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ON THE ASYMPTOTIC DISTRIBUTION OF THE POWERS OF $S \times S$-MATRICES

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1. Introduction

It is the aim of this paper to study the asymptotic distribution modulo 1 of the sequence of powers of a real (or complex) $s \times s$-matrix A from a metrical point of view; more generally for a strictly monotone sequence $(p(n))_{n=1}^{\infty}$ of positive integers the sequence $(A^{p(n)})_{n=1}^{\infty}$ is considered. Obviously a real $s \times s$-matrix can be regarded as an element of $\mathbb{R}^{s^2}$ (by componentwise identification). Similarly a complex $s \times s$-matrix (real symmetric, triangular or Hermitian $s \times s$-matrix) can be regarded as an element of $\mathbb{R}^{2s^2}$ ($\mathbb{R}^{(s(s+1)/2)}$, $\mathbb{R}^{(s(s+1)/2)}$, $\mathbb{R}^{s^2}$). So the sequence $(A^{p(n)})_{n=1}^{\infty}$ of matrices can be regarded as a sequence $(x_n)_{n=1}^{\infty}$ with elements $x_n \in \mathbb{R}^d$.

In the theory of uniform distribution the discrepancy (see [6] and [8]) of a sequence (x_n) with elements $x_n \in \mathbb{R}^d$ is defined by

$$D_N(x_n) = \sup_I \left| \frac{1}{N} \sum_{n=1}^{N} \chi_I(\{x_n\}) - \mu(I) \right|, \tag{1.1}$$

where I runs through all subintervals of the d-dimensional unit interval $[0, 1)^d$; χ_I denotes the indicator function of I; $\mu(I)$ the Lebesgue measure of I and $\{x_n\}$ the fractional part of x_n, componentwise. $(x_n)_{n=1}^{\infty}$ is uniformly distributed modulo 1, if and only if

$$\lim_{N \to \infty} D_N(x_n) = 0. \tag{1.2}$$

In [7] it is proved that the sequence (x^n) is uniformly distributed for almost all real numbers x with $|x| \geq 1$ (in the sense of the usual Lebesgue measure). This result was generalized to the sequence (z^n)

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of complex numbers by Le Veque [10], and to the sequence (z^n) of quaternions in [15]; in [11] an estimation of the discrepancy is established. Furthermore special types of matrix-sequences $(A^{p(n)})_{n=1}^{\infty}$ are investigated in the papers [5], [12] and [13].

We use the abbreviations $\Lambda(a) = \max|\lambda_i|$ and $\lambda(A) = \min|\lambda_i|$, where the maximum and minimum is taken over all eigenvalues of the matrix A; our main theorem says.

THEOREM I: Let $p(n)$ *be a strictly monotone sequence of positive integers. Then for almost all real* $s \times s$*-matrices* A *with* $\Lambda(A) \geq 1$ (*in the sense of the* s^2*-dimensional Lebesgue measure*) *and all* $\epsilon > 0$ *a positive constant* $C(A, \epsilon)$ *exists such that the discrepancy* $D_N(A^{p(n)})$ *can be estimated by*

$$D_N(A^{p(n)}) \leq C(A, \epsilon) N^{-1/2} (\log N)^{s^2+(3/2)+\epsilon}. \tag{1.3}$$

REMARK 1.4: For complex $s \times s$-matrices A one can prove similarly the following estimate for almost all matrices A with $\Lambda(A) \geq 1$ (in the sense of the $2s^2$-dimensional Lebesgue measure):

$$D_N(A^{p(n)}) \leq C(A, \epsilon) N^{-1/2} (\log N)^{2s^2+(3/2)+\epsilon}. \tag{1.4}$$

Furthermore we prove

THEOREM II: *Let* $p(n)$ *be given as above and* $\mathfrak{R}$ *the family of all real* $s \times s$*-matrices. A having at least one real eigenvalue with modulus larger than* 1. *Then for almost all* $A \in \mathfrak{R}$ (*in the sense of the* s^2*-dimensional Lebesgue measure*) *the discrepancy of* $(A^{p(n)})_{n=1}^{N}$ *can be estimated by*

$$D_N(A^{p(n)}) \leq C(A) N^{-1/2} (\log N)^{s^2+1/2} (\log\log N)^{1/2}. \tag{1.5}$$

Similarly we can prove

THEOREM III: *Let* $p(n)$ *be given as above. Then for almost all real symmetric* $s \times s$*-matrices* A *with* $\Lambda(A) \geq 1$ (*or almost all* $s \times s$*-Hermitian matrices* A with $\Lambda(A) \geq 1$, *respectively*) *a positive constant* $C(A)$ *exists such that*

$$D_N(A^{p(n)}) \leq C(A) N^{-1/2} (\log N)^{d(s)+1/2\,(\log\log N)^{1/2}}, \tag{1.6}$$

where $d(s) = \frac{s(s+1)}{2}$ in the case of real symmetric and $d(s) = s^2$ in the case of Hermitian matrices.

For real triangular matrices we obtain

THEOREM IV: *Let $p(n)$ be given as above. Then for almost all real $s \times s$-triangular matrices with $\lambda(A) \geq 1$ (in the sense of the $\frac{s(s+1)}{2}$ dimensional Lebesgue measure) a positive constant $C(A)$ exists such that*

$$D_N(A^{p(n)}) \leq C(A) N^{-1/2} (\log N)^{(s^2+s+1)/2} (\log \log N)^{1/2}. \tag{1.7}$$

In the case of complex triangular matrices we only obtain an estimate as in Theorem I:

THEOREM V: *Let $p(n)$ be given as above. Then for almost all complex $s \times s$-triangular matrices A with $\lambda(A) \geq 1$ (in the sense of the $s(s+1)$-dimensional Lebesgue measure) and all $\epsilon > 0$ a positive constant $C(S, \epsilon)$ exists such that*

$$D_N(A^{p(n)}) \leq C(A, \epsilon) N^{-1/2} (\log N)^{s2+s+3/2+\epsilon}. \tag{1.8}$$

We want to remark that the asymptotic distribution of $(A^{p(n)})_{n=1}^{\infty}$ is completely described for the class of all $s \times s$-matrices by Theorem I from a metrical point of view; in the case $\Lambda(A) < 1$ the sequence $(A^{p(n)})$ tends to zero and so it cannot be uniformly distributed.

2. Auxiliary results

In this chapter we state some auxiliary results, the first of it is a generalization of Chintchin's metric result on diophantine approximation going back to Sprindžuk and Kovalevskaja.

2.1 PROPOSITION: *For almost all elements $a,b,c,d \in \mathbb{R}^s$, $a = (a_i)$, $b = (b_i)$, $c = (c_i)$, $d = (d_i)$, $\langle a, d \rangle = \langle b, c \rangle = 0$ (with $\langle a, d \rangle = \Sigma_i a_i d_i$, the ordinary inner product), there exists a constant $C = C(a, b, c, d)$ such that the following inequalities hold for all non-zero integer vectors*

(t_{ij}):

(i) $$\left|\sum_{i,j=1}^{s} a_i b_j t_{ij}\right| \geq C(\max|t_{ij}|)^{-s^2}$$

(ii) $$\left|\sum_{1\leq i\leq j\leq s} a_i a_j t_{ij}\right| \geq C(\max|t_{ij}|)^{-s^2/2}$$

(iii) $$\left|\sum_{i,j=1}^{s} (a_i b_j + c_i d_j) t_{ij}\right| \geq C(\max|t_{ij}|)^{-2s^2-9}.$$

REMARK: Since c and d do not appear in (i) and (ii), the statement means simply that (i) and (ii) hold for almost all (a, b) and almost all a respectively. In (ii) one clearly needs $\max_{i\leq j} |t_{ij}| > 0$.

PROOF: (i) and (ii) follow from [14], chapter 2, theorem 8, p. 106 together with Chintchin's transfer principle [9], §45, theorem 6, p. 392.

We will now sketch a proof for (iii). It is based on the following observations: if $x \in \mathbb{R}^s$ is arbitrary, $\alpha, \beta \in \mathbb{R}$, then:

$$m_s\left(\left\{(y_i) \in \mathbb{R}^s\colon |y_i| \leq 1, \left|\sum_{i=1}^{s} x_i y_i + \beta\right| < \alpha\right\}\right) \leq (2s)^s \alpha\ (\max|x_i|)^{-1} \tag{2.2}$$

(m_s denotes the ordinary Lebesgue measure on $\mathbb{R}^s$; the estimate holds because the set is contained in a parallelepiped of height $2\alpha(\max|x_i|)^{-1}$ and base lengths $(2s)^{1/2}$).

Assume that $c_1, d_1 \neq 0$ are fixed. Then $\langle a, d\rangle = \langle b, c\rangle = 0$ implies

$$a_1 = -\sum_{i=2}^{s} (d_i/d_1) a_i, \qquad b_1 = -\sum_{j=2}^{s} (c_j/c_1) b_j.$$

It follows that:

$$\sum_{i,j=1}^{s} a_i b_j t_{ij} = \sum_{i=2}^{s} a_i \left(\sum_{j=2}^{s} b_j (t_{ij} - (d_i/d_1) t_{1j} - (c_j/c_1)(t_{i1} + (d_i/d_1) t_{11}))\right).$$

We consider only the case where all numbers a_i, b_j, c_i, d_j have modulus not greater than 1. If $\epsilon > 0$ is arbitrary, it follows from Chintchin's classical result that for almost all d there exists a constant $C_1 = C_1(d) \leq 1$, such that $|t_{i1} - (d_i/d_1) t_{11}| \geq C_1 |t_{11}|^{-1-\epsilon}$ if $t_{11} \neq 0$. We fix such a vector d. By (2.2)

$$m_{s-1}(\{(c_j)_{j=2}^s \in \mathbb{R}^{s-1} : |c_j| \le 1,\ |t_{ij} - (d_i/d_1)t_{1j} - (c_j/c_1)(t_{i1} - (d_i/d_1)t_{11})| < \\ < (\max|t_{ij}|)^{-6-2\epsilon}\}) \le \\ \le (2s)^s(\max|t_{ij}|)^{-6-2\epsilon}\,|c_1|^{-1}|t_{i1} - (d_i/d_1)t_{11}|^{-1} \le \\ \le (2s)^s|c_1|^{-1}C_1(\max|t_{ij}|)^{-5-\epsilon} \text{ if } (t_{i1}, t_{11}) \ne (0,0).$$

If $t_{i1} = t_{11} = 0$ and $(t_{ij}, t_{1j}) \ne (0,0)$, then the set is empty for $\max(|t_{ij}|, |t_{1j}|) \ge C_1^{1/5}$.

By the Borel-Cantelli-lemma, we conclude that

$$\{(c_j)_{j=2}^s \in \mathbb{R}^{s-1}: |c_j| \le 1, |t_{ij} - (d_i/d_1)t_{1j} - (c_j/c_1)(t_{i1} - (d_i/d_1)t_{11})| \\ < (\max|t_{ij}|)^{-6-2\epsilon} \text{ for infinitely many integers } (t_{11}, t_{1j}, t_{i1}, t_{ij})\}$$

has zero measure in $\mathbb{R}^{s-1}$. This means that for almost all c there exists a constant $C_2 = C_2(c, d)$, such that

$$|t_{ij} - (d_i/d_1)t_{1j} - (c_j/c_1)(t_{i1} - (d_i/d_1)t_{11})| \ge C_2(\max|t_{ij}|)^{-6-2\epsilon}$$

if the coefficients do not vanish. We fix such a vector c and by repeating this argument we deduce that for almost all b there exists a constant $C_3 = C_3(b, c, d)$, such that

$$\left|\sum_{j=2}^s b_j(t_{ij} - (d_i/d_1)t_{1j} - (c_j/c_1)(t_{i1} - (d_i/d_1)t_{11}))\right| \ge C_3(\max|t_{ij}|)^{-s^2-7-3\epsilon}$$

if the coefficients do not vanish. In the last step, it follows that for almost all b there exists a constant $C_4 = C_4(a, b, c, d)$, such that

$$\left|\sum_{i=2}^s a_i\left(\sum_{j=2}^s b_j(t_{ij} - (d_i/d_1)t_{1j} - (c_j/c_1)(t_{i1} + (d_i/d_1)t_{11}))\right) + \sum_{i,j=1}^s c_i d_j t_{ij}\right| \ge \\ \ge C_4(\max|t_{ij}|)^{-2s^2-8-4\epsilon}.$$

Taking $\epsilon = 1/4$ gives the result.

2.2 Proposition: *Let M_1, M_2 be C^∞-manifolds (with countable base), $f: M_1 \to M_2$ a surjective mapping, $E = \{x \in M_1 : (df)_x$ is not surjective$\}$. Then the following holds:*

(i) *If B_1 is a subset of M_1 such that $M_1 \backslash B_1$ is negligible, then $M_2 \backslash f(B_1)$ is also negligible.*

(ii) *If E is negligible and B_2 is a subset of M_2 such that $M_2 \backslash B_2$ is negligible, then $M_1 \backslash f^{-1}(B_2)$ is also negligible.*

PROOF: *If* $x \notin E$, then there exists charts (U, φ) $((V, \psi)$ respectively) in M_1 (M_2 respectively) with $x \in U$, such that $f(U) = V$ and $\psi \circ f \circ \varphi^{-1}(u_1, \ldots, u_m) = (u_1, \ldots, u_n)$, $m = \dim M_1$ and $n = \dim M_2$ (see [1], 16.7.4). $\varphi(U \backslash B_1)$ is a null set and, by Fubini's theorem, it follows that

$$\psi \circ f(U) \backslash \psi \circ f(B_1 \cap U) = \psi \circ f \circ \varphi^{-1}(\varphi(U \backslash B_1))$$

is a null set, too. Consequently $f(U) \backslash f(B_1)$ is a null set. Similary one shows that $U \backslash f^{-1}(B_2)$ is a null set. Since there exists a countable family of charts (U, φ) $((V, \psi)$ respectively) with the properties mentioned above and covering M_1 (or M_2 respectively), we deduce that $(M_2 \backslash f(E)) \backslash f(B_1)$ and $(M_1 \backslash E) \backslash f^{-1}(B_2)$ are negligible. Finally by Sard's theorem ([1], 16.23.1), $f(E)$ is a null set in M_2 and in the second case E is a null set in M_1. This proves that $M_2 \backslash f(B_1)$ and $M_1 \backslash f^{-1}(B_2)$ are negligible.

REMARK: The assumption that E is negligible is satisfied, if M_1 and M_2 are connected, analytic manifolds and f is analytic. Indeed, considering again local charts (U, φ), the set $U \cap E$ can be represented as the set of common zeroes of a finite number of analytic functions (the subdeterminants of order n of df). If U is connected, it follows that $U \cap E$ can be non-negligible only if $U \subseteq E$. But since M_1 is connected one would get (by repeating this argument) that $E = M_1$ and this contradicts Sard's theorem.

Now we formulate two well known theorems without proof, the first of it is the inequality of Erdös–Turan–Koksma in $\mathbb{R}^d$ (see [2] and [8], p. 116). We use the abbreviations $e(x) = e^{2\pi i x}$ and $r(h) = \prod_{i=1}^{d} \max(|h_i|, 1)$, $\|h\| = \max_{i=1,\ldots,d} |h_i|$ for the d-dimensional integer vector $h \in \mathbb{Z}^d \backslash \{0\}$; $\langle .\,, . \rangle$ denotes the ordinary inner product on $\mathbb{R}^d$.

2.3 PROPOSITION: *Let (x_n) be a sequence with elements $x_n \in \mathbb{R}^d$, then for all positive integers H the following estimation of discrepancy holds*:

$$D_N(x_n) \leq c_d \Big(H^{-1} + \sum_{0 < \|h\| \leq H} r(h)^{-1} \Big| \frac{1}{N} \sum_{n=1}^{N} e(\langle h, x_n \rangle) \Big| \Big)$$

with an absolute constant c_d.

The following result is a special of a theorem of Gal and Koksma (see [3] and [4]).

2.4 PROPOSITION: *Assume that for all M, $N \in \mathbb{N}_0$ a function $y \mapsto F(M,N)(y)$ is defined on the interval $[a,b]$ such that $F(M,N)$ belongs to the class $\mathfrak{L}^2[a,b]$ and for all $(M,N,N_1) \in \mathbb{N}_0^3$ the following inequality holds*:

(i) $$|F(M,N)| \leq |F(M,N_1)| + |F(M+N_1, N-N_1)|;\ (N_1 \leq N).$$

Furthermore, we assume the existence of a constant C_1 (independent of M, N) such that

(ii) $$\int_a^b |F(M,N)(y)|^2\, dy \leq C_1 N \cdot (\log N)^\sigma.$$

Then for all $\delta > 0$ and for almost all $y \in (a,b)$ there exists a constant $C_2 = C_2(y,\delta)$ such that

$$|F(0,N)(y)| \leq C_2 N^{1/2} (\log N)^{1/2(\sigma+3+\delta)}.$$

3. Proof of theorem I

First we show that we can restrict our investigations to the matrices A with pairwise distinct eigenvalues λ_i. Let $\mathfrak{M} \subset \mathbb{R}^{s^2}$ be the set of $s \times s$-matrices A with $\Lambda(A) > 1$ and pairwise distinct eigenvalues $\lambda_i \neq \lambda_j$ for $i \neq j$. The matrix $A = (a_{ij})$ is contained in $\mathfrak{M}$ if $\Delta(a_{ij}) = 0$, where Δ is the discriminant of the characteristic polynomial of A. Δ is a polynomial in the s^2 variables a_{ij} $(i,j = 1,\ldots,s)$. If we suppose that Δ vanishes identically, then all $s \times s$-matrices A would have two equal eigenvalues. Therefore Δ is different from the null polynomial and $\Delta = 0$ defines an at most $(s^2 - 1)$-dimensional algebraic manifold, and so $\mathfrak{M}$ is an open set and its complement $\mathbb{R}^{s^2} \backslash \mathfrak{M}$ an s^2-dimensional null set.

For a matrix $A \in \mathfrak{M}$ we denote by $\lambda_1, \ldots, \lambda_\rho$ the real eigenvalues of A and by $\lambda_{\rho+\kappa} = \bar{\lambda}_{\rho+\kappa+\tau}$ $(1 \leq \kappa \leq \tau,\ \rho + 2\tau = s)$ the (pairwise conjugate) complex eigenvalues of A. For fixed ρ (and τ) $\mathfrak{M}_\rho$ denotes the subset of $\mathfrak{M}$ with ρ real eigenvalues and $\Lambda(A) > 1$; Y denotes a matrix of the type

(3.1) $$Y = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_\rho & & & \\ & & & \Lambda_{\rho+1} & & \\ & & & & \ddots & \\ & 0 & & & & \Lambda_{\rho+\tau} \end{pmatrix}$$

with

$$\Lambda_{\rho+\kappa} = \begin{pmatrix} r_{\rho+\kappa} \cos \varphi_{\rho+\kappa} & r_{\rho+\kappa} \sin \varphi_{\rho+\kappa} \\ -r_{\rho+\kappa} \sin \varphi_{\rho+\kappa} & r_{\rho+\kappa} \cos \varphi_{\rho+\kappa} \end{pmatrix} \quad (1 \le \kappa \le \tau)$$

and

$$\begin{aligned} r_{\rho+\kappa} &= |\lambda_{\rho+\kappa}|, \ \varphi_{\rho+\kappa} = \arg \lambda_{\rho+\kappa} \in (0, \pi), && \text{if } \operatorname{Im} \lambda_{\rho+\kappa} > 0, \\ r_{\rho+\kappa} &= -|\lambda_{\rho+\kappa}|, \ \varphi_{\rho+\kappa} = -\arg \lambda_{\rho+\kappa} \in (0, \pi), && \text{if } \operatorname{Im} \lambda_{\rho+\kappa} < 0. \end{aligned}$$

The set of all such matrices with $\max\{|\lambda_1|, \ldots, |\lambda_\rho|, |r_{\rho+1}|, \ldots, |r_{\rho+\tau}|\} > 1$ and $\varphi_{\rho+\kappa} \in (0, \pi)$ (for $1 \le \kappa \le \tau$) is denoted by $\mathfrak{K}_\rho$. Now for all $A \in \mathfrak{M}_\rho$ a matrix $Y \in \mathfrak{K}_\rho$ exists such that $\alpha = X^{-1}YX$ with a certain invertible transformation matrix X; we have for $\nu \in \mathbb{N}$

(3.2)
$$A = X^{-1}Y^\nu X = \sum_{i=1}^{\rho} \lambda_i^\nu P_i(X) + \sum_{i=\rho+1}^{\rho+\tau} (P_i(X) \cos \nu\varphi_i + Q_i(X) \sin \nu\varphi_i) r_i^\nu$$

with $P_i(X) = X^{-1}I_iX$, where I_i denotes a matrix as in (3.1) and

$$\lambda_i = 1, \ \lambda_j = 0 \text{ (for } j \ne i), \ \Lambda_j = 0 \text{ (all } j) \text{ if } 1 \le i \le \rho.$$

or

$$\Lambda_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \lambda_j = 0 \text{ (for all } j), \ \Lambda_j = 0 \text{ (for } j \ne i) \text{ if } \rho + 1 \le i \le \rho + \tau;$$

and $Q_i(X) = X^{-1}J_iX$ $(\rho + 1 \le i \le \rho + \tau)$, where J_i denotes a matrix as in (3.1) and

$$\Lambda_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \lambda_j = 0 \text{ (for all } j), \ \Lambda_j = 0 \text{ (for } j \ne i).$$

We define for an integer-valued $s \times s$-matrix $G = (g_{ij})$ (different from the null matrix) and a fixed X:

$$\begin{aligned} &\alpha_i(G, X) = \operatorname{tr}(P_i(X) \cdot G) && \text{for } 1 \le i \le \rho + \tau \\ (3.3) \quad &\beta_i(G, X) = \operatorname{tr}(Q_i(X) \cdot G) && \text{for } \rho + 1 \le i \le \rho + \pi, \\ &\gamma_i(G, X) = (\alpha_i(G, X)^2 + \beta_i(G, X)^2)^{1/2} && \text{for } \rho + 1 \le i \le \rho + \tau. \end{aligned}$$

Now the family of all invertible transformation matrices X is an s^2-dimensional C^∞-manifold; we denote it by $\mathfrak{M}^*$; of course $\mathfrak{K}_\rho$ is an

s-dimensional manifold. The mapping $\vartheta_\rho : \mathfrak{M}^* \times \mathfrak{K}_\rho \to \mathfrak{M}_\rho$ defined by

$$\vartheta_\rho(X, Y) = X^{-1}YX = A \tag{3.4}$$

is an analytic mapping of the product manifold $\mathfrak{M}^* \times \mathfrak{K}_\rho$ onto the manifold $\mathfrak{M}_\rho$.

For $1 \leq i \leq \rho$ we consider the mapping $\eta_i : \mathfrak{M}^* \to \mathbb{R}^{2s}$, defined by

$$\eta_i(X) = (a, b), \tag{3.5}$$

where $a = (a_k) \in \mathbb{R}^s$ is the i-th row of the matrix X and $b = (b_k) \in \mathbb{R}^s$ the i-th column of X^{-1}. The image $\mathfrak{M}^{(i)} = \eta_i(\mathfrak{M}^*)$ is open in $\mathbb{R}^{2s}$ and the mapping $\eta_i : \mathfrak{M}^* \to \mathfrak{M}^{(i)}$ is surjective and analytic. Similarly we consider for $\rho + 1 \leq i \leq \rho + \tau$ the mapping $\eta_i : \mathfrak{M}^* \to \mathbb{R}^{4s}$ defined by

$$\eta_i(X) = (a, b, c, d) \tag{3.6}$$

where $a = (a_k)$ is the $(2i - 1 - \rho)$-th row of X,

$c = (c_k)$ is the $(2i - \rho)$-th row of X

$b = (b_k)$ is the $(2i - 1 - \rho)$-th column of X^{-1},

and

$d = (d_k)$ is the $(2i - \rho)$-th column of X^{-1}.

$\eta_i : \mathfrak{M}^* \to \mathfrak{M}^{(i)}$ is surjective and analytic; $\mathfrak{M}^{(i)} = \{(a, b, c, d) : \langle a, d\rangle = \langle b, c\rangle = 0\}$ is the image of $\mathfrak{M}^*$ as above.

An immediate consequence of Proposition 2.1 and Proposition 2.2 is the following

3.7 Lemma: *For almost all $X \in \mathfrak{M}^*$ (in the sense of the s^2-dimensional Lebesgue measure) there exists a positive constant $C(X)$ such that*

(i) $$|\alpha_i(G, X)| \geq C(X)\|G\|^{-5s^2}$$

(ii) $$|\gamma_i(G, X)| \geq C(X)\|G\|^{-5s^2}$$

with $G = \max_{i,j}|g_{ij}| : \mathfrak{S}$ denotes the set of all $X \in \mathfrak{M}$ such that (i) and (ii) are valid.

Proof: Let $1 \leq i \leq \rho$, then $\alpha_i(G, X) = \Sigma_{i,j=1}^{s}\, a_i b_j g_{ji}$ with $\eta_i(X) =$

(a, b) and $a = (a_k)$, $b = (b_k)$ as in (3.5). By Proposition 2.1 we obtain

$$|\alpha_i(G, X)| \geq C(a, b)\|G\|^{-5s^2}$$

for almost all $(a, b) \in \mathbb{R}^{2s}$. Now we obtain by Proposition 2.2, because $\mathfrak{M}^{(i)}$ is open in $\mathbb{R}^{2s}$:

$$|\alpha_i(G, X)| \geq C(X)\|G\|^{-5s^2}.$$

for almost all $X \in \mathfrak{M}^{(i)}$.

In the case $\rho + 1 \leq i \leq \rho + \tau$ the lemma follows by similar arguments using Proposition 2.1 (iii) because of

$$\alpha_i(G, X) = \sum_{i,j} (a_i b_j + c_i d_j) g_{ij}$$

and

$$\beta_i(G, X) = \sum_{i,j} (a_i d_j - b_i c_j) g_{ij}$$

We consider a matrix $A = X^{-1}YX$, such that the modulus of at least one complex eigenvalue is larger than 1 (the case that the modulus of a real eigenvalue is larger than 1 is discussed in chapter 4); w.l.o.g. let this eigenvalue be $\lambda_{\rho+1} = (r_{\rho+1}, \varphi_{\rho+1})$. In the following we shall prove.

3.8 Proposition: *Let $X \in \mathfrak{S}$ (see lemma 3.7) be a fixed transformation matrix and all eigenvalues λ_i $(i \neq \rho + 1)$ and $r_{\rho+1}$ are fixed too. Then for all $\epsilon > 0$ a positive constant $\tilde{C} = \tilde{C}(X, Y, \epsilon)$ exists such that*

$$D_N(X, Y) \leq \tilde{C} N^{-1/2} (\log N)^{s^2+3/2+\epsilon} \tag{3.9}$$

for almost all $\varphi_{\rho+1} \in [0, \pi]$; there $D_N(X, Y)$ denotes the discrepancy of $(A^{p(n)})$ for $A = X^{-1}YX$.

First we show that Theorem I is an immediate consequence of Proposition 3.8. Let $\ddot{\mathfrak{U}}_\rho$ denote the subset of $\mathfrak{M}^* \times \mathfrak{R}_\rho$ such that an estimate (3.9) holds. Because of

$$\ddot{\mathfrak{U}}_\rho = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{M=1}^{\infty} \bigcap_{N=M}^{\infty} \Big\{(X, Y) \in \mathfrak{M}^* \times \mathfrak{R}_\rho : D(N, X, Y) N^{1/2} (\log N)^{-s^2-(3/2)-1/m} < \frac{1}{k}\Big\}, \tag{3.10}$$

where the function $(X, Y) \mapsto D(N, X, Y)$ is measurable it follows that $\check{\mathfrak{U}}_\rho$ is a measurable subset of $\mathfrak{M}^* \times \mathfrak{R}_\rho$. Let $I = Q \times (0, \pi)$ be an $s^2 + s$-dimensional interval with $I \subset \mathfrak{M}^* \times \mathfrak{R}_\rho$ and $B = I \cap \check{\mathfrak{U}}_\rho$; then B is measurable and we can apply Fubini's Theorem to the indicator function 1_C, where $C = I \backslash B$:

$$(3.11) \qquad \iint\limits_I 1_C(Z, \varphi_{\rho+1})\, d(Z, \varphi_{\rho+1}) = \int_Q \left(\int_0^\pi 1_C(Z, \varphi_{\rho+1})\, d\varphi_{\rho+1} \right) dZ.$$

Now we consider a countable covering of $\mathfrak{M}^* \times \mathfrak{R}_\rho$ consisting of such intervals $I = Q \times (0, \pi)$; by (3.8) we have

$$(3.12) \qquad D(N, X, Y) \leq C_1(X, Y, \epsilon) N^{-1/2} (\log N)^{s^2 + (3/2) + \epsilon}$$

for almost all $(X, Y) \in I$. Obviously the mapping $\vartheta_\rho : (X, Y) \mapsto X^{-1}YX = A$ has all properties required in Proposition 2.2 and so

$$(3.13) \qquad D(N, A) \leq C_1(A, \epsilon)\, N^{-1/2} (\log N)^{s^2 + (3/2) + \epsilon}$$

for almost all $A \in \vartheta_\rho(I) \subset \mathfrak{M}_\rho$. Since the countable union of null sets is $X^{-1}YX = A$ has all properties required in Proposition 2.2 and so

$$D(N, A) \leq C(A, \epsilon)\, N^{-1/2} (\log N)^{s^2 + (3/2) + \epsilon}$$

for almost all $A \in \mathfrak{M}_\rho$.

Because of $\mathfrak{M} = \bigcup_{\rho=1}^{s} \mathfrak{M}_\rho$ the proof of Theorem I is complete.

In the following we give a proof of Proposition 3.8. For this purpose we consider the Weyl sum as a function of $\varphi_{\rho+1}$:

$$(3.14) \quad S(N, G, Y)(\varphi_{\rho+1}) = \sum_{n=1}^{N} e\left(\sum_{i=1}^{\rho} \alpha_i \lambda_i^{p(n)} + \sum_{i=\rho+1}^{\rho+\tau} r_i^{p(n)} \gamma_i \cos(p(n)\varphi_i + y_i) \right)$$

with α_i, β_i, γ_i as in (3.3) and $y_i = y_i(G, X)$ with

$$\sin y_i = -\frac{\beta_i(G, X)}{\gamma_i(G, X)}, \qquad \cos y_i = \frac{\alpha_i(G, X)}{\gamma_i(G, X)}.$$

The following lemma leads to an estimate of the Weyl sum:

LEMMA 3.15: *Let* $G = (g_{ij})$ *be an integer valued matrix and* $1 \leq \|G\| = \max|g_{ij}| \leq \sqrt{N}$; *furthermore let* $\lambda_{\rho+1} = (r_{\rho+1}, \varphi_{\rho+1})$ *be a complex* *eigenvalue with* $1 < u \leq r_{\rho+1} \leq v$ *and* $X \in \mathfrak{S}$. *Then two positive constants* $C_1(u, X)$ and $C_2(u, x)$ *independent of G exist such that*

$$\left| \int_0^\pi e(\gamma_{\rho+1}(G)(r_{\rho+1}^{p(k)} \cos(p(k)\rho_{\rho+1} + y_{\rho+1}) - r_{\rho+1}^{p(l)} \cos(p(l)\varphi_{\rho+1} + y_{\rho+1}))) \, d\varphi_{\rho+1} \right| \leq C_1(u, X) \cdot N^{-1}$$

for all k, l with $|k - l| \geq C_2(u, X)$, $\sqrt{N} \leq k \leq N$, $N \geq N_0(u, X)$.

PROOF: In order to simplify the notation we omit in the following the index and write γ, φ, y, r instead of $\varphi_{\rho+1}, y_{\rho+1}, r_{\rho+1}, \gamma_{\rho+1}(G)$. Furthermore we define

$$g(\varphi) = g_{kl}(\varphi) = r^{p(k)} \cos(p(k)\varphi + y) - r^{p(l)} \cos(p(l)\varphi + y).$$

An elementary calculation shows for $f(\varphi) := \dfrac{d}{d\varphi}(g(\varphi)) : (k > l$ w.l.o.g.)

$$p(k)^2 f^2(\varphi) + f'^2(\varphi) \geq r^{2p(k)} p(k)^4 \left(1 - 4 \frac{p(l)}{p(k)} r^{p(l)-p(k)}\right); \quad \text{(see [9])}$$

hence for $k - l > \dfrac{\log 8}{\log r}$

$$p(k)^2 f^2(\varphi) + f'^2(\varphi) > \frac{1}{2} p(k)^4 r^{2p(k)}. \tag{3.16}$$

Easily an estimation of the numbers of zeroes of f and f' in $[0, \pi]$ can be established. f and f' are polynomials ($\neq 0$) of degree $0(p(k))$ in $\sin \varphi$ and $\cos \varphi$ with the property

(3.17) The number of zeroes of f and f' in $[0, \pi]$ is $0(p(k))$.

Now we dissect the interval $(0, \pi)$ into two disjoint sets U and V defined by

$$U := \{\varphi \in (0, \pi) : |f(\varphi)| < \gamma^{-1/2} k^{4s^2+2} p(k)\}, \quad V = (0, \pi) \backslash U. \tag{3.18}$$

U is an open set and therefore the union of $0(p(k))$ (compare (3.17)) open intervals I_j. We obtain by (3.16):

$$|f'(\varphi)| \geq r^{p(k)} p(k)^2 \left(\frac{1}{2} - \gamma^{-1} k^{8s^2+4} r^{-2p(k)}\right)^{1/2};$$

hence for all $k \geq \sqrt{N}$, $N \geq N_0(u, X)$ by 3.7

$$|f'(\varphi)| \geq r^{p(k)} p(k)^2 \left(\frac{1}{2} - C_3 N^{13s^2+4} r^{-2p(k)}\right)^{1/2},$$

and so $|f'(\varphi) \geq \frac{1}{4} r^{p(k)} p(k)^2$. Using (3.18) we get for the measure of I_j

$$\mu(I_j) = \sup_{x_1, x_2 \in I_j} |x_1 - x_2| \leq \sup_{I_j} (|f(x_1)| + |f(x_2)|) \cdot |f'(x_3)|^{-1} \leq \leq C_4 \gamma^{-1/2} k^{4s^2+2} p(k)^{-1} r^{-p(k)}, \tag{3.19}$$

and with 3.7: $\mu(I_j) = 0(k^{8s^2+2} p(k)^{-1} r^{-p(k)})$. Now each I_j contains 0 or π or a zero of f, because f is monotone on I_j; so

$$\left| \int_U e(\gamma g(\varphi)\, d\varphi \right| = 0(k^{8s^2+2} r^{-p(k)}) = 0(N^{-1}) \tag{3.20}$$

by (3.17). Now we dissect V in $0(p(k))$ intervals I'_j such that f has constant sign on I'_j and is monotone there. By the second mean value theorem and 3.16 we obtain

$$\left| \int_{I'_j} e(\gamma g(\varphi))\, d\varphi \right| \leq C_4 \gamma^{-1/2} k^{-4s^2-2} \cdot p(k)^{-1}; \tag{3.21}$$

hence together with 3.7.

$$\left| \int_V e(\gamma g(\varphi))\, d\varphi \right| = 0(k^{-4s^2-2} N^{2s^2}) = 0(N^{-1}). \tag{3.22}$$

and the lemma is proved.

Because of

$$|S(N, G, Y)|^2 = \sum_{k,l} e \left\{ \sum_{i=1}^{\rho} \alpha_i(G)(\lambda_i^{p(k)} - \lambda_i^{p(l)}) + \right.$$
$$\left. + \sum_{i=\rho+1}^{\rho+\tau} \gamma_i(G)(r_i^{p(k)} \cos(p(k)\varphi_i + y_i) - r_i^{p(l)} \cos(p(l)\varphi_i + y_i)) \right\}$$

we obtain for an integer valued matrix G with $1 \leq \|G\| \leq \sqrt{N}$

$$\int_0^{\pi} |S(N, G, Y)|^2\, d\varphi_{\rho+1} \leq C \cdot N \tag{3.23}$$

for $[u, v] \cap (-1, 1) = \emptyset$ with a positive constant C by estimating the exponential terms for $|k-l| < C_2(u, X)$ and $k < \sqrt{N}$ trivially by 0(1) and by Lemma 3.15 otherwise.

Now we apply Cauchy's inequality and obtain for all integer valued matrices G, G' with $0 < \|G\|, \|G'\| \leq \sqrt{N}$:

$$\text{(3.24)} \qquad \begin{aligned} &\int_0^\pi |S(N, G, Y) \cdot S(N, G', Y)| \, d\varphi_{\rho+1} = 0(N), \\ &\int_0^\pi |S(N, G, Y) \, d\varphi_{\rho+1} = 0(N^{1/2}). \end{aligned}$$

We obtain by setting $H = \sqrt{N}$ and taking the square of $D_N(X, Y)$ and the integal ($r(G) = \Pi_{i,j} \max(|g_{ij}|, 1)$):

(3.25)

$$\begin{aligned} \int_0^\pi N^2 D_N(X, Y)^2 \, d\varphi_{\rho+1} &= O(N) + O(N^{1/2}) \\ &\times \sum_{0<\|G\|\leq\sqrt{N}} r(G)^{-1} \int_0^\pi |S(N, G, Y)| \, d\varphi_{\rho+1} + O(1) \\ &\times \sum_{0<\|G\|,\|G'\|\leq\sqrt{N}} \sum r(G)^{-1} r(G')^{-1} \int_0^\pi |S(N, G, Y) S(N, G', Y)| \, d\varphi_{\rho+1} \\ &= O(N) + O(N)\Big(\sum_{1\leq j\leq\sqrt{N}} \frac{1}{j}\Big)^{s^2} + O(N)\Big(\sum_{1\leq j\leq\sqrt{N}} \frac{1}{j}\Big)^{2s^2} = O(N(\log N)^{2s^2}). \end{aligned}$$

This estimation is independent of the sequence $p(n)$, so it remains valid for all "shifted" sequence $p'(n) := p(M+n)$ with $M \in N$; for the discrepancy $D(M, N, X, Y)$ of such sequence we obtain

$$\text{(3.26)} \qquad \int_0^\pi N^2 D(M, N, X, Y)^2 \, d\varphi_{\rho+1} \leq C(X) N (\log N)^{2s^2}.$$

Now we apply Proposition 2.4 with $F(M, N)(Y) := D(M, N, X, Y)$ and it follows (2.4(i) is valid for $\sigma = 2s^2$ and 2.4 (ii) trivially is valid for the discrepancy):

$$\text{(3.27)} \qquad D_N(X, Y) \leq C(X, Y, \epsilon) N^{-1/2} (\log N)^{s^2+(3/2)+\epsilon}$$

for almost all $\varphi_{\rho+1} \in (0, \pi)$; so Proposition 3.8 is proved.

4. Proof of theorem II

We now consider the case that A possesses at least one real eigenvalue with modulus larger than 1. W.l.o.g. we may assume the real eigenvalues of A to be ordered according to the magnitude of their moduli so that $|\lambda_1| > 1$. As in the previous chapter it suffices to prove inequality (1.5.) for a fixed matrix X, fixed eigenvalues $\lambda_2, \ldots, \lambda_s$ and for almost all $\lambda = \lambda_1$ from a compact interval I disjoint to $[-1, 1]$ containing none of the other eigenvalues $\lambda_2, \ldots, \lambda_s$. So the Weyl sum in (3.14) can be expressed in the form

$$S(N, G, Y) = \sum_{n=1}^{N} e(f(n, \lambda_1)) \tag{4.1}$$

with

$$f(n, \lambda_1) := \alpha_1(G, X)\lambda_1^{p(n)} + C \tag{4.2}$$

where the constant C does not depend on λ_1 and therefore is fixed for fixed $X, G, \lambda_2, \ldots, \lambda_s$. Employing a method developed by Erdös and Koksma in [2] we now define (dropping the index 1 of λ_1 for short)

$$h_\sigma(n, \lambda) := f(\sigma + (n-1)q, \lambda) \tag{4.3}$$

for $n = 1, 2, \ldots, N_\sigma := [(N-\sigma)q^{-1}] + 1$, $\sigma = 1, \ldots, q$, where the positive integer q is defined (for sufficiently large N) by

$$q = q(N) = [3 \log\log N (\log|\lambda_0|)^{-1}], \tag{4.4}$$

λ_0 being the element of I with smallest modulus. We further define another positive integer w by

$$w = w(N) = [\log N] \tag{4.5}$$

and get by a straightforward calculation

$$\begin{aligned} \left|\sum_{n=1}^{N} e(h_\sigma(n, \lambda))\right|^{2w} &= \sum_{[n_1, \ldots, n_w]} P[n_1, \ldots, n_w]^2 + \\ &+ 2 \sum_{[n_1, \ldots, n_w] > [m_1, \ldots, m_w]} \sum P[n_1, \ldots, n_w] p[m_1, \ldots, m_w] \cos(2\pi\psi(\lambda)). \end{aligned} \tag{4.6}$$

Here $[n_1, \ldots, n_w]$ denotes the quivalence class of all w-tuples which can be obtained from the special w-tuple $(n_1, \ldots, n_w)$ with $n_1 \leq \cdots \leq n_w \leq N_\sigma$ by a permutation of the entries; $P[n_1, \ldots, n_w]$ is the cardinality of this equivalence class. $[n_1, \ldots, n_w] > [m_1, \ldots, m_w]$ means that for some $k \in \{1, \ldots, w\}$ we have $n_k > m_k$, $n_j = m_j$ for $k < j \leq w$

and $\psi(\lambda)$ is defined for each pair $([n_1, \ldots, n_w], [m_1, \ldots, m_w])$ by

(4.7)
$$\psi(\lambda) := \sum_{j=1}^{w} (h_\sigma(n_j, \lambda) - h_\sigma(m_j, \lambda)).$$

In order to establish a lower estimate for $|\psi'(\lambda)|$ on I we first conclude from (4.2), (4.3), (4.4) and (4.5)

(4.8)
$$|h'_\sigma(m, \lambda) h'_\sigma(m-1, \lambda)^{-1}| \geq |\lambda|^q (\log N)^2 > 2w$$

for sufficiently large N. So we get for $[m_1, \ldots, m_w] > [m'_1, \ldots, m'_w]$:

(4.9)
$$\begin{aligned} |\psi'(\lambda)| &\geq |h'_\sigma(m_k, \lambda)| - (2w-1)|h'_\sigma(m_k - 1, \lambda)| > h'_\sigma(m_k - 1, \lambda) = \\ &= |f'(\sigma + (m_k - 2)q, \lambda)| \geq |\alpha_1(G, X)| \geq C(X) N^{-k(s)} \end{aligned}$$

with $k(s) = 4s^2$ for $\|G\| \leq N^{1/2}$ by lemma 3.7 (of course we may suppose that the fixed matrix X is not chosen from the null set for which inequality (i) of this lemma is false.) So we have for arbitrary fixed $[n_1, \ldots, n_w]$

(4.10)
$$\begin{aligned} \sum_{[m_1, \ldots, m_w] < [n_1, \ldots, n_w]} (\min_I |\psi'(\lambda)|)^{-1} &\leq 2C(X)^{-1} N^{k(s)} \sum_{j=1}^{N_\sigma^w} \frac{1}{j} \\ &\leq C_1 N^{k(s)} w (\log N). \end{aligned}$$

(by a similar argument as in [8], p. 35, (4.4)) and therefore the second mean-value theorem yields from (4.6) (the monotony of $\psi'(\lambda)$ on I follows by repeating the above argument for $|\psi''(\lambda)|$):

(4.11)
$$\begin{aligned} \int_I \left| \sum_{n=1}^{N_q} e(h_\sigma(n, \lambda)) \right|^{2w} d\lambda &\leq m(I) w! N_\sigma^w + 2C_1 N^{k(s)} N_\sigma^w w! w (\log N) \\ &\leq C_2 w! w N^{k(s)} N_\sigma^w (\log N). \end{aligned}$$

Here $m(I)$ denotes the Lebesgue measure of I and the obvious combinatorical facts

(4.12)
$$P[n_1, \ldots, n_w] \leq w!$$

(4.13)
$$\sum_{1 \leq n_1 < \cdots \leq n_w \leq N_\sigma} P[n_1, \ldots, n_w] = N_\sigma^w$$

have been used. We now consider subsets of I defined by

$$(4.14)\qquad \mathfrak{M}(N, G, \sigma) := \left\{\lambda \in I : \left|\sum_{n=1}^{N_q} e(h_\sigma(n, \lambda)\right| \geq N^{1/2}\chi(N_\sigma)\right\}$$

with $\chi(x) := (\log x^2)^{1/2} e^{(1/4)(2k(s)+s^2+5)}$. For their Lebesgue measure we get by (4.11)

$$m(\mathfrak{M}(N, G, \sigma))N_\sigma^w\chi(N_\sigma)^{2w} \leq C_2 w! wN^w \log N \Rightarrow$$

$$(4.15)\qquad m(\mathfrak{M}(N, G, \sigma)) \leq C_2 w^w N^{k(s)} \log N\chi([Nq^{-1}])^{-2w}.$$

Forming the union

$$(4.16)\qquad \mathfrak{M}(N) := \bigcup_{\sigma=1}^{q} \bigcup_{0<\|G\|\leq\sqrt{N}} \mathfrak{M}(N, G, \sigma)$$

we get for its measure

$$(4.17)\qquad m(\mathfrak{M}(N)) \leq C_3 q N^{k(s)+(1/2)s^2} \log N\, w^w \chi(N^{1/2})^{-2w}$$

and therefore by the definitions of $q = q(N)$, $w = w(N)$ and χ after a short calculation

$$m(\mathfrak{M}(B)) < C_4 N^{-2}$$

for sufficiently large N. So the series $\Sigma_N m(\mathfrak{M}(N))$ converges and the Borel-Cantelli-lemma implies that for almost all $\lambda \in I$ there exists a positive integer $N_0(\lambda)$ such that for all integers $N > N_0(\lambda)$ the inequality

$$(4.18)\qquad \left|\sum_{n=1}^{N_q} e(h_\sigma(n, \lambda))\right| < N^{1/2}\chi(N_\sigma)$$

holds for all $\sigma = 1, \ldots, q(N)$ and for all matrices G with integer entries and with $0 < \|G\| < N^{1/2}$. We finally conclude by (4.1) and (4.3)

$$|S(N, G, Y)| = \left|\sum_{\sigma=1}^{q} \sum_{n=1}^{N_\sigma} e(h_\sigma(n, \lambda))\right| \leq q \max_q(N_\sigma^{1/2}\chi(N_\sigma)) \leq$$

$$(4.19)\qquad \leq 2q(Nq^{-1})^{1/2}\chi(N) \leq CN^{1/2}(\log N)^{1/2}(\log\log N)^{1/2}$$

for $N > N_0(\lambda)$. The inequality of Erdös–Turan (with $H = [N^{1/2}]$) now immediately yields the desired result.

5. Remarks on the Theorems III, IV and V

The further results formulated in the introduction can be proved in a completely analagous way by minor modifications of the proofs given in the previous sections. To establish theorem III (on real symmetric resp. complex Hermitatian matrices) one can adopt the proof of chapter 4 (all eigenvalues being real in this case.) The only necessary change is to replace the lower estimate for $|\alpha_1(G, X)|$ of lemma 3.7 by the corresponding result following from part (ii) of proposition 2.2. Obviously the value of the exponent $k(s)$ (which changes in this case) is of no importance.

On the theorems IV and V (dealing with real resp. complex triangular matrices) it should be pointed out that the more stringent condition $\lambda(A) \geq 1$ (with $\lambda(A) = \min |\lambda_j| = \min |a_{jj}|$) is necessary because otherwise in the sequence of powers of A at least one component (corresponding to the k-th diagonal entry, where $|a_{kk}| < 1$) converges to zero and so the sequence is not even dense in $[0, 1]^{d(s)}$. On the other hand the coefficients $\alpha_i(G, X)$ occuring in the Weyl sum (3.8) now reduce to

$$\alpha_i(G, X) = \sum_{j \geq i,\, k \leq i} a_j b_k g_{kj} \tag{5.1}$$

where (a_j), (b_k) are again the i-th row of X resp. the i-th column of X^{-1} (X, X^{-1} being also (upper) triangular matrices in this case) and $G = (g_{kj})$ is an (upper) triangular matrix not vanishing identically with integral entries. Let g_{kj} be an integer with $g_{kj} \neq 0$, then $k \leq j$ and so there exists i with $k \leq i, j \geq i$. So at least this $\alpha_i(G, X)$ does not vanish identically and therefore can be estimated as in lemma 3.7. The rest of the proof follows identically the lines of section 4 (for theorem IV) resp. (after an analogous reasoning on the coefficients in the Weyl sum) the method of section 3 for theorem V.

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