COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 45, nº 1 (1982), p. 115-141

<http://www.numdam.org/item?id=CM_1982__45_1_115_0>

COMPOSITIO MATHEMATICA, Vol. 45, Fasc. 1, 1981, pag. 115–141

$\mathscr{C}_p$-*E*-MOVABLE AND $\mathscr{C}$-*E*-CALM COMPACTA AND THEIR IMAGES*

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to George

Abstract

In this paper we shall introduce classes of $\mathscr{C}_p$-e-movable and $\mathscr{C}$-e-calm compacta which generalize both the class of compact absolute neighborhood retracts and the class of locally n-connected compacta. We prove a number of characterizations of these classes and thus get as corollaries new methods of recognizing compact ANR's that are inspired by Borsuk's shape theory. With this approach we identify certain classes of maps related to refinable and approximately right invertible maps which preserve many invariants of the theory of retracts.

1. Introduction

The notion of an absolute neighborhood retract (ANR) was extensively studied by many authors ever since 1931 when Borsuk [2] defined it. There are many characterizations of ANR's (see [3], [20]) but recent theorems and current open problems in the infinite-dimen-

* *AMS(MOS) Subject Classification (1970)*: Primary 54F40.
Key words and phrases: absolute neighborhood retract, LC^n, $\mathscr{C}_p$-e-movable, $\mathscr{C}$-e-calm, fundamental e-sequence, refinable map, approximately right invertible map, $\mathscr{C}_p$-e-surjection, $\mathscr{C}_p$-e-bundle.
**This paper was completed during the author's visit to the University of Oklahoma at Norman on leave from the University of Zagreb.

0010–437X/81010115–27$00.20/0

sional topology [18] suggest the need for still more methods of recognizing ANR's particularly among infinite-dimensional spaces.

The main theorems in the present paper give new necessary and sufficient conditions for a compactum X to be an ANR. Our conditions can be described as positional or shape theoretic because they depend on properties of embeddings of X into an ANR.

The key idea of our approach is taken from [8] and is here applied to strong movability ([4], p. 263) instead of to movability. By requiring control on the size of maps and homotopies in the definition of strong movability we introduce a more restrictive positional property called strong e-movability. It turns out that a compactum X is an ANR iff X is strongly e-movable. This is clearly analogous to Borsuk's theorem ([4], p. 264) which says that a compactum X is an FANR iff X is strongly movable. We shall investigate in this paper strongly e-movable compacta (and hence compact ANR's) using the above analogy. A similar method gives us a class of e-calm compacta from the class of calm compacta [5]. The two classes are closely related to each other. Ever strongly e-movable compactum is e-calm but the converse is an open question.

In order to cover at the same time ANR's and LC^n compacta we shall take a slightly more general point of view (without obscuring notation and results but with an obvious danger of diminishing the interest of prejudiced readers) by introducing $\mathscr{C}_p$-e-movable and $\mathscr{C}$-e-calm compacta. Here $\mathscr{C}_p$ is an arbitrary class of pairs of metrizable spaces and $\mathscr{C}$ is an arbitrary class of topological spaces. Of course, when $\mathscr{C}_p$ is the class of all pairs of compact ANR's and $\mathscr{C}$ is the class of all compact ANR's, then a compactum is $\mathscr{C}_p$-e-movable ($\mathscr{C}$-e-calm) iff it is strongly e-movable (e-calm).

The paper is organized as follows. In §2 we collect definitions and set out our notation. The §3 studies $\mathscr{C}_p$-e-movable compacta. The §4 is concerned with $\mathscr{C}$-e-calmness. In §5 we consider classes of maps which preserve $\mathscr{C}_p$-e-movability. Our results give partial answers to problems (CE2) and (CE3) in [18].

We assume that the reader is familiar with the theory of retracts [3], [20], shape theory [4], and elements of infinite-dimensional topology [12].

2. Preliminaries and notation

Throughout the paper, if not stated otherwise, $\mathscr{C}$ and $\mathscr{D}$ will be arbitrary classes of topological spaces while $\mathscr{C}_p$ and $\mathscr{D}_p$ will be arbitrary classes of pairs of metrizable spaces. By $\mathscr{C}^n$ ($\mathscr{C}_p^n$) we denote all $K \in \mathscr{C}$ ($(K, K_0) \in \mathscr{C}_p$) with (covering) dimension $\dim K \leqq n$. We reserve $\mathscr{P}$, $\mathscr{K}$, $\mathscr{S}$, $\mathscr{P}_p$, $\mathscr{K}_p$ and $\mathscr{S}_p$ for classes of all compact ANR's, all finite CW-complexes, all finite simplicial complexes, all pairs of compact ANR's, all pairs of finite CW-complexes, and all pairs of finite simplicial complexes, respectively.

A map $f: K \to Z$ is called a *$\mathscr{C}$-map* provided $K \in \mathscr{C}$. Similarly, a map of pairs $f:(K, K_0) \to (Z, Z_0)$ is a *$\mathscr{C}_p$-map* if $(K, K_0) \in \mathscr{C}_p$.

We shall say that maps f and g of a space Z into a metric space (Y, d) are *ϵ-close* provided $d(f(z), g(z)) < \epsilon$ for every $z \in Z$. If Z is a subset of Y and $f: Z \to Y$ is ϵ-close to the inclusion $i_{Z,Y}$ of Z into Y, we call f an *ϵ-map*.

Fundamental sequences $\underline{f} = \{f_k, A, B\}_{M,N}$ and $\underline{g} = \{g_k, A, B\}_{M,N}$ are *ϵ-close* if for some neighborhood U of A in M restrictions $f_k \mid U$ and $g_k \mid U$ are ϵ-close maps for almost all k. A fundamental sequence $\underline{f} = \{f_k, A, B\}_{M,M}$ is an *ϵ-fundamental sequence* provided there is a neighborhood U of A in M such that $f_k \mid U$ is an ϵ-map for almost all k.

Two maps $f, g: Z \to Y$ of a space Z into a metric space (Y, d) are *ϵ-homotopic* (and we write $f \simeq^\epsilon g$) if there is a homotopy $h_t: Z \to Y$, $(0 \leqq t \leqq 1)$, between f and g (called an *ϵ-homotopy*) such that h_0 and h_t are ϵ-close for all $t \in I = [0, 1]$.

Fundamental sequences $\underline{f} = \{f_k, A, B\}_{M,N}$ and $\underline{g} = \{g_k, A, B\}_{M,N}$ are *ϵ-homotopic* (in notation, $\underline{f} \simeq^\epsilon \underline{g}$) provided for every neighborhood V of B in N there is a neighborhood U of A in M with $f_k \mid U$ ϵ-homotopic in V to $g_k \mid U$ for almost all k.

Let X be a subset of a metric space M, let U and V, $V \subset U$, be open subsets of M which contain X, and let $\epsilon > 0$ and $\delta > 0$ be given. Then $\mathscr{C}_p^\epsilon(U, V; X)$ and $\mathscr{C}_h^\epsilon(V, \delta; X)$ will denote the following statements.

$\boxed{\mathscr{C}_p^\epsilon(U, V; X)}$ For every neighborhood of W of X in M there is a neighborhood W_0 of X in M, $W_0 \subset V \cap W$, such that for every $\mathscr{C}_p$-map $f:(K, K_0) \to (V, W_0)$ there is an ϵ-homotopy $f_t: K \to U$, $(0 \leqq t \leqq 1)$, with $f_0 = f$, $f_1(K) \subset W$, and $f_1 \mid K_0 = f \mid K_0$.

$\boxed{\mathscr{C}_h^\epsilon(V, \delta; X)}$ For every neighborhood W of X in M there is a neighborhood W_0 of X in M, $W_0 \subset V \cap W$, such that every two $\mathscr{C}$-maps $f, g: K \to W_0$ which are δ-homotopic in V are ϵ-homotopic in W.

For a compact ANR M and an $\epsilon > 0$, let $\Gamma(M, \epsilon)$ denote the set of all $\delta > 0$ such that, for any two δ-close maps $f, g: X \to M$ defined on a metrizable space X and any δ-homotopy $j_t: A \to M$, $(0 \leqq t \leqq 1)$, defined on a closed subspace A of X with $j_0 = f \mid A$ and $j_1 = g \mid A$, there exists an ϵ-homotopy $h_t : X \to M$, $(0 \leqq t \leqq 1)$, such that $h_0 = f$, $h_1 = g$, and $h_t \mid A = j_t$ for every $t \in I$ ([20], p. 112).

For a map $f : A \to B$ between metric spaces, let $\Lambda(f, \epsilon)$ be the set of all $\delta > 0$ with the property that $d(x, y) < \delta$ in A implies $d(f(x), f(y)) < \epsilon$ in B.

Let A and B be compacta lying in AR spaces M and N, respectively. A fundamental sequence $\underline{f} = \{f_k, A, B\}_{M,N}$ is called a *fundamental e-sequence* [8] provided the family $\{f_k\}$ satisfies the following condition. For every $\epsilon > 0$ there is a $\delta > 0$ and there is a neighborhood V of A in M such that $\delta \in \Lambda(f_k \mid V, \epsilon)$ for almost all k.

In our proofs in §§3–5 we shall always consider compacta as subsets of the Hilbert cube Q unless explicitely required otherwise.

3. $\mathscr{C}_p$-e-movable compacta

By requiring that the homotopies in the definition of (weak) $\mathscr{C}_p$-movability in [6] are small we shall introduce a class of $\mathscr{C}_p$-e-movable compacta which generalizes the notions of an absolute neighborhood retract and a locally n-connected compactum.

(3.1) Definition: A compactum X is *$\mathscr{C}_p$-e-movable* if for some (and hence for every) embedding of X into an ANR M the following holds. For each neighborhood U of X in M and every $\epsilon > 0$ there is a neighborhood V of X in M, $V \subset U$, such that $\mathscr{C}_p^\epsilon(U, V; X)$ is true.

The $\mathscr{P}_p$-e-movable compacta are also called *strongly e-movable.*

(3.2) Proposition: *For a compactum X in a compact Q-manifold M and a class $\mathscr{C}_p$ of pairs of metrizable spaces the following are equivalent.*

(i) *For each neighborhood U of X in M and every $\epsilon > 0$ there is a neighborhood V of X in M, $V \subset U$, such that for every neighborhood W of X in M there is a neighborhood W_0 of X in M, $W_0 \subset V \cap W$, with the property that for every $\mathscr{C}_p$-map $f : (K, K_0) \to (V, W_0)$ there is an ϵ-homotopy $f_t : K \to U$, $(0 \leqq t \leqq 1)$, with $f_0 = f$, $f_1(K) \subset W$, and $f_t \mid K_0 = f \mid K_0$ for all $t \in I$.*

(ii) *X is $\mathscr{C}_p$-e-movable.*

(iii) *For every $\epsilon > 0$ there is a neighborhood V of X in M such that for every neighborhood W of X in M there is a neighborhood W_0 of X in M, $W_0 \subset V \cap W$, with the property that every $\mathscr{C}_p$-map $f : (K, K_0) \to (V, W_0)$ is ϵ-close to a map $f' : K \to W$ which agrees with f on K_0.*

(iv) *For every* $\epsilon > 0$ *there is a neighborhood* V *of* X *in* M *such that for every neighborhood* W *of* X *in* M *and an arbitrary* $\delta > 0$ *there is a neighborhood* W_0 *of* X *in* M, $W_0 \subset V \cap W$, *with the property that every* $\mathscr{C}_p$*-map* $f:(K, K_0) \to (V, W_0)$ *is* ϵ*-close to a map* $f': K \to W$ *such that* $f' \mid K_0$ *and* $f \mid K_0$ *are* δ*-close.*

PROOF: Clearly, (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (iv). Hence, it remains to see that (iv) $\Rightarrow$ (i). Let a neighborhood U of X in M and an $\epsilon > 0$ be given. We can assume that U is a compact ANR. Pick an $\eta \in \Gamma(U, \epsilon/3)$, $0 < \eta < \epsilon/6$. Then every neighborhood V of X in M, $V \subset U$, which satisfies (iv) for η will also satisfy (i).

Indeed, let W be an arbitrary compact ANR neighborhood of X in M and let $\delta \in \Gamma(W, \eta)$. Pick a neighborhood W_0 of X in M, $W_0 \subset V \cap W$, such that for every $\mathscr{C}_p$-map $f:(K, K_0) \to (V, W_0)$ there is a map $f': K \to W$ which is η-close to f and $f' \mid K_0$ is δ-close to $f \mid K_0$. By the choice of δ and η, there is an η-homotopy $G_t: K_0 \to W$ and an $(\epsilon/3)$-homotopy $F_t: K \to U$, $(0 \leqq t \leqq 1)$, with $G_0 = f \mid K_0$, $G_1 = f' \mid K_0$, $F_0 = f$, $F_1 = f'$, and $F_t \mid K_0 = G_t$ for all $t \in I$. On the product $K_0 \times I \times I$ define a map E into W by the formula $E(x, t, s) = G_{t(1-s)}(x)$. Let $D: K \times \{1\} \times I \to W$ be an 2η-homotopy obtained by applying the homotopy extension theorem in W to a map of $K \times \{1\}$ into W which maps $(x, 1)$ into $f'(x)$ and a partial homotopy $E \mid K_0 \times \{1\} \times I$. Finally, we apply the homotopy extension theorem in U to a map $F: K \times I \to U$ and a partial homotopy on the closed subset $K \times \{0\} \cup K_0 \times I \cup K \times \{1\}$ of $K \times I$ defined as f on each level of $K_0 \times \{0\} \times I$, as E on $K_0 \times I \times I$, and as D on $K \times \{1\} \times I$ and get an ϵ-homotopy $H^*: K \times I \times I \to U$. The restriction $H = H^* \mid K \times I \times \{1\}$ shows that (i) holds for U and V.

Let $\mathscr{C}_p$ and $\mathscr{D}_p$ be two classes of pairs of metrizable spaces. The class $\mathscr{C}_p$ *approximately dominates* the class $\mathscr{D}_p$ provided for every $(K, K_0) \in \mathscr{D}_p$ and every open cover $\mathscr{U}$ of K there exists an $(L, L_0) \in \mathscr{C}_p$ and there exist maps of pairs $f:(K, K_0) \to (L, L_0)$ and $g:(L, L_0) \to (K, K_0)$ such that $g \circ f$ is $\mathscr{U}$-close to id_K (i.e., for every $x \in K$, some member of $\mathscr{U}$ contains both x and $g(f(x))$). By an argument similar to the proof of Theorem 1 in [24], one can prove.

(3.3) Each of the classes $\mathscr{P}_p$, $\mathscr{K}_p$, and $\mathscr{S}_p$ approximately dominates the other two. Similarly, each of the classes $\mathscr{P}_p^n$, $\mathscr{K}_p^n$, and $\mathscr{S}_p^n$ approximately dominates the other two.

(3.4) THEOREM: *If a class* $\mathscr{C}_p$ *of pairs of metrizable spaces ap-*

proximately dominates another such class $\mathscr{D}_p$ and a compactum X is $\mathscr{C}_p$-e-movable, then X is also $\mathscr{D}_p$-e-movable.

PROOF: Consider X as a subset of Q and let $\epsilon > 0$ be given. Select a neighborhood V of X in Q such that (3.2)(iv) holds for $\epsilon/2$ and the class $\mathscr{C}_p$. We claim that V will also satisfy (3.2)(iv) for ϵ and the class $\mathscr{D}_p$.

Indeed, let W be an arbitrary neighborhood of X in Q and let $\delta \in (0, \epsilon)$. Let $\mathscr{V}$ be an open cover of V with sets of diameter $< \delta/2$. Pick a neighborhood W_0, $W_0 \subset V \cap W$, with the property that every $\mathscr{C}_p$-map $\psi : (L, L_0) \to (V, W_0)$ is $(\epsilon/2)$-close to a map $\psi' : L \to W$ such that $\psi' \mid L_0$ is $(\delta/2)$-close to $\psi \mid L_0$.

Consider a $\mathscr{D}_p$-map $\varphi : (K, K_0) \to (V, W_0)$. Observe that $\mathscr{U} = \varphi^{-1}(\mathscr{V})$ is an open cover of K. Since the class $\mathscr{C}_p$ approximately dominates the class $\mathscr{D}_p$, there is an $(L, L_0) \in \mathscr{C}_p$ and maps of pairs $f : (K, K_0) \to (L, L_0)$ and $g : (L, L_0) \to (K, K_0)$ such that $g \circ f$ is $\mathscr{U}$-close to id_K. Put $\psi = \varphi \circ g$ and select $\psi' : L \to W$ as above. One can easily check that φ is ϵ-close to $\psi' \circ f : K \to W$ and that $\varphi \mid K_0$ is δ-close to $\psi' \circ f \mid K_0$.

(3.5) EXAMPLE: In ([16], p. 199) R. Edwards suggests the idea of an ϵ-version of Siebenmann's open regular neighborhood theory [29]. The basic property that a compactum X in a manifold M with these neighborhoods would satisfy is the following compression property: For any $\epsilon > 0$ there is a $\delta > 0$ and an ambient ϵ-isotopy $h_t : M \to M$, $(0 \leqq t \leqq 1)$, having support in the open ϵ-neighborhood $N_\epsilon(X)$ of X in M and fixing some neighborhood of X, such that $h_1(N_\delta(X))$ lies arbitrarily close to X. It is clear that such an X must be $\mathscr{C}_p$-e-movable for every class $\mathscr{C}_p$ (and hence, by (3.6) below, an ANR).

(3.6) THEOREM: *A compactum X is an ANR iff X is $\mathscr{S}_p$-e-movable (or, equivalently, iff X is strongly e-movable).*

PROOF: Let X be a compact ANR in the Hilbert cube Q. Let N be a neighborhood of X in Q for which there is a retraction $r : N \to X$. Observe that for every $\epsilon > 0$ there is a neighborhood N_ϵ of X in N such that $r \mid N_\epsilon$ is an ϵ-map. For a given $\epsilon > 0$, $V = N_\epsilon$ will satisfy (3.2)(iv) because $W_0 = N_{\delta^*}$, where $\delta^* \leqq \delta$ is so small that $N_{\delta^*} \subset V \cap W$, is the required neighborhood for a neighborhood W of X in Q and a $\delta > 0$. Hence, X is $\mathscr{C}_p$-e-movable for every class $\mathscr{C}_p$ of metrizable pairs.

Conversely, suppose that X is an $\mathscr{S}_p$-e-movable compactum in Q. By (3.3) and (3.4), X is also $\mathscr{P}_p$-e-movable. Hence, by (3.2) (iii), there is a sequence $V_1 \supset V_2 \supset \cdots$ of compact ANR neighborhoods of X in Q such that $X = \bigcap_{n=1}^{\infty} V_n$ and for every $n = 1, 2, \ldots$ there is a $(1/2^n)$-

map $r_n : V_n \to V_{n+1}$ which is the identity on X. Then $r = \lim_{n\to\infty} r_n \circ \cdots \circ r_1$ is a retraction of V_1 onto X. Hence, X is an ANR.

(3.7) THEOREM: *A compactum X is LC^{n-1} iff X is $\mathscr{S}_p^n$-e-movable.*

PROOF: Assume that X is an $\mathscr{S}_p^n$-e-movable compactum in Q. We shall prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that δ-close maps f, $g : S^k \to X$, $0 \leqq k \leqq n-1$, are ϵ-homotopic (in X). This clearly implies that X is LC^{n-1}.

Let an $\epsilon > 0$ be given. Select a decreasing sequence $V_0 = Q \supset V_1 \supset V_2 \supset \cdots$ of compact ANR neighborhoods of X in Q such that $X = \bigcap_{n=0}^{\infty} V_n$ and $(\mathscr{S}_p^n)^{\epsilon/2^{i+1}}(V_{i-1}, V_i; X)$ holds for each $i > 0$. Pick a $\delta \in \Gamma(V_1, \epsilon/2)$.

Suppose that f, $g : S^k \to X$ are δ-close maps, $0 \leqq k \leqq n-1$. Let $H : S^k \times I \to V_1$ be an $(\epsilon/2)$-homotopy joining f and g. By the choice of V_1, we see that there is a map $H_1 : S^k \times I \to V_2$ which is $(\epsilon/2^2)$-close to H and $H_1 \mid S^k \times \{0, 1\} = H \mid S^k \times \{0, 1\}$. Now, by the choice of V_2, $(\epsilon/2^3)$-close to H_1 there is a map $H_2 : S^k \times I \to V_3$ satisfying $H_2 \mid S^k \times \{0, 1\} = H_1 \mid S^k \times \{0, 1\}$. Continuing in this way, we can construct an ϵ-homotopy $H_\infty : S^k \times I \to X$ between f and g.

Conversely, assume that X is an LC^{n-1} compactum in Q. Let $\mathscr{W}$ be an open cover of Q with sets of diameter $< \epsilon/2$. Since X is an LC^{n-1} compactum, the inclusion $i : X \to Q$ is a strong local connection in dimension $n-1$ [21]. Hence, there is a refinement $\mathscr{V}$ of $\mathscr{W}$ for which the assertion $E(\mathscr{V}, \mathscr{W}, n)$ holds [21]. In other words, given an at most n-dimensional finite simplicial complex K, a subcomplex L of K, and maps $g : L \to X$ and $h : K \to Q$ where $h \mid L = i \circ g$ and h maps every simplex σ of K into some member of the collection $\{V \in \mathscr{V} \mid V \cap X \neq \emptyset\}$, then there is an extension $h' : K \to X$ of g such that for every simplex σ of K, some element of $\mathscr{W}$ contains $i \circ h'(\sigma) \cup h(\sigma)$. Let $V = \bigcup\{V' \in \mathscr{V} \mid V' \cap X \neq \emptyset\}$. Then the neighborhood V satisfies (3.2)(iv) for the class $\mathscr{S}_p^n$.

Consider a compact ANR neighborhood W of X in V and a $\delta \in (0, \Gamma(W, \epsilon/2))$. Choose a neighborhood W_0 of X inside $V \cap W$ with respect to δ in the same way as V was chosen with respect to $\epsilon/2$. Then every $\mathscr{S}_p^n$-map $f : (K, K_0) \to (V, W_0)$ is $(\epsilon/2)$-close to a map $f_1 : (K, K_0) \to (V, X)$ because $f \mid K_0$ is $(\epsilon/2)$-homotopic in V to a map $f_1' : K_0 \to X$ which is δ-close to $f \mid K_0$ so that we can use the homotopy extension theorem to get f_1 from f_1'. But, by the choice of V, f_1 is $(\epsilon/2)$-close to a map $f_2 : K \to X$ with $f_2 \mid K_0 = f_1'$. Hence, f is ϵ-close to f_2 and $f \mid K_0$ is δ-close to $f_2 \mid K_0$.

Theorems (3.4), (3.6), and (3.7) and ([20], pp. 122 and 156) suggest the following characterization of $\mathcal{S}_p$-e-movable and $\mathcal{S}_p^n$-e-movable compacta (and hence of compact ANR's and LC^{n-1} compacta).

Let K denote a finite simplicial polytope and K_0 a subpolytope of K which contains all vertices of K. By a *partial ϵ-realization* of K in a metric space Y defined on K_0, we mean a map $f: K_0 \to Y$ such that, for every closed simplex σ of K, $\operatorname{diam} f(K_0 \cap \sigma) < \epsilon$. In case $K_0 = K$, then f will be called a *full ϵ-realization* of K in Y.

(3.8) THEOREM: *A compactum X in Q is $\mathcal{S}_p$-e-movable ($\mathcal{S}_p^n$-e-movable) iff for every $\epsilon > 0$ there is a $\delta > 0$ such that for every neighborhood W of X in Q there exists a smaller neighborhood W_0 of X in Q with the property that every partial δ-realization of any finite simplicial polytope K (which has no simplex of dimension $> n+1$) in W_0 extends to a full ϵ-realization of K in W.*

PROOF: Suppose that X is an $\mathcal{S}_p$-e-movable compactum in Q and let $\epsilon > 0$. Select a compact ANR neighborhood V of X in Q such that (3.2)(iii) holds for V and $\epsilon/3$. By ([20], p. 122), there is a $\delta > 0$ such that every partial δ-realization of any finite simplicial polytope K in V extends to a full $(\epsilon/3)$-realization of K in V.

Let W be an arbitrary neighborhood of X in Q. Pick a neighborhood W_0 of X in Q, $W_0 \subset V \cap W$, using (3.2)(iii). Consider a partial δ-realization $f: K_0 \to W_0$ of a finite simplicial polytope K in W_0. Let $f': K \to V$ be a full $(\epsilon/3)$-realization of K in V. But, there is a map $f'': K \to W$ extending f which is $(\epsilon/3)$-close to f'. Hence, f extends to a full ϵ-realization f'' of K in W.

Conversely, assume that X satisfies the condition in the statement of the theorem. For an $\epsilon > 0$, pick a δ, $0 < \delta < \epsilon$, with respect to $\epsilon/3$ using the assumption. Let $V = N_{\delta/3}(X)$. We claim that V satisfies (3.2)(iii).

Indeed, if W is a neighborhood of X in Q, choose a neighborhood W_0 of X in Q, $W_0 \subset V \cap W$, such that every partial δ-realization of any finite simplicial polytope K in W_0 extends to a full $(\epsilon/3)$-realization of K in W. Consider an $\mathcal{S}_p$-map $f: (K, K_0) \to (V, W_0)$. Without loss of generality we can assume that $\operatorname{diam} f(\sigma) < \delta/3$ for every closed simplex σ of K. For every vertex v of K not in K_0 pick a point $f'(v) \in X$ which is $(\delta/3)$-close to $f(v)$. Hence, we can define a partial δ-realization $f': K_0 \cup K^{(0)} \to W_0$ of K in W_0 ($K^{(0)}$ denotes the 0-dimensional skeleton of K). Let $f'': K \to W$ be an extension of f' to a full $(\epsilon/3)$-realization of K in W. Clearly, f'' is ϵ-close to f and $f'' \mid K_0 = f \mid K_0$.

The proof of the second statement in the theorem is analogous. All we have to do is restrict to polytopes which have no simplices of dimension $>n+1$.

The following characterization is motivated by Chapman's complement theorem [11] and shows that $\mathscr{C}_p$-e-movability of a Z-set in Q can be described in terms of certain properties of its complement.

A noncompact, locally compact, metric space (M, d) is *$\mathscr{C}_p$-e-movable at ∞* provided for every compactum A in M and every $\epsilon>0$ there is a compactum B in M, $B \supset A$, such that $\mathscr{C}_p^\epsilon(M-A, M-B; \infty)$ holds (i.e., such that for every compactum C in M there is a compactum D in M which contains $B \cup C$ and has the property that for every $\mathscr{C}_p$-map $f:(K, K_0)\to(M-B, M-D)$ there is an ϵ-homotopy $f_t: K \to M-A$, $0 \leqq t \leqq 1$, with $f_0=f$, $f_1(K) \subset M-C$, and $f_1 \mid K_0 = f \mid K_0$).

(3.9) THEOREM: *Let $\mathscr{C}_p$ be a class of pairs of compact metric spaces and let d be a metric on the Hilbert cube Q. A Z-set X in Q is $\mathscr{C}_p$-e-movable iff $(Q-X, d \mid (Q-X)\times(Q-X))$ is $\mathscr{C}_p$-e-movable at ∞.*

PROOF: Suppose first that X is a $\mathscr{C}_p$-e-movable Z-set in Q. Let $M=Q-X$ and let a compactum A in M and an $\epsilon>0$ be given. Choose an open neighborhood V of X in Q, $V \subset Q-A$, such that $\mathscr{C}_p^{\epsilon/2}(Q-A, V; X)$ is true and put $B=Q-V$. We claim that $\mathscr{C}_p^\epsilon(M-A, M-B; \infty)$ holds.

Indeed, if C is a compactum in M, let $W=Q-C$ and select an open neighborhood W_0 of X in Q, $W_0 \subset W \cap V$, with respect to W using $\mathscr{C}_p^{\epsilon/2}(Q-A, V; X)$. Let $D=Q-W_0$. Note that $D \supset B \cup C$.

Consider a $\mathscr{C}_p$-map $f:(K, K_0)\to(M-B, M-D)$. Since $M-B \subset V$ and $M-D \subset W_0$, there is an $(\epsilon/2)$-homotopy $f_t: K \to Q-A$, $0 \leqq t \leqq 1$, with $f_0=f$, $f_1(K) \subset W$, and $f_1 \mid K_0 = f \mid K_0$. Observe that $\eta = d(f_1(K), C)>0$. But, since X is a Z-set, by Lemma (4.1) in [11], there is a $\min\{\epsilon/2, \eta\}$-homotopy $\lambda_t: Q \to Q$, $0 \leqq t \leqq 1$, with $\lambda_0 = \mathrm{id}_Q$, $\lambda_t(Q) \subset Q-X$ for all $t>0$, and $\lambda_t(Q-A) \subset Q-A$ and $\lambda_t \mid f(K_0) = \mathrm{id}$ for all $t \in [0, 1]$. Define $h_t: K \to M-A$ by $h_t(x) = (\lambda_t \circ h_t)(x)$ for $x \in K$ and $t \in [0, 1]$. Clearly, h_t is an ϵ-homotopy in $M-A$, $h_0=f$, $h_1(K) \subset M-C$, and $h_1 \mid K_0 = f \mid K_0$.

Conversely, assume that X is a Z-set in Q and that $(M, d \mid M \times M)$ is $\mathscr{C}_p$-e-movable at ∞, where $M=Q-X$. Let an open neighborhood U of X in Q and an $\epsilon>0$ be given. Put $A=Q-U$ and choose a compactum B in M, $B \supset A$, such that $\mathscr{C}_p^{\epsilon/3}(M-A, M-B; \infty)$ is true. Let $V=Q-B$. We claim that $\mathscr{C}_p^\epsilon(U, V; X)$ holds.

Indeed, let W be an arbitrary open neighborhood of X in Q. Choose another open neighborhood W_1 of X in Q such that $W_1 \subset$

$\bar{W}_1 \subset W$. Let $C = Q - W_1$ and let $\eta_1 = d(Q - W, W_1)$. Select a compactum D in M, $D \supset B \cup C$, with respect to C using $\mathscr{C}_p^{\epsilon/3}(M - A, M - B; \infty)$ and put $W_0 = Q - D$.

For a $\mathscr{C}_p$-map $f:(K, K_0) \to (V, W_0)$, let $\eta_2 = d(f(K), B)$. By Lemma (4.1) in [11] again, there is a $\min\{\epsilon/3, \eta_1, \eta_2\}$-homotopy $\lambda_t : Q \to Q$, $0 \leqq t \leqq 1$, with $\lambda_0 = \mathrm{id}_Q$, $\lambda_t(Q) \subset Q - X$ for all $t > 0$, and $\lambda_t(W_0) \subset W_0$ for all $t \in [0, 1]$. Let $f_t : K \to M - A$, $1 \leqq t \leqq 2$, be an $(\epsilon/3)$-homotopy satisfying $f_1 = \lambda_1 \circ f$, $f_2(K) \subset M - C$, and $f_2 \mid K_0 = \lambda \circ f \mid K_0$. Let $g_t : K \to W$, $2 \leqq t \leqq 3$, be an extension to an $(\epsilon/3)$-homotopy of the partial homotopy $\lambda_{t-2} \circ f \mid K_0$ on K_0 of the map $\lambda_1 \circ f = f_2$. Clearly, the join $h_t : K \to U, 0 \leqq t \leqq 3$, of homotopies $\lambda_t \circ f$ $(0 \leqq t \leqq 1)$, f_t $(1 \leqq t \leqq 2)$, and g_t $(2 \leqq t \leqq 3)$ is an ϵ-homotopy in U which satisfies $h_0 = f$, $h_3(K) \subset W$, and $h_3 \mid K_0 = f \mid K_0$.

The next result generalizes the well-known theorem: A compactum X is an ANR iff for every $\epsilon > 0$, X is ϵ-dominated by an ANR ([20], p. 140).

Let $\mathscr{C}$ be a class of compacta in Q. We say that the class $\mathscr{C}$ *strongly e-dominates* a compactum X in Q if for every $\epsilon > 0$ there is $Y \in \mathscr{C}$ and fundamental e-sequences $\underline{f} = \{f_k, X, Y\}_{Q,Q}$ and $\underline{g} = \{g_k, Y, X\}_{Q,Q}$ such that $\underline{g} \circ \underline{f}$ and $\underline{id}_X = \{\mathrm{id}_Q, X, X\}_{Q,Q}$ are ϵ-homotopic.

(3.10) Theorem: *If a class of $\mathscr{C}_p$-e-movable compacta in Q strongly e-dominates a compactum X in Q, then X is also $\mathscr{C}_p$-e-movable.*

Proof: Let a neighborhood U of X in Q and an $\epsilon > 0$ be given. Select a $\mathscr{C}_p$-e-movable compactum Y in Q and fundamental e-sequences $\underline{f} = \{f_k, X, Y\}_{Q,Q}$ and $\underline{g} = \{g_k, Y, X\}_{Q,Q}$ such that $\underline{g} \circ \underline{f}$ and $\underline{id}_X$ are $(\epsilon/4)$-homotopic. Then pick a neighborhood U^* of X in Q and an index k_0 such that $g_k \circ f_k \mid U^*$ is $(\epsilon/4)$-homotopic in U to the inclusion $U^* \to U$ for all $k \geqq k_0$. Next we choose a neighborhood U' of Y in Q, an $\epsilon' > 0$, and a $k_1 \geqq k_0$ such that $g_k \mid U'$ maps ϵ'-close points of U' into $(\epsilon/4)$-close points of U^*. Since Y is $\mathscr{C}_p$-e-movable, there is a neighborhood V' of Y in U' such that (3.2)(i) holds for U', V', and ϵ'. Finally, pick a neighborhood V of X in U^* and an index $k_2 \geqq k_1$ with the property that $f_k(V) \subset V'$ for all $k \geqq k_2$. We claim that $\mathscr{C}_p^{\epsilon}(U, V; X)$ is true.

Suppose that W is an arbitrary open neighborhood of X in Q. Let $k_3 \geqq k_2$ and a neighborhood W' of Y in Q be such that $g_k(W') \subset W$ for all $k \geqq k_3$. Now, we pick a neighborhood W_0' of Y inside $V' \cap W'$ using the choice of V'. At last, we take a neighborhood W_0 of X, $W_0 \subset V \cap W$, and an index $k \geqq k_3$ so that $f_k(W_0) \subset W_0'$ and $g_k \circ f_k \mid W_0$ is $(\epsilon/4)$-homotopic in W to the inclusion $W_0 \to W$.

Consider a $\mathscr{C}_p$-map $\varphi:(K, K_0)\to(V, W_0)$. For a $\mathscr{C}_p$-map $\psi = f_k\circ\varphi:(K, K_0)\to(V', W_0')$, there is an ϵ'-homotopy $G: K\times[1/3, 2/3]\to U'$ with $G_{1/3}=\psi$, $G_{2/3}(K)\subset W'$, and $G_t \mid K_0 = \psi \mid K_0$ for all $t\in[1/3, 2/3]$. Then $H = g_k\circ G: K\times[1/3, 2/3]\to U^*$ is an $(\epsilon/4)$-homotopy satisfying $H_{1/3}=g_k\circ f_k\circ\varphi$, $H_{2/3}(K)\subset W$, and $H_t \mid K_0 = g_k\circ f_k\circ\varphi \mid K_0$ for all $t\in[1/3, 2/3]$. The choice of U^* and k_0 gives us an $(\epsilon/4)$-homotopy $D: K\times[0, 1/3]\to U$ with $D_0=\varphi$ and $D_{1/3}=g_k\circ f_k\circ\varphi$. Similarly, there is an $(\epsilon/4)$-homotopy $E'\times[2/3, 1]\to W$ such that $E'_{2/3}=g_k\circ f_k\circ\varphi \mid K_0$ and $E'_1=\varphi \mid K_0$. By applying the homotopy extension theorem we see that there is also an $(\epsilon/2)$-homotopy $E: K\times[2/3, 1]\to W$ satisfying $E_{2/3}=H_{2/3}$ and $E \mid K_0\times[2/3, 1]=E'$. The join of homotopies D, H, and E shows that $\mathscr{C}_p^\epsilon(U, V; X)$ holds.

In the statement of the next theorem we shall use the notion of a strong fundamental convergence defined as follows. Let $X[A]$ denote all compacta in a metric space X shape equivalent to a compactum A. A sequence $\{A_n\}_{n=1}^\infty$ in $X[A]$ converges *strongly fundamentally* to a compactum $A_0\in X[A]$ if for some (and hence for every) AR space M which contains X the following holds. For every $\epsilon>0$ there is an index n_0 such that for every $n\geqq n_0$ there exist ϵ-fundamental sequences $\underline{f}^n=\{f_k^n, A_n, A_0\}_{M,M}$ and $\underline{g}^n=\{g_k^n, A_0, A_n\}_{M,M}$ with $\underline{g}^n\circ\underline{f}^n\simeq^\epsilon \underline{id}_{A_n}$ and $\underline{f}^n\circ\underline{g}^n\simeq^\epsilon \underline{id}_{A_0}$.

Observe that on the hyperspace of all e-movable compacta [8] in $X(A)$ (the set of all compacta in X homotopy equivalent to a compactum A), the strong fundamental convergence agrees with the strong homotopy convegence defined in [7] as follows. A sequence $\{A_n\}_{n=1}^\infty$ in $X(A)$ converges *strongly homotopically* to a compactum $A_0\in X(A)$ provided for every $\epsilon>0$ there is an index n_0 such that for every $n\geqq n_0$ there exist ϵ-maps $f^n: A_n\to A_0$ and $g^n: A_0\to A_n$ with $g^n\circ f^n\simeq^\epsilon \mathrm{id}_{A_n}$ and $f^n\circ g^n\simeq^\epsilon \mathrm{id}_{A_0}$. Clearly, the strong homotopy convergence implies the strong fundamental convergence.

(3.11) THEOREM: *Let X be a metric space and let A be a compactum. If a sequence $\{A_n\}_{n=1}^\infty$ of $\mathscr{C}_p$-e-movable compacta in $X[A]$ converges strongly fundamentaly to a compactum $A_0\in X[A]$, then A_0 is also $\mathscr{C}_p$-e-movable.*

PROOF: Without loss of generality, we can assume that X is a subset of Q [10]. For a given $\epsilon>0$, select an index n for which there exist $(\epsilon/5)$-fundamental sequences $\underline{f}=\{f_k, A_0, A_n\}_{Q,Q}$ and $\underline{g}=\{g_k, A_n\, A_0\}_{Q,Q}$ such that $\underline{g}\circ\underline{f}\simeq^{\epsilon/5}\underline{id}_{A_0}$. Next we choose a neighborhood V' of A_n in Q which satisfies (3.2)(iii) for $\epsilon/5$ in place of ϵ. Finally, pick a neighborhood V of A_0 in Q and an index k_0 such that $f_k \mid V$ is an $(\epsilon/5)$-map of V into V' for all $k \geqq k_0$. We claim that (3.2)(iii) holds for V.

Indeed, let W be an arbitrary open neighborhood of A_0 in Q. Take a neighborhood W' of A_n in Q and a $k_1 \geqq k_0$ so that $g_k \mid W'$ is an $(\epsilon/5)$-map of W' into W for all $k \geqq k_1$. Inside $V' \cap W'$ we pick a neighborhood W_0' of A_n using the way in which V' was chosen. Then select a neighborhood W_0 of A_0 in Q, $W_0 \subset V \cap W$, and a $k \geqq k_1$ such that $f_k(W_0) \subset W_0'$ and $g_k \circ f_k \mid W_0$ is $(\epsilon/5)$-homotopic in W to the inclusion of W_0 into W.

Consider a $\mathscr{C}_p$-map $\varphi : (K, K_0) \to (V, W_0)$. The composition $f_k \circ \varphi$ is a $\mathscr{C}_p$-map into (V', W_0'). Hence, there is a map $\psi : K \to W'$ which is $(\epsilon/5)$-close to $f_k \circ \varphi$ and which agrees with $f_k \circ \varphi$ on K_0. Then $g_k \circ \psi : K \to W$ is $(3\epsilon/5)$-close to φ and $g_k \circ \psi \mid K_0$ is $(\epsilon/5)$-homotopic in W to $\varphi \mid K_0$. Applying the homotopy extension theorem it follows that $(2\epsilon/5)$-close to $g_k \circ \psi$ and therefore ϵ-close to φ there is a map $\varphi' : K \to W$ which agrees with φ on K_0.

The strong homotopy convergence preserves, by (3.11), ANR's and LC^n compacta. It is interesting that it also preserves local contractibility which we were unable to describe as a positional property analogous to $\mathscr{C}_p$-e-movability.

(3.12) THEOREM: *Let X be a metric space and let A be a compactum. If a sequence $\{A_n\}_{n=1}^{\infty}$ of locally contractible compacta in $X(A)$ converges strongly homotopically to a compactum $A_0 \in X(A)$, then A_0 is also locally contractible.*

PROOF: A routine proof is left to the reader.

Let $\mathscr{C}_p$ and $\mathscr{D}_p$ be classes of pairs of topological spaces. A compactum X is *$(\mathscr{C}_p, \mathscr{D}_p)$-e-tame* if for some (and hence for every) embedding of X into an ANR M the following holds. For each neighborhood U of X in M and every $\epsilon > 0$ there is a neighborhood V of X in M, $V \subset U$, with the property that for each neighborhood W of X in M and any $\delta > 0$ there is a neighborhood W_0 of X in M, $W_0 \subset V \cap W$, such that for every $\mathscr{C}_p$-map $f : (K, K_0) \to (V, W_0)$ one can find a pair $(L, L_0) \in \mathscr{D}_p$ and maps $g : (K, K_0) \to (L, L_0)$ and $h : (L, L_0) \to (U, W)$ with $h \circ g \mid K_0$ δ-close to $f \mid K_0$ and $h \circ g$ ϵ-close to f.

(3.13) LEMMA: *A compactum of dimension $\leqq n$ is $(\mathscr{S}_p, \mathscr{S}_p^{n+1})$-e-tame.*

PROOF: Let X be a compactum of dimension $\leqq n$. Then X can be represented as the inverse limit of an inverse sequence $\sigma = \{X_i, f_{i+1}^i\}_{i=1}^{\infty}$ of polyhedra of dimension $\leqq n$. But, the infinite mapping cylinder Map(σ) of σ [13] can be compactified to an ANR by adding a copy of

X. Hence, X is clearly $(\mathscr{P}_p, \mathscr{P}_p^{n+1})$-$e$-tame. By a result analogous to Theorem (3.4), it follows that X is $(\mathscr{S}_p, \mathscr{S}_p^{n+1})$-$e$-tame.

Combining the above lemma with theorems (3.6) and (3.7) we see that the last result in this section includes as a special case the fact that an n-dimensional LC^n compactum is an ANR ([20], p. 168).

(3.14) THEOREM: *If a compactum X is $(\mathscr{C}_p, \mathscr{D}_p)$-$e$-tame and $\mathscr{D}_p$-e-movable, then X is also $\mathscr{C}_p$-e-movable.*

PROOF: Consider X as a subset of Q. For a given $\epsilon > 0$, pick a neighborhood U of X in Q such that U satisfies (3.2)(iii) with respect to $\mathscr{D}_p$ and $\epsilon/2$. Then select a neighborhood V of X in Q, $V \subset U$, for $\epsilon/2$ using $(\mathscr{C}_p, \mathscr{D}_p)$-$e$-tameness of X. We claim that (3.2)(iv) holds for V.

Indeed, let a neighborhood W of X in Q and a $\delta > 0$ be given. First take a neighborhood W_0' of X inside $V \cap W$ such that every $\mathscr{D}_p$-map $h:(L, L_0) \to (U, W_0')$ is $(\epsilon/2)$-close to a map $h': L \to W$ which agrees with h on L_0. Finally, pick a required neighborhood W_0 of X, $W_0 \subset W_0'$, so that for every $\mathscr{C}_p$-map $f:(K, K_0) \to (V, W_0)$ there is a pair $(L, L_0) \in \mathscr{D}_p$ and maps $g:(K, K_0) \to (L, L_0)$ and $h:(L, L_0) \to (U, W_0')$ with $h \circ g \mid K_0$ δ-close to $f \mid K_0$ and $h \circ g$ $(\epsilon/2)$-close to f.

4. $\mathscr{C}$-e-calm compacta

In this section we shall study the class of $\mathscr{C}$-e-calm compacta which one gets from the class of $\mathscr{C}$-calm compacta [5] by requiring control on the size of homotopies appearing in the definition of $\mathscr{C}$-calmness.

(4.1) DEFINITION: A compactum X is *$\mathscr{C}$-e-calm* if for some (and hence for every) embedding of X into an ANR M the following holds. For every $\epsilon > 0$, there is a neighborhood V of X in M and a $\delta > 0$ such that $\mathscr{C}_h^\epsilon(V, \delta; X)$ is true. The $\mathscr{P}$-e-calm compacta are simply called *e-calm.*

(4.2) PROPOSITION: *A compactum X in a Q-manifold M is $\mathscr{C}$-e-calm iff for every $\epsilon > 0$ there is a $\delta > 0$ with the property that for every neighborhood W of X in M there is a smaller neighborhood W_0 of X in M such that δ-close $\mathscr{C}$-maps into W_0 are ϵ-homotopic in W.*

PROOF: Suppose that X is $\mathscr{C}$-e-calm. For an $\epsilon > 0$, select a compact ANR neighborhood V of X in M and a $\delta' > 0$ such that $\mathscr{C}_h^\epsilon(V, \delta'; X)$ holds. Pick a $\delta > 0$ with the property that δ-close maps into V are δ'-homotopic in V. Then δ is clearly a required number. The converse implication is obvious (just put $V = M$).

(4.3) EXAMPLE: A compact ANR is $\mathscr{C}$-e-calm and an LC^n compactum is $\mathscr{C}^n$-e-calm for every class $\mathscr{C}$. The key open question concerning $\mathscr{C}$-e-calm compacta is whether e-calm compacta coincide with compact ANR's ($\mathscr{P}^n$-e-calm compacta coincide with LC^n compacta).

The properties of $\mathscr{C}$-e-calm compacta are similar to properties of $\mathscr{C}_p$-e-movable compacta. We shall only state theorems about $\mathscr{C}$-e-calmness and leave to the reader their proofs if they are similar to the proofs of the corresponding results about either $\mathscr{C}_p$-e-movability in §3 or $\mathscr{C}$-e-movability in [8].

(4.4) THEOREM: *If a class $\mathscr{C}$ approximately dominates a class $\mathscr{D}$* [8] *and a compactum X is $\mathscr{C}$-e-calm, then X is also $\mathscr{D}$-e-calm.*

(4.5) THEOREM: *If a class of $\mathscr{C}$-e-calm compacta in Q strongly e-dominates a compactum X in Q, then X is also $\mathscr{C}$-e-calm.*

(4.6) THEOREM: *Let $\mathscr{C}$ be a class of compact metric spaces. A Z-set in Q is $\mathscr{C}$-e-calm iff $M = Q - X$ is $\mathscr{C}$-e-calm at ∞ (i.e., iff for every $\epsilon > 0$ there is a compactum B in M and a $\delta > 0$ with the property that for every compactum C in M one can find a compactum D in M, $D \supset B \cup C$, such that two $\mathscr{C}$-maps f, $g: K \to M - D$ which are δ-homotopic in $M - B$ are also ϵ-homotopic in $M - C$).*

(4.7) THEOREM: *Let X be a metric space and let A be a compactum. If a sequence $\{A_n\}_{n=1}^{\infty}$ of $\mathscr{C}$-e-calm compacta in $X[A]$ converges strongly fundamentaly to a compactum $A_0 \in X[A]$, then A_0 is also $\mathscr{C}$-e-calm.*

(4.8) THEOREM ([10], (4.1)): *If a compactum A in Q is e-calm, then for every $\epsilon > 0$ there is a $\delta > 0$ such that δ-close fundamental sequences $\underline{f} = \{f_k, X, A\}_{M,Q}$ and $\underline{g} = \{g_k, X, A\}_{M,Q}$ defined on an arbitrary compactum X in an AR space M are ϵ-homotopic.*

The following theorem is analogous to the characterization of an FANR as a compactum which is both movable and calm [10].

(4.9) THEOREM: (a) *A compactum X is strongly e-movable iff X is both e-movable* [8] *and e-calm (or, equivalently, a compactum X is an ANR iff X is an approximate absolute neighborhood retract* [14] *and e-calm).*

(b) *A compactum X is LC^{n-1} iff X is both $\mathscr{S}^n$-e-movable and $\mathscr{S}^n$-e-calm.*

PROOF: (a) Suppose that a compactum X in Q is e-movable and e-calm. For a given $\epsilon > 0$, select a δ, $0 < \delta < \epsilon/2$, such that for every neighborhood W of X in Q there is a neighborhood W_0, $W_0 \subset W$, of X in Q with the property that δ-close $\mathscr{S}$-maps into W_0 are $(\epsilon/2)$-homotopic in W. Then we pick a neighborhood V of X in Q such that every $\mathscr{S}$-map $f: K \to V$ is δ-close to a map $f': K \to X$ ([8], (4.2)). We claim that V satisfies (3.2)(iii) for the class $\mathscr{S}_p$.

Indeed, consider an open neighborhood W of X in Q. Inside $V \cap W$ we pick a neighborhood W_0 using the way in which δ was chosen. Then an $\mathscr{S}_p$-map $f:(K, K_0) \to (V, W_0)$ is δ-close to a map $f': K \to X$. The restrictions $f \mid K_0$ and $f' \mid K_0$ are δ-close maps of K_0 into W_0. It follows that they are $(\epsilon/2)$-homotopic in W. By the homotopy extension theorem, there is a map $f'': K \to W$ which is $(\epsilon/2)$-close to f' and which agrees with f on K_0. Hence, f is ϵ-close to f''.

The other implication and the proof of (b) are obvious.

The above proof shows that a compactum X which is both $\mathscr{C}'_p$-e-movable and $\mathscr{C}''_p$-e-calm must be $\mathscr{C}_p$-e-movable. Here $\mathscr{C}'_p = \{K \mid \exists K_0$ such that $(K, K_0) \in \mathscr{C}_p\}$ and $\mathscr{C}''_p = \{K_0 \mid \exists K$ such that $(K, K_0) \in \mathscr{C}_p\}$. This can be slightly improved by the introduction of a class of internally $\mathscr{C}$-e-calm compacta. It also suggests other "internal" properties like internal $\mathscr{C}_p$-e-movability (see also the proof of (5.2) below).

A compactum X is *internally $\mathscr{C}$-e-calm* if for some (and hence for every) embedding of X into an ANR M the following holds. For every $\epsilon > 0$ there is a $\delta > 0$ such that δ-close $\mathscr{C}$-maps into X are ϵ-homotopic in every neighborhood of X in M.

(4.10) LEMMA: *A $\mathscr{C}$-e-movable compactum X* [8] *is $\mathscr{C}$-e-calm iff X is internally $\mathscr{C}$-e-calm.*

PROOF: Suppose that a compactum X in Q is both $\mathscr{C}$-e-movable and internally $\mathscr{C}$-e-calm. For a given $\epsilon > 0$, pick a $\delta > 0$ such that 2δ-close $\mathscr{C}$-maps into X are $(\epsilon/3)$-homotopic in every neighborhood of X in Q. We claim that X satisfies the condition in (4.2).

Indeed, let W be an arbitrary compact ANR neighborhood of X in Q. Let $\eta \in \Gamma(W, \epsilon/3)$, $0 < \eta < \delta/2$. Since X is also $\mathscr{C}$-e-movable, there is a neighborhood W_0 of X in Q, $W_0 \subset W$, such that every $\mathscr{C}$-map $f: K \to W_0$ is η-close to a map $f': K \to X$.

Now, if $\varphi, \psi: K \to W_0$ are δ-close $\mathscr{C}$-maps, then there are $\mathscr{C}$-maps $\varphi', \psi': K \to X$ with φ' η-close to φ and ψ' η-close to ψ. It follows that (φ, φ'), (φ', ψ'), and (ψ', ψ) are three pairs of $(\epsilon/3)$-homotopic (in W) maps. Hence, φ and ψ are ϵ-homotopic in W.

Just how close are $\mathscr{S}^n$-e-calm compacta to LC^{n-1} compacta is best illustrated by the following theorem.

(4.11) THEOREM: *A compactum X in Q is $\mathscr{S}^n$-e-calm iff for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that for every neighborhood W of X in Q we can find a smaller neighborhood W_0 of X in Q with the property that any map $f: S^k \to W_0$ of the k-dimensional sphere into W_0 $(0 \leqq k \leqq n)$ with* $\operatorname{diam} f(S^k) < \delta$ *extends to a map $f^*: B^{k+1} \to W$ of the $(k+1)$-dimensional cell into W with* $\operatorname{diam} f^*(B^{k+1}) < \epsilon$.

PROOF: Suppose that X is $\mathscr{S}^n$-e-calm. Then X satisfies the above condition by (4.2) because a map whose image has diameter $<\epsilon$ is ϵ-close to a constant map.

Conversely, let X have the property described in the statement of the theorem and let an $\epsilon > 0$ be given. Put $\delta_1 = \epsilon$ and define inductively a δ_k, $0 < \delta_k < \delta_{k-1}/3$, by $\delta_k = \delta(\delta_{k-1}/3)$ for $k = 2, 3, \ldots, n$. It can be proved easily by induction that $\delta = \delta_n$ has the property described in (4.2) for the class $\mathscr{S}^n$.

5. Images of $\mathscr{C}_p$-e-movable and $\mathscr{C}$-e-calm compacta

In this section we shall consider the problem of identifying classes of maps which preserve $\mathscr{C}_p$-e-movability and $\mathscr{C}$-e-calmness. Our approach allows to get some new necessary and sufficient conditions for the image of a compact ANR under either an approximately right invertible [18] or a refinable [17] map to be an (F)ANR.

A map $f: X \to Y$ between compact metric spaces is *approximately right invertible* (ARI) [18] provided there is a sequence of maps $\{h_n: Y \to X\}$ and a null-sequence $\{\epsilon_n\}$ of positive reals such that $f \circ h_n$ is ϵ_n-close to id_Y. If, in addition, there is a sequence $\{g_n: X \to Y\}$ of surjective maps such that $h_n \circ g_n$ is ϵ_n-close to id_X, then f is called *approximately invertible* (AI). Observe that a map $f: X \to Y$ of an e-movable compactum X onto Y is refinable [17] iff it is AI.

The main open problem about ARI and AI maps is whether the images of ANR's under these maps are ANR's. It is not even known if the images are FANR's ([18], Problems (CE2) and (CE3)). It is this later question that we study first.

Let X and Y be compacta in AR spaces M and N, respectively. A sequence $f = \{f_n: (M, X) \to (N, Y)\}$ of maps of pairs is called a *net from X into Y (in M, N)* provided for every neighborhood V of Y in N there is a neighborhood U of X in M such that $f_n(U) \subset V$ for almost all n. If each f_n maps X onto Y, then f is a *net from X onto Y (in M, N)*.

A map $f: X \to Y$ between compact metric spaces is *strongly approximately right invertible* (SARI) iff for some (and hence for all) AR spaces M and N which contain X and Y, respectively, there is a net $\boldsymbol{h} = \{h_n : (N, Y) \to (M, X)\}$ from Y into X and a null-sequence $\{\epsilon_n\}$ of positive reals such that $f \circ h_n \mid Y$ is ϵ_n-close to id_Y. If, in addition, there is a net $\boldsymbol{g} = \{g_n : (M, X) \to (N, Y)\}$ from X onto Y such that $h_n \circ g_n \mid X$ is ϵ_n-close to id_X, then f is called *strongly approximately invertible* (SAI).

Clearly, every SARI map is ARI and every SAI map is AI.

(5.1) Lemma: (a) *Every ARI map $f: X \to Y$ of a compactum X onto an $AANR_N$ (an approximate absolute neighborhood retract in the sense of Noguchi* [26]) *Y is SARI.*

(b) *Every AI map $f: X \to Y$ of an $SAANR_N$ (a surjective approximate absolute neighborhood retract in the sense of Noguchi* [27]) *X onto an $AANR_N$ Y is SAI.*

(5.2) Theorem: (a) *If $f: X \to Y$ is an SARI map of an $AANR_N$ X onto Y, then Y is also an $AANR_N$.*

(b) *Let $f: X \to Y$ be an SAI map. Then X is an $AANR_N$ iff Y is an $AANR_N$.*

Proof: (a) Recall that a compactum X is an AANR_N iff X is an AANR_C (i.e., an e-movable compactum) and an FANR [1]. Since Y is an AANR_C by Corollary (6.5) in [8], it clearly suffices to prove that Y is *internally calm* (i.e., that in some ANR space N which contains Y there is a neighborhood V of Y in N such that $\mathscr{P}$-maps $\varphi, \psi: K \to Y$ which are homotopic in V are homotopic in every neighborhood of Y in N). Assume that X and Y are subsets of Q and pick a net $\boldsymbol{h} = \{h_n : (Q, Y) \to (Q, X)\}$ from Y into X and a null-sequence $\{\epsilon_n\}$ such that $f \circ h_n \mid Y$ is ϵ_n-close to id_Y. Choose an extension $f^*: Q \to Q$ of f and a neighborhood V' of X in Q such that $\mathscr{P}$-maps $\varphi', \psi': K \to X$ which are homotopic in V' are homotopic in every neighborhood of X in Q. Then select a neighborhood V of Y in Q and an index n_0 so that $h_n(V) \subset V'$ for all $n \geqq n_0$.

Consider $\mathscr{P}$-maps $\varphi, \psi: K \to Y$ which are homotopic in V. Let W be an arbitrary compact ANR neighborhood of Y in Q. Choose a neighborhood W' of X in Q that is mapped into W by f^* and an $\epsilon > 0$ and an $n \geqq n_0$ such that ϵ-close maps into W are homotopic in W and $\epsilon_n < \epsilon$. Then $\mathscr{P}$-maps $\varphi' = h_n \circ \varphi : K \to X$ and $\psi' = h_n \circ \psi : K \to X$ are homotopic in V'. Hence, they are homotopic in W' so that

$f \circ h_n \circ \varphi : K \to Y$ and $f \circ h_n \circ \psi : K \to Y$ are homotopic in W. But, since $f \circ h_n \circ \varphi$ and φ are ϵ_n-close, they are also homotopic in W. Similarly, $f \circ h_n \circ \psi$ is homotopic in W to ψ. Thus, φ and ψ are homotopic in W.

(b) We must only show that the domain X of an SAI map $f : X \to Y$ onto an AANR_N Y is an AANR_N because the other implication follows from (a). X is an AANR_C by Theorem (4.6) in [8]. Hence, just like in the proof of (a), it suffices to prove that X is internally calm.

Assume again that X, $Y \subset Q$ and pick a net $\mathbf{h} = \{h_n : (Q, Y) \to (Q, X)\}$ from Y into X, a net $\mathbf{g} = \{g_n : (Q, X) \to (Q, Y)\}$ from X onto Y, and a null-sequence $\{\epsilon_n\}$ such that $f \circ h_n \mid Y$ is ϵ_n-close to id_Y and $h_n \circ g_n \mid X$ is ϵ_n-close to id_X. Let a neighborhood V' of Y in Q has the property that $\mathcal{P}$-maps into Y which are homotopic in V' are homotopic in every neighborhood of Y in Q. Choose a neighborhood V of X in Q and an index n_0 such that $g_n(V) \subset V'$ for all $n \geqq n_0$.

Consider $\mathcal{P}$-maps $\varphi, \psi : K \to X$ which are homotopic in V. Let W be an arbitrary neighborhood of X in Q. Pick an $\epsilon > 0$ such that ϵ-close maps into W are homotopic in W. Then select a neighborhood W' of Y in Q and an $n \geqq n_0$ with $\epsilon_n < \epsilon$ and $h_n(W') \subset W$. Maps $g_n \circ \varphi$ and $g_n \circ \psi$ are homotopic in V' and therefore also in W'. Hence, $h_n \circ g_n \circ \varphi$ and $h_n \circ g_n \circ \psi$ are homotopic in W. But, since $h_n \circ g_n \circ \varphi$ is ϵ_n-close to φ and $h_n \circ g_n \circ \psi$ is ϵ_n-close to ψ, φ and ψ are homotopic in W.

(5.3) Corollary: (a) *An image of an ANR under an ARI map f is an FANR iff f is SARI.*

(b) *An image of an SAANR_N under an SARI map is an SAANR_N.*

(c) *Let $f : X \to Y$ be an SAI map. Then Y is an SAANR_N iff X is an SAANR_N.*

Proof: (a) Combine [8], (6.5), [1], (5.1), and (5.2).

(b) and (c). Recall [9] that a compactum X is an SAANR_N iff X is an FANR and an SAANR_C (a surjective approximate absolute neighborhood retract in the sense of Clapp; or, equivalently, a quasi-ANR [27]) and that refinable maps preserve SAANR_C's [28].

The problem of identifying classes of maps which preserve $\mathcal{C}_p$-e-movability naturally leads to the notion of a $\mathcal{C}_p$-e-surjective fundamental sequence (or a $\mathcal{C}_p$-e-surjection).

(5.4) Definition: A fundamental sequence $\underline{f} = \{f_k, A, B\}_{M,N}$ is called a *$\mathcal{C}_p$-e-surjection* provided $\underline{f}$ is the fundamental e-sequence and it satisfies the following condition. For every neighborhood U' of A in M and every $\epsilon > 0$ there is a neighborhood V of B in N with the property that for every neighborhood W' of A in M, $W' \subset U'$, and every $\delta > 0$ there is a neighborhood W_0 of B in N, $W_0 \subset V$, and an index k_0 such that for every $k \geqq k_0$ and every $\mathscr{C}_p$-map $\varphi:(K, K_0) \rightarrow (V, W_0)$ there is a map $\tilde{\varphi}:(K, K_0) \rightarrow (U', W')$ with $f_k \circ \tilde{\varphi}$ ϵ-close to φ and $f_k \circ \tilde{\varphi} \mid K_0$ δ-close to $\varphi \mid K_0$.

The following theorem describes some properties of $\mathscr{C}_p$-e-surjections.

(5.5) Theorem: (a) *If a class $\mathscr{C}_p$ approximately dominates a class $\mathscr{D}_p$ and $\underline{f} = \{f_k, A, B\}_{M,N}$ is a $\mathscr{C}_p$-e-surjection, then $\underline{f}$ is also a $\mathscr{D}_p$-e-surjection.*

(b) *Let $\underline{f} = \{f_k, A, B\}_{M,N}$ be a $\mathscr{D}_p$-e-surjection of a compactum A into a $(\mathscr{C}_p, \mathscr{D}_p)$-e-tame compactum B. Then $\underline{f}$ is also a $\mathscr{C}_p$-e-surjection.*

(c) *If $\underline{f} = \{f_k, A, B\}_{M,N}$ is an $\mathscr{S}^n_p$-e-surjection for every $n \geqq 0$ and B is an ANR, then $\underline{f}$ is an $\mathscr{S}_p$-e-surjection.*

(d) *The composition of two $\mathscr{C}_p$-e-surjections is a $\mathscr{C}_p$-e-surjection.*

(e) *If a fundamental sequence generated by a map is a $\mathscr{C}_p$-e-surjection, then every other fundamental sequence generated by that map is also a $\mathscr{C}_p$-e-surjection.*

Proof: The proof is left to the reader.

In order to describe some examples of $\mathscr{C}_p$-e-surjections we shall need $\mathscr{C}$-e-surjections and $\mathscr{C}_p$-e-bundles.

A fundamental sequence $\underline{f} = \{f_k, A, B\}_{M,N}$ is a *$\mathscr{C}$-e-surjection* [8] provided $\underline{f}$ is the fundamental e-sequence and for every neighborhood U' of A in M and every $\epsilon > 0$ there is a neighborhood V of B in N, a neighborhood W' of A in M, $W' \subset U'$, and an index k_0 such that for every $k \geqq k_0$ and every $\mathscr{C}$-map $\varphi: K \rightarrow V$ there is a map $\tilde{\varphi}: K \rightarrow W'$ with $f_k \circ \tilde{\varphi}$ ϵ-close to φ.

One easily shows that every fundamental sequence generated by either an ARI or a refinable map is a $\mathscr{C}$-e-surjection for every class $\mathscr{C}$ and that a map $f: A \rightarrow B$ of an e-movable compactum A onto B is a $\mathscr{P}$-e-surjection iff f is an ARI map.

(5.6) Lemma: *Let $\mathscr{C}_p$ be a class of pairs of metrizable spaces. If $\underline{f} = \{f_k, A, B\}_{M,N}$ is a $\mathscr{C}'_p$-e-surjection and B is a $\mathscr{C}_p$-e-movable compactum, then $\underline{f}$ is a $\mathscr{C}_p$-e-surjection.*

Proof: Let a neighborhood U' of A in M and an $\epsilon > 0$ be given. Let V be a neighborhood of B in N which satisfies (3.2)(iii) with respect to $\epsilon/2$. Consider a neighborhood W' of A in M, $W' \subset U'$, and a δ, $0 < \delta < \epsilon/2$. Since $\underline{f}$ is the $\mathscr{C}'_p$-e-surjection, there is a neighborhood

W of B in N, a neighborhood W'' of A in M, $W'' \subset W'$, and an index k_0 such that for every $\mathscr{C}_p'$-map $\varphi' : K \to W$ and every $k \geqq k_0$ there is a map $\tilde{\varphi}' : K \to W''$ with $f_k \circ \tilde{\varphi}'$ δ-close to φ'. Finally, the required neighborhood W_0 of B in N is chosen so that $W_0 \subset V \cap W$ and so that every $\mathscr{C}_p$-map $\varphi : (K, K_0) \to (V, W_0)$ is $(\epsilon/2)$-close to a map $\varphi' : K \to W$ which agrees with φ on K_0.

Let A and B be compacta in AR spaces M and N, respectively, and let $f : A \to B$ be a map. An extension $F : M \to N$ of f is called a **-extension* provided $F^{-1}(B) = A$. A map $f : A \to B$ is a ***$\mathscr{C}_p$-e-bundle*** if there are AR spaces M and N containing A and B, respectively, and a *-extension $F : M \to N$ of f which has the following *$\mathscr{C}_p$-e-lifting property near* (A, B). For every neighborhood U' of A in M and every $\epsilon > 0$ there is a neighborhood V of B in N, a neighborhood V' of A in M, $V' \subset U'$, and a $\delta > 0$ such that for every pair $(K, K_0) \in \mathscr{C}_p$ and maps $\varphi : K \to V$ and $\psi_0 : K_0 \to V'$ with $F \circ \psi_0$ δ-close to $\varphi \mid K_0$ there is a map $\psi : K \to U'$ such that $\psi \mid K_0$ is ϵ-close to ψ_0 and $F \circ \psi$ is ϵ-close to φ.

One readily proves that the choice of spaces M and N and the *-extension F in the above definition is immaterial. If $\mathscr{C}_p$ and $\mathscr{D}_p$ are classes of pairs of compacta, $\mathscr{C}_p$ approximately dominates $\mathscr{D}_p$, and $f : A \to B$ is a $\mathscr{C}_p$-e-bundle, then f is also a $\mathscr{D}_p$-e-bundle. The composition of $\mathscr{C}_p$-e-bundles is a $\mathscr{C}_p$-e-bundle.

A map $f : X \to Y$ will be called *$\mathscr{C}$-trivial* if f maps X onto Y and each pre-image $Z = f^{-1}(y)$ $(y \in Y)$ is a $\mathscr{C}$-trivial compactum [10] (i.e., in some, and hence in every, ANR space M containing Z, for every neighborhood U of Z in M there is a smaller neighborhood V of Z in M such that every $\mathscr{C}$-map into V is null-homotopic in U). Observe that cell-like maps coincide with $\mathscr{S}$-trivial maps.

Let Σ^n denote the class of all at most n-dimensional spheres.

(5.7) LEMMA: *A map $f : A \to B$ between compacta is a Σ^n-trivial map iff f is an $\mathscr{S}_p^{n+1}$-e-bundle.*

PROOF: Suppose that A and B are Z-sets in Q and that f is the $\mathscr{S}_p^{n+1}$-e-bundle. Let $F : Q \to Q$ be a *-extension of f. Then F has the $\mathscr{S}_p^{n+1}$-e-lifting property near (A, B). Clearly, f must map A onto B. Let $b \in B$ and let W be a compact ANR neighborhood of $Z = f^{-1}(b)$ in Q. Pick an $\epsilon > 0$ such that $F^{-1}(N_{2\epsilon}(b)) \subset W$ and ϵ-close maps into W are homotopic in W. Then select a neighborhood V of B in Q, a neighborhood V' of A in Q, and a $\delta > 0$ with respect to ϵ and $U' = Q$ using the fact that F has the $\mathscr{S}_p^{n+1}$-e-lifting property near (A, B). Let K be a Hilbert cube neighborhood of b in Q with $b \in$

$\operatorname{int} K \subset K \subset N_\epsilon(b) \cap V$. Let $W_0 = F^{-1}(\operatorname{int} K) \cap V'$. We claim that every Σ^n-map $\varphi : S^k \to W_0$ $(0 \leqq k \leqq n)$ is null-homotopic in W.

Indeed, $F \circ \varphi : S^k \to K$ extends to a map $G : B^{k+1} \to K$. By the choice of V and V', there is a map $\tilde{G} : B^{k+1} \to Q$ such that $F \circ \tilde{G}$ is ϵ-close to G and $\tilde{G} \mid S^k$ is ϵ-close to φ. Since $G(B^{k+1}) \subset K \subset N_\epsilon(b)$, $\tilde{G}(B^{k+1}) \subset F^{-1}(N_{2\epsilon}(b)) \subset W$. Hence, φ is null-homotopic in W.

The converse follows from Lemma (8.3) in [15].

(5.8) LEMMA: *A map $f : A \to B$ between compacta is a hereditary shape equivalence (HSE)* [22], [18], *iff f is an $\mathscr{S}_p$-e-bundle.*

PROOF: Suppose first that A and B are Z-sets in Q and that f is the $\mathscr{S}_p$-e-bundle. Let $F : Q \to Q$ be a *-extension of f, let D be a closed subset of B, and let $C = F^{-1}(D) = f^{-1}(D)$. Then F has the $\mathscr{P}_p$-e-lifting property near (A, B). We shall prove that $f \mid C : C \to D$ is a shape equivalence. This will follow provided we prove that for every compact ANR neighborhood D_1 of D in Q and every compact ANR neighborhood C_1 of C in Q such that $F(C_1) \subset D_1$ there is a compact ANR neighborhood D_2 of D in Q, a compact ANR neighborhood C_2 of C in Q, and a map $\psi : D_2 \to C_1$ which makes the following diagram homotopy commutative. The horizontal maps i and j on the diagram are inclusions.

$$\begin{tikzcd} C_1 \arrow[d, "F"'] & C_2 \arrow[l, "j"'] \arrow[d, "F"] \\ D_1 & D_2 \arrow[l, "i"] \arrow[ul, "\psi"'] \end{tikzcd}$$

Pick compact ANR neighborhoods D_1' and D_1'' of D in Q, $D_1'' \subset \operatorname{int} D_1' \subset D_1' \subset \operatorname{int} D_1$, and an $\epsilon_1 \in \Gamma(C_1, 1)$ such that $F^{-1}(D_1') \subset C_1$ and $N_{\epsilon_1}(D_1'') \subset D_1'$. Choose an open neighborhood V_1' of A in Q, an open neighborhood V_1 of B in Q, and a $\delta_1 > 0$ with respect to $U' = Q$ and ϵ_1 using the fact that F has the $\mathscr{P}_p$-e-lifting property near (A, B). Inside $V_1 \cap (\operatorname{int} D_1'')$ take a compact ANR neighborhood D_1''' of D in Q and select an $\epsilon_2 > 0$ such that $N_{2\epsilon_2}(D_1''') \subset V_1 \cap D_1''$. Let $\epsilon \in \Gamma(D_1'', \epsilon_2)$. Pick an open neighborhood V_2 of B in Q, an open neighborhood V_2' of A in Q, and a $\delta_2 > 0$ with respect to V_1' and ϵ again using the $\mathscr{P}_p$-e-lifting property of F near (A, B). Let D_2 be a compact ANR neighborhood of D in Q which lies in $V_2 \cap D_1'''$ and let C_2 be a compact ANR neighborhood of C in Q such that $F(C_2) \subset D_2$ and $C_2 \subset V_1' \cap C_1$.

The choice of V_2 implies that the inclusion $\varphi : D_2 \to V_2$ lifts to a map $\psi : D_2 \to V_1'$ with $F \circ \psi$ being ϵ-close to φ. Since $\epsilon \leqq \epsilon_2$ and $\varphi(D_2) \subset D_1'''$,

$F \circ \psi(D_2) \subset N_{\epsilon_2}(D_1''') \subset D_1''$ so that $\psi(D_2) \subset F^{-1}(D_1'') \subset F^{-1}(D_1') \subset C_1$. Also, since $F \circ \psi$ and i are ϵ-close maps into D_1'', they are ϵ_2-homotopic in D_1''. Finally, $F \circ \psi \circ F \mid C_2$ and $F \mid C_2$ are ϵ-close maps of C_2 into D_1''. Hence, there is an ϵ_2-homotopy $H: C_2 \times I \to D_1''$ with $H_0 = F \circ \psi \circ F \mid C_2$ and $H_1 = F \mid C_2$. Observe that $H(C_2 \times I) \subset N_2\delta_2(D_1''') \subset V_1$. Let $g: C_2 \times \{0, 1\} \to V_1'$ be given as $\psi \circ F$ on $C_2 \times \{0\}$ and as the inclusion $\chi: C_2 \to V_1'$ on $C_2 \times \{1\}$. Then $F \circ g = H \mid C_2 \times \{0, 1\}$. By the choice of V_1 and V_1', there is a map $G: C_2 \times I \to Q$ with G_0 ϵ_1-close to $\psi \circ F \mid C_2$, G_1 ϵ_1-close to χ, and $F \circ G$ is ϵ_1-close to H. These three conditions imply that G can be actually completed to a homotopy in C_1 between $\psi \circ F \mid C_2$ and j. This proves that the upper triangle in the diagram is indeed homotopy commutative.

The converse implication follows easily from Kozlowski's theorem which says that a HSE between Z-sets in Q extends to a relative homeomorphism of Q onto itself and the standard property of a Z-set (see Lemma 4.1 in [11]) which allows to replace maps into Q with nearby maps whose images are disjoint with the Z-set.

(5.9) Remark: It can be easily proved that a map $f: A \to B$ of compact ANR's is a fine homotopy equivalence [19] iff f is a $\mathscr{C}_p$-e-bundle for every class $\mathscr{C}_p$ of metrizable pairs.

For a class $\mathscr{C}$ of metrizable spaces. let $\mathscr{C}_{h0}$ denote the class of all pairs $(K \times [0, 1], K \times \{0\})$ where $K \in \mathscr{C}$.

(5.9) Example: A map $f: A \to B$ between compacta is a shape fibration [25] iff f is an $\mathscr{S}_{h0}$-e-bundle.

Proof: Consider A and B as Z-sets in Q and choose a *-extension $F: Q \to Q$ of f. Pick a decreasing sequence $A_1 \supset A_2 \supset \cdots$ of compact ANR neighborhoods of A in Q and a decreasing sequence $B_1 \supset B_2 \supset \cdots$ of compact ANR neighborhoods of B in Q such that $A = \bigcap_{i>0} A_i$, $B = \bigcap_{i>0} B_i$, and $F(A_i) \subset B_i$ for every $i > 0$. Clearly, the statement that F has the $\mathscr{S}_{h0}$-e-lifting property near (A, B) is equivalent to the statement that the level map $\underline{f} = \{F \mid A_i\}: \{A_i\} \to \{B_i\}$ between ANR-sequences $\{A_1 \leftarrow A_2 \leftarrow \cdots\}$ and $\{B_1 \leftarrow B_2 \leftarrow \cdots\}$ satisfies the AHLP with respect to the class $\mathscr{S}$ [25].

The next lemma gives additional examples of $\mathscr{C}_p$-e-bundles. In view of Remark (5.9), it includes as a special case the statement that a refinable map between ANR's is a fine homotopy equivalence [20], [23].

(5.10) Lemma: *An AI map $f: A \to B$ of a compactum A onto a $\mathscr{C}_p$-e-movable compactum B is a $\mathscr{C}_p$-e-bundle.*

PROOF: Regard A and B as Z-sets in Q and let $F:Q\to Q$ be a *-extension of f. We shall prove that F has the $\mathscr{C}_p$-e-lifting property near (A, B). Let a compact ANR neighborhood U' of A in Q and an $\epsilon>0$ be given. Pick a compact ANR neighborhood V of B in Q such that $\mathscr{C}_p^{\epsilon/3}(Q, V; B)$ holds. Let $\eta\in\Gamma(V, \epsilon/3)$ and put $\delta=\eta/2$. Then we pick maps $h:B\to A$ and $g:A\to B$ with $f\circ h$ $(\epsilon/3)$-close to id_B, with g δ-close to f, and with $h\circ g$ ϵ-close to id_A. Let $H:Q\to Q$ and $G:Q\to Q$ be extensions of h and g, respectively. Then select a neighborhood W of B in Q and a neighborhood $\tilde V$ of A in Q such that $G\mid\tilde V$ is δ-close to $F\mid\tilde V$, $H(W)\subset U'$, $F\circ H\mid W$ is $(\epsilon/3)$-close to id_W, and $H\circ G\mid\tilde V$ is ϵ-close to $\mathrm{id}_{\tilde V}$. Finally, pick a neighborhood W_0 of B in Q, $W_0\subset W\cap V$, with respect to W using $\mathscr{C}_p^{\epsilon/3}(Q, V; B)$ and put $V'=\tilde V\cap G^{-1}(W_0)$.

Consider a pair $(K, K_0)\in\mathscr{C}_p$ and maps $\psi_0:K_0\to V'$ and $\varphi:K\to V$ with $F\circ\psi_0$ δ-close to $\varphi\mid K_0$. Since $F\circ\psi_0$ and $G\circ\psi_0$ are δ-close maps, there is an $(\epsilon/3)$-homotopy in V joining $\varphi\mid K_0$ and $G\circ\psi_0$. Hence, by the homotopy extension theorem, there is a map $\varphi':K\to V$ which is $(\epsilon/3)$-close to φ and which agrees with $G\circ\psi_0$ on K_0. Observe that $G\circ\psi_0(K_0)\subset W_0$. The choice of V and W_0 implies that φ' is $(\epsilon/3)$-close to a map $\varphi'':K\to W$ where $\varphi''\mid K_0=G\circ\psi_0$. Let $\psi=H\circ\varphi''$. Since $F\circ\psi=F\circ H\circ\varphi''$ is $(\epsilon/3)$-close to φ'', $F\circ\psi$ is ϵ-close to φ. On the other hand, $\psi\mid K_0=H\circ\varphi''\mid K_0=H\circ G\circ\psi_0$ is ϵ-close to ψ_0 because $H\circ G\mid V'$ is ϵ-close to $\mathrm{id}_{V'}$.

(5.11) LEMMA: *Let $f:A\to B$ be a $\mathscr{C}_p$-e-bundle between Z-sets in Q and let $F:Q\to Q$ be a *-extension of f. If the fundamental sequence $\underline{f}=\{f_k=F, A, B\}_{Q,Q}$ is the $\mathscr{C}_p''$-e-surjection, then $\underline{f}$ is also both the $\mathscr{C}_p'$-e-surjection and the $\mathscr{C}_p$-e-surjection.*

PROOF: Let a compact ANR neighborhood U' of A in Q and an $\epsilon>0$ be given Let $\eta=\min\{\epsilon/2, \Gamma(U', \Lambda(F, \epsilon/2))\}$. Pick a neighborhood V of B in Q, a neighborhood V' of A in Q, $V'\subset U'$, and a $\gamma>0$ with respect to U' and η using the fact that F has the $\mathscr{C}_p$-e-lifting property near (A, B). Let W' be an arbitrary neighborhood of A in Q and let $\delta>0$. Choose a neighborhood W_0 of B in Q, $W_0\subset V$, and a neighborhood Z' of A in Q, $Z'\subset W'\cap V'$, such that for every $\mathscr{C}_p''$-map $\varphi_0:K_0\to W_0$ there is a map $\psi_0:K_0\to Z'$ with $F\circ\psi_0$ $\min\{\gamma, \delta\}$-close to φ_0.

Consider a $\mathscr{C}_p$-map $\varphi:(K, K_0)\to(V, W_0)$. The restriction $\varphi_0=\varphi\mid K_0$ is a $\mathscr{C}_p''$-map into W_0. Let ψ_0 be chosen as above. Since φ_0 and $F\circ\psi_0$ are γ-close, there is a map $\psi:K\to U'$ such that $\psi\mid K_0$ is η-close to ψ_0 and $F\circ\psi$ is η-close to φ. Using the homotopy extension theorem, we see that there is a $\Lambda(F, \epsilon/2)$-homotopy $H_t:K\to U'$, $0\leqq t\leqq 1$, with

$H_0 = \psi$, and $H_1 \mid K_0 = \psi_0$. But, then $F \circ H_t$ is an $(\epsilon/2)$-homotopy between $F \circ \psi$ and $F \circ H_1$ with $F \circ H_1 \mid K_0 = F \circ \psi_0$. Hence, $H_1 : (K, K_0) \to (U', W')$ is a $\mathscr{C}_p$-map such that $F \circ H_1 \mid K_0$ is δ-close to φ_0 and $F \circ H_1$ is ϵ-close to φ.

The proof that $\underline{f}$ is the $\mathscr{C}'_p$-e-surjection is left to the reader.

REMARK: Observe that, in view of (5.6), every retraction onto an ANR is a $\mathscr{C}_p$-e-surjection for every class $\mathscr{C}_p$. Hence, the point preimages of $\mathscr{C}_p$-e-surjections can be quite arbitrary (in particular, they need not have trivial shape). This, however, is not true for $\mathscr{C}_p$-e-bundles.

A simpler way of recognizing $\mathscr{C}_p$-e-surjections is offered in the next lemma.

(5.12) LEMMA: *Let A and B be compacta lying in compact ANR's P and R, respectively, let $f : A \to B$ be a map, and let $f^* : P \to R$ be a *-extension of f. If f^* is ARI, then every fundamental sequence $\underline{f} = \{f_k, A, B\}$ generated by the map f is a $\mathscr{C}_p$-e-surjection for every class $\mathscr{C}_p$.*

PROOF: By (5.5)(e), it suffices to construct a fundamental sequence $\underline{f}$ generated by f which is a $\mathscr{C}_p$-e-surjection. Let $F = Cf^* : CP \to CR$ be the cone of the map f^* and identify A and B with $A \times \{0\} \subset M = CP = P \times [0, 1]/P \times \{1\}$ and $B \times \{0\} \subset N = CR = R \times [0, 1]/R \times \{1\}$, respectively. Then F is an ARI *-extension of f and M and N are compact AR's. We claim that $\underline{f} = \{f_k = F, A, B\}_{M,N}$ is a $\mathscr{C}_p$-e-surjection.

Let a neighborhood U' of A in M and an $\epsilon > 0$ be given. Since F is ARI and $F^{-1}(B) = A$, there is a decreasing sequence $V_1 \supset V_2 \supset \cdots$ of neighborhoods of B in N, a decreasing sequence $U' = V'_1 \supset V'_2 \supset \cdots$ of neighborhoods of A in M, and a sequence of maps $G_i : N \to M$ $(i = 1, 2, \ldots)$ such that $B = \cap_{i>0} V_i$, $A = \cap_{i>0} V'_i$, $F^{-1}(V_i) \subset V'_i$, $G_j(V_i) \subset V'_i$, and $F \circ G_j$ is $(\epsilon/2^i)$-close to id_N for all $j \geqq i$ and all $i > 0$. Then $V = V_1$ is a neighborhood of B in N that we are looking after.

Indeed, let W' be a neighborhood of A in M and let $\delta > 0$. Pick an index k such that V'_k is contained in W' and $\epsilon/2^k < \delta$. Put $W_0 = V_k$.

For a $\mathscr{C}_p$-map $\varphi : (K, K_0) \to (V, W_0)$, let $\tilde{\varphi} = G_k \circ \varphi : (K, K_0) \to (U', W')$. Then $F \circ \tilde{\varphi} = F \circ G_k \circ \varphi$ is δ-close to φ.

(5.13) THEOREM: *If $\underline{f} = \{f_k, A, B\}_{Q,Q}$ is a $\mathscr{C}_p$-e-surjection and A is a $\mathscr{C}_p$-e-movable compactum, then B is also $\mathscr{C}_p$-e-movable.*

PROOF: For a given $\epsilon > 0$, select a neighborhood Z' of A in Q, an

$\eta > 0$, and an index k_0 such that $\eta \in \Lambda(f_k \mid Z', \epsilon/2)$ for all $k \geqq k_0$. Then we pick a neighborhood U' of A in $Q, U' \subset Z'$, satisfying (3.2)(iii) with respect to η. Next, we choose a neighborhood V of B in Q as in the definition (5.4) but with respect to $\epsilon/2$. We claim that V has the property described in (3.2)(iv).

Indeed, let W be an arbitrary neighborhood of B in Q and let $\delta > 0$. Choose a neighborhood W'_* of A in Q and an index $k_1 \geqq k_0$ such that $f_k(W'_*) \subset W$ for all $k \geqq k_1$. Then take a neighborhood W' of A in Q, $W' \subset U' \cap W'_*$, such that every $\mathscr{C}_p$-map $\psi:(K, K_0) \to (U', W')$ is η-close to a map $\psi^*: K \to W'_*$ which agrees with ψ on K_0. Finally, select a neighborhood W_0 of B in Q, $W_0 \subset V \cap W$, and a $k_2 \geqq k_1$ as in (5.4) (using the way in which V was chosen).

Consider a $\mathscr{C}_p$-map $\varphi:(K, K_0) \to (V, W_0)$ and an index $k \geqq k_2$. The choice of V, W_0, and k_2 implies that there is a map $\psi:(K, K_0) \to (U', W')$ with $f_k \circ \psi$ $(\epsilon/2)$-close to φ and $f_k \circ \psi \mid K_0$ δ-close to $\varphi \mid K_0$. But, the map ψ is η-close to a map $\psi^*: K \to W'_*$ which agrees with ψ on K_0. Then $f_k \circ \psi^*$ is $(\epsilon/2)$-close to $f_k \circ \psi$ and $f_k \circ \psi^* \mid K_0 = f_k \circ \psi \mid K_0$. Hence, $f_k \circ \psi^*: K \to W$ is ϵ-close to φ and $f_k \circ \psi^* \mid K_0$ is δ-close to $\varphi \mid K_0$.

(5.14) Corollary: *Let $f: A \to B$ be a surjective map between Z-sets in Q, let $F: Q \to Q$ be a *-extension of f, and let $\underline{f} = \{f_k = F, A, B\}_{Q,Q}$ be a fundamental sequence generated by f.*

(a) *If A is an LC^n compactum and f is a Σ^{n-1}-trivial map, then B is also an LC^n compactum* [21].

(b) *If A is $\mathscr{C}_p$-e-movable and f is a HSE, then B is also $\mathscr{C}_p$-e-movable. In particular, if A is an ANR and f is a cell-like map, then B is an ANR iff f is a HSE* [22].

(c) *If A is $\mathscr{C}_p$-e-movable and f is ARI, then B will be $\mathscr{C}_p$-e-movable iff $\underline{f}$ is a $\mathscr{C}_p$-e-surjection. In particular, if A is an ANR and f is ARI, then B will be an ANR iff $\underline{f}$ is an $\mathscr{S}_p$-e-surjection.*

(d) *If A is an ANR and f is ARI, then B will be an ANR iff there are ANR's M and N in Q containing A and B, respectively, and an ARI *-extension $F: M \to N$ of f.*

Proof: (a) follows from (5.7), (5.11), and (5.13); the first statement in (b) follows from (5.8), (5.11), and (5.13) and the second requires also (5.5)(c); (c) follows from (5.6) and (5.13); and (d) follows from (5.12) and (5.13).

The question as to which maps preserve (internal) $\mathscr{C}$-e-calmness seems to be even more difficult. At present we can prove only the following.

(5.15) THEOREM: (a) *If A is an (internally) $\mathscr{C}$-e-calm compactum and B is a retract of A, then B is also (internally) $\mathscr{C}$-e-calm.*

(b) *If A is an (internally) $\mathscr{C}_p$-e-movable compactum and B is a retract of A, then B is also (internally) $\mathscr{C}_p$-e-movable.*

PROOF: A routine proof is left to the reader.

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(Oblatum 8-VIII-1980 & 8-I-1981 & 27-III-1981)

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