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THE ZEROS OF CERTAIN POINCARÉ SERIES

R.A. Rankin

1. Introduction

K. Wohlfahrt [6] showed in 1964 that the only zeros of the Eisenstein series E_k for the modular group lie on transforms of the unit circle when $4 \leq k \leq 26$, and conjectured that this holds for all $k \geq 4$. The range of k was extended to $k \leq 34$ and $k = 38$ in [4], but in [2] F.K.C. Rankin and H.P.F. Swinnerton-Dyer proved Wohlfahrt's conjecture for all k by a simple argument. The purpose of the present paper is to show that similar properties hold for a wide class of meromorphic modular forms belonging to the modular group. In particular, it is shown that, if $G_k(z, m)$ ($m \in \mathbb{Z}$) is the m th Poincaré series of weight k , then for $m \leq 1$ all its finite zeros in the standard fundamental region lie on the lower arc A , while for $m > 1$ at most $m - 1$ of these zeros do not lie on A ; for $m = 0$ this reproduces the result of [2].

Throughout the paper I shall be concerned with meromorphic modular forms of even positive weight $k \geq 4$ on the upper half-plane $H = \{z: \text{Im } z > 0\}$ for the modular group

$$\Gamma(1) = \text{SL}(2, \mathbb{Z}).$$

The vector space of all such forms is denoted by M_k . Thus, if $f \in M_k$, f has a Fourier series expansion of the form

$$f(z) = \sum_{n=-N}^{\infty} a_n e^{2\pi i n z}, \quad (1.1)$$

which is convergent when $\text{Im } z$ is sufficiently large. The subspace of

M_k consisting of forms $f \in M_k$ that are holomorphic on \mathbb{H} is denoted by H_k ; for such forms the series (1.1) converges for all $z \in \mathbb{H}$. The subspace of H_k consisting of cusp forms, for which we can take $N = -1$, is denoted by C_k .

If $f \in M_k$, then f has at most a finite number of poles in any fundamental region. From the work of Petersson [1] it follows that $f(z)$ can be represented as the sum of Poincaré series

$$f(z) = \frac{1}{2} \sum_{c,d} \frac{R(e^{2\pi i Tz})}{(cz + d)^k} =: G_k(z; R). \quad (1.2)$$

Here

$$Tz = \frac{az + b}{cz + d},$$

where

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1),$$

and the summation is over all pairs of coprime integers c, d ; for each such pair we choose a single $T \in \Gamma(1)$ with $[c, d]$ as bottom row. Here $R(t)$ is a suitably chosen rational function of t . The series is absolutely and uniformly convergent on every compact subset of \mathbb{H} free of poles of f .

To illustrate this result we take two special cases. In the first place, take

$$R(t) = t^m \quad (m \in \mathbb{Z})$$

and put

$$G_k(z; R) = G_k(z, m) \quad (1.3)$$

in this case. For $m > 0$, $G_k(z, m) \in C_k$ and C_k is spanned by those $G_k(z, m)$ for which

$$0 < m \leq \frac{k}{12};$$

see [5], Theorem 6.2.1. For $m = 0$, $G_k(z, 0)$ is an Eisenstein series,

being usually denoted by $E_k(z)$. For $m < 0$, $G_k(z, m) \in H_k$ and has a pole of order m at ∞ .

As a second example take

$$R(t) = (t - q)^{-n} \quad (n \in \mathbb{N}), \quad (1.4)$$

where $0 < |q| < 1$ and

$$q = e^{2\pi iw} \quad (w \in \mathbb{H}).$$

Then $G_k(z; R)$ has poles of order n at those points of H congruent to w modulo $\Gamma(1)$.

By taking R to be an appropriate linear combination of the rational functions described in the previous two paragraphs we see that any function $f \in M_k$ can be expressed in the form (1.2).

If $f \in M_k$, then $f^K \in M_k$, where

$$f^K(z) = \overline{f(-\bar{z})};$$

see §8.6 of [5]. Moreover, $f^K = f$ if and only if f has real Fourier coefficients. Such a form we call a *real modular form* and denote by M_k^* the subset of M_k consisting of such forms; similarly for H_k^* and C_k^* . These are clearly vector spaces over the real field \mathbb{R} . Note that, if $f \in M_k$, then both

$$f + f^K \text{ and } i(f - f^K)$$

are in M_k^* , so that there is, in a sense, no loss of generality in confining attention to real modular forms.

If $f \in M_k^*$ and if f has a zero or pole at a point $z \in \mathbb{H}$, then it has another of the same order at $-\bar{z}$. Further, if the rational function R has real coefficients, then clearly $G_k(z; R) \in M_k^*$; we call such a function R a *real rational function*. When representing real modular forms as Poincaré series $G_k(z; R)$ we shall restrict our attention to real rational functions R . Such a function has the property that

$$R(\bar{t}) = \overline{R(t)} \quad (1.5)$$

for all $t \in \mathbb{C}$.

An arbitrary modular form may have a zero at any point of H . However, we shall show that there is a wide class of real forms that have all their zeros on transforms of the arc

$$S = \left\{ z = e^{i\theta} : \frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3} \right\}$$

This is already known to be true for the Eisenstein series E_k ; see [2].

2. General results

We denote by F the standard fundamental region for $\Gamma(1)$. This is the subset of \mathbb{C} consisting of all points $z \in \mathbb{H}$ for which either

$$|z| > 1, -\frac{1}{2} < \operatorname{Re} z < 0$$

or

$$|z| \geq 1, 0 \leq \operatorname{Re} z \leq \frac{1}{2},$$

and we regard ∞ as belonging to F . F is bounded on its lower side by the arc S , but only half this arc, namely

$$A = \left\{ z = e^{i\theta} : \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \right\} \quad (2.1)$$

is contained in F .

If $f \in M_k$ and has N zeros and P poles in F , counted with appropriate multiplicities, then

$$N - P = \frac{k}{12}; \quad (2.2)$$

see [5], Theorem 4.1.4. Here zeros or poles at i are counted with weight $\frac{1}{2}$, while those at $\rho = e^{\pi i/3}$ are counted with weight $\frac{1}{3}$.

Let

$$L_i = \{z \in \mathbb{H} : z = iy, y > 1\}, \quad (2.3)$$

and

$$L_\rho = \{z \in \mathbb{H} : z = \frac{1}{2} + iy, y > \frac{1}{2}\sqrt{3}\}, \quad (2.4)$$

so that L_i , A and L_ρ form the boundary in \mathbb{H} of the right-hand half of F .

For $k \geq 4$ we express k in the form

$$k = 12l + s, \quad (2.5)$$

where

$$l = \dim C_k \geq 0 \quad (2.6)$$

and

$$s = 4, 6, 8, 10, 0 \text{ or } 14. \quad (2.7)$$

If f is holomorphic at i and ρ , then, since

$$\frac{k}{12} = l + \frac{s}{12},$$

we see that we must have

$$\frac{s}{12} = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3} + \frac{1}{2}, 0 \text{ and } \frac{2}{3} + \frac{1}{2} \quad (2.8)$$

in the six cases, respectively. Accordingly the total weighted order of the zeros of f at i and ρ is at least $s/12$ in each case.

Let $G_k(z; R)$ be defined as in (1.2) and suppose that this function is holomorphic on the arc A . We wish to count the number of its zeros on A . For this purpose it is convenient to consider points on the larger arc S and put

$$F_k(\theta, R) = e^{ki\theta/2} G_k(e^{i\theta}, R), \quad (2.9)$$

where $\theta \in [(\pi/3), (2\pi/3)] = I$, say. If we pair the terms of the Poincaré series corresponding to c, d and d, c , and use (1.5), we see that $F_k(\theta, R)$ is real for $\theta \in I$.

Further,

$$F_k(\theta; R) = 2 \operatorname{Re} g_k(\theta; R) + F_k^*(\theta; R), \quad (2.10)$$

where

$$g_k(\theta; R) = e^{(1/2)ki\theta} R(e^{2\pi ie^{i\theta}}) \quad (2.11)$$

and $F_k^*(\theta; R)$ consists of those terms of the series defining F_k for which $c^2 + d^2 \geq 2$. Note that $g_k(\theta; R)$ arises from the terms with $c, d = \pm 1, 0$ and $0, \pm 1$.

As θ increases from $\pi/3$ to $2\pi/3$ the point

$$t = e^{2\pi i e^{i\theta}} = e^{-2\pi \sin \theta + 2\pi i \cos \theta}$$

describes in a clockwise direction a curve γ beginning at

$$-r_0 = -e^{-\pi\sqrt{3}},$$

which encircles the origin, passing through the point

$$r_1 = e^{-2\pi}$$

and returning to $-r_0$. The curve γ is pear-shaped and symmetric about the real axis. It has a cusp at $-r_0$, the two tangents there making angles of $\pm \pi/3$ with the positive real axis. The curve γ and its interior D_γ are entirely contained in the unit disc

$$D = \{t \in \mathbb{C} : |t| < 1\}.$$

Moreover there is a one-to-one correspondence between points $t = e^{2\pi iz}$ in D_γ and points z of F for which $|z| > 1$.

We now assume that R has no zero or pole on γ and that it has N_γ zeros and P_γ poles in D_γ , counted with the appropriate multiplicities. Then the variation in the argument of $e^{ik\theta/2}R(t)$ as t describes S , i.e. as θ goes from $\pi/3$ to $2\pi/3$ is clearly

$$2\pi \left\{ P_\gamma - N_\gamma + \frac{k}{12} \right\},$$

by the Argument Principle.

Because of the symmetry of γ about the real axis, the variation in the argument of $e^{ik\theta/2}R(t)$ as t describes A , i.e. as θ goes from $\pi/3$ to $\pi/2$, is half this amount, namely

$$\pi \left(P_\gamma - N_\gamma + \frac{k}{12} \right)$$

Now

$$g_k(\pi/3; R) = e^{ik\pi/6}R(-r_0)$$

and

$$g_k(\pi/2; R) = e^{ik\pi/4}R(r_1).$$

Thus we may take

$$\arg g_k(\pi/3; R) = \pi \left(n_0 + \frac{k}{6} \right), \arg g_k(\pi/2; R) = \pi \left(n_1 + \frac{k}{4} \right),$$

where n_0 and n_1 are integers and

$$n_1 - n_0 = P_\gamma - N_\gamma. \quad (2.12)$$

Now suppose that $G_k(z; R)$ has N_R zeros and P_R poles in F , counted with appropriate multiplicities and weights. Then

$$N_R - P_R = \frac{k}{12} = l + \frac{s}{12}. \quad (2.13)$$

We are now ready to prove our main theorems. These apply to rational functions R with certain properties. We shall say that R has property P_k if (i) R is a real rational function, (ii) all the poles of R lie in D_γ , R has no zeros on γ , (iii) $l \geq N_\gamma - P_\gamma$, and (iv)

$$|F_k^*(\theta; R)| < 2|R(e^{2\pi i e^{i\theta}})| \quad (2.14)$$

for $\theta \in I_0 = [\pi/3, \pi/2]$. Note that (2.14) ensures that $G_k(z; R)$ does not vanish identically.

THEOREM 1: *Suppose that R has property P_k . Then the Poincaré series $G_k(z; R)$ has at least $N_R - N_\gamma$ zeros at points of A .*

PROOF: Note that N_γ is an integer, but N_R need not be. Further, by our assumptions, $P_R = P_\gamma$. It can be checked in each of the six cases that the interval $[n_0 + k/6, n_1 + k/4]$ contains exactly

$$n_1 - n_0 + k + 1$$

integers N . Note that $n_1 - n_0 + l \geq 0$ by (2.12) and condition (iii).

At the corresponding points $N\pi$, $g_k(\theta; R)$ takes alternately the values $\pm |g_k(\theta; R)|$, so that it follows by continuity from (2.10), (14) that $F_k^*(\theta; R)$ vanishes at least once in each of the $n_1 - n_0 + k$ subintervals between these points. Hence $G_k(z; R)$ has at least

$$n_1 - n_0 + k = P_\gamma - N_\gamma + l$$

zeros at interior points of A and therefore by (2.13), at least

$$P_\gamma - N_\gamma + l + \frac{s}{12} = N_R - N_\gamma$$

zeros on A .

As an immediate corollary we have

THEOREM 2. *Suppose that R has property P_k and that it does not vanish in D_γ . Then all the zeros of $G_k(z; R)$ in F lie on A . They are all simple zeros except that, when $k \equiv 2 \pmod{6}$, there are of necessity double zeros at $\rho = e^{\pi i/3}$.*

PROOF: For $N_\gamma = 0$ and we see that in (2.8), $\frac{2}{3}$ occurs only for $k \equiv 2 \pmod{6}$.

THEOREM 3: *Suppose that R has property P_k and that it has exactly one zero in D_γ , which is at the origin and is simple. Suppose also that R is bounded on $D - D_\gamma$. Then $G_k(z; R)$ has a simple zero at ∞ . All its other zeros in F lie on A and are simple except that, when $k \equiv 2 \pmod{6}$, there are double zeros at ρ .*

For it is easy to see that $G_k(z; R)$ has a zero at ∞ whenever $R(0) = 0$.

3. Applications

Before the theorems of the previous section can be applied, it is necessary to put condition (2.14) of property P_k into a more usable form. For our present purposes fairly crude estimates suffice, although we shall require more refined approximations in §4.

For $c^2 + d^2 \geq 2$ and $z = e^{i\theta} \in A$,

$$\text{Im } Tz = \frac{\sin \theta}{c^2 + d^2 + 2cd \cos \theta} = \psi_T(\theta), \tag{3.1}$$

say. Now it is easily checked, since $|cd| \geq 1$, that

$$c^2 + d^2 + 2cd \cos \theta \geq \frac{2}{\sqrt{3}} \sin \theta \quad (\pi/3 \leq \theta \leq \pi/2) \tag{3.2}$$

and hence

$$\psi_T(\theta) \leq \frac{\sqrt{3}}{2}. \tag{3.3}$$

Accordingly,

$$|e^{2\pi i Tz}| \geq e^{-\pi\sqrt{3}} = r_0 \quad (c^2 + d^2 \geq 2).$$

Define

$$M_R = \sup\{|R(t)|: r_0 \leq |t| \leq 1\}.$$

Note that M is finite by condition (ii) of P_k since, at any pole t of R , $|t| < r_0$.

Accordingly we have

$$|F_k^*(\theta; R)| \leq M_R \sum |c e^{i\theta} + d|^{-k}, \tag{3.4}$$

where, in the summation we take

$$c > 0, c^2 + d^2 \geq 2, (c, d) = 1. \tag{3.5}$$

Now

$$(c^2 + d^2 + 2cd \cos \theta)^{-k/2} + (c^2 + d^2 - 2cd \cos \theta)^{-k/2}$$

has, for $\theta \in I_0$, a maximum value when $\theta = \pi/3$ of

$$(c^2 + cd + d^2)^{-k/2} + (c^2 - cd + d^2)^{-k/2}$$

and accordingly

$$|F_k^*(\theta; R)| \leq M_R \sum (c^2 + cd + d^2)^{-k/2}, \tag{3.6}$$

subject to the same conditions (3.5). The series on the right is, apart from the omission of the terms with $c^2 + d^2 = 1$, a well-known Epstein zeta-function and we therefore have

$$|F_k^*(\theta; R)| \leq 2M_R \alpha_k, \tag{3.7}$$

where

$$\alpha_k = \frac{3Z_3(k/2)\zeta(k/2)}{2\zeta(k)} - 1. \tag{3.8}$$

Here ζ is the Riemann zeta-function and, for $s > 1$, $Z_3(s)$ is the Dirichlet L -series

$$Z_3(s) = 1 - 2^{-s} + 4^{-s} - 5^{-s} + 7^{-s} - 8^{-s} + \dots$$

α_k is a decreasing function of k . We have

$$\alpha_4 \leq 0.795, \alpha_6 \leq 0.568, \alpha_8 \leq 0.520, \alpha_{10} \leq 0.507, \alpha_{12} \leq 0.503,$$

while

$$\alpha_{24} \leq 0.500003$$

and for large k

$$\alpha_k = \frac{1}{2} + \frac{1}{2}3^{1-k/2} + O(7^{-k/2}).$$

Accordingly, condition (2.14) will be satisfied if

$$M_R \alpha_k < |R(e^{2\pi i e^{\theta}})| \quad (\theta \in I_0). \quad (3.9)$$

We now make a number of applications of these results.

CASE 1: Take

$$R(t) = t^{-m}, \text{ where } m \in \mathbb{Z}, m \geq 0,$$

so that $M_R = e^{mm\sqrt{3}}$, while

$$|R(e^{2\pi i e^{\theta}})| = e^{2\pi m \sin \theta} \geq e^{mm\sqrt{3}},$$

so that (3.9) is satisfied because $\alpha_k < 1$.

Since $P_\gamma = m$ and $N_\gamma = 0$ it is clear that property P_k holds. We deduce that the Poincaré series $G_k(z, m)$ has all its zeros in F on A and that they are all simple except as specified in Theorem 2. This includes the case $m = 0$ considered in [2].

CASE 2: Let

$$R(t) = \frac{g_n(t)}{f_m(t)},$$

where f_m and g_n are real polynomials with leading coefficients 1 and

of degrees m and n , respectively, where $m \geq n$. They therefore possess a total of $m + n$ non-leading coefficients all of which are real. We assume that the zeros of f_m and g_n lie in D_γ . Property P_k then holds.

We deduce from Theorem 1 that, provided that

$$\inf\{|R(t)|: t \in \gamma\} > \alpha_k \sup\{|R(t)|: r_0 \leq |t| \leq 1\}, \tag{3.10}$$

the Poincaré series $G_k(z; R)$ has at least $N_R - N_\gamma$ zeros on A . Now (3.10) is satisfied when $f_m(t) = t^m, g_n(t) = t^n$ by Case 1. Because of continuity and the compactness of the sets involved, there exists a neighbourhood U of the origin in \mathbb{R}^{m+n} such that, if the non-leading coefficients of f_m and g_n lie in U , then $G_k(z; R)$ has at least $N_R - N_\gamma$ zeros on A .

In particular, if $n = 1, g_n(t) = t$ and $m \geq 1$, it follows from Theorem 3 that, on some neighbourhood V of the origin in \mathbb{R}^m containing the non-leading coefficients of $f_m, G_k(z; R)$ has the properties stated in that theorem, provided that $f_n(0) \neq 0$.

CASE 3: We examine in greater detail the special case when

$$R(t) = (t - q)^{-m},$$

where $m \in \mathbb{N}$ and $q \in \mathbb{R} \cap D_\gamma$. Accordingly

$$-r_0 < q < r_1.$$

Note that

$$r_0 = 4.3334 \times 10^{-3}, r_1 = 1.8674 \times 10^{-3}.$$

Then, for $t \in \gamma$,

$$|t - q| \leq \max\{q + r_0, r_1 - q\} = r_3 + |q + r_2|,$$

where

$$r_2 = \frac{1}{2}(r_0 - r_1), r_3 = \frac{1}{2}(r_0 + r_1).$$

Also, for $|t| \geq r_0$

$$|t - q| \geq r_0 - |q|.$$

Accordingly (3.10) is satisfied whenever

$$\frac{r_0 - |q|}{r_3 + |q + r_2|} > \beta(k, m) = \alpha_k^{1/m}, \tag{3.11}$$

and we have

$$\frac{1}{2} < \beta = \beta(k, m) < 1.$$

Condition (3.11) is easily seen to be equivalent to

$$-\frac{r_0 - \beta r_1}{1 + \beta} < q < \frac{1 - \beta}{1 + \beta} r_0.$$

Thus, when q lies in this interval, all the zeros of $G_k(z; \mathbf{R})$ lie on A , for all $k \geq 4$.

4. Application to cusp forms

In what follows we take

$$R(t) = t^m \quad (m \in \mathbf{N})$$

so that, by (1.3),

$$G_k(z; \mathbf{R}) = G_k(z, m).$$

We assume that

$$k = 24 \text{ or } k \geq 28. \tag{4.1}$$

For $G_k(z, m)$ vanishes identically for $k = 4, 6, 8, 10, 14$, while, for $k = 12, 16, 18, 20, 22, 26$, the location of its zeros is known, since

$$G_k(z, m) = B_{k,m} \Delta(z) E_{k-12}(z),$$

where E_{k-12} is an Eisenstein series ($E_0 = 1$) and $B_{k,m}$ is a constant. It is known that the functions $G_k(z, m)$ ($0 < m \leq l$) span C_k and therefore do not vanish identically.

It is necessary to assume in what follows that

$$0 < m \leq l - 1. \tag{4.2}$$

By (2.12) and (4.2),

$$n_1 - n_0 + l = l - m \geq 1,$$

so that the interval $[n_0 + k/6, n_1 + k/4]$ contains $l - m + 1 \geq 2$ integers and condition (iii) of property P_k holds.

To obtain the results we wish to prove we must examine the function $F_k^*(\theta; R)$ in greater detail than previously. We consider first the terms with

$$\pm (c, d) = (-1, 1) \text{ and } (1, 1).$$

These give contributions

$$\frac{(-1)^{m+(1/2)k} e^{-\pi m \cot (1/2)\theta}}{(2 \sin \frac{1}{2}\theta)^k} \text{ and } \frac{(-1)^m e^{-\pi m \tan (1/2)\theta}}{(2 \cos \frac{1}{2}\theta)^k}.$$

Write

$$g_1(\theta) = 2 \sin \theta - \cot \frac{1}{2}\theta, \quad g_2(\theta) = 2 \sin \theta - \tan \frac{1}{2}\theta$$

and put

$$G_1(\theta) = \frac{\exp\{\pi m g_1(\theta)\}}{(2 \sin \frac{1}{2}\theta)^k}, \quad G_2(\theta) = \frac{\exp\{\pi m g_2(\theta)\}}{(2 \cos \frac{1}{2}\theta)^k}$$

for $\pi/3 \leq \theta \leq \pi/2$. Then

$$2G_1'(\theta) \sin^2 \frac{1}{2}\theta = G_1(\theta)[\pi m \{1 + 2 \cos \theta(1 - \cos \theta)\} - \frac{1}{2}k \sin \theta].$$

The expression in square brackets decreases as θ increases taking its maximum value of $\frac{3}{4}(2\pi m - k\sqrt{3})$ at $\theta = \frac{1}{3}\pi$. This value is negative, so that $G_1'(\theta) \leq 0$ and therefore

$$G_1(\theta) \leq G_1(\frac{1}{3}\pi) = 1 \quad (\theta \in I_0). \tag{4.3}$$

Also

$$G_2'(\theta) \cos^2 \frac{1}{2}\theta = G_2(\theta)[\pi m \{\cos \theta(1 + \cos \theta) - \frac{1}{2}\} + \frac{1}{4}k \sin \theta],$$

which is positive since $k \geq 12m$. Hence

$$G_2(\theta) \leq G_2(\frac{1}{2}\pi) = \frac{e^{\pi m}}{2^{k/2}}. \tag{4.4}$$

We have $c^2 + d^2 \geq 5$ for the remaining values of c, d summed over in $F_k(\theta; R)$, and $\psi_T(\theta) \geq 0$; see (3.1). Hence, as in §3, an upper bound for the remaining terms is given by

$$\begin{aligned} \delta_k &= \frac{1}{2} \sum_{c^2+d^2>5} (c^2 + d^2 + cd)^{-k/2} \\ &= 3 \left\{ \frac{Z_3(\frac{1}{2}k)\zeta(\frac{1}{2}k)}{\zeta(k)} - 1 - 3^{-1-k/2} \right\}. \end{aligned} \quad (4.5)$$

We have

$$\delta_{24} = 10^{-6} \times 3.764,$$

and, by using the approximations

$$\begin{aligned} Z_3(x) &\leq 1 - 2^{-x} + 4^{-x}, \\ \zeta(x) &\leq 1 + 2^{-x} + 3^{-x} + \frac{3^{1-x}}{x-1}, \\ \{\zeta(2x)\}^{-1} &\leq 1 - 2^{-2x}, \end{aligned}$$

we find that

$$\delta_k \leq 3^{-k/2} \left(2 + \frac{18}{k-2} \right). \quad (4.6)$$

The only condition of property P_k that remains to be checked is (2.14), which now takes the form

$$e^{-2\pi m \sin \theta} \left\{ 1 + \frac{e^{\pi m}}{2^{k/2}} \right\} + \delta_k < 2e^{-2\pi m \sin \theta}.$$

For this to hold we require that

$$\frac{e^{\pi m}}{2^{k/2}} + \delta_k e^{2\pi m} < 1$$

which, by (4.6), since $12(m+1) \leq k$, reduces to

$$e^{-\pi} \left(\frac{1}{2} e^{\pi/6} \right)^{k/2} + e^{-2\pi} \left(\frac{1}{3} e^{\pi/3} \right)^{k/2} \left(2 + \frac{18}{k-2} \right) < 1.$$

For $k \geq 24$ the left-hand side is less than 0.00849 so that condition (2.11) is satisfied.

THEOREM 4: *Suppose that $l = \dim C_k \geq 1$ and that $0 < m \leq l$. Then $G_k(z, m)$ has at least $\frac{1}{12}k - m$ zeros on A and at least one at ∞ . In particular, all the zeros of $G_k(1, m)$ in F are simple, except for a double zero at $\rho = e^{\pi i/3}$, when $k \equiv 2 \pmod{6}$. One of these simple zeros is at ∞ and the others lie on A .*

In view of the preceding analysis we need only remark that the theorem is trivial when $m = l$ since in that case there are at least $s/12$ zeros at i and ρ .

5. Cusp forms of weight 24

Theorem 4 gives an exact estimate of the number of zeros of $G_k(z, m)$ on A only when $m = 1$. For $m > 1$ only a lower bound is given. It would be of interest to have more precise information about the location of zeros when $m > 1$. In this section we examine the first such case, which arises when

$$k = 24, l = 2, m \geq 2.$$

The space C_{24} has a basis consisting of the newforms

$$f_j(z) = \sum_{n=1}^{\infty} \lambda_j(n) e^{2\pi i n z} \quad (z \in H; j = 1, 2), \tag{5.1}$$

where the coefficients $\lambda_j(n)$ are the eigenvalues corresponding to Hecke's operator T_n . We have (see [3], (9.7))

$$f_j = \Delta[E_{12} + \{\mu + (-1)^{j-1}\nu\}\Delta] \quad (j = 1, 2), \tag{5.2}$$

where

$$\mu = \frac{324204}{691}, \nu = 12 \sqrt{144169} = 12\eta,$$

and

$$\eta = \sqrt{144169} = 379.69593.$$

It follows from (5.1, 2) that

$$\lambda_j(2) = 540 + (-1)^{j-1}\nu$$

so that

$$\lambda_1 = \lambda_1(2) = 5096.3512, \lambda_2 = \lambda_2(2) = -4016.3512.$$

For later purposes we also require the values

$$\mu_1 = \lambda_1(3) = 169740 - 576\eta = -48964.855,$$

and

$$\mu_2 = \lambda_2(3) = 169740 + 576\eta = 388444.85.$$

In all these and later estimates the last digit may be in doubt.

Write

$$g_k(z, m) = m^{k-1}G_k(z, m) \quad (m \in \mathbb{N}).$$

Then

$$g_k(z, m) = g_k(z, 1)|T_m,$$

where T_m is Hecke's operator; see [3]. If we write

$$g_{24}(z, 1) = \xi_1 f_1(z) + \xi_2 f_2(z),$$

then

$$g_n(z) := g_{24}(z, n) = \xi_1 \lambda_1(n) f_1(z) + \xi_2 \lambda_2(n) f_2(z). \quad (5.3)$$

We are particularly interested in the location of the zeros of g_2 , and therefore require to evaluate ξ_1 and ξ_2 .

From [3] (p. 205) we know that ξ_1 and ξ_2 are positive and that (see [3], equation (7.4), with $q = 20$, $r = 4$, $k = 24$)

$$\xi_j = \frac{\zeta(20)\{\Lambda_{j1}\beta(20, 4; 1) + \Lambda_{j2}\beta(20, 4; 2)\}}{\alpha_4 \phi_{24}^{(j)}(23) \phi_{24}^{(j)}(20)}. \quad (5.4)$$

Here

$$\Lambda_{11} = \lambda_2/(\lambda_2 - \lambda_1), \Lambda_{12} = -1/(\lambda_2 - \lambda_1),$$

$$\Lambda_{21} = -\lambda_1/(\lambda_2 - \lambda_1), \Lambda_{22} = 1/(\lambda_2 - \lambda_1),$$

and α_n is the coefficient of $e^{2\pi iz}$ in the Fourier expansion of $E_n(z)$, and not the quantity defined in (3.8). Also

$$\begin{aligned} \beta(20, 4, 1) &= \alpha_{20} + \alpha_4 - \alpha_{24} = 240.07005, \\ \beta(20, 4, 2) &= 9\alpha_4 + \alpha_4\alpha_{20} + 524289\alpha_{20} - 8388609\alpha_{24} \\ &= 37161.979. \end{aligned}$$

Finally

$$\phi_{24}^{(j)}(s) = \sum_{n=1}^{\infty} \lambda_j(n)n^{-s} = \prod_p \{1 - \lambda_j(p)p^{-s} + p^{23-2s}\}^{-1},$$

where the product is carried out over all prime numbers p .

By using the values of $\lambda_j(2)$ and $\lambda_j(3)$ and Deligne's bounds

$$|\lambda_j(p)| \leq 2p^{23/2}$$

for $p > 3$ we find that

$$\begin{aligned} \phi_{24}^{(1)}(20) &= 1.00486, \quad \phi_{24}^{(2)}(20) = 0.99629, \\ \phi_{24}^{(1)}(23) &= 1.000607, \quad \phi_{24}^{(2)}(23) = 0.999525, \end{aligned}$$

which leads to the values

$$\xi_1 = 0.45537, \quad \xi_2 = 0.54471$$

and

$$\xi_1\lambda_1 + \xi_2\lambda_2 = 133. \tag{5.5}$$

From (5.3, 5) we see that g_2 has a simple zero at ∞ . We now show that it does not vanish on A , but that its remaining zero in F lies on L_ρ , the right-hand boundary of F , and is simple. For this purpose we put

$$g_n(z) = \Delta(z)h_n(z),$$

where $h_n \in H_{12}$ and

$$h_n = \{\xi_1\lambda_1(n) + \xi_2\lambda_2(n)\}(E_{12} + \mu\Delta) + \{\xi_1\lambda_1(n) - \xi_2\lambda_2(n)\}\nu\Delta.$$

Then h_2 has exactly one zero, which is simple, in F . Since $h_2 \in H_{12}^*$ this zero lies either on A or on L_i or on L_ρ . To check this we require the

values of E_{12} and Δ at the points i and ρ . We have

$$E_4(i) = e_4 > 0, E_4(\rho) = 0, E_6(i) = 0, E_6(\rho) = e_6 > 0$$

so that

$$\begin{aligned} 1728\Delta(i) &= e_4^3, 1728\Delta(\rho) = -e_6^2, \\ 691E_{12}(i) &= 441e_4^3, 691E_{12}(\rho) = 250e_6^2. \end{aligned}$$

See (6.1.14–16) of [5]; in (6.1.16) the denominator should be 762048. It follows that

$$144e_6^{-2}h_2(\rho) = -(1723008 + 384\eta)\xi_1 - (1723008 - 348\eta)\xi_2 < 0;$$

for $384\eta < 1723008$.

Since h_2 is real on L_ρ and $h_2(\infty) = \xi_1\lambda_1 + \xi_2\lambda_2 > 0$, by (5.5), it follows that h_2 has a simple zero at a point of L_ρ . It can be shown in a similar way that the same is true for h_4 . On the other hand, h_3 has a simple zero on L_i . Observe that L_ρ forms part of the set of transforms of the unit circle under $\Gamma(1)$, whereas L_i does not.

Finally, by using the fact that f_1 and f_2 are orthogonal and the asymptotic formula for $\sum_{n \leq x} \lambda_i^2(n)$ we can show that for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that g_n does not vanish on A . A similar result holds with A replaced by L_i and L_ρ .

REFERENCES

- [1] H. PETERSSON: Theorie der automorphen Formen beliebiger reeller Dimension und ihre Darstellung durch eine neue Art Poincaréscher Reihen. *Math. Ann.* 103 (1930) 369–436.
- [2] F.K.C. RANKIN and H.P.F. SWINNERTON-DYER: On the zeros of Eisenstein series. *Bull. London Math. Soc.* 2 (1970) 169–170.
- [3] R.A. RANKIN: The scalar product of modular forms. *Proc. London Math. Soc.* (3) 2 (1952) 198–217.
- [4] R.A. RANKIN: The zeros of Eisenstein series. *Publications Ramanujan Inst.* 1 (1969) 137–144.
- [5] R.A. RANKIN: *Modular forms and functions*. Cambridge Univ. Press, 1977.
- [6] K. WOHLFAHRT: Über die Nullstellen einiger Eisensteinreihen, *Math. Nachr.* 26 (1964) 381–383.

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