = 0$, so E would be F-redundant. Q.E.D.

§4. D-dimension and numerical type

THEOREM 2: *Let D be a divisor on a surface X. Then we have the following relation between $\kappa(D, X)$ and the numerical type of D.*

TABLE I:

Numerical type of D		$\kappa(D, X)$	Structure of X
(a)		$-\infty$	
(b)		$-\infty$	irregular
		0	P is a torsion
(c)	$PK < 0$	1	ruled
	$PK = 0$	$-\infty$	$\chi(\mathcal{O}_X) \leq 0$
		0	rational or elliptic ruled
		1	elliptic
	$PK > 0$	≤ 1	
(d)		2	

Here P denotes the numerically effective part in the Zariski decomposition of D.

PROOF: We prove the assertions separately for each type (See Introduction, for the general principle).

Type (b). If $\kappa(P, X) \geq 0$, we find an effective divisor $Z \in |mP|$ for a positive integer m such that mP is integral. Since $P \approx 0$, we must have

$Z = 0$. We infer that $\kappa(P, X) = 0$ if and only if P is a torsion. When X is a regular surface, the condition $P \approx 0$ always implies that P is a torsion.

Type (d). By Proposition 1, $\kappa(P, X) = 2 \Leftrightarrow P^2 > 0$.

Type (c). If $PK < 0$, the Riemann-Roch theorem yields $\kappa(P, X) = 1$. Similarly, if $PK = 0$, we get $h^0(mP) \geq \chi(\mathcal{O}_X)$ for all large integers m such that mP is integral (cf. the proof of Proposition 1). So if $\chi(\mathcal{O}_X) \geq 1$, we have $\kappa(P, X) \geq 0$. Now assume $\kappa(P, X) \geq 0$. There exists an effective divisor $F \in |mP|$ for some positive integer m such that mP is integral. Let $F = \sum n_i E_i$ be the irreducible decomposition. Since P is numerically effective and $P^2 = 0$, we have $F^2 = 0$, $FE_i \geq 0$ for all i. So $FE_i = 0$ for all i. Thus F is a curve of fibre type with $KF = 0$. The other assertions for type(c) follow from Propositions 3 and 4. Q.E.D.

EXAMPLE 3: We give examples of D of type(c), $PK = 0$ and $\kappa(D, X) = -\infty$. Take a product of an elliptic curve C and a curve B. Choose two points x, x' in C so that $\delta = x - x'$ is not a torsion as a divisor. Put $b = x \times B$, $b' = x' \times B$ and $f = C \times y$ where y is a point in B. Let $D = f + b - b'$. Since $D \approx$ f, D $= P$ in the Zariski decomposition. Obviously D is of type(c) and $PK = 0$. Using the fact $\mathcal{O}(D) \cong p_1^*\mathcal{O}(\delta) \otimes p_2^*\mathcal{O}(y)$ where the p_i are projections, we get an isomorphism

$$H^0(X, \mathcal{O}(mD)) \cong H^0(C, \mathcal{O}(m\delta)) \otimes H^0(B, \mathcal{O}(my)).$$

Since $h^0(m\delta) = 0$ for $m > 0$, we conclude that $\kappa(D, X) = -\infty$.

Let Δ be a pseudo effective Q-divisor on X. We deal with the divisor $K + \Delta$. Suppose that $K + \Delta$ is of type(c). Let $K + \Delta = P + N$ be the Zariski decomposition. Since $P^2 = PN = 0$, we get $PK = -P\Delta \leq 0$ (Lemma 1). The following result is an immediate consequence of Table I.

COROLLARY: *Suppose that X is a regular surface with $\kappa(X) \geq 0$. For a pseudo effective* Q*-divisor Δ, $\kappa(K + \Delta, X)$ is determined by the numerical type of $K + \Delta$.*

In general this is not the case even for effective Δ. For instance, let X, C, C' be the surface and the curves in Theorem 1, (ii-2). If we put $\Delta = 2C$, $K + \Delta$ is of type(b) but $\kappa(K + \Delta, X) = -\infty$. If $\Delta = nC$, for $n \geq 3$, then $K + \Delta$ is of type(c) but $\kappa(K + \Delta, X) = -\infty$. If $\Delta = nC + n'C'$, for $n \geq 1$, $n' \geq 1$, $n + n' \geq 3$, then $K + \Delta$ is of type(c) but $\kappa(K + \Delta, X) = 0$. Note that these Δ are not reduced.

In what follows we consider the case in which Δ is a reduced curve.

PROPOSITION 8: *If Δ is a reduced curve on a surface X, then we have $\kappa(K + \Delta, X) \geq 0$ if and only if $K + \Delta$ is pseudo effective.*

PROOF: Suppose that $K + \Delta$ is pseudo effective. If $\kappa(X) \geq 0$, obviously $\kappa(K + \Delta, X) \geq 0$. If X is a rational surface, since $\chi(\mathcal{O}_X) = 1$, we get $\kappa(K + \Delta, X) \geq 0$ (by Table I). If X is a non-rational ruled surface, since $K + \Delta$ is pseudo effective, $(K + \Delta)f \geq 0$ for a fibre f of the ruled fibration of X. On the other hand $Kf = -2$, so $\Delta f \geq 2$. We obtain $\kappa(K + \Delta, X) \geq 0$ by the following

LEMMA 12: *Let X be a non-rational ruled surface. Let Δ be a reduced curve on X and f a fibre of the ruled fibration of X. If $\Delta f \geq 2$, then $\kappa(K + \Delta, X) \geq 0$. If furthermore $\Delta f \geq 3$, then $\kappa(K + \Delta, X) \geq 1$.*

PROOF: This fact is in principle a part of the so called "Addition Formula" of logarithmic Kodaira dimensions (See [6]). For the sake of covenience, we sketch a proof. Let $\Phi: X \to B$ be the ruled fibration of X. It suffices to see the case in which Δ contains no fibre components. Put $n = \Delta f$. By the inequality (*), we get

$$h^0(K + \Delta) \geq h^0(\omega_\Delta) - q(X).$$

Step I. Let k be the number of the irreducible components of Δ. Clearly $h^0(\omega_\Delta) \geq kq(X)$, hence $h^0(K + \Delta) \geq (k - 1)q(X)$. So $h^0(K + \Delta) \geq 2$ except if $k = 1$, or if $k = 2$, $q(X) = 1$.

Step II. *Case*(i) $k = 1$, $q(X) \geq 2$. By the Hurwitz' theorem, $h^0(K + \Delta) \geq (n - 1)(q(X) - 1) \geq n - 1$, which proves the assertion.

Case (ii). $k = 1$, $q(X) = 1$. We may assume that Δ is a non-singular elliptic curve. The map $\Phi: \Delta \to B$ is an étale covering of degree n. Take the fibre product $\tilde{X} = \Delta \times_B X$. The associated map $\pi: \tilde{X} \to X$ is also étale and $\tilde{\Delta} = \pi^*\Delta$ consists of distinct n sections of the ruled surface $\tilde{X}$. Clearly $\tilde{K} + \tilde{\Delta} \sim \pi^*(K + \Delta)$ where the $\tilde{K}$ is a canonical divisor of $\tilde{X}$. By the result in Step I, we get

$$\kappa(K + \Delta, X) = \kappa(\tilde{K} + \tilde{\Delta}, \tilde{X}) \geq \begin{cases} 0 & \text{if} \quad n = 2, \\ 1 & \text{if} \quad n \geq 3. \end{cases}$$

Case (iii). $k = 2$, $q(X) = 1$. One can similarly check this case. Q.E.D.

PROPOSITION 9: *If Δ is a reduced curve on a surface X, then we have $\kappa(K + \Delta, X) = 0$ if and only if $K + \Delta$ is pseudo effective and its numerically effective part P is a torsion.*

PROOF: We have only to verify the only if part. Suppose $\kappa(K + \Delta, X) = 0$. Let $K + \Delta = P + N$ be the Zariski decomposition. In view of Table I, it suffices to show the non-existence of type(c). Assume now $P^2 = PK = 0$, $P \not\approx 0$. By hypothesis, there exists an effective divisor $F \in |mP|$ for some positive integer m such that mP is integral. Then F is a curve of fibre type with $KF = 0$ (cf. the proof of Theorem 2). For X and F, we let X_0 and F_0 be the same as given in Proposition 2 (See also §3). We have $K \sim \mu^* K_0 + G$ where the G is a sum of exceptional curves for μ. Furthermore $F = \mu^* F_0$ and F_0 is again a curve of fibre type with $K_0 F_0 = 0$, $\kappa(F_0, X_0) = 0$. By construction, F_0 contains no F_0-redundant exceptional curves. In Theorem 1, such a curve is classified.

Case (i). X_0 is a rational surface. We have $F_0 = nC_0$ where the C_0 is an indecomposable curve of canonical type satisfying $C_0 \sim -K_0$. If we put $C = \mu^* C_0$, then $mG + m\Delta \sim (m + n)C + mN$ and hence the equality holds:

$$mG + m\Delta = (m + n)C + mN.$$

It follows that Δ is never reduced. In fact, if C' is the strict transform of C_0, Δ must contain $2C'$.

Case (ii). X_0 is an elliptic ruled surface. We have either $F_0 = nC_0$, or $F = nC_0 + n'C_0'$ where the C_0 and the C_0' are sections of the ruled fibration of X_0. We infer that $Ff \geq 1$ for a fibre f. Note that $m(K + \Delta) \sim F + mN$. Since $Kf = -2$, we get $\Delta f \geq 2 + m^{-1}$, hence $\Delta f \geq 3$. So Lemma 12 gives $\kappa(K + \Delta, X) \geq 1$, a contradiction. Q.E.D.

In case $\kappa(K + \Delta, X) = 1$, there exists a ruled fibration or an elliptic fibration on X according as $PK < 0$ or $PK = 0$ (Table I). Summarizing, we obtain

THEOREM 3: *Let Δ be a reduced curve on a surface X. Then the dimension $\kappa(K + \Delta, X)$ is determined by the numerical type of $K + \Delta$. Namely, we have the following*

TABLE II:

Numerical Type of $K + \Delta$	(a)	(b)	(c)	(d)
$\kappa(K + \Delta, X)$	$-\infty$	0	1	2

REMARK: This fact holds to be true including the case $\Delta = 0$, which is the framework of Enriques classification of surfaces ([10]). When Δ has only normal crossings as singularities, $\kappa(K + \Delta, X)$ is called the logarithmic Kodaira dimension of the complement of Δ. In this case, Proposition 9 is first given by Kawamata ([7], see also [9]). Our proof can be regarded as a clarification of this phenomenon.

CONCLUDING REMARK: Most arguments in the paper also work in characteristic $p > 0$. In the proof of Theorem 2, we must have assumed $\mathrm{char}(k) \neq 2, 3$ only at the end of the proof of Lemma 9. On the other hand, in the proof of Theorems 1 and 3, we used the hypothesis $\mathrm{char}(k) = 0$ (especially in Propositions 6, 7 and Lemma 12).

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