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# GLUED CONVEXITY SPACES, FRAMES AND GRAPHS

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## Abstract

The notion of a convex sum space, which is – roughly speaking – a convexity structure on the union of sets, was introduced by Gerard Sierksma. We will consider another possibility of summing convexity spaces in what will be called a normal gluing, using semi-extreme sets. In the special case of ordinary convexity on $\mathbb{R}^d$, necessary and sufficient conditions are given in terms of facets. In this connection, we finally study a non-trivial case where line segments are glued together to a frame and study it as an embedding problem in graph theory.

## 1. Normal gluings

A convexity space is a pair $(X, \mathscr{C})$ with $\mathscr{C}$ a collection of subsets of a set $X$ such that $\phi$, $X \in \mathscr{C}$ and $\mathscr{C}$ is closed under intersections. The members of $\mathscr{C}$ are called convex sets. The $\mathscr{C}$-hull of $S$ in $X$ is defined by $\mathscr{C}(S) = \cap \{A \mid A \in \mathscr{C}, S \subset A\}$. In order to study the sharpness of relationships between the Caratheodory, Helly, Radon and Sierksma (or exchange) numbers (for definitions and results we refer to [6]), starting with a finite number of convexity spaces $(X_1, \mathscr{C}_1), \ldots, (X_n, \mathscr{C}_n)$, Sierksma [7] has introduced the notion of convex sum space and has studied the various numbers in case the universal sets $X_1, \ldots, X_n$ are disjoint. In [1], [2] and [3] we looked at the problem in case the universal sets are not disjoint and gave other possibilities to construct convexity on $\cup_{i=1}^{n} X_i$. Throughout this paper we will consider again the construction as introduced in [2] and define $\mathscr{G} \equiv \mathscr{G}(\mathscr{C}_1, \ldots, \mathscr{C}_n)$ on $X_1 \cup \ldots \cup X_n$ by

$$\mathscr{G} = \{A \subset X_1 \cup \ldots \cup X_n \mid A \cap X_i \in \mathscr{C}_i \text{ for } i = 1, \ldots, n\}. \tag{1}$$

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Clearly $(\bigcup_{i=1}^{n} X_i, \mathscr{G})$ is a convexity space and we say that it is obtained by *gluing the spaces* $(X_1, \mathscr{C}_1), \ldots, (X_n, \mathscr{C}_n)$. In case the universal sets are disjoint, we have, see [2],

$$\mathscr{G} = \mathscr{C}_1 + \mathscr{C}_2 + \ldots + \mathscr{C}_n = \left\{ \bigcup_{i=1}^{n} C_i \mid C_i \in l_i;\ i = 1, \ldots, n \right\},$$

which is precisely Sierksma's convex sum space in that case.

In Degreef and Fourneau [4] it is shown that, in general, there always exists an ordinal $\mu$ such that, for every subset $S$ of $X_1 \cup \ldots \cup X_n$, $S^\mu = \mathscr{G}(S)$, where, setting $S = S^0$,

$$S^\mu = \begin{cases} \bigcup_{i=1}^{n} \mathscr{C}_i(S^{\mu-1} \cap X_i) & \text{if } \mu \text{ is not a limit ordinal,} \\ \bigcup_{1 \leq \nu < \mu} S^\nu & \text{if } \mu \text{ is a limit ordinal.} \end{cases}$$

We call the gluing *normal* iff for each $S$ in $X_1 \cup \ldots \cup X_n$,

$$\mathscr{G}(S) = \bigcup_{i=1}^{n} \mathscr{C}_i(S \cap X_i).$$

The following example shows that gluings are *not* normal in general. Take $X_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\}$, $X_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 2\}$, $\mathscr{C}_i = \text{conv}|_{X_i} = \{C \cap X_i \mid C \in \text{conv}\}$ with $i = 1, 2$ and conv is the ordinary convexity structure on $\mathbb{R}^2$. Let $S = \{(-1, 4), (1, 0), (3, 1)\}$.

Clearly $\mathscr{C}_1(S \cap X_1) \cup \mathscr{C}_2(S \cap X_2)$ is not an element of $\mathscr{G}$ and so certainly cannot be the $\mathscr{G}$-hull of $S$.

In the next example, the gluing is normal. Let $X_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\}$, $X_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0\}$, $\mathscr{C}_i = \text{conv}|_{X_i}$ with $i = 1, 2$. One can verify that for each subset $S$ of $X_1 \cup X_2 = \mathbb{R}^2$, $\mathscr{G}(S) = \mathscr{C}_1(S \cap X_1) \cup \mathscr{C}_2(S \cap X_2)$.

Note that gluings are normal in case the universal sets are pairwise disjoint. Throughout this paper we restrict ourselves to two convexity spaces $(X_1, \mathscr{C}_1)$, $(X_2, \mathscr{C}_2)$ and assume in general that $X_1 \cap X_2 \neq \phi$. In the next theorem the concept of semi-extreme set is used; see Sierksma [8], p. 14: a nonempty subset $T$ of $A$ in $X$ is called *a semi-extreme set* of $A$ iff $T \cap \mathscr{C}(A \setminus T) = \phi$. If $\{p\}$ is a semi-extreme set, $p$ is called *a semi-extreme point.*

THEOREM 1: *Let $(X_1, \mathscr{C}_1)$ and $(X_2, \mathscr{C}_2)$ be convexity spaces such that each* *subset of $X_1 \cap X_2$ is a semi-extreme set of $X_1$ and $X_2$. Then the gluing $(X_1 \cup X_2, \mathscr{G})$ is normal.*

PROOF: We have to show that for each subset $S$ of $X_1 \cup X_2$, $\mathscr{C}_1(S \cap X_1) \cup \mathscr{C}_2(S \cap X_2)$ is convex, i.e. $[\mathscr{C}_1(S \cap X_1) \cup \mathscr{C}_2(S \cap X_2)] \cap X_i$ is an element of $\mathscr{C}_i$; $i = 1, 2$. Take $i = 1$. If $\mathscr{C}_2(S \cap X_2) \cap X_1 = \phi$ we are done. So we may assume that $\mathscr{C}_2(S \cap X_2) \cap X_1 = T \neq \emptyset$. We show that $T \subset S \cap X_2$. Suppose $T \not\subset S \cap X_2$, i.e. there exists a nonempty subset $T_1$ of $T$ such that $T_1 \cap (S \cap X_2) = \emptyset$, hence $S \cap X_2 \subset X_2 \backslash T_1$ and therefore $\mathscr{C}_2(S \cap X_2) \subset \mathscr{C}_2(X_2 \backslash T_1)$.

So $T_1 \subset \mathscr{C}_2(X_2 \backslash T_1)$ and then $T_1 \cap \mathscr{C}_2(X_2 \backslash T_1) \neq \emptyset$ which contradicts the assumption that each subset of $X_1 \cap X_2$ is a semi-extreme subset of $X_2$. So $T \subset S \cap X_2$. As $T \subset X_1$ we have $T \subset S \cap X_1$, so

$$T = \mathscr{C}_2(S \cap X_2) \cap X_1 \subset \mathscr{C}_1(S \cap X_1),$$

which means that $[\mathscr{C}_1(S \cap X_1) \cup \mathscr{C}_2(S \cap X_2)] \cap X_1 \in \mathscr{C}_1$. □

The preceding theorem yields sufficient conditions for $(X_1, \mathscr{C}_1)$ and $(X_2, \mathscr{C}_2)$ in order to have a normal gluing. The second example above shows that, in general, these conditions certainly are not necessary. The question now is: is it possible to show that, with certain assumptions on $(X_1, \mathscr{C}_1)$ and $(X_2, \mathscr{C}_2)$, they are necessary? The following theorem gives a positive answer. The proof is left to the reader, just as the proof of Theorem 3.

THEOREM 2: *Let $(X_1, \mathscr{C}_1)$ and $(X_2, \mathscr{C}_2)$ be convexity spaces with the following property. For $T_1 \subset T \subset X_1 \cap X_2$ there is for each $i = 1, 2$ a convex set $C_T \in \mathscr{C}_i$ such that $C_T \subset X_i \backslash T$ and*

(a) *$(T_1 \cup \tilde{T}) \cup C_T \notin \mathscr{C}_i$, with $\tilde{T} = (X_1 \cap X_2) \backslash T$,*

(b) *$\tilde{T} \cup C_T \in \mathscr{C}_i$.*

*Then are equivalent:*

(1) *each subset of $X_1 \cap X_2$ is a semi-extreme set of $X_1$ and $X_2$,*

(2) *the gluing $(X_1 \cup X_2, \mathscr{G})$ is normal.*

In case $X_1 \cap X_2$ is just one element, we have the following.

THEOREM 3: *Let $(X_1, \mathscr{C}_1)$ and $(X_2, \mathscr{C}_2)$ be convexity spaces such that $X_1 \cap X_2 = \{p\}$. If for some $i \in \{1, 2\}$, there is a subset $C_i$ of $X_i \backslash \{p\}$, $C_i \in \mathscr{C}_i$ and $\{p\} \cup C_i \notin \mathscr{C}_i$, then are equivalent:*

(1) *$p$ is a semi-extreme point of $X_j$ $(j \neq i)$,*

(2) *the gluing $(X_1 \cup X_2, \mathscr{G})$ is normal.*

To end this part of the paper, we return to the situation in our first examples, namely to $(\mathbb{R}^n, \text{conv})$. Let $X$ be a nonempty convex set of $\mathbb{R}^n$. A convex subset $F$ of $X$ is called *a facet of X* iff for each $x, y \in X$ such that the open segment $]x, y[$ and $F$ have a nonempty intersection, $[x, y]$ is contained in $F$. Making use of the classical theorem of Caratheodory, we have the following theorem; we will assume of course $X_1 \cap X_2$ not to be an element of $\{\phi, X_1, X_2\}$, since otherwise the gluing is always normal.

THEOREM 4: *Let $X_i$ be a nonempty convex set in $\mathbb{R}^n$ and $\mathscr{C}_i = conv|_{X_i}$; $i = 1, 2$. In case $X_1 \cup X_2$ is on a straight line, the gluing $(X_1 \cup X_2, \mathscr{G})$ is always normal. In case $X_1 \cup X_2$ is not on a straight line, the gluing $(X_1 \cup X_2, \mathscr{G})$ is normal iff $X_1 \cap X_2$ is a facet of $X_1$ and $X_2$.*

PROOF: We only give the proof of the last part of the theorem. Suppose $X_1 \cap X_2$ to be a facet of $X_1$ and $X_2$. We have to show that the gluing $(X_1 \cup X_2, \mathscr{G})$ is normal. In order to do so, we'll show that for each subset $S$ of $X_1 \cup X_2$, $\text{conv}(S \cap X_2) \cap X_1 = \text{conv}(S \cap X_1) \cap X_2 = \text{conv}(S \cap X_1 \cap X_2)$. This clearly is sufficient. We'll prove that $\text{conv}(S \cap X_1) \cap X_2 = \text{conv}(S \cap X_1 \cap X_2)$, the other equality being completely similar.

The fact that $\text{conv}(S \cap X_1 \cap X_2)$ is contained in $\text{conv}(S \cap X_1) \cap X_2$ is trivial. Suppose that, on the other hand, $\text{conv}(S \cap X_1) \cap X_2$ is not contained in $\text{conv}(S \cap X_1 \cap X_2)$, i.e. there exists

$$x \in [\text{conv}(S \cap X_1) \cap X_2] \backslash \text{conv}(S \cap X_1 \cap X_2).$$

Using Caratheodory's theorem, there is a finite subset $\{x_1, x_2, \ldots, x_c\}$ of $S \cap X_1$ such that $x \in \text{conv}\{x_1, \ldots, x_c\}$. Since $x \notin \text{conv}(S \cap X_1 \cap X_2)$ we have for a certain number of the $x_i$'s, say $x_1, \ldots, x_k$, that $x_i \notin X_2$. But then there always exists $y \in \text{conv}\{x_1, \ldots, x_k\}$ and $z \in \text{conv}\{x_{k+1}, \ldots, x_c\}$ such that $x \in \text{conv}\{y, z\} = [y, z]$. Consequently, $x \in ]y, z[ \cap (X_1 \cap X_2)$. Finally, since $X_1 \cap X_2$ is a facet of $X_1$, $[y, z] \subset X_1 \cap X_2$. This is impossible, since $y \notin X_2$. So $\text{conv}(S \cap X_1) \cap X_2 \subset \text{conv}(S \cap X_1 \cap X_2)$. □

## 2. Frames: an embedding problem

In this section a non-trivial example of a normal gluing of a finite number of convexity spaces is given. Consider $(X_i, \mathscr{C}_i)$, $i = 1, \ldots, n$, where $X_i$ is a closed segment and $\mathscr{C}_i = \text{conv}|_{X_i}$. A normal gluing $(\bigcup_{i=1}^n X_i, \mathscr{G})$ where two segments have either an empty intersection or one endpoint in common, is called *a frame* $(X, \mathscr{G})$, with $X = \bigcup_{i=1}^n X_i$. The endpoints of the segments are the semi-extreme points of the frame. The number of semi-extreme points of a frame will be called *the order of the frame. The graph associated with a frame* $(X, \mathscr{G})$, denoted, by $G(X, \mathscr{G})$, is the graph with vertices the semi-extreme points of $(X, \mathscr{G})$ and edges the pairs of semi-extreme points $\{p, q\}$ such that $\mathscr{G}(p, q)$ is a segment. Clearly, each graph without isolated points is, up to an isomorphism, the graph of a frame. We say that *a frame* $(X, \mathscr{G})$ *can be embedded in* $(\mathbb{R}^n$, *conv*) if there exists an injection $\phi$: $X \to \mathbb{R}^n$ such that for each subset $S$ of $X$, $\phi[\mathscr{G}(S)] = \phi(X) \cap \operatorname{conv}[\phi(S)]$. Note that each frame of order $d$ can be embedded in $(\mathbb{R}^{d-1}, \text{conv})$. One may ask whether it is possible to do better and characterize which frames are embeddable in which $(\mathbb{R}^n, \text{conv})$. The following theorem gives the answer to this problem.

THEOREM 5: *A frame* $(X, \mathscr{G})$ *is*

(1) *embeddable in* $(\mathbb{R}, conv)$ *iff* $G(X, \mathscr{G})$ *is a line segment;*
(2) *embeddable in* $(\mathbb{R}^2, conv)$ *iff* $G(X, \mathscr{G})$ *is either a cycle or a part of a cycle;*
(3) *embeddable in* $(\mathbb{R}^3, conv)$ *iff* $G(X, \mathscr{G})$ *is planar;*
(4) *always embeddable in* $(\mathbb{R}^4, conv)$.

PROOF:

(1) and (2) are left to the reader.
(3) Let $G(X, \mathscr{G})$ be a planar graph. If there are at most 3 vertices, there is nothing to prove. In the other case, we may assume without loss of generality the graph to be maximal. Then the graph is 3-connected and using the Steinitz-Rademacher theorem, it is the 1-skeleton of a convex 3-dimensional polyhedron. The last conclusion gives us the embedding. If $G(X, \mathscr{G})$ is not a planar graph, using Kuratowski's theorem, there is a subgraph homeomorphic with $K_5$ or $K_{3,3}$. It will be sufficient to show that neither $K_5$ nor $K_{3,3}$, considered as normal gluings, is embeddable in $(\mathbb{R}^3, \text{conv})$. $K_{3,3}$ is left to the reader, since the arguments are the same as for $K_5$. What $K_5$ is about, there are two possibilities. Denoting the set of vertices by $A = \{a_1, \ldots, a_5\}$, one of the vertices, say $a_5$, may belong to the convex hull of the other ones. This clearly gives not an embedding. In the other case, the vertices are "convex independent". Then, by Radon's theorem, it is always possible to find vertices, say $a_1, a_2$, such that $\operatorname{conv}\{a_1, a_2\}$ and $\operatorname{conv}[A \backslash \{a_1, a_2\}]$ have a nonempty intersection. Let $x$ be an element in the intersection. We will denote the vertices of the "gluing $K_5$" by $A' = \{a'_1, \ldots, a'_5\}$, such that $\phi(a'_i) = a_i$ for each $i$.

If $x$ is an element of one of the three closed line segments defined by $\{a_3, a_4, a_5\}$, say of $[a_3, a_4]$, we consider the set $\{a_1, a_2, a_3\}$. $\phi[\mathscr{G}(\{a'_1, a'_2, a'_3\})]$ is given by the union of three closed line segments defined by $\{a_1, a_2, a_3\}$. $\phi(K_5) \cap \operatorname{conv}[\phi(\{a'_1, a'_2, a'_3\})]$ on the other hand contains the closed segment $[a_3, x]$, and consequently they are different. If $x$ is an element of the "open triangle" $a_3\, a_4\, a_5$, we consider $\{a_3, a_4, a_5\}$. $\phi[\mathscr{G}(\{a'_3, a'_4, a'_5\})]$ is given by the union of the three closed line segments defined by $\{a_3, a_4, a_5\}$.

$\phi(K_5) \cap \operatorname{conv}[\phi(\{a'_3, a'_4, a'_5\})]$ on the other hand contains $x$. So once again they are different and we have no embedding.

(4) It suffices to consider the so-called moment-curve in $\mathbb{R}^4$, with parametrization $x(t) = (t, t^2, t^3, t^4)$. Then we know that the convex hull of 2 (resp. 1) extreme points of the cyclic polytope $C(v, 4)$, which is the convex hull of $v \geq 5$ points of the moment curve, is a 1-face (resp. 0-face) of $C(v, 4)$ ([5]). This is sufficient to embed each frame in $(\mathbb{R}^4, \operatorname{conv})$. □

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