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THE CHARACTERIZATION OF STRICTLY PARABOLIC SPACES

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*Dedicated to
the memory of
Aldo Andreotti*

1. Introduction

As shown in [16], a strictly parabolic *manifold* of dimension m is biholomorphically isometric either to $\mathbf{C}^m$ or to a ball in $\mathbf{C}^m$. Here we will prove that a strictly parabolic *complex space* is biholomorphically isometric either to an affine algebraic cone or to a truncated affine algebraic cone.

Let M be a locally compact Hausdorff space. Let τ be a non-negative, continuous function on M. Define

$$M_* = \{x \in M \mid \tau(x) > 0\} \tag{1.1}$$

and $\Delta = \operatorname{supp} \sqrt{\tau} \leq +\infty$. For each $r \geq 0$, define

$$M[r] = \{x \in M \mid \tau(x) \leq r^2\} \quad M(r) = \{x \in M \mid \tau(x) < r^2\} \tag{1.2}$$

$$M\langle r\rangle = \{x \in M \mid \tau(x) = r^2\} = M[r] - M(r). \tag{1.3}$$

Then τ is said to be an *exhaustion* with *maximal radius* Δ if and only if $\sqrt{\tau} < \Delta$ on M and if $M[r]$ is compact for every $r \in \mathbf{R}$ with $0 \leq r < \Delta$. Here we call $M[0]$ the *center* of τ. Also $M[r]$ and $M(r)$ are called the *closed* and *open pseudo-balls* of radius r of τ and $M\langle r\rangle$ is the *pseudosphere* of radius r of τ.

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Let M be a (reduced) complex space of pure dimension m. Let $\mathfrak{R}(M)$ be the set of regular points of M and let $\mathfrak{S}(M)$ be the set of singular points of M. The exterior derivative splits into $d = \partial + \bar{\partial}$ and twists to $d^c = (i/4\pi)(\bar{\partial} - \partial)$. A non-negative function τ of class C^∞ on M with $M_* \neq \emptyset$ is said to be *weakly parabolic* on M if and only if

$$(1.4) \qquad dd^c \log \tau \geq 0 \qquad (dd^c \log \tau)^m \equiv 0$$

on M_*. A weakly parabolic function τ is said to be *parabolic* if $(dd^c\tau)^m \not\equiv 0$ on each branch of M. If M is a *complex manifold*, then a weakly parabolic function τ on M is said to be *strictly parabolic* on M if and only if $dd^c\tau > 0$ on M.

If M is a *complex space*, a weakly parabolic function is said to be *strictly parabolic* on M, if τ is strictly parabolic on $\mathfrak{R}(M)$ and if for every point $b \in \mathfrak{S}(M)$ there exists a biholomorphic map $\mathfrak{p}: U \to U'$ of an open neighborhood U of b onto an analytic subset U' of an open subset G of $\mathbf{C}^n$ and if there exists a non-negative function $\tilde{\tau}$ of class C^∞ on G such that the following conditions are satisfied.

1. On U we have $\tau = \tilde{\tau} \circ \mathfrak{p}$.
2. On G we have $dd^c\tilde{\tau} > 0$.
3. For each $p \in U \cap M_*$ there exists an open neighborhood V_p on $\mathfrak{p}(p)$ in G such that $\tilde{\tau} > 0$ on V_p and such that $dd^c \log \tilde{\tau} \geq 0$ on V_p.

We call $\mathfrak{p}: U \to U'$: a *chart* of M at b and $\tilde{\tau}$ a *strictly parabolic extension of τ at b*. Our conditions (1)–(3) are mild stability requirements.

(M, τ) is said to be a *strictly parabolic space of dimension m and of maximal radius Δ* and τ is said to be a *strictly parabolic exhaustion of maximal radius Δ* of M if M is an irreducible complex space of dimension m, if τ is strictly parabolic on M and if τ is an exhaustion of M with maximal radius Δ. If, in addition, M is a complex manifold, we call (M, τ) a *strictly parabolic manifold.*

On $\mathbf{C}^n$, define a norm by $\|z\|^2 = |z_1|^2 + \cdots + |z_n|^2$ if $z = (z_1, \ldots, z_n)$. If $0 < \Delta \leq \infty$, define $\mathbf{C}^n(\Delta) = \{\mathfrak{z} \in \mathbf{C}^n \mid \|\mathfrak{z}\| < \Delta\}$. Take $m > 0$ and define $\tau_0: \mathbf{C}^m \to \mathbf{R}$ by $\tau_0(z) = \|z\|^2$. Then $(\mathbf{C}^m(\Delta), \tau_0)$ is a strictly parabolic manifold of dimension m and of maximal radius Δ. In [16], the following classification theorem was proved.

THEOREM I: *Let (M, τ) be a strictly parabolic manifold of dimension m and of maximal radius Δ. Then there exists a biholomorphic map $h: \mathbf{C}^m(\Delta) \to M$ such that $\tau_0 = \tau \circ h$.*

Thus h is a biholomorphic isometry. Originally an additional requirement was needed [17], which was eliminated by Dan Burns.

If M is permitted to have singularities, more examples of strictly parabolic spaces exist. An analytic subset K of $\mathbf{C}^n$ is said to be an *affine algebraic cone* with vertex 0 in $\mathbf{C}^n$ if $\mathbf{C}z \subseteq K$ for every $z \in K$. Let K be an irreducible affine algebraic cone of dimension m with vertex 0 in $\mathbf{C}^n$. Define $\tau_0: K \to \mathbf{R}$ by $\tau_0(z) = \|z\|^2$ for all $z \in K$. Take Δ with $0 < \Delta \leq \infty$ and define $K(\Delta) = K \cap \mathbf{C}^n(\Delta)$. Then $(K(\Delta), \tau_0)$ is a strictly parabolic space of dimension m and maximal radius Δ.

THEOREM II: *Let (M, τ) be a strictly parabolic space of dimension m and maximal radius Δ. Then there exists an irreducible, affine algebraic cone K of dimension m with vertex* 0 *in* $\mathbf{C}^n$ for some $n \geq m$ *and a biholomorphic map $h: K(\Delta) \to M$ such that $\tau_0 = \tau \circ h$.*

Thus the affine algebraic cones and truncated cones are the only strictly parabolic spaces up to a biholomorphic isometry. In Theorem II we show first that $M[0]$ consists of one and only one point O_M. Then $\mathbf{C}^n$ can be taken as the holomorphic tangent space $\mathfrak{T}$ at O_M and K as the Whitney tangent cone of M at O_M in $\mathfrak{T}$. Then h is the restriction of the exponential map from $\mathfrak{T}$ to the cone.

The proof of Theorem I given in [16] does not extend directly to Theorem II because the singularities of M provide considerable difficulties. Extensive changes have to be made. The proof of Theorem II is based on the notions of vector fields on complex spaces and their integral curves. Since no satisfactory explanation seems to exist in the literature, these concepts are introduced in section 2 and their required properties are proved there.

2. Vector fields and integral curves

(a) *Charts.* Let M be a (reduced) complex space. Let $\mathfrak{R}(M)$ be the set of regular points of M. Let $\mathfrak{S}(M)$ be the set of singular points of M. A holomorphic map $\mathfrak{p}: U \to G$ of an open subset $U \neq \emptyset$ of M into a pure dimensional complex manifold G is said to be a *chart* of M if and only if $U' = \mathfrak{p}(U)$ is analytic in G and if $\mathfrak{p}: U \to U'$ is biholomorphic. If $a \in U$, then $\mathfrak{p}$ is said to be a *chart at a*. There is a chart at every point of M. If $U = U'$ is identified, such that $\mathfrak{p}: U \to U'$ is the identity, then the inclusion $\mathfrak{p}: U \to G$ is called an *embedded chart* and we also write $U \subseteq G$. If $\mathfrak{p}: U \to G$ is a chart and if G is an open subset of $\mathbf{C}^n$, then $\mathfrak{p} = (p^1, \ldots, p^n)$ where each $p^j: U \to \mathbf{C}$ is a holomorphic function. Then $p^1, \ldots, p^n$ are called *embedding coordinates* of M on U.

A chart $\mathfrak{p}: U \to G$ is called a *patch*, if $U' = \mathfrak{p}(U)$ is open in G.

W.l.o.g., we can assume that $U' = G$. If $U = U' = G$ are identified, such that $\mathfrak{p}$ becomes the identity, then U is called an *embedded patch*. If $\mathfrak{p}: U \to G$ is a patch and if G is open in $\mathbf{C}^n$, then $\mathfrak{p} = (p^1, \ldots, p^n)$ where each $p^j: U \to \mathbf{C}$ is holomorphic. Then $p^1, \ldots, p^n$ are called *coordinates* of M on U. There exists a patch at $a \in M$ if and only if a is a regular point of M.

Take $a \in M$. Then $e_a = \operatorname{Min}\{\dim G \mid \mathfrak{p}: U \to G \text{ chart at } a\}$ is called the embedding dimension at a. Obviously $e_a \geq \dim_a G$. A chart $\mathfrak{p}: U \to G$ at $a \in M$ is said to be *neat* at a if and only if $\dim G = e_a$. Let $\mathfrak{O}_a$ be the ring of germs of local holomorphic functions at $a \in M$. Let $\mathfrak{m}$ be the maximal ideal in $\mathfrak{O}_a$. Then $\mathfrak{T}_a = \mathfrak{m}/\mathfrak{m}^2$ is a vector space of dimension e_a over $\mathbf{C}$ called the *holomorphic tangent space* of M at a. There exists a neat chart $\mathfrak{p}: U \to G$ at a where G is an open subset of $\mathfrak{T}_a$. If $\mathfrak{p}: U \to G$ and $\mathfrak{q}: V \to H$ are neat charts at a, then there exist open neighborhoods W of a in $U \cap V$ and G_0 of $\mathfrak{p}(W)$ in G and H_0 of $\mathfrak{q}(W)$ in H and there exists a biholomorphic map $f: G_0 \to H_0$ such that $\mathfrak{q} = f \circ \mathfrak{p}$ on W. If $\mathfrak{p}: U \to G$ is a chart a, then there exist open neighborhoods V of a in M and G_0 of $\mathfrak{p}(V)$ in G and a smooth, pure e_a-dimensional complex submanifold E in G_0 such that E is closed in G_0 such that $\mathfrak{p}(V) \subseteq E$ and such that $\mathfrak{p}: V \to E$ is a neat chart at a.

The transition from one patch to another is rather simple. However the transition from one chart to another chart is more complicated. Suppose that $\mathfrak{p}: U \to G$ and $\mathfrak{q}: V \to H$ and $\mathfrak{r}: W \to N$ are charts at a where $\mathfrak{q}$ is neat at a. Then $\dim H = e_a$ and $\dim G = n \geq e_a$ and $\dim H = p \geq e_a$. Define $s = n - e_a \geq 0$ and $q = p - e_a \geq 0$. Then we can construct the following *transition diagram* (2.1), where

$$
\begin{tikzcd}
F & & V_0 \arrow[ll, "\mathfrak{r}_0"'] \arrow[rr, "\mathfrak{p}_0"] \arrow[d, "\mathfrak{q}_0"] & & E \arrow[d, "j"] \\
N_0 \arrow[from=u, "k"'] & H_0 \times D \arrow[l, "\delta"'] & H_0 \arrow[l, "\kappa"'] \arrow[r, "\iota"] \arrow[ull, "\beta"] \arrow[urr, "\alpha"'] & H_0 \times B \arrow[r, "\gamma"] & G_0
\end{tikzcd}
\tag{2.1}
$$

(1) V_0 is an open neighborhood of a in M with $V_0 \subseteq U \cap V \cap W$.

(2) H_0 is an open neighborhood of $\mathfrak{q}(V_0)$ in H such that $\mathfrak{q}(V_0) = H_0 \cap \mathfrak{q}(V)$ is analytic in H_0. Then $\mathfrak{q}: V_0 \to H_0$ is a neat chart a.

(3) G_0 is an open neighborhood of $\mathfrak{p}(V_0)$ in G such that $\mathfrak{p}(V_0) = G_0 \cap \mathfrak{p}(V)$ is analytic in G_0. Then $\mathfrak{p}: V_0 \to G_0$ is a chart at a.

(4) N_0 is an open neighborhood of $\mathfrak{r}(V_0)$ in N such that $\mathfrak{r}(V_0) = N_0 \cap \mathfrak{r}(V)$ is analytic in N_0. Then $\mathfrak{r}: V_0 \to N_0$ is a chart at a.

(5) E is a smooth e_a-dimensional complex submanifold of G_0 such that E is closed in G_0. Moreover $\mathfrak{p}(V_0)$ is contained and analytic in E and the restriction $\mathfrak{p}_0: V_0 \to E$ of $\mathfrak{p}$ is a neat chart at a.

(6) F is a smooth e_a-dimensional complex submanifold of N_0 such that F is closed in N_0. Moreover $\mathfrak{r}(V_0)$ is contained and analytic in F and the restriction $\mathfrak{r}_0: V_0 \to F$ of $\mathfrak{r}$ is a neat chart at a.

(7) $j: E \to G_0$ is the inclusion map. Then $\mathfrak{p} = j \circ \mathfrak{p}_0: V_0 \to G_0$.

(8) $k: F \to N_0$ is the inclusion map. The $\mathfrak{r} = k \circ \mathfrak{r}_0: V_0 \to N_0$.

(9) $\alpha: H_0 \to E$ is a biholomorphic map such that $\mathfrak{p}_0 = \alpha \circ \mathfrak{q}_0$.

(10) $\beta: H_0 \to F$ is a biholomorphic map such that $\mathfrak{r}_0 = \beta \circ \mathfrak{q}_0$.

(11) B is an open neighborhood of $0 \in \mathbf{C}^s$ and $\iota: H_0 \to H_0 \times B$ is defined by $\iota(x) = (x, 0)$ where $0 \in \mathbf{C}^s$ and $x \in H_0$.

(12) D is an open neighborhood of $0 \in \mathbf{C}^q$ and $\kappa: H_0 \to H_0 \times D$ is defined by $\kappa(x) = (x, 0)$ where $0 \in \mathbf{C}^q$ and $x \in H_0$.

(13) $\gamma: H_0 \times B \to G_0$ is a biholomorphic map such that $\gamma \circ \iota = j \circ \alpha$.

(14) $\delta: H_0 \times D \to N_0$ is a biholomorphic map such that $\delta \circ \kappa = k \circ \beta$.

If only $\mathfrak{p}: U \to G$ and $\mathfrak{q}: V \to H$ are given, take $\mathfrak{r} = \mathfrak{p}$, $W = U$ and $N = G$. We obtain the diagram

$$
\begin{tikzcd}
V_0 \arrow[rr, "\mathfrak{p}_0"] \arrow[dd, "\mathfrak{q}_0"'] & & E \arrow[dd, "j"] \\
& & \\
H_0 \arrow[r, "\iota"] \arrow[uurr, "\alpha"] & H_0 \times B \arrow[r, "\gamma"] & G_0
\end{tikzcd}
\tag{2.2}
$$

such that (1), (2), (3), (5), (7), (9), (11) and (13) hold.

(*b*) *Maps of class* C^k. For $0 \leq k \leq \infty$, C^k means k-times continuously differentiable, for $k = \rho$, C^ρ means real analytic, for $k = \omega$, C^ω means holomorphic.

Let M and N be complex spaces. A map $f: M \to N$ is said to be *of class* C^k if and only if for each $a \in M$ there are charts $\mathfrak{p}: U \to G$ of M and a and $\mathfrak{q}: V \to H$ of N at $f(a)$ such that $f(U) \subseteq V$ and such that there exists a map $\tilde{f}: G \to H$ of class C^k such that $\tilde{f} \circ \mathfrak{p} = \mathfrak{q} \circ f$ on

U. Let $f: M \to N$ be a map of class C^k. Let $\mathfrak{p}: U \to G$ be a chart of M at a and let $\mathfrak{q}: V \to H$ be a chart of N at $f(a)$. Then there exist open neighborhoods U_0 of a in U, G_0 of $\mathfrak{p}(a)$ in G and a map $\tilde{f}: G_0 \to H$ of class C^k such that $\mathfrak{p}(U_0) \subseteq G_0$ and $\tilde{f} \circ \mathfrak{p} = \mathfrak{q} \circ f$ on U_0.

Let M be a complex space. Let N be a differentiable manifold of class C^∞. Take $0 \leq k \leq \infty$. A map $f: M \to N$ is said to be *of class C^k* if and only if for each $a \in M$, there is an open neighborhood U of a and a chart $\mathfrak{q}: V \to H$ of N at $f(a)$ such that $f(U) \subseteq V$ and such that map of class C^k. Let $\mathfrak{p}: U \to G$ be a chart of M at a. Then there exist open neighborhoods U_0 of a in U and G_0 of $\mathfrak{p}(a)$ in G and a map $\tilde{f}: G_0 \to N$ of class C^k such that $\mathfrak{p}(U_0) \subseteq G_0$ and $\tilde{f} \circ \mathfrak{p} = f$ on U_0.

Let M be a differentiable manifold of class C^∞. Let N be a complex space. Take $0 \leq k \leq \infty$. A map $f: M \to N$ is said to be *of class C^k* if and only if for each $a \in M$, there is an open neighborhood U and a chart $\mathfrak{q}: V \to H$ of N at $f(a)$ such that $f(U) \subseteq V$ and such that $\mathfrak{q} \circ f: U \to H$ is of class C^k. Let $f: M \to N$ be a map of class C^k. Take $a \in M$ and let $\mathfrak{q}: V \to H$ be a chart of N at $f(a)$ such that $f(U) \subseteq V$, then $\mathfrak{q} \circ f: U \to H$ is of class C^k.

In any of these cases, the composition of maps of class C^k is a map of class C^k. If $f: M \to N$ is bijective and if f and f^{-1} are of class C^k, then f is said to be a diffeomorphism of class C^k. For more information on maps, functions and differential forms of class C^k on complex spaces see Tung [19].

(*c*) *Vector fields.* Let M be a complex manifold. Let $T(M)$ be the real tangent bundle of M. Then $T^c(M) = T(M) \oplus iT(M)$ is the complexified tangent bundle. Let $\mathfrak{T}(M)$ be the holomorphic tangent bundle and let $\bar{\mathfrak{T}}(M)$ be the conjugate holomorphic tangent bundle. Then

$$T^c(M) = \mathfrak{T}(M) \oplus \bar{\mathfrak{T}}(M) \tag{2.3}$$

$$\eta_0: T^c(M) \to \mathfrak{T}(M) \qquad \eta_1: T^c(M) \to \bar{\mathfrak{T}}(M) \tag{2.4}$$

are the projections which restrict to bundle isomorphisms $\eta_0: T(M) \to \mathfrak{T}(M)$ and $\eta_1: T(M) \to \bar{\mathfrak{T}}(M)$ over $\mathbf{R}$. A bundle isomorphism $J: T^c(M) \to T^c(M)$ over $\mathbf{C}$ called the associated almost complex structure is defined such that $J \mid \mathfrak{T}(M)$ is the multiplication by i and $J \mid \bar{\mathfrak{T}}(M)$ is the multiplication by $-i$. Then $-J \circ J$ is the identity. If $x \in M$ and $v \in T^c_x(M)$ then

$$\eta_0(x) = \tfrac{1}{2}(v - iJv) \qquad \eta_1(x) = \tfrac{1}{2}(v + iJv). \tag{2.5}$$

Hence $\eta_0 \circ J = i\eta_0$ and $\eta_1 \circ J = -i\eta_1$. The sections of $T(M)$, $T^c(M)$, $\mathfrak{T}(M)$ and $\bar{\mathfrak{T}}(M)$ are called respectively real vector fields, complex vector fields, vector fields of type $(1,0)$, and vector fields of type $(0,1)$.

Let M be a complex space. Consider a vector field Y of class C^k on $\mathfrak{R}(M)$. Let $\mathfrak{p}: U \to G$ be a chart on M. A vector field $\tilde{Y}$ of class C^k on G is said to be an *extension* of Y to G if and only if

$$\tilde{Y}(\mathfrak{p}(x)) = \mathfrak{p}_*(Y(x)) \qquad \forall x \in \mathfrak{R}(U). \tag{2.6}$$

The vector field Y of class C^k on $\mathfrak{R}(M)$ is said to be a vector field of class C^k on M if and only if for every point $a \in M$ there exists a chart $\mathfrak{p}: U \to G$ at a and an extension $\tilde{Y}$ of Y to G. If Y is real, or of type $(1,0)$, or of type $(0,1)$ the extension can be taken likewise. Obviously, it suffices to require such an extension at the singular points only. If we assume that $\mathfrak{p}: U \to G$ is an embedded chart, then (2.6) reads as

$$\tilde{Y} \mid \mathfrak{R}(U) = \mathfrak{p}_*(Y \mid \mathfrak{R}(U)). \tag{2.7}$$

Obviously, if the extension to G exists, the extension is uniquely defined on $\mathfrak{R}(U)$ and by continuity on U.

LEMMA 2.1: *Let Y be a vector field of class C^k on the complex space M. Let $\mathfrak{p}: U \to G$ be a chart of M. Take any $a \in U$. Then there exists an open neighborhood G_a of $\mathfrak{p}(a)$ in G and a vector field $\tilde{Y}_a$ on G_a such that $U_a = \mathfrak{p}^{-1}(G_a)$ is an open neighborhood a and such that $\tilde{Y}_a$ is an extension of Y to G_a. Moreover, if $0 \leq k \leq \infty$, then there exists an extension $\tilde{Y}$ of Y on G.*

PROOF: Take $a \in U$. Then there exists a chart $\mathfrak{r}: W \to N$ of M at a and an extension $\hat{Y}$ of Y to N. Also select a neat chart $\mathfrak{q}: V \to H$ at a. Now, construct the transition diagram (2.1). Observe that $V_0 = \mathfrak{p}^{-1}(G_0)$. Take $G_a = G_0$; then $U_a = V_0$. Let $\pi: H_0 \times D \to H_0$ be the projection. Then $\pi \circ \kappa$ is the identity on H_0. There exists uniquely a vector field $\hat{Y}_1$ on $H_0 \times D$ such that $\delta_* \hat{Y}_1(x) = \hat{Y}(\delta(x))$ for all $x \in H_0 \times D$. Also there exists a vector field $\hat{Y}_2$ such that $\hat{Y}_2(x) = \pi_* \hat{Y}_1(\kappa(x))$ for all $x \in H_0$. There exists a vector field $\hat{Y}_3$ on $H_0 \times \{0\}$ as a subset of $H_0 \times B$ such that $\hat{Y}_3(\iota(x)) = \iota_*(\hat{Y}_2(x))$ for all $x \in H_0$. There is a vector field $\hat{Y}_4$ on $H_0 \times B$ such that $\hat{Y}_4 \mid H_0 \times \{0\} = \hat{Y}_3$. There exists a vector field $\tilde{Y}_a$ on $G_a = G_0$ such that $\tilde{Y}_a(\gamma(x)) = \gamma_* \hat{Y}_4(x)$ for all

$x \in H_0 \times B$. Take $x \in \mathfrak{R}(U_a) = \mathfrak{R}(V_0)$. Then

$$\begin{aligned}\tilde{Y}_a(\mathfrak{p}(x)) &= \tilde{Y}_a(j(\mathfrak{p}_0(x))) = \tilde{Y}_a(\gamma(\iota(\mathfrak{q}(x)))) = \gamma_*(\hat{Y}_4(\iota(\mathfrak{q}(x))))\\ &= \gamma_*(\hat{Y}_3(\iota(\mathfrak{q}(x)))) = \gamma_*\iota_*(\hat{Y}_2(\mathfrak{q}(x)))\\ &= \gamma_*\iota_*\pi_*(\hat{Y}_1(\kappa(\mathfrak{q}(x))))\\ &= \gamma_*\iota_*\pi_*\delta_*^{-1}(\hat{Y}(\delta(\kappa(\mathfrak{q}(x))))) = \gamma_*\iota_*\pi_*\delta_*^{-1}(\hat{Y}(k(\mathfrak{r}_0(x))))\\ &= \gamma_*\iota_*\pi_*\delta_*^{-1}(\hat{Y}(\mathfrak{r}(x))) = \gamma_*\iota_*\pi_*\delta_*^{-1}\mathfrak{r}_*(Y(x))\\ &= \gamma_*\iota_*\pi_*\kappa_*\mathfrak{q}_*(Y(x)) = \gamma_*\iota_*\mathfrak{q}_*(Y(x)) = j_*\mathfrak{p}_{0*}(Y(x))\\ &= \mathfrak{p}_*(Y(x)).\end{aligned}$$

Hence $\tilde{Y}_a$ is an extension of Y to G_a.

Assume that $0 \le k \le \infty$. Let P be a countable subset of U such that $\{G_a\}_{a\in P}$ is a locally finite family covering $\mathfrak{p}(U)$. Let $\{\lambda_a\}_{a\in P}$ be a partition of unity. Here $\lambda_a : G \to \mathbf{R}$ is of class C^∞ with compact support in G_a such that

$$\sum_{a\in P} \lambda_a(\mathfrak{p}(x)) = 1 \quad \text{for all } x \in U.$$

Define $\tilde{\tilde{Y}}_a = \lambda_a \tilde{Y}_a$ on G_a and $\tilde{\tilde{Y}}_a = 0$ on $G - G_a$. Then $\tilde{\tilde{Y}}_a$ is of class C^k on G. Since the covering is a locally finite family, a vector field $\tilde{Y}$ of class C^k on G is defined by

$$\tilde{Y} = \sum_{a\in P} \tilde{\tilde{Y}}_a.$$

Take $x \in \mathfrak{R}(U)$. Define $P(x) = \{a \in P \mid \mathfrak{p}(x) \in G_a\}$. Then

$$\begin{aligned}\tilde{Y}(\mathfrak{p}(x)) &= \sum_{a\in P} \tilde{\tilde{Y}}_a(\mathfrak{p}(x)) = \sum_{a\in P(x)} \lambda_a(\mathfrak{p}(x))\tilde{Y}_a(\mathfrak{p}(x))\\ &= \sum_{a\in P(x)} \lambda_a(\mathfrak{p}(x))\mathfrak{p}_*(Y(x)) = \sum_{a\in P} \lambda_a(\mathfrak{p}(x))\mathfrak{p}_*(Y(x))\\ &= \mathfrak{p}_*(Y(x)).\end{aligned}$$

Hence $\tilde{Y}$ is an extension of Y to G. Q.E.D.

If g is any function, its partial derivatives in respect to local coordinates $z^1, \ldots, z^m$ are denoted by

$$g_\mu = \frac{\partial g}{\partial z^\mu} \qquad g_{\bar\nu} = \frac{\partial g}{\partial \bar z^\nu}. \tag{2.8}$$

However not every lower index will signify a partial derivative. If so, it will be clear from the context. Einstein's summation convention will be used. Greek indices run from 1 to m and latin indices run from 1 to n.

Let Y be a vector field of class C^k on the complex space M. Let $\mathfrak{p}: U \to G$ be a chart on M. Let $\tilde{Y}$ be an extension of Y to G. Assume that G is open in $\mathbf{C}^n$. Then $\mathfrak{p} = (p^1, \ldots, p^n)$. Then functions $\tilde{Y}^j$ and $\tilde{X}^j$ of class C^k exist on G such that

$$\tilde{Y} = \tilde{Y}^j \frac{\partial}{\partial z^j} + \tilde{X}^j \frac{\partial}{\partial \bar{z}^j} \tag{2.9}$$

where $z^1, \ldots, z^n$ are the coordinate functions on $\mathbf{C}^n$. Let $\mathfrak{b}: V \to V'$ be a patch on $\mathfrak{R}(U)$ where V' is open in $\mathbf{C}^m$. Then $\mathfrak{b} = (v^1, \ldots, v^m)$. Functions Y^μ and X^μ of class C^k exist on V such that

$$Y = Y^\mu \frac{\partial}{\partial v^\mu} + X^\mu \frac{\partial}{\partial \bar{v}^\mu}. \tag{2.10}$$

Then

$$\mathfrak{p}_*(Y) = Y^\mu p^j_\mu \frac{\partial}{\partial z^j} + X^\mu \bar{p}^j_\mu \frac{\partial}{\partial \bar{z}^j} \quad \text{on } \mathfrak{p}(V). \tag{2.11}$$

Since $\mathfrak{p}_*(Y(x)) = \tilde{Y}(\mathfrak{p}(x))$ for all $x \in \mathfrak{R}(U)$, we have

$$\tilde{Y}^J \circ \mathfrak{p} = Y^\mu p^j_\mu \qquad \tilde{X}^j \circ \mathfrak{p} = X^\mu p^j_\mu \quad \text{on } V. \tag{2.12}$$

If X and Y are vector fields of class C^k on M and if f is a function of class C^k on M, then fX and $X + Y$ are vector fields of class C^k on M.

(*d*) *Integral curves.* Let Y be a real vector field of class C^∞ on a complex space M. A curve $\phi: \mathbf{R}(\alpha, \beta) \to M$ of class C^∞ with $-\infty \leq \alpha < \beta \leq \infty$ is said to be an *integral curve* of Y on M if the following condition is satisfied:

Take any $t_0 \in \mathbf{R}(\alpha, \beta)$. Then there exists a chart $\mathfrak{p}: U \to G$, a real extension vector field $\tilde{Y}$ of Y on G of class C^∞ and an interval $\mathbf{R}(\alpha_0, \beta_0)$ with $t_0 \in \mathbf{R}(\alpha_0, \beta_0) \subseteq \mathbf{R}(\alpha, \beta)$ such that $\phi(t) \in U$ for all $t \in \mathbf{R}(\alpha_0, \beta_0)$ and such that $(\mathfrak{p} \circ \phi)^\cdot(t) = \tilde{Y}(\mathfrak{p}(\phi(t)))$ for all $t \in \mathbf{R}(\alpha_0, \beta_0)$.

If $\mathfrak{p}: U \to G$ is an embedded chart, then $U \subseteq G$ and $\mathfrak{p}$ is the inclusion map. Hence we are permitted to write $\dot{\phi}(t) = \tilde{Y}(\phi(t))$.

LEMMA 2.2: *Let Y be a real vector field of class C^∞ on the complex space M. Let $\phi : \mathbf{R}(\alpha, \beta) \to M$ be an integral curve of Y. Let $\mathfrak{p} : U \to G$ be a chart on M. Let $\tilde{Y}$ be a real extension of Y on G of class C^∞. Assume that $-\infty \leq \alpha \leq \alpha_0 < \beta_0 \leq \beta \leq \infty$ is given such that $\phi(t) \in U$ for all $t \in \mathbf{R}(\alpha_0, \beta_0)$. Then*

$$(\mathfrak{p} \circ \phi)^{\cdot}(t) = \tilde{Y}(\mathfrak{p}(\phi(t))) \quad \textit{for all } t \in \mathbf{R}(\alpha_0, \beta_0). \tag{2.13}$$

PROOF: Take $t_0 \in \mathbf{R}(\alpha_0, \beta_0)$. Define $a = \phi(t_0)$. Then there exists a chart $\mathfrak{r} : W \to N$ at a and an extension $\hat{Y}$ of Y to N. Moreover, numbers α_1, β_1 exist such that $\alpha_0 \leq \alpha_1 < t_0 < \beta_1 \leq \beta_0$ and such that $(\mathfrak{r} \circ \phi)^{\cdot}(t) = \hat{Y}(\mathfrak{r}(\phi(t)))$ for all $t \in \mathbf{R}(\alpha_1, \beta_1)$. Also select a neat chart $\mathfrak{q} : V \to H$ at a. Now, construct the transition diagram (2.1). Take α_2 and β_2 such that $\alpha_1 \leq \alpha_2 < t_0 < \beta_2 \leq \beta_1$ and such that $\phi(t) \in V_0$ for all $t \in \mathbf{R}(\alpha_2, \beta_2)$. Let $\pi : H_0 \times D \to H_0$ be the projection. Then $\pi \circ \kappa$ is the identity on H_0. There exist vector fields $\hat{Y}_1$ on $H_0 \times D$, $\hat{Y}_2$ on H_0, $\hat{Y}_3$ on $H_0 \times \{0\}$, $\hat{Y}_4$ on $H_0 \times B$, $\hat{Y}_5$ on G_0 such that $\delta_* \hat{Y}_1 = \hat{Y} \circ \delta$, $\hat{Y}_2 = \pi_* \hat{Y}_1 \circ \kappa$, $\hat{Y}_3 \circ \iota = \iota_* \hat{Y}_2$, $\hat{Y}_4 \mid H_0 \times \{0\} = \hat{Y}_3$, $\hat{Y}_5 \circ \gamma = \gamma_* \hat{Y}_4$. Then $\hat{Y}_5 \circ \mathfrak{p} = \mathfrak{p}_* Y = \tilde{Y} \circ \mathfrak{p}$ on V_0. For $t \in \mathbf{R}(\alpha_2, \beta_2)$ we have

$$\begin{aligned}
\tilde{Y}(\mathfrak{p}(\phi(t))) &= \hat{Y}_5(\mathfrak{p}(\phi(t))) = \gamma_* \hat{Y}_4(\gamma^{-1}(j(\mathfrak{p}_0(\phi(t))))) \\
&= \gamma_* \hat{Y}_4(\iota(\mathfrak{q}_0(\phi(t)))) = \gamma_* \hat{Y}_3(\iota(\mathfrak{q}_0(\phi(t)))) \\
&= \gamma_* \iota_* \hat{Y}_2(\mathfrak{q}_0(\phi(t))) = \gamma_* \iota_* \pi_* \hat{Y}_1(\kappa(\mathfrak{q}_0(\phi(t)))) \\
&= \gamma_* \iota_* \pi_* (\delta^{-1})_* \hat{Y}(\delta(\kappa(\mathfrak{q}_0(\phi(t))))) \\
&= \gamma_* \iota_* \pi_* (\delta^{-1})_* \hat{Y}(k(\mathfrak{r}_0(\phi(t)))) \\
&= \gamma_* \iota_* \pi_* (\delta^{-1})_* \hat{Y}(\mathfrak{r}(\phi(t))) = \gamma_* \iota_* \pi_* (\delta^{-1})_* (\mathfrak{r} \circ \phi)^{\cdot}(t) \\
&= (\gamma \circ \iota \circ \pi \circ \delta^{-1} \circ k \circ \mathfrak{r}_0 \phi)^{\cdot}(t) \\
&= (\gamma \circ \iota \circ \pi \circ \kappa \circ \mathfrak{q}_0 \circ \phi)^{\cdot}(t) \\
&= (\gamma \circ \iota \circ \mathfrak{q}_0 \circ \phi)^{\cdot}(t) = (j \circ \mathfrak{p}_0 \circ \phi)^{\cdot}(t) = (\mathfrak{p} \circ \phi)^{\cdot}(t).
\end{aligned}$$

Since this holds for a neighborhood of any $t_0 \in \mathbf{R}(\alpha_0, \beta_0)$, the claim (2.13) is proved. Q.E.D.

LEMMA 2.3: *Let Y be a real vector field of class C^∞ on the complex space M. Let $\phi : \mathbf{R}(\alpha, \beta) \to M$ and $\psi : \mathbf{R}(\alpha, \beta) \to M$ be integral curves of Y. Assume that $t_0 \in \mathbf{R}(\alpha, \beta)$ exists such that $\phi(t_0) = \psi(t_0)$. Then $\phi = \psi$.*

PROOF: The set $C = \{t \in \mathbf{R}(\alpha, \beta) \mid \phi(t) = \psi(t)\}$ is closed in $\mathbf{R}(\alpha, \beta)$ with $t_0 \in C$. Take $t_1 \in C$. A chart $\mathfrak{p} : U \to G$ at $\phi(t_1) = \psi(t_1)$ and numbers α_0, β_0 exist such that $\alpha \leq \alpha_0 < t_1 < \beta_0 \leq \beta$, and such that $\phi(t) \in U$

and $\psi(t) \in U$ for all $t \in \mathbf{R}(\alpha_0, \beta_0)$. Also an extension $\tilde{Y}$ of Y on G exists such that $(\mathfrak{p} \circ \psi)(t) = \tilde{Y}(\mathfrak{p}(\psi(t)))$ and $(\mathfrak{p} \circ \phi)(t) = \tilde{Y}(\mathfrak{p}(\phi(t)))$ holds for all $t \in \mathbf{R}(\alpha_0, \beta_0)$. Now $\mathfrak{p}(\psi(t_1)) = \mathfrak{p}(\phi(t_1))$ with $t_1 \in \mathbf{R}(\alpha_0, \beta_0)$ implies $\mathfrak{p}(\psi(t)) = \mathfrak{p}(\phi(t))$ for all $t \in \mathbf{R}(\alpha_0, \beta_0)$. Hence $\psi(t) = \phi(t)$ for all $t \in \mathbf{R}(\alpha_0, \beta_0)$ which implies $\mathbf{R}(\alpha_0, \beta_0) \subseteq C$. The set $C \neq \emptyset$ is open and closed in $\mathbf{R}(\alpha, \beta)$. Hence $C = \mathbf{R}(\alpha, \beta)$. Q.E.D.

Let Y be a real vector field of class C^∞ on the complex space M. A map

$$\phi : \mathbf{R}(-\epsilon, \epsilon) \times W \to M \tag{2.14}$$

of class C^∞ is said to be a *local one parameter group of diffeomorphisms associated to* Y if and only if these conditions are satisfied.

(1) An open subset $W \neq \emptyset$ of M and $0 < \epsilon \leq \infty$ are given.

(2) For each $p \in W$, the curve $\phi(\square, p) : \mathbf{R}(-\epsilon, \epsilon) \to M$ is an integral curve of Y with $\phi(0, p) = p$.

(3) For each $t \in \mathbf{R}(-\epsilon, \epsilon)$ the image $W_t = \phi(t, W)$ is open and $\phi(t, \square) : W \to W_t$ is a diffeomorphism of class C^∞.

(4) If $p \in W$ and if t, s and $t + s$ belong to $\mathbf{R}(-\epsilon, \epsilon)$ and if $\phi(s, p) \in W$, then

$$\phi(t + s, p) = \phi(t, \phi(s, p)). \tag{2.15}$$

If $a \in W$, then ϕ is said to be a *local one parameter group of diffeomorphisms at* a. If $W = M$ and $\epsilon = \infty$, then $\phi : \mathbf{R} \times M \to M$ is said to be *global*.

PROPOSITION 2.4: *Let Y be a real vector field of class C^∞ on the complex space M of pure dimension m. Take $a \in M$. Then there exists a local one parameter group of diffeomorphisms associated to Y at a.*

PROOF: Take an embedded chart $\mathfrak{p} : U \to G \subseteq \mathbf{C}^n$ at a. Let $\tilde{Y}$ be an extension of Y to G. Then $a \in U \subseteq G$ and $\tilde{Y} \mid \mathfrak{R}(U) = Y \mid \mathfrak{R}(U)$. There exists an open connected neighborhood H of a in G and a number $\epsilon > 0$ such that there is a local one parameter group of diffeomorphisms

$$\phi : \mathbf{R}(-\epsilon, \epsilon) \times H \to G$$

associated to $\tilde{Y}$. An injective, local diffeomorphism

$$\Phi : \mathbf{R}(-\epsilon, \epsilon) \times H \to \mathbf{R}(-\epsilon, \epsilon) \times G$$

is defined by $\Phi(t,x)=(t,\phi(t,x))$. Hence $N=\Phi(\mathbf{R}(-\epsilon,\epsilon)\times H)$ is open and

$$\Phi:\mathbf{R}(-\epsilon,\epsilon)\times H\to N$$

is a diffeomorphism. Let H_0 be an open neighborhood of a such that $\bar{H}_0$ is compact and contained in H. Take $\epsilon_0\in\mathbf{R}(0,\epsilon)$. Then $N_0=\Phi(\mathbf{R}(-\epsilon_0,\epsilon_0)\times H_0)$ is open and $\bar{N}_0=\Phi(\mathbf{R}[-\epsilon_0,\epsilon_0]\times\bar{H}_0)$ is a compact subset of N. The set $\mathfrak{S}(U)$ of singular points of U has at most complex dimension $m-1$. Therefore $S=(\mathbf{R}(-\epsilon_0,\epsilon_0)\times\mathfrak{S}(U))\cap N_0$ has finite $(2m-1)$-dimensional Hausdorff measure. Also $T=\Phi^{-1}(S)$ has finite $(2m-1)$-dimensional Hausdorff measure. The projection $\pi:\mathbf{R}(-\epsilon_0,\epsilon_0)\times H_0\to H_0$ is a Lipschitz map with Lipschitz constant 1. By Federer [6], 2.10.11 $\pi(T)$ has finite $(2m-1)$-dimensional Hausdorff measure in H_0. Observe that $V=H_0\cap U$ is an open neighborhood of a in M and that V has pure complex dimension m. Therefore $V_0=\mathfrak{R}(V)-\pi(T)$ is dense in V. Also $\mathbf{R}(-\epsilon_0,\epsilon_0)\times V_0$ is dense in $\mathbf{R}(-\epsilon_0,\epsilon_0)\times V$.

Take any $p\in V_0$. A number $\epsilon_1\in\mathbf{R}(0,\epsilon_0)$ and an integral curve $\psi:\mathbf{R}(-\epsilon_1,\epsilon_1)\to\mathfrak{R}(V)$ of Y exist such that $\psi(0)=p$. Then ψ is also an integral curve of the extension $\tilde{Y}$. Consequently $\phi(t,p)=\psi(t)\in\mathfrak{R}(V)\subseteq\mathfrak{R}(U)$ for all $t\in\mathfrak{R}(-\epsilon_1,\epsilon_1)$. A maximal number $\epsilon_2\in\mathbf{R}(0,\epsilon_0)$ exists such that $\phi(t,p)\in\mathfrak{R}(U)$ for all $t\in\mathbf{R}(-\epsilon_2,\epsilon_2)$. Then $0<\epsilon_1\leq\epsilon_2\leq\epsilon_0$. Assume that $\epsilon_2<\epsilon_0$. Then $\phi(\eta\epsilon_2,p)\in\mathfrak{S}(U)$ where $\eta=+1$ or $\eta=-1$. Hence

$$\Phi(\eta\epsilon_2,p)\in(\mathbf{R}(-\epsilon_0,\epsilon_0)\times\mathfrak{S}(U))\cap N_0=S.$$

Thus $(\eta\epsilon_2,p)\in T$ and $p\in\pi(T)$ against the choice of p. Therefore $\epsilon_2=\epsilon_0$ and $\phi(t,p)\in\mathfrak{R}(U)$ for all $t\in\mathbf{R}(-\epsilon_0,\epsilon_0)$ and every $p\in V_0$. Since $\phi:\mathbf{R}(-\epsilon_0,\epsilon_0)\times V\to G$ is continuous, where $\mathfrak{R}(U)\subseteq U\subseteq G$ and where U is closed in G and since $\mathbf{R}(-\epsilon_0,\epsilon_0)\times V_0$ is dense in $\mathbf{R}(-\epsilon_0,\epsilon_0)\times V$, we obtain $\phi(\mathbf{R}(-\epsilon_0,\epsilon_0)\times V)\subseteq U$. A map

$$\phi:\mathbf{R}(-\epsilon_0,\epsilon_0)\times V\to U$$

of class C^∞ is defined.

Let W be an open neighborhood of a in M such that $\bar{W}$ is a compact subset of $V=U\cap H_0$. An open subset H_1 of H_0 exists such that $W=U\cap H_1$ and such that $\bar{H}_1$ is a compact subset of H_0. Since $\phi(0,p)=p$ for all $p\in H$, a number $\epsilon_3\in\mathbf{R}(0,\epsilon_0)$ exists such that $\phi(t,p)\in H_0$ for all $t\in\mathbf{R}(-\epsilon_3,\epsilon_3)$ and $p\in H_1$. If $t\in\mathbf{R}(-\epsilon_3,\epsilon_3)$ and $p\in W$, then $\phi(t,p)\in U\cap H_0=V$.

Take $t \in \mathbf{R}(-\epsilon_3, \epsilon_3)$ and define $W_t = \phi(t, W)$. Then a bijective map $\chi : W \to W_t$ is defined by $\chi(x) = \phi(t, x)$ for all $x \in W$. Here $H_{1t} = \phi(t, H_1)$ is open in H_0 and $W_t \subset U \cap H_0 = V$. A map $\rho : V \to U$ of class C^∞ is defined by $\rho(x) = \phi(-t, x)$ for all $x \in V$. We have $W_t \subseteq U \cap H_{1t} \subseteq U \cap H_0 = V$. Take $q \in U \cap H_{1t}$. Then $q = \phi(t, x)$ for some $x \in H_1$. Since $x \in H$ since $\phi(t, x) \in H$ since $t \in \mathbf{R}(-\epsilon, \epsilon)$ and $-t \in \mathbf{R}(-\epsilon, \epsilon)$, we have $\phi(-t, q) = \phi(-t, \phi(t, x)) = x$. Hence $x = \rho(q) \in U \cap H_1 = W$. Therefore $q = \phi(t, x) \in W_t$. Consequently $W_t = U \cap H_{1t}$ is open in V and thus in M. The map $\chi : W \to W_t$ is bijective and of class C^∞. If $q \in W_t$, then $x \in W$ exists such that $q = \phi(t, x) = \chi(x)$ where $x \in H_1$; hence $x = \rho(q)$ and $x = \chi^{-1}(q)$. Consequently $\chi^{-1} = \rho \mid W_t : W_t \to W$ is bijective and of class C^∞. The map $\phi(t, \square) = \chi : W \to W_t$ is a diffeomorphism of class C^∞ for each $t \in \mathbf{R}(-\epsilon_3, \epsilon_3)$. For each $p \in W$, the curve $\phi(\square, p) : \mathbf{R}(-\epsilon_3, \epsilon_3) \to M$ is an integral curve of Y with $\phi(0, p) = p$. If $p \in W$, if t, s, and $t + s$ belong to $\mathbf{R}(-\epsilon_3, \epsilon_3)$ and if $\phi(s, p) \in W$, then $p \in H$ and $\phi(s, p) \in H$. Consequently $\phi(t + s, p) = \phi(t, \phi(s, p))$. Also W is an open neighborhood of a in M. Hence $\phi : \mathbf{R}(-\epsilon_3, \epsilon_3) \times W \to M$ is a local one parameter group of diffeomorphisms at a, associated to Y. Q.E.D.

Let Y be a real vector field on the complex space M. An integral curve $\phi : \mathbf{R}(\alpha, \beta) \to M$ of Y is said to be *maximal*, if for every integral curve $\psi : \mathbf{R}(\gamma, \delta) \to M$ of Y with $-\infty \leq \gamma \leq \alpha < \beta \leq \delta \leq \infty$ with $\psi \mid \mathbf{R}(\alpha, \beta) = \phi$ we have $\gamma = \alpha$ and $\beta = \delta$. An integral curve $\phi : \mathbf{R}(\alpha, \beta) \to M$ of Y is said to be *complete* if $\alpha = -\infty$ and $\beta = +\infty$. Obviously a complete integral curve is maximal.

LEMMA 2.5: *Let Y be a real vector field on the complex space M. Let $\phi : \mathbf{R}(\alpha, \beta) \to M$ be a maximal integral curve of Y. Let $\psi : \mathbf{R}(\gamma, \delta) \to M$ be an integral curve of Y. Assume that $t_0 \in \mathbf{R}(\alpha, \beta) \cap \mathbf{R}(\gamma, \delta)$ exists with $\phi(t_0) = \psi(t_0)$. Then $\alpha \leq \gamma < \delta \leq \beta$ and $\psi(t) = \phi(t)$ for all $t \in \mathbf{R}(\gamma, \delta)$.*

PROOF: We determine α_0, α_1, β_0, β_1 uniquely by

$$t_0 \in \mathbf{R}(\alpha_1, \beta_1) = \mathbf{R}(\alpha, \beta) \cap \mathbf{R}(\gamma, \delta) \qquad \mathbf{R}(\alpha_0, \beta_0) = \mathbf{R}(\alpha, \beta) \cup \mathbf{R}(\gamma, \delta).$$

Then $\phi \mid \mathbf{R}(\alpha_1, \beta_1)$ and $\psi \mid \mathbf{R}(\alpha_1, \beta_1)$ are integral curves of Y with $\phi(t_0) = \psi(t_0)$. Lemma 2.3 implies $\phi(t) = \psi(t)$ for all $t \in \mathbf{R}(\alpha_1, \beta_1)$. Hence one and only one integral curve $\chi : \mathbf{R}(\alpha_0, \beta_0) \to M$ of Y is defined by $\chi \mid \mathbf{R}(\alpha, \beta) = \phi$ and $\chi \mid \mathbf{R}(\gamma, \delta) = \psi$. By maximality $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Hence $\mathbf{R}(\gamma, \delta) \subseteq \mathbf{R}(\alpha, \beta)$. Q.E.D.

PROPOSITION 2.6: *Let Y be a real vector field of class C^∞ on the pure m-dimensional complex space M. Take $p \in M$ and $t_0 \in \mathbf{R}$. Then there exists one and only one maximal integral curve $\phi : \mathbf{R}(\alpha, \beta) \to M$ of Y with $t_0 \in \mathbf{R}(\alpha, \beta)$ and $\phi(t_0) = p$.*

PROOF: Proposition 2.4 implies the existence of an integral curve $\psi : \mathbf{R}(-\epsilon, \epsilon) \to M$ of Y with $0 < \epsilon \in \mathbf{R}$ and $\psi(0) = p$. An integral curve $\chi : \mathbf{R}(t_0 - \epsilon, t_0 + \epsilon) \to M$ of Y with $\chi(t_0) = p$ is defined by $\chi(t) = \psi(t - t_0)$. Let A be the set of all $a \in \mathbf{R}$ with $a \leq t_0 - \epsilon$ such that there exists an integral curve $\phi_a : \mathbf{R}(a, t_0 + \epsilon) \to M$ of Y with $\phi_a(t_0) = p$. Let B be the set of all $b \in \mathbf{R}$ with $b \geq t_0 + \epsilon$ such that there exists an integral curve $\phi_b : \mathbf{R}(t_0 - \epsilon, b) \to M$ of Y with $\phi_b(t_0) = p$. If $a \in A$ and $b \in B$, then $\phi_a \mid \mathbf{R}(t_0 - \epsilon, t_0 + \epsilon)$, $\phi_b \mid \mathbf{R}(t_0 - \epsilon, t_0 + \epsilon)$ and χ are integral curves of Y with $\phi_a(t_0) = \phi_b(t_0) = \chi(t_0) = p$. Hence $\phi_a(t) = \phi_b(t) = \chi(t)$ for all $t \in \mathbf{R}(t_0 - \epsilon, t_0 + \epsilon)$. Hence an integral curve $\phi_{ab} : \mathbf{R}(a, b) \to M$ of Y is defined by $\phi_{ab}(t) = \phi_a(t)$ if $t \in \mathbf{R}(a, t_0 + \epsilon)$ and $\phi_{ab}(t) = \phi_b(t)$ if $t \in \mathbf{R}(t_0 - \epsilon, b)$. Define $a = \inf A$ and $\beta = \sup A$. If $a \in A$, $a' \in A$ and $b \in B$ and $b' \in B$ with $\alpha \leq a' \leq a < t_0 < b \leq b' \leq \beta$. Then ϕ_{ab} and $\phi_{a'b'} \mid \mathbf{R}(a, b)$ are integral curves of Y with $\phi_{ab}(t_0) = p = \phi_{a'b'}(t_0)$. Hence $\phi_{ab}(t) = \phi_{a'b'}(t)$ for all $t \in \mathbf{R}(a, b)$. Therefore one and only one integral curve $\phi : \mathbf{R}(\alpha, \beta) \to M$ of Y exists such that $\phi \mid \mathbf{R}(a, b) = \phi_{ab}$ whenever $a \in A$ and $b \in B$. Then $\phi(t_0) = p$. If $-\infty \leq \gamma \leq \alpha < \beta \leq \delta \leq \infty$ and if $\omega : \mathbf{R}(\gamma, \delta) \to M$ is an integral curve of Y with $\omega \mid \mathbf{R}(a, b) = \phi$, then $\omega(t_0) = p$ and $\gamma \in A$ and $\delta \in B$. Hence $\alpha \leq \gamma$ and $\delta \leq \beta$ which implies $\alpha = \gamma$ and $\delta = \beta$. Hence ϕ is maximal. Let $\tilde{\phi} : \mathbf{R}(\tilde{\alpha}, \tilde{\beta}) \to M$ be a maximal integral curve of Y with $t_0 \in \mathbf{R}(\tilde{\alpha}, \tilde{\beta})$ and $\tilde{\phi}(t_0) = p$. Lemma 2.5 implies $\alpha \leq \tilde{\alpha} < \tilde{\beta} \leq \beta$ and $\tilde{\alpha} \leq \alpha < \beta \leq \tilde{\beta}$. Hence $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \beta$. Also Lemma 2.5 and Lemma 2.3 imply $\phi(t) = \tilde{\phi}(t)$ for all $t \in \mathbf{R}(\alpha, \beta) = \mathbf{R}(\tilde{\alpha}, \tilde{\beta})$. Q.E.D.

LEMMA 2.7: *Let Y be a real vector field on the pure m-dimensional complex space M. Let $\phi : \mathbf{R}(\alpha, \beta) \to M$ be a maximal integral curve of Y. Assume that there exists a compact subset K of M such that $\phi(t) \in K$ for all $t \in \mathbf{R}(\alpha, \beta)$. Then ϕ is complete*; i.e. *$\alpha = -\infty$ and $\beta = +\infty$.*

PROOF: Assume that $\beta < +\infty$. There exists a sequence $\{t_\nu\}_{\nu \in \mathbf{N}}$ such that $t_\nu \in \mathbf{R}(\alpha, \beta)$ for all $\nu \in \mathbf{N}$ and such that $t_\nu \to \beta$ and $\phi(t_\nu) \to p$ for $\nu \to \infty$. Take a local one parameter group of diffeomorphisms

$$\psi : \mathbf{R}(-\epsilon, \epsilon) \times U \to M$$

associated to Y where $0<\epsilon \in \mathbf{R}$ and $p \in U$. Let V and U_0 be open neighborhoods of p such that $\bar{U}_0$ is compact and such that

$$p \in V \subseteq \bar{V} \subset U_0 \subseteq \bar{U}_0 \subset U.$$

A number $\epsilon_0 = \mathbf{R}(0, \epsilon)$ exists such that $\alpha < \beta - \epsilon_0$ and such that

$$\bar{V} \subset U_t = \psi(t, U_0) \subseteq \psi(t, \bar{U}_0) = \bar{U}_t \subset U$$

for all $t \in \mathbf{R}(-\epsilon_0, \epsilon_0)$. Take $\lambda \in \mathbf{N}$ such that $0 > t_\lambda - \beta = r_\lambda > -\epsilon_0$ and such that $\phi(t_\lambda) \in V$. Then $\phi(t_\lambda) \in U_{r_\lambda}$. Hence $p_\lambda \in U_0$ exists such that $\phi(t_\lambda) = \psi(r_\lambda, p_\lambda)$. Hence $\phi(t) = \psi(t - \beta, p_\lambda)$ for all $t \in \mathbf{R}(\beta - \epsilon_0, \beta)$. An integral curve $\chi : \mathbf{R}(\alpha, \beta + \epsilon_0) \to M$ of Y is defined by $\chi(t) = \phi(t)$ for all $t \in \mathbf{R}(\alpha, \beta)$ and $\chi(t) = \psi(t - \beta, p_\lambda)$ for all $t \in \mathbf{R}(\beta - \epsilon_0, \beta + \epsilon_0)$. Because ϕ is maximal, this is impossible. Hence $\beta = +\infty$. Similarly, $\alpha = -\infty$ is proved. Q.E.D.

A real vector field Y on a complex space M is said to be *complete*, if there exists a global one parameter group $\phi : \mathbf{R} \times M \to M$ associated to Y.

PROPOSITION 2.8: *Let Y be a real vector field on the pure m-dimensional complex space M. Assume that for every point $p \in M$ there exists a complete integral curve $\phi_p : \mathbf{R} \to M$ with $\phi_p(0) = p$. Then a global one parameter group $\phi : \mathbf{R} \times M \to M$ associated to Y is defined by $\phi(t, p) = \phi_p(t)$ for all $(t, p) \in \mathbf{R} \times M$. In particular, Y is complete.*

PROOF: The set N of all $(t, p) \in \mathbf{R} \times M$ such that ϕ is of class C^∞ at (t, p) is open in $\mathbf{R} \times M$.

Take $p_0 \in M$. A local one parameter group $\psi : \mathbf{R}(-\epsilon, \epsilon) \times U \to M$ of diffeomorphisms associated to Y exists such that $p_0 \in U$. If $p \in U$, then $\psi(\square, p) : \mathbf{R}(-\epsilon, \epsilon) \to M$ is an integral curve of Y with $\psi(0, p) = p$. Hence $\phi(t, p) = \phi_p(t) = \psi(t, p)$ for all $t \in \mathbf{R}(-\epsilon, \epsilon)$ and for each $p \in U$. Therefore ϕ is of class C^∞ on $\mathbf{R}(-\epsilon, \epsilon) \times U$. We see that $(0, p_0) \in N$.

Define $t_0 = \sup\{t \in \mathbf{R} \mid t \geq 0 \text{ and } \mathbf{R}[0, t_0] \times \{p_0\} \subset N\}$. *Assume that* $t_0 < \infty$. Then $\phi(t_0, p_0) = q_0 \in M$. There exists a local one parameter group $\chi : \mathbf{R}(-\eta, \eta) \times Z \to M$ of diffeomorphisms at $q_0 \in Z$ associated to Y with $0 < \eta < t_0$. Let V and Z_0 be open neighborhoods of q_0 such that $\bar{Z}_0$ is compact and such that

$$q_0 \in V \subset \bar{V} \subset Z_0 \subset \bar{Z}_0 \subset Z.$$

A number $\epsilon_0 \in \mathbf{R}(0, \eta)$ exists such that

$$\bar{V} \subset Z_t = \psi(t, Z_0) \subset \psi(t, \bar{Z}_0) = \bar{Z}_t \subset Z$$

for all $t \in \mathbf{R}(-\epsilon_0, \epsilon_0)$. Take $t_1 \in \mathbf{R}(t_0 - \epsilon_0, t_0)$ such that $\phi(t_1, p_0) \in V$. Since $(t_1, p_0) \in N$, an open neighborhood W of p_0 exists such that $\{t_1\} \times W \subset N$ and such that $\phi(t_1, p) \in V$ for all $p \in W$. Define $r = t_1 - t_0$. Then $-\epsilon_0 < r < 0$. Also $\rho = \psi(r, \square): Y_0 \to Y_r$ is a diffeomorphism with $V \subseteq Z_r$. Hence

$$g = \chi^{-1} \circ \phi(t_1, \square): W \to Y_0$$

is a map of class C^∞. If $p \in W$, then $\phi(t_1, p) = \psi(t_1 - t_0, g(p))$. Hence

$$\phi(t, p) = \psi(t - t_0, g(p)) \qquad \forall t \in \mathbf{R}(t_0 - \epsilon, t_0 + \epsilon) \text{ and } p \in W.$$

Therefore ϕ is of class C^∞ on $\mathbf{R}(t_0 - \epsilon, t_0 + \epsilon) \times W$ and $\mathbf{R}(t_0 - \epsilon, t_0 + \epsilon) \times W \subseteq N$. In particular, $\mathbf{R}[0, t_0 + \epsilon] \times \{p_0\} \subseteq N$, which contradicts the definition of t_0. Therefore $t_0 = +\infty$.

We have shown that $\mathbf{R}[0, +\infty] \times M \subseteq N$. A symmetric argument shows that $\mathbf{R}(-\infty, 0] \times M \subseteq N$. Hence $\mathbf{R} \times M = N$ and $\phi: \mathbf{R} \times M \to M$ is of class C^∞.

Take $s \in \mathbf{R}$ and $p \in M$. Integral curves $\zeta: \mathbf{R} \to M$ and $\lambda: \mathbf{R} \to M$ of Y are defined by $\zeta(t) = \phi(t, \phi(s, p))$ and $\lambda(t) = \phi(t + s, p)$ for all $t \in \mathbf{R}$ with $\zeta(0) = \phi(0, \phi(s, p)) = \phi(s, p) = \lambda(0)$. Hence $\zeta = \lambda$ on $\mathbf{R}$ and $\phi(t + s, p) = \phi(t, \phi(s, p))$ for all $t \in \mathbf{R}$.

Take $t \in \mathbf{R}$. The map $\phi_t = \phi(t, \square): M \to M$ is of class C^∞ where ϕ_0 is the identity and $\phi_{-t} \circ \phi_t = \phi_0 = \phi_t \circ \phi_{-t}$. Hence $\phi_t: M \to M$ is a diffeomorphism with $\phi_t^{-1} = \phi_{-t}$. Consequently, ϕ is a global one parameter group of diffeomorphisms associated to Y. Q.E.D.

Now Lemma 2.7 and Proposition 2.8 imply

PROPOSITION 2.9: *Let Y be a real vector field of class C^∞ on a pure m-dimensional complex space M. Assume that each maximal integral curve of Y is contained in some compact subset of M. Then Y is complete.*

3. Strictly parabolic functions

Differential forms on complex spaces are explained in Tung [19]. Let M be a complex space of pure dimension M. Let τ be a non-negative function of class C^∞ on M. Define

$$(3.1)\qquad M_* = \{x \in M \mid \tau(x) > 0\}$$
$$M = M - M[0] = \{x \in M \mid \tau(X) = 0\}.$$

Assume that $M_* \neq \emptyset$. The function τ is said to be *weakly parabolic* on M if

$$(3.2)\qquad dd^c \log \tau \geq 0 \qquad (dd^c \log \tau)^m \equiv 0 \quad \text{on } M_*.$$

Hence $\log \tau$ is plurisubharmonic and satisfies the complex Monge–Ampère equation on M_*. A weakly parabolic function τ is said to be *parabolic* if $(dd^c\tau)^m \not\equiv 0$ on each branch of M. If M is a complex manifold, a weakly parabolic function τ on M is said to be *strictly parabolic* on M if $dd^c\tau > 0$ on M.

If M is a pure m-dimensional *complex space*, a weakly parabolic function τ is said to be *strictly parabolic* on M, if τ is strictly parabolic on $\mathfrak{R}(M)$ and if for every $a \in \mathfrak{S}(M)$ there exists a chart $\mathfrak{p}: U \to G$ at a and a non-negative function θ of class C^∞ on G satisfying these conditions.

1. On U we have $\tau = \theta \circ \mathfrak{p}$.
2. On G we have $dd^c\theta > 0$.
3. For each $p \in U \cap M_*$ there exists an open neighborhood V_p of $\mathfrak{p}(p)$ in G such that $\theta > 0$ and $dd^c \log \tau \geq 0$ on V_p. Here θ is called a *strictly parabolic extension* of τ to G. Note that (1) and (2) is the standard definition for $dd^c\tau$ to be positive at a, and that (3) in itself is the standard definition for $dd^c \log \tau$ to be non-negative at p. Thus we require that these extension properties are satisfied by the same function θ. Trivially, a strictly parabolic extension exists at every regular point of M.

LEMMA 3.1: *Let τ be a strictly parabolic function on a pure m-dimensional complex space M. Take $a \in M$. Let $\mathfrak{p}: U \to G$ be a chart of M at a. Then there exists an open neighborhood G_0 of a in G and a strictly parabolic extension θ of τ of G_0 for the chart $\mathfrak{p}: U_0 \to G_0$ with $\mathfrak{p}^{-1}(G_0) = U_0$.*

PROOF: There exists a chart $\mathfrak{r}: W \to N$ of M at a and a strictly parabolic extension $\hat{\theta}$ of τ to N. Select a neat chart $\mathfrak{q}: V \to H$ at a. Now construct the transition diagram (2.1). Let $\pi: H_0 \times B \to H_0$ and $\chi: H_0 \times B \to B \subseteq \mathbf{C}^s$ be the projections. Define $\tilde{\theta} = \hat{\theta} \circ \delta \circ \kappa$ on H_0 and

$\theta_0 : B \to \mathbf{R}_+$ by $\theta_0(z) = \|z\|^2$ for $z \in B$. A non-negative function θ of class C^∞ is defined on G_0 by

$$\theta = \tilde{\theta} \circ \pi \circ \gamma^{-1} + \theta_0 \circ \chi \circ \gamma^{-1}.$$

We shall see that θ is the desired extension. First we have

$$\begin{aligned} \theta \circ \mathfrak{p} = \theta \circ j \circ \mathfrak{p}_0 &= \tilde{\theta} \circ \pi \circ \iota \circ \mathfrak{q}_0 + \theta_0 \circ \chi \circ \iota \circ \mathfrak{q}_0 \\ &= \hat{\theta} \circ \delta \circ \kappa \circ \mathfrak{q}_0 + \theta_0(0) \\ &= \hat{\theta} \circ k \circ \mathfrak{r}_0 = \hat{\theta} \circ \mathfrak{r} = \tau \quad \text{on } V_0. \end{aligned}$$

Since $dd^c\hat{\theta} > 0$ on N_0, we have $dd^c\tilde{\theta} = \kappa^*\delta^*(dd^c\hat{\theta}) > 0$ on H_0. Also $dd^c\theta_0 > 0$ on B. Therefore

$$\pi^*(dd^c\tilde{\theta}) + \chi^*(dd^c\theta_0) > 0 \quad \text{on } H_0 \times B$$

$$dd^c\theta = (\gamma^{-1})^*(\pi^*(dd^c\theta) + \chi^*(dd^c\theta_0)) > 0 \quad \text{on } G_0.$$

Take $p \in V_0 \cap M_*$. Then $\hat{\theta}(\mathfrak{r}(p)) = \tau(p) > 0$. An open neighborhood V_p of $\mathfrak{r}(p)$ in N_0 exists such that $\hat{\theta} > 0$ and $dd^c \log \hat{\theta} \geq 0$ on $\hat{V}_p$. Then $\tilde{V}_p = \kappa^{-1}\delta^{-1}(\hat{V}_p)$ is open in H_0 and $\tilde{\theta} > 0$ on $\tilde{V}_p$. Also

$$dd^c \log \tilde{\theta} = dd^c \log \hat{\theta} \circ \delta \circ \kappa = \kappa^*\delta^*(dd^c \log \hat{\theta}) \geq 0 \quad \text{on } \tilde{V}_p.$$

Also $dd^c \log \theta_0 \geq 0$ on $B - \{0\}$. Define $u = \tilde{\theta} \circ \pi$ and $w = \theta_0 \circ \chi$ and $v = u + w > 0$ on $\tilde{V}_p \times B$. Here $dd^c \log u \geq 0$ on $\tilde{V}_p \times B$ and $dd^c \log w \geq 0$ on $\tilde{V}_p \times (B - \{0\})$. On $\tilde{V}_p \times (B - \{0\})$, we have

$$0 \leq u^2 dd^c \log u = u dd^c u - du \wedge d^c u$$

$$0 \leq w^2 dd^c \log w = w dd^c w - dw \wedge d^c w$$

$$\begin{aligned} 0 &\leq u^2 w^2 d \log \frac{w}{u} \wedge d^c \log \frac{w}{u} \\ &= (u dw - w du) \wedge (u d^c w - w d^c u) \\ &= u^2 dw \wedge d^c w + w^2 du \wedge d^c u - uw dw \wedge d^c u + uw du \wedge d^c w \\ &\leq u^2 w dd^c w + w^2 u dd^c u - uw(du \wedge d^c w + dw \wedge d^c u). \end{aligned}$$

By continuity, on $\tilde{V}_p \times B$ we have

$$\begin{aligned} uwv^2 dd^c \log v &= uw(vdd^c v - dv \wedge d^c v) \\ &= uw((u+w)(dd^c u + dd^c w) - (du + dw) \wedge (d^c u + d^c w)) \\ &= uw(udd^c u - du \wedge d^c u) + uw(wdd^c w - dw \wedge d^c w) \\ &\quad \times uw^2 dd^c u + wu^2 dd^c w - uw(du \wedge d^c w + dw \wedge d^c u) \\ &\geq 0. \end{aligned}$$

Hence $dd^c \log v \geq 0$ on $\tilde{V}_p \times B$. we have $\theta = u \circ \gamma^{-1} + w \circ \gamma^{-1} = v \circ \gamma^{-1}$. Hence $dd^c \log \theta = (\gamma^{-1})^*(dd^c \log v) \geq 0$ on $\gamma(\tilde{V}_p \times B) = V_p$ where V_p is an open neighborhood of $\mathfrak{p}(p)$ in G_0 with $\theta > 0$ on V_p. Therefore θ is a strictly parabolic extension of τ to G_0. Q.E.D.

LEMMA 3.2: *Let M be a complex space of pure dimension m. Let τ be a strictly parabolic function on M. Then $M[0]$ does not contain any non-empty open subset of M and $\mathfrak{R}(M_*)$ is dense in M.*

PROOF: Assume that there exists an open, non-empty subset U of M such that $U \subseteq M[0]$. Then $\mathfrak{R}(U) \neq \emptyset$ and $\tau \equiv 0$ on $\mathfrak{R}(U)$ while $dd^c \tau > 0$ on $\mathfrak{R}(U)$, which is impossible. Hence the interior of $M[0]$ is empty and M_* is dense in M. Since $\mathfrak{R}(M_*)$ is dense in M_*, the set $\mathfrak{R}(M_*)$ is dense in M. Q.E.D.

LEMMA 3.3: *Let M be a complex manifold of pure dimension m. Let τ be a positive function of class C^2 on M. Take $a \in M$. Assume that $dd^c \tau > 0$ at a. Define $\omega = (dd^c \log \tau)(a)$. Then we conclude*

(1) *ω has at most one zero eigenvalue and at least $m - 1$ positive eigenvalues.*

(2) *$\omega^m = 0$ if and only if $\omega \geq 0$ but not $\omega > 0$.*

(3) *$\omega^m = 0$ if and only if ω has exactly one zero eigenvalue.*

(4) *If $\omega \geq 0$, then $\omega^m = 0$ if and only if there exists $0 \neq v \in \mathfrak{T}_a(M)$ such that $\omega(v, \bar{v}) = 0$.*

PROOF: There exist local coordinates $z^1, \ldots, z^m$ at a such that

$$dd^c \tau(a) = \frac{i}{2\pi} \sum_{\mu=1}^{m} dz^\mu \wedge d\bar{z}^\mu$$

$$\omega = (dd^c \log \tau)(a) = \frac{i}{2\pi} \sum_{\mu=1}^{m} \lambda_\mu dz^\mu \wedge d\bar{z}^\mu.$$

Then

$$\tau(a)^2\omega = \frac{i}{2\pi}\sum_{\mu=1}^{m}\sum_{\nu=1}^{m}(\tau(a)\tau_{\mu\bar\nu}(a) - \tau_\mu(a)\tau_{\bar\nu}(a))dz^\mu \wedge d\bar z^\nu$$

$$\omega^m = \left(\frac{i}{2\pi}\right)^m m!\lambda_1 \dots \lambda_m dz^1 \wedge d\bar z^1 \wedge \cdots \wedge dz^m \wedge d\bar z^m$$

where $\tau_{\mu\bar\nu}(a) = 0$ if $\mu \neq \nu$ and $\tau_{\mu\bar\mu}(a) = 1$. Therefore

$$\tau(a)^2\lambda_\mu = \tau(a) - |\tau_\mu(a)|^2 \quad \text{for all } \mu \in \mathbf{N}[1, m]$$

$$0 = \tau_\mu(a) - \tau_{\bar\nu}(a) \quad \text{if } 1 \leq \mu < \nu \leq m.$$

(1) If $\tau_\mu(a) = 0$ for all $\mu \in \mathbf{N}[1, m]$, then $\lambda_\mu = 1/\tau(a) > 0$ for all $\mu \in \mathbf{N}[1, m]$. If $\partial\tau(a) \neq 0$, we can assume that $\tau_m(a) \neq 0$. Then $\tau_\mu(a) = 0$ for all $\mu \in \mathbf{N}[1, m-1]$ and $\lambda_\mu = 1/\tau(a) > 0$ for all $\mu \in \mathbf{N}[1, m)$. Hence there are at least $(m-1)$ positive eigenvalues and at most one eigenvalue is zero.

(2) If $\omega^m = 0$, one eigenvalue is zero and the other eigenvalues are positive. Therefore $\omega \geq 0$ but not $\omega > 0$. If $\omega \geq 0$ but not $\omega > 0$, then one eigenvalue is zero and the others are positive. If this is so, then $\omega^m = 0$. Hence (2) and (3) are proved.

(4) If $\omega \geq 0$, all eigenvalues are non-negative. If $\omega^m = 0$, then $\lambda_\mu = 0$ for one and only one $\mu \in \mathbf{N}[1, m]$. Define $v = (\partial/\partial z^\mu)(a)$. Then $\omega(v, \bar v) = 0$. If $0 = v \in \mathfrak{T}_a(M)$ exists such that $\omega(v, \bar v) = 0$, then ω is not positive. By (2) $\omega^m = 0$. Q.E.D.

Proposition 3.4: *Let M be a pure m-dimensional complex space. Let τ be a strictly parabolic function on M. Let $U \subseteq G$ be an embedded chart where G has pure dimension n. Assume that there is given a strictly parabolic extension θ of τ on G. Then*

$$(dd^c \log\theta)^n = 0 \quad \text{on } U \cap M_*. \tag{3.3}$$

Proof: Take $a \in U \cap M_*$. An open neighborhood V of a in G exists such that $\theta > 0$ and $\tilde\omega = dd^c \log\theta \geq 0$ on V. Define $\omega = dd^c \log\tau$ on $\mathfrak{R}(M_*)$. Take $p \in V \cap \mathfrak{R}(M)$. Then $p \in \mathfrak{R}(M_*)$ and $\omega(p) \geq 0$ and $\omega^m(p) = 0$. A vector $0 \neq v \in \mathfrak{T}_p(M)$ exists such that $\omega(p, v, \bar v) = 0$. Consider $\mathfrak{T}_p(M)$ as a linear subspace of $\mathfrak{T}_p(G)$. Then $\tilde\omega(p, v, \bar v) = \omega(p, v, \bar v) = 0$. Since $dd^c\theta > 0$ and $\tilde\omega \geq 0$ at p, Lemma 3.3 implies $\tilde\omega^n(p) = 0$. Since $V \cap \mathfrak{R}(M)$ is dense in $V \cap M$, we obtain $\tilde\omega^n(p) = 0$ for all $p \in V \cap M$. Consequently, $\tilde\omega^n(a) = 0$. Q.E.D.

It is remarkable that the strictly parabolic extension θ satisfies the Monge–Ampère equation (3.3), but observe, the Monge-Ampère equation is satisfied on $U \cap M_*$ only!

Let M be a complex space of pure dimension m. Let τ be a strictly parabolic function on M. The Monge–Ampère equation $(dd^c \log \tau)^m = 0$ on M_* implies

$$(3.4) \qquad \tau(dd^c\tau)^m = m d\tau \wedge d^c\tau \wedge (dd^c\tau)^{m-1}$$

which holds on $\mathfrak{R}(M)$ by continuity. If $z^1, \ldots, z^m$ are local coordinates on a patch of $\mathfrak{R}(M)$, then

$$(3.5) \qquad \partial\tau = \tau_\mu dz^\mu \quad \bar\partial\tau = \tau_{\bar\nu} d\bar z^\nu \quad dd^c\tau = \frac{i}{2\pi}\tau_{\mu\bar\nu} dz^\mu \wedge d\bar z^\nu$$

where the matrix $(\tau_{\mu\bar\nu})$ is invertible. Let $(\tau^{\bar\nu\mu})$ be the inverse matrix. Then (3.4) translates into

$$(3.6) \qquad \tau = \tau_{\bar\nu}\tau^{\bar\nu\mu}\tau_\mu.$$

Let $\mathfrak{p}: U \to G$ be an embedded chart of M where G is open in $\mathbf{C}^n$. The coordinate functions on $\mathbf{C}^n$ are denoted by $w^1, \ldots, w^n$. On $U \cap M_*$, the Monge–Ampère equation $(dd^c \log \theta)^n = 0$ translates into

$$(3.7) \qquad \theta(dd^c\theta)^n = n d\theta \wedge d^c\theta \wedge (dd^c\theta)^{n-1}.$$

Since $U \cap M_*$ is dense in U, the identity (3.7) holds on U by continuity. On G we have

$$(3.8) \qquad \partial\theta = \theta_j dw^j \quad \bar\partial\theta = \theta_{\bar k} dw^{\bar k} \quad dd^c\theta = \frac{i}{2\pi}\theta_{j\bar k} dw^j \wedge d\bar w^k$$

where the matrix $(\theta_{j\bar k})$ is invertible. Let $\theta^{\bar k j}$ be the inverse matrix. Then (3.7) translates into

$$(3.9) \qquad \theta = \theta_{\bar k}\theta^{\bar k j}\theta_j \quad \text{on } U.$$

This implies trivially:

LEMMA 3.5: *Let M be a pure m-dimensional complex space. Let τ be a strictly parabolic function on M. Let $\mathfrak{p}: U \to G$ be an embedded chart of M where G has pure dimension n. Assume that there is given* *a strictly parabolic extension θ of τ on G. Take $a \in U$. Then $a \in M[0]$ if and only if $d\theta(a) = 0$.*

Let M be a complex space. Take $a \in M$. Let $\mathfrak{p}: U \to G$ be an embedded chart at a where G is open in $\mathbf{C}^n$. Let K be the Whitney tangent cone of M at a in $\mathbf{C}^n$. Then $w \in K$ if and only if there exists a number $t \geq 0$ and a sequence $\{w_\lambda\}_{\lambda \in \mathbf{N}}$ with $a \neq w_\lambda \in U$ such that

$$w_\lambda \to a \quad \text{and} \quad t\frac{w_\lambda - a}{\|w_\lambda - a\|} \to w \quad \text{for } \lambda \to \infty. \tag{3.10}$$

Then K is an analytic in $\mathbf{C}^n$ (Whitney [22] Chapter 7, Theorems 4.D and 2.E). If M is pure m-dimensional, then K is pure m-dimensional.

Let G be an open subset of $\mathbf{C}^n$. Let $A(G)$ be the algebra of complex valued functions of class C^∞ on G. Take $a \in G$. Then

$$\mathfrak{m}_a = \mathfrak{m}_a(G) = \{f \in A(G) \mid f(a) = 0\}. \tag{3.11}$$

is an ideal in $A(G)$. Let $w^1, \ldots, w^n$ be the coordinate functions on $\mathbf{C}^n$. If $a = (a^1, \ldots, a^n)$ and if G is convex, then $\mathfrak{m}_a$ is generated by $w^1 - a^1, \ldots, w^n - a^n$. If $p \in \mathbb{N}$ and $q \in \mathbb{Z}[0, p]$ and if $f \in \mathfrak{m}_a^p$, then any q^{th} partial derivative of f belongs to $\mathfrak{m}_a^{p-q}$. If $K \neq \emptyset$ is a compact subset of G and if $f \in \mathfrak{m}_a^p$, then there exists a constant $c > 0$ such that

$$|f(w)| \leq c\|w - a\|^p \quad \text{for all } w \in K. \tag{3.12}$$

LEMMA 3.6: *Let M be a complex space of pure dimension m. Let τ be a strictly parabolic function on M. Take $a \in M[0]$. Let $\mathfrak{p}: U \to G$ be an embedded chart of M at a such that G is open in $\mathbf{C}^n$ with $a = 0 \in \mathbf{C}^n$. Let K be the Whitney tangent cone of M at a in $\mathbf{C}^n$. Assume that a strictly parabolic extension θ of τ on G is given. Then there exists $R \in \mathfrak{m}_0(G)^3$ such that*

$$\theta(w) = \operatorname{Re}(\theta_{jk}(0)w^j w^k) + \theta_{j\bar{k}}(0)w^j \bar{w}^k + R(w) \tag{3.13}$$

for all $w \in G$. Moreover, if $w \in K$, then

$$\theta_{jk}(0)w^j = 0 \quad \textit{for all } k \in \mathbf{N}[1, n]. \tag{3.14}$$

PROOF: The existence of $R \in \mathfrak{m}_0(G)^3$ and the representation (3.13) follow from Taylor's Theorem since $\theta(0) = \theta_j(0) = 0$ for $j = 1, \ldots, n$. Take $w \in K$. According to Whitney [22] Chapter 7, Theorem 3.C, page 218, a curve $\gamma:\mathbf{R}(-\epsilon,\epsilon)\to U$ of class C^1 exists such that $\gamma(0)=0$ and $\gamma'(0)=w$. Substituting γ into (3.13) implies

$$\lim_{0<t\to 0} t^{-2}\theta(\gamma(t)) = \operatorname{Re}(\theta_{jk}(0)w^j w^k) + \theta_{j\bar{k}}(0)w^j\bar{w}^k$$

$$\lim_{0<t\to 0} t^{-1}\theta_j(\gamma(t)) = \theta_{jk}(0)w^k + \theta_{j\bar{k}}(0)\bar{w}^k$$

$$\lim_{0<t\to 0} \theta_{j\bar{k}}(\gamma(t)) = \theta_{j\bar{k}}(0)$$

$$\lim_{0<t\to 0} \theta^{\bar{k}j}(\gamma(t)) = \theta^{\bar{k}j}(0).$$

If $t\in\mathbf{R}(-\epsilon,\epsilon)$, then $\gamma(t)\in U$. Substituting γ into (3.9) implies

$$\lim_{0<t\to 0} t^{-2}\theta(\gamma(t)) = (\theta_{\bar{k}\bar{h}}(0)\bar{w}^h + \theta_{\bar{k}h}(0)w^h)\theta^{\bar{k}j}(0)(\theta_{ju}(0)w^u + \theta_{j\bar{u}}(0)\bar{w}^u).$$

Define the matrices $B=(\theta_{jk}(0))$ and $H=(\theta_{j\bar{k}}(0))$. Then

$$\tfrac{1}{2}wB{}^t w + \tfrac{1}{2}\bar{w}\bar{B}{}^t\bar{w} + wH{}^t\bar{w} = (\bar{w}\bar{B} + wH)H^{-1}(B{}^t w + H{}^t\bar{w})$$

$$2\bar{w}\bar{B}H^{-1}B{}^t w + \bar{w}\bar{B}{}^t\bar{w} + wB{}^t w = 0.$$

If $w\in K$, then $iw\in K$. Hence

$$2\bar{w}\bar{B}H^{-1}Bw - \bar{w}\bar{B}{}^t\bar{w} - wB{}^t w = 0.$$

Therefore

$$wB{}^t w + \bar{w}\bar{B}{}^t\bar{w} = 0.$$

If $w\in K$, then $(1+i)w\in K$. Hence

$$wB{}^t w - \bar{w}\bar{B}{}^t\bar{w} = 0.$$

We obtain

$$wB{}^t w = 0 \quad \text{for all } w\in K.$$

Define $y = wB$. Then ${}^t y = {}^t B{}^t W = B{}^t w$. Then

$$w\bar{y}H^{-1t}y + \bar{y}{}^t\bar{w} + y{}^t w = 0$$

where $y{}^t w = 0$. Hence $\bar{y}H^{-1t}y = 0$ which implies $y=0$. Q.E.D.

PROPOSITION 3.7: *Let M be a complex space of pure dimension m. Let τ be a strictly parabolic function on M. Then every point of $M[0]$ is isolated.*

PROOF: Take $a \in M[0]$. Let $\mathfrak{p}: U \to G$ be an embedded chart at a, where G is open in $\mathbf{C}^n$ and $a = 0 \in \mathbf{C}^n$. Moreover, we can assume that there exists a strictly parabolic extension θ of τ on G. Let K be the Whitney tangent cone of M at a in $\mathbf{C}^n$. Then (3.13) and (3.14) hold. Assume that a is an accumulation point of $M[0]$. Then there exists a sequence $\{w_\lambda\}_{\lambda \in \mathbf{N}}$ of points $0 = a \neq w_\lambda \in M[0]$ such that $w_\lambda \to a$ for $\lambda \to \infty$. By taking a subsequence, we may assume that $v_\lambda = w_\lambda/\|w_\lambda\| \to v \in K$ for $\lambda \to \infty$. Then $\|v\| = 1$. Hence $v \neq 0$. Now 3.13 implies

$$0 = \theta(w_\lambda)\|w_\lambda\|^{-2} = \mathrm{Re}(\theta_{jk}(0)v_\lambda^j v_\lambda^k) + \theta_{jk}(0)v_\lambda^j \bar{v}_\lambda^k + 0(1)$$

for $\lambda \to \infty$. Hence

$$0 = \mathrm{Re}(\theta_{jk}(0)v^j v^k) + \theta_{j\bar{k}}(0)v^j\bar{v}^k = \theta_{j\bar{k}}(0)v^j\bar{v}^k > 0.$$

Contradiction! Therefore a is an isolated point of $M[0]$. Q.E.D.

Let M be a complex space of pure dimension m. Let τ be a strictly parabolic function on M. Then $dd^c\tau > 0$ is the associated form of a Kaehler metric κ on $\mathfrak{R}(M)$. Therefore real vector fields

$$F = f + \bar{f} = \tfrac{1}{2}\,\mathrm{grad}\,\tau \quad \text{on } \mathfrak{R}(M) \tag{3.15}$$

$$Y = \frac{1}{\sqrt{\tau}}F = \mathrm{grad}\sqrt{\tau} \quad \text{on } \mathfrak{R}(M_*) \tag{3.16}$$

are defined, where f is the component of type $(1,0)$ of F. Let $z^1, \ldots, z^m$ be local coordinates on a patch U of $\mathfrak{R}(M)$. As shown in [18] (3.20)–(3.23) on U we have

$$f = f^\mu \frac{\partial}{\partial z^\mu} \qquad f^\mu = \tau_{\bar{\nu}}\tau^{\bar{\nu}\mu} \tag{3.17}$$

$$\tau = f^\mu\tau_\mu = \bar{f}^\nu\tau_{\bar{\nu}} = f^\mu\tau_{\mu\bar{\nu}}\bar{f}^\nu. \tag{3.18}$$

LEMMA 3.8: *Let M be a complex space of pure dimension m. Let τ be a strictly parabolic function on M. Let $\mathfrak{p}: U \to G$ be an embedded chart where G is open in $\mathbf{C}^n$. Let $w^1, \ldots, w^n$ be the coordinate functions on $\mathbf{C}^n$. Assume that a strictly parabolic extension θ of τ is* *given on G. Then*

$$\partial\theta = \theta_j dw^j \quad \bar\partial\theta = \theta_{\bar k} d\bar w^k \quad dd^c\theta = \frac{i}{2\pi}\theta_{j\bar k} dw^j \wedge d\bar w^k \tag{3.19}$$

on G. The matrix $(\theta_{j\bar k})$ *is invertible. Let* $(\theta^{\bar k j})$ *be the inverse matrix. On G define*

$$\tilde f = \tilde f^j \frac{\partial}{\partial w^j} \qquad \tilde f^j = \theta_{\bar k}\theta^{\bar k j}. \tag{3.20}$$

Then we have

$$\tilde f^j \theta_{j\bar k} = \theta_{\bar k} \quad \text{on } G \tag{3.21}$$

$$\tilde f^j \theta_j = \bar{\tilde f}^k \theta_{\bar k} = \tilde f^j \theta_{j\bar k} \bar{\tilde f}^k = \theta = \tau \quad \text{on } U. \tag{3.22}$$

On G define

$$\Lambda_{j\bar k} = \theta\theta_{j\bar k} - \theta_j\theta_{\bar k}. \tag{3.23}$$

Take any $x \in U \cap M_*$ *and define*

$$\mathfrak{A}_x = \left\{ Z = Z^j \frac{\partial}{\partial w^j} \in \mathfrak{T}_x(\mathbf{C}^n) \,\middle|\, Z^j \Lambda_{j\bar k}(x) \bar Z^k = 0 \right\}. \tag{3.24}$$

Then

$$\mathfrak{A}_x = \mathbf{C}\tilde f(x) = \{\lambda \tilde f(x) \mid \lambda \in \mathbf{C}\}. \tag{3.25}$$

PROOF: On G we have

$$\tilde f^j \theta_{j\bar k} = \theta_{\bar h}\theta^{\bar h j}\theta_{j\bar k} = \theta_{\bar k}.$$

On U we have

$$\tilde f^j \theta_j = \theta_{\bar k}\theta^{\bar k j}\theta_j = \theta \qquad \bar{\tilde f}^k \theta_{\bar k} = \theta$$

$$\tilde f^j \theta_{j\bar k} \bar{\tilde f}^k = \theta$$

$$\tilde f^j \Lambda_{j\bar k} = \theta \tilde f^j \theta_{j\bar k} - \tilde f^j \theta_j \theta_{\bar k} = \theta\bar\theta_k - \theta\bar\theta_k = 0.$$

Therefore(3.21) and (3.22) are proved. Take $x \in U \cap M_*$ and define

$$\mathfrak{B}_x = \left\{ Z = Z^j \frac{\partial}{\partial w^j} \in \mathfrak{T}_x(\mathbf{C}^n) \mid Z^j \Lambda_{j\bar{k}}(x) = 0 \right\}.$$

Then $\mathfrak{B}_x$ is a linear subspace of $\mathfrak{T}_x(\mathbf{C}^n)$ with $\mathbf{C}\tilde{f}(x) \subseteq \mathfrak{B}_x \subseteq \mathfrak{A}_x$. Take $Z \in \mathfrak{B}_x$ and define $\lambda = (1/\theta(x))Z^j\theta_j(x) \in \mathbf{C}$. Then

$$0 = (Z^j - \lambda\tilde{f}^j(x))\Lambda_{j\bar{k}}(x) = \theta(x)(Z^j - \lambda\tilde{f}^j(x))\theta_{j\bar{k}}(x).$$

Since $\theta(x) > 0$ and $(\theta_{j\bar{k}}(x))$ is an invertible matrix, we obtain $Z^j = \lambda\tilde{f}^j(x)$. Therefore $Z = \lambda\tilde{f}(x) \in \mathbf{C}\tilde{f}(x)$. We have shown that $\mathbf{C}\tilde{f}(x) = \mathfrak{B}_x \subseteq \mathfrak{A}_x$. Take $Z \in \mathfrak{A}_x$. Take any $X \in \mathfrak{T}_x(\mathbf{C}^n)$. Take any $\zeta \in \mathbf{C}$. Then

$$\begin{aligned} 0 &\leq 2\pi\theta(x)^2(dd^c \log \theta)(x, Z + \zeta X, \overline{i(Z + \zeta X)}) \\ &= (Z^j + \zeta X^j)\Lambda_{j\bar{k}}(x)(\bar{Z}^k + \bar{\zeta}\bar{X}^k) \\ &= \zeta X^j\Lambda_{j\bar{k}}(x)\bar{Z}^k + \bar{\zeta}Z^j\Lambda_{j\bar{k}}(x)\bar{X}^k + |\zeta|^2 X^j\Lambda_{jk}(x)\bar{X}^k. \end{aligned}$$

Take $\phi \in \mathbf{R}$ and $t > 0$ and substitute $\zeta = te^{i\phi}$. Divide by t and let t converge to zero. This yields

$$0 \leq e^{i\phi}X^j\Lambda_{j\bar{k}}(x)\bar{Z}^k + e^{-i\phi}Z^j\Lambda_{j\bar{k}}(x)\bar{X}^k.$$

Replacing ϕ by $\phi + \pi$ implies

$$0 \leq -e^{i\phi}X^j\Lambda_{j\bar{k}}(x)\bar{Z}^k - e^{-i\phi}Z^j\Lambda_{j\bar{k}}(x)\bar{X}^k.$$

Hence

$$0 = e^{i\phi}X^j\Lambda_{jk}(x)\bar{Z}^k + e^{-i\phi}Z^j\Lambda_{j\bar{k}}(x)\bar{X}^k.$$

Take $\phi = 0$ and $\phi = \pi/2$ and compare. This yields $Z^j\Lambda_{j\bar{k}}(x)\bar{X}^k = 0$ for all $X \in \mathfrak{T}_x(\mathbf{C}^n)$. Hence $Z^j\Lambda_{j\bar{k}}(x) = 0$ for all $k \in \mathbf{N}[1, n]$. Thus $Z \in \mathfrak{B}_x$. We have $\mathfrak{A}_x = \mathfrak{B}_x = \mathbf{C}\tilde{f}(x)$. Q.E.D.

Now, we are able to establish the fundamental result that F is of class C^∞ on M. Also we identify extensions of F.

THEOREM 3.9: *Let M be a pure m-dimensional complex space. Let τ be a strictly parabolic function on M. Then the vector fields F and f defined in* (3.15) *are of class C^∞ on M. Also the vector field Y defined in* (3.16) *is of class C^∞ on M_*. If $\mathfrak{p}: U \to G$ is an embedded chart where G is open in $\mathbf{C}^n$, if θ is a strictly parabolic extension of τ to G, if*

$\tilde{f}$ is defined by (3.20), then $\tilde{f}$ is an extension of class C^∞ of f onto G and $\tilde{F} = \hat{f} + \bar{\tilde{f}}$ is an extension of class C^∞ of F onto G.

PROOF: Let $w^1, \ldots, w^n$ be the coordinate functions on $\mathbf{C}^n$. Write $\mathfrak{p} = (p^1, \ldots, p^n)$ where $p^1, \ldots, p^n$ are the embedding coordinates. Let $\mathfrak{p} = (z^1, \ldots, z^m): V \to V'$ be a patch of M with $V \subseteq \mathfrak{R}(U)$. On V define

$$\hat{f}^j = f^\mu p^j_\mu \qquad \hat{f} = \hat{f}^j \frac{\partial}{\partial w^j}. \tag{3.26}$$

Then $\hat{f} = \tilde{f} \mid V$ has to be proved. If $x \in V$ and $\theta(x) = 0$, then $x \in M[0]$ and $\tau_{\bar{\nu}}(x)$ for all $\nu \in \mathbf{N}[1, m]$ and $\theta_{\bar{k}}(x) = 0$ for all $\nu \in \mathbf{Z}[1, \eta]$. Therefore $f(x) = 0 = \tilde{f}(x)$ and $\hat{f}(x) = 0$. Hence $\hat{f}(x) = \tilde{f}(x)$. Therefore we can assume that $\theta > 0$ on V. Now $\tau = \theta \circ \mathfrak{p}$ implies

$$\theta_j p^j_\mu = \tau_\mu \qquad \theta_{j\bar{k}} p^j_\mu \bar{p}^k_\nu = \tau_{\mu\bar{\nu}} \tag{3.27}$$

on V. Define $\Lambda_{j\bar{k}}$ by (3.23). Then

$$\begin{aligned} \hat{f}^j \Lambda_{j\bar{k}} \bar{\hat{f}}^k &= \theta f^\mu p^j_\mu \theta_{j\bar{k}} \bar{p}^k_\nu \bar{f}^\nu - f^\mu p^j_\mu \theta_j \bar{f}^\nu \bar{p}^k_\nu \theta_{\bar{k}} \\ &= \tau f^\mu \tau_{\mu\bar{\nu}} \bar{f}^\nu - f^\mu \tau_\mu \bar{f}^\nu \tau_{\bar{\nu}} = \tau^2 - \tau^2 = 0. \end{aligned}$$

By Lemma 3.8 a function $\lambda : V \to \mathbf{C}$ exists such that $\hat{f} = \lambda \tilde{f}$ on V. We have

$$\lambda\tau = \lambda\theta = \lambda \tilde{f}^j \theta_j = \hat{f}^j \theta_j = f^\mu p^j_\mu \theta_j = f^\mu \tau_\mu = \tau > 0$$

on V. Therefore $\lambda = 1$ and $\hat{f} = \tilde{f}$ on V. Q.E.D.

Here $\tilde{f}$ and $\tilde{F}$ are the *extensions* of f and F *associated to* θ.

Identify $\partial/\partial w_j = \mathfrak{e}_j = (\delta_{j1}, \ldots, \delta_{jn})$. Then $\tilde{f}: G \to \mathbf{C}^n$ becomes a vector function. Then we want to study the behavior of $\tilde{f}$ near a point a in the center.

LEMMA 3.10: *Let M be a complex space of pure dimension m. Let τ be a strictly parabolic function on M. Take $a \in M[0]$. Let $\mathfrak{p}: U \to G$ be an embedded chart at a where G is an open neighborhood of $a = 0 \in \mathbf{C}^n$. Let θ be a strictly parabolic extension of τ on G. Let $\tilde{f}$ be the extension of f associated to θ. Define*

$$b_{jk} = \theta_{jk}(0) \qquad b^j_{\bar{h}} = \bar{b}_{hk} \theta^{\bar{k}j}(0) \tag{3.28}$$

(3.29) $$\mathfrak{b}:\mathbf{C}^n \to \mathbf{C}^n \quad by \quad \mathfrak{b}(w) = (b^1_{\bar{j}}\bar{w}^j, \ldots, b^n_{\bar{j}}\bar{w}^j).$$

Then $\mathfrak{b}$ *is an antilinear map. Define*

(3.30) $$\tilde{G} = \{(t, w) \in \mathbf{R} \times \mathbf{C}^n \mid tw \in G\}.$$

Then there exists a function $R:\tilde{G} \to \mathbf{C}^n$ *of class* C^∞ *such that*

(3.31) $$\tilde{f}(tw) = t\mathfrak{b}(w) + tw + t^2R(t, w)$$

for all $(t, w) \in \tilde{G}$. *Moreover, if* K *is the Whitney tangent cone of* M *at* a *in* $\mathbf{C}^n$, *then*

(3.32) $$\mathfrak{b}(w) = 0 \quad \textit{for all } w \in K.$$

PROOF: If $w \in K$, then (3.14) implies

$$b^j_{\bar{h}}\bar{w}^h = \theta^{\bar{k}j}(0)\bar{b}_{hk}\bar{w}^h = 0.$$

Hence $\mathfrak{b}(w) = 0$. Define $\mathfrak{m}_0 = \mathfrak{m}_a(G)$ as in (3.11). A function $\hat{R}:G \to \mathbf{R}$ of class C^∞ with $\hat{R} \in \mathfrak{m}_0^3$ exists such that we have the Taylor expansion

$$\theta(w) = \tfrac{1}{2}b_{jk}w^jw^k + \tfrac{1}{2}\bar{b}_{jk}\bar{w}^j\bar{w}^k + \theta_{j\bar{k}}(0)w^j\bar{w}^k + \hat{R}(w).$$

Therefore

$$\begin{aligned} \theta_{\bar{k}}(w) &= \bar{b}_{jk}\bar{w}^j + \theta_{j\bar{k}}(0)w^j + \hat{R}_{\bar{k}}(w) && \text{with } \hat{R}_{\bar{k}} \in \mathfrak{m}_0^2 \\ \theta_{j\bar{k}}(w) &= \theta_{j\bar{k}}(0) + \hat{R}_{j\bar{k}}(w) && \text{with } \hat{R}_{j\bar{k}} \in \mathfrak{m}_0 \\ \theta^{\bar{k}j}(w) &= \theta^{\bar{k}j}(0) + \tilde{R}^{\bar{k}j}(w) && \text{with } \tilde{R}_{j\bar{k}} \in \mathfrak{m}_0. \end{aligned}$$

Hence we obtain

$$\tilde{f}^j(w) = \theta_{\bar{k}}(w)\theta^{\bar{k}j}(w) = \bar{b}^j_h\bar{w}^h + w^j + \hat{R}^j(w)$$

where

$$\hat{R}^j(w) = \bar{R}_k(w)\theta^{\bar{k}j}(w) + \bar{b}_{hk}\bar{w}^h\tilde{R}^{\bar{k}j}(w) + \theta_{h\bar{k}}(0)w^h\tilde{R}^{\bar{k}j}(w).$$

Hence $\hat{R}^j \in \mathfrak{m}_0^2$. Therefore $R = (R^1, \ldots, R^n):\tilde{G} \to \mathbf{C}$ of class C^∞ exists such that $t^2R^j(t, w) = \hat{R}^j(tw)$. Then

$$\tilde{f}(tw) = t\mathfrak{b}(w) + tw + t^2R(t, w). \qquad \text{Q.E.D.}$$

We need a number of estimates and identities to establish that the integral curves of Y are geodesics. We will make the following *general assumptions.*

(A1) *Let M be a complex space of pure dimension m.*

(A2) *Let τ be a strictly parabolic function on M.*

(A3) *Let f, F and Y be the vector fields of class C^∞ on M respectively M_* defined by* (3.15) *and* (3.16).

(A4) *Let $\mathfrak{p}: U \to G$ be an embedded chart where G is open in $\mathbf{C}^n$. Let $w^1, \ldots, w^n$ be the coordinate functions on G.*

(A5) *Let θ be a strictly parabolic extension of τ to G.*

(A6) *Let $\tilde{f}$, $\tilde{F} = f + \bar{\tilde{f}}$ and $\tilde{Y} = (1/\sqrt{\theta})\tilde{F}$ be the extension of f, F and Y on G, associated to θ and defined by* (3.20) *and Theorem* 3.9. Naturally, $\tilde{Y}$ is defined only on $\{w \in G \mid \theta(w) > 0\}$.

LEMMA 3.11: *Assume (A1)–(A6). Take $p \in U \cap M_*$. Then there exists an open neighborhood V_p of p in G such that*

$$\theta_{\bar{k}}\theta^{\bar{k}j}\theta_j \le \theta \quad \text{on } V_p \tag{3.33}$$

$$\tilde{f}^j\theta_j = \tilde{f}^j\theta_{j\bar{k}}\bar{\tilde{f}}^k = \theta_{\bar{k}}\bar{\tilde{f}}^k \le \theta \quad \textit{on } V_p. \tag{3.34}$$

PROOF: An open neighborhood V_p of p in G exists such that $\theta > 0$ and $dd^c \log \theta \ge 0$ on V_p. Therefore

$$0 \le \theta^2 dd^c \log \theta = \theta dd^c\theta - d\theta \wedge d^c\theta$$

$$0 \le \theta^{n+1}(dd^c \log \theta)^n = \theta(dd^c\theta)^n - n d\theta \wedge d^c\theta \wedge (dd^c\theta)^{n-1}$$

$$n d\theta \wedge d^c\theta \wedge (dd^c\theta)^{n-1} \le \theta(dd^c\theta)^n$$

on V_p. Define $T = \det(\theta_{j\bar{k}})$ and let $T^{j\bar{k}}$ be the minor determinants. Then

$$(dd^c\theta)^n = \left(\frac{i}{2\pi}\right)^n n!T dw^1 \wedge d\bar{w}^1 \wedge \cdots \wedge dw^n \wedge d\bar{w}^n$$

$$n d\theta \wedge d^c\theta \wedge (dd^c\theta)^{n-1} = \left(\frac{i}{2\pi}\right)^n n!\theta_j T^{j\bar{k}}\theta_{\bar{k}} dw^1 \wedge d\bar{w}^1 \wedge \cdots \wedge dw^n \wedge d\bar{w}^n.$$

Therefore

$$\theta_j T^{j\bar{k}} \theta_{\bar{k}} \leq T\theta.$$

Observe that $\theta^{\bar{k}j} = T^{j\bar{k}}/T$. Hence we obtain (3.33). Also we have

$$\tilde{f}^j \theta_{j\bar{k}} \bar{\tilde{f}}^k = \tilde{f}^j \theta_{j\bar{k}} \theta^{\bar{k}h} \theta_h = \tilde{f}^j \theta_j = \theta_{\bar{k}} \theta^{\bar{k}j} \theta_j \leq \theta.$$

Conjugation implies

$$\bar{\tilde{f}}^k \theta_{\bar{k}} = \tilde{f}^j \theta_j \leq \theta \qquad \text{Q.E.D.}$$

LEMMA 3.12: *Assume (A1)–(A6). Let $z^1, \ldots, z^m$ be local coordinates of $\mathfrak{R}(M)$ on an open subset Z of $\mathfrak{R}(M)$. Then*

$$\theta_{\bar{k}\bar{h}} \theta^{\bar{k}j} \theta_j + \theta_{\bar{k}} \theta^{\bar{k}j}_{\bar{h}} \theta_j = 0 \quad \text{on } U \tag{3.35}$$

$$\tau_{\bar{\nu}\bar{\lambda}} \tau^{\bar{\nu}\mu} \tau_\mu + \tau_{\bar{\nu}} \tau^{\bar{\nu}\mu}_{\bar{\lambda}} \tau_\mu = 0 \quad \text{on } Z. \tag{3.36}$$

PROOF: Differentiation of (3.6) yields

$$\begin{aligned} \tau_{\bar{\lambda}} &= \tau_{\bar{\nu}\bar{\lambda}} \tau^{\bar{\nu}\mu} \tau_\mu + \tau_{\bar{\nu}} \tau^{\bar{\nu}\mu}_{\bar{\lambda}} \tau_\mu + \tau_{\bar{\nu}} \tau^{\bar{\nu}\mu} \tau_{\mu\bar{\lambda}} \\ &= \tau_{\bar{\nu}\bar{\lambda}} \tau^{\bar{\nu}\mu} \tau_\mu + \tau_{\bar{\nu}} \tau^{\bar{\nu}\mu}_{\bar{\lambda}} \tau_\mu + \tau_{\bar{\lambda}} \end{aligned}$$

which implies (3.36). Since (3.9) holds on U only, (3.35) cannot be proved by the same method. Because $\theta_j = 0$ for all $j \in \mathbb{N}[1, n]$ on $U \cap M[0]$, (3.35) is trivially correct on $U \cap M[0]$. Take $p \in U \cap M_*$. An open neighborhood V_p of p in G exists such that $\theta > 0$ on V_p and such that (3.33) and (3.34) hold on V_p. On V_p define

$$g = \theta - \theta_{\bar{k}} \theta^{\bar{k}j} \theta_j \geq 0.$$

Then $g \mid U \cap V_p = 0$. Thus g assumes a minimum at every point of $U \cap V_p$. Hence $g_{\bar{h}} = 0$ on $U \cap V_p$ for all $h \in \mathbb{N}[1, n]$, which implies

$$\begin{aligned} 0 &= \theta_{\bar{h}} - \theta_{\bar{k}\bar{h}} \theta^{\bar{k}j} \theta_j - \theta_{\bar{k}} \theta^{\bar{k}j}_{\bar{h}} \theta_j - \theta_{\bar{k}} \theta^{\bar{k}j} \theta_{j\bar{h}} \\ &= \theta_{\bar{h}} - \theta_{\bar{k}\bar{h}} \theta^{\bar{k}j} \theta_j - \theta_{\bar{k}} \theta^{\bar{k}j}_{\bar{h}} \theta_j - \theta_{\bar{h}} \end{aligned}$$

on $U \cap V_p$, which implies (3.35) on $U \cap V_p$. Together we obtain (3.35) on U. Q.E.D.

LEMMA 3.13: *Assume (A1)–(A6). Let $z^1, \ldots, z^m$ be local coor-*

dinates on an open subset Z of $\Re(M)$. Then

$$\tilde{f}^{j}_{\bar{k}} - \bar{\tilde{f}}^{k} = 0 \quad \text{on } U \qquad f^{\mu}_{\bar{\nu}}\bar{f}^{\bar{\nu}} = 0 \quad \text{on } Z. \tag{3.37}$$

PROOF: The connection of the Kaehler metric $dd^c\theta$ on G is given by

$$\tilde{\Gamma}^{\bar{q}}_{\bar{p}\bar{r}} = \theta_{\bar{p}\bar{r}a}\theta^{\bar{q}a} = -\theta_{\bar{p}a}\theta^{\bar{q}a}_{\bar{r}} = \tilde{\Gamma}^{\bar{q}}_{\bar{r}\bar{p}}. \tag{3.38}$$

Hence, on U we have

$$\begin{aligned}
\tilde{f}^{j}_{\bar{k}}\bar{\tilde{f}}^{k} &= (\theta_{\bar{p}}\theta^{\bar{p}j})_{\bar{k}}\theta_a\theta^{\bar{k}a} \\
&= \theta_{\bar{p}\bar{k}}\theta^{\bar{p}j}\theta_a\theta^{\bar{k}a} + \theta_{\bar{p}}\theta^{\bar{p}j}_{\bar{k}}\theta_a\theta^{\bar{k}a} \\
&= -\theta_{\bar{k}}\theta^{\bar{k}a}_{\bar{p}}\theta_a\theta^{\bar{p}j} + \theta_{\bar{p}}\theta^{\bar{p}j}_{\bar{k}}\theta_a\theta^{\bar{k}a} \\
&= -\theta_{\bar{k}}\theta^{\bar{q}a}\theta_{\bar{q}b}\theta^{\bar{k}b}_{\bar{p}}\theta_a\theta^{\bar{p}j} + \theta_{\bar{p}}\theta^{\bar{q}j}\theta_{\bar{q}b}\theta^{\bar{p}b}_{\bar{k}}\theta_a\theta^{\bar{k}a} \\
&= \theta_{\bar{k}}\theta^{\bar{q}a}\tilde{\Gamma}^{\bar{k}}_{\bar{q}\bar{p}}\theta_a\theta^{\bar{p}j} - \theta_{\bar{p}}\theta^{\bar{q}j}\Gamma^{\bar{p}}_{\bar{q}\bar{k}}\theta_a\theta^{\bar{k}a} \\
&= \theta_{\bar{p}}\theta^{\bar{k}a}\tilde{\Gamma}^{\bar{p}}_{\bar{k}\bar{q}}\theta_a\theta^{\bar{q}j} - \theta_{\bar{p}}\theta^{\bar{q}j}\Gamma^{\bar{p}}_{\bar{q}\bar{k}}\theta_a\theta^{\bar{k}a} = 0
\end{aligned}$$

where we have changed the summation index notation in the first term by $k \to p$, $p \to q$, $q \to k$. Since Z can be viewed as an embedded chart with $z^1, \ldots, z^m$ as embedding coordinates into Z' in $\mathbf{C}^m$, the identity on Z follows trivially. Q.E.D.

Assume (A1)–(A6). Let J be the almost complex structure on $\Re(M)$. Then

$$JF = if - i\bar{f} \tag{3.39}$$

is a vector field on $\Re(M)$ which is of class C^∞ on M. An extension on G is provided by

$$J\tilde{F} = i\tilde{f} - i\bar{\tilde{f}}. \tag{3.40}$$

PROPOSITION 3.14: *Assume* (*A*1)–(*A*6). *Then*

$$[F, JF] = 0 \quad \text{on } \Re(M) \qquad [\tilde{F}, J\tilde{F}] = 0 \quad \text{on } U. \tag{3.41}$$

PROOF: Define $\tilde{H} = [\tilde{F}, J\tilde{F}]$. Then

$$\tilde{H} = \tilde{H}^k \frac{\partial}{\partial w^k} + \bar{\tilde{H}}^k \frac{\partial}{\partial \bar{w}^k}$$

on G. On U we have

$$\tilde{H}^k = i\tilde{f}^j\tilde{f}^k_j + i\bar{\tilde{f}}^j\tilde{f}^k_{\bar{j}} - i\tilde{f}^j\tilde{f}^k_j + i\bar{\tilde{f}}^j\tilde{f}^k_{\bar{j}} = 2i\bar{\tilde{f}}^j\tilde{f}^k_{\bar{j}} = 0.$$

Hence $[\tilde{F}, J\tilde{F}] = 0$. The same computation proves $[F, JF] = 0$. Q.E.D.

PROPOSITION 3.15: *Assume (A1)–(A6). Let $\phi : \mathbf{R}(\alpha, \beta) \to M_*$ be an integral curve on Y. If $\alpha \leq \alpha_0 < \beta_0 \leq \beta$ and if $\phi : \mathbf{R}(\alpha_0, \beta) \to \mathfrak{R}(M_*)$, then $\phi \mid \mathbf{R}(\alpha_0, \beta_0)$ is geodesic in respect to the Kaehler metric on $\mathbf{R}(M)$ defined by $dd^c\tau > 0$. If $\alpha \leq \alpha_1 < \beta_1 \leq \beta$ and if $\phi : \mathbf{R}(\alpha_0, \beta_0) \to U \cap M_*$, then $\phi \mid \mathbf{R}(\alpha_1, \beta_1)$ is geodesic in respect to the Kaehler metric on G defined by $dd^c\theta > 0$.*

PROOF: On $\mathbf{R}(\alpha_1, \beta_1)$ we regard ϕ as a map into G with

$$\sqrt{\theta \circ \phi}\,\dot{\phi}^k = \tilde{f}^k \circ \phi \tag{3.42}$$

$$\begin{aligned} \frac{d}{dt}\theta \circ \phi &= (\theta_k \circ \phi)\dot{\phi}^k + (\theta_{\bar{k}} \circ \phi)\dot{\bar{\phi}}^k \\ &= (1/\sqrt{\theta \circ \phi})((\theta_k \circ \phi)\tilde{f}^k \circ \phi + (\theta_{\bar{k}} \circ \phi)\bar{\tilde{f}}^k \circ \phi) \\ &= 2\sqrt{\theta \circ \phi} \end{aligned}$$

$$\frac{d}{dt}\sqrt{\theta \circ \phi} = 1 \quad \text{on } \mathbf{R}(\alpha_1, \beta_1). \tag{3.43}$$

Hence differentiating (3.42) we obtain

$$\begin{aligned} \dot{\phi}^k + \sqrt{\theta \circ \phi}\,\ddot{\phi}^k &= (\tilde{f}^k_j \circ \phi)\dot{\phi}^j + (\tilde{f}^k_{\bar{j}} \circ \phi)\dot{\bar{\phi}}^j \\ &= (\tilde{f}^k_j \circ \phi)\dot{\phi}^j + (1/\sqrt{\theta \circ \phi})(\tilde{f}^k_{\bar{j}} \circ \phi)\bar{\tilde{f}}^j \circ \phi. \end{aligned}$$

Now, (3.37) and (3.38) imply

$$\begin{aligned} \dot{\phi}^k + \sqrt{\theta \circ \phi}\,\ddot{\phi}^k &= (\tilde{f}^k_j \circ \phi)\dot{\phi}^j \\ &= (\theta_{\bar{a}j} \circ \phi)(\theta^{\bar{a}k} \circ \phi)\dot{\phi}^j + (\theta_{\bar{a}} \circ \phi)(\theta^{\bar{a}k}_j \circ \phi)\dot{\phi}^j \\ &= \dot{\phi}^k - (\theta_{\bar{a}} \circ \phi)(\tilde{\Gamma}^k_{bj} \circ \phi)(\theta^{\bar{a}b} \circ \phi)\dot{\phi}^j \\ &= \dot{\phi}^k - (\tilde{\Gamma}^k_{bj} \circ \phi)(\tilde{f}^b \circ \phi)\dot{\phi}^j \\ &= \dot{\phi}^k - \sqrt{\theta \circ \phi}(\tilde{\Gamma}^k_{bj} \circ \phi)\dot{\phi}^b\dot{\phi}^j \end{aligned}$$

which implies

$$\ddot{\phi}^k + \tilde{\Gamma}^k_{bj}\dot{\phi}^b\dot{\phi}^j = 0 \quad \text{on } \mathbf{R}(\alpha_1, \beta_1).$$

Therefore $\phi:\mathbf{R}(\alpha_1, \beta_1) \to M_* \cap U$ is geodesic for $dd^c\theta > 0$. The same calculation for τ in a local coordinate system on $\mathfrak{R}(M)$ shows that $\phi:\mathbf{R}(\alpha_0, \beta_0) \to \mathfrak{R}(M_*)$ is geodesic in respect to $dd^c\tau > 0$ on $\mathfrak{R}(M_*)$. Q.E.D.

Dan Burns first pointed out to me the result of Proposition 3.15 in the manifold case. The proof given here is new and uses only direct local calculations.

LEMMA 3.16: *Assume* (*A*1)–(*A*3). *Let* $\phi:\mathbf{R}(\alpha, \beta) \to M$ *be an integral curve of JF. Then* $\tau \circ \phi:\mathbf{R}(\alpha, \beta) \to \mathbf{R}$ *is constant.*

PROOF: Take $t_0 \in \mathbf{R}(\alpha, \beta)$. Then we can construct the assumptions (A4)–(A6) such that $\phi(t_0) \in U$. Numbers α_0, β_0 exist such that $\alpha \leq \alpha_0 < t_0 < \beta_0 \leq \beta$ and such that $\phi(\mathbf{R}(\alpha_0, \beta_0)) \subset \underset{\cdot}{U}$. Consider $\phi|\mathbf{R}(\alpha_0, \beta_0)$ as a map into G. Then $\dot{\phi} = J\tilde{F} \circ \phi = i\tilde{f} \circ \phi - i\bar{\tilde{f}} \circ \phi$. On $\mathbf{R}(\alpha_0, \beta_0)$ we have

$$\begin{aligned}\frac{d}{dt}\tau \circ \phi = \frac{d}{dt}\theta \circ \phi &= (\theta_j \circ \phi)\dot{\phi}^j + (\theta_{\bar{j}} \circ \phi)\dot{\bar{\phi}}^j \\ &= i(\theta_j \circ \phi)(\tilde{f}^j \circ \phi) - i(\theta_{\bar{j}} \circ \phi)(\bar{\tilde{f}}^j \circ \phi) \\ &= i(\theta \circ \phi) - i(\theta \circ \phi) = 0.\end{aligned}$$

Consequently $d/dt\,(\tau \circ \phi) = 0$ on $\mathbf{R}(\alpha, \beta)$. Hence $\tau \circ \phi$ is constant. Q.E.D.

4. The gradient flow

Let M be a locally compact Hausdorff space. Let τ be a non-negative, continuous function on M. For each $r \geq 0$ define

$$M[r] = \{x \in M \mid \tau(x) \leq r^2\} \qquad M(r) = \{x \in M \mid \tau(x) < r^2\} \tag{4.1}$$

$$M\langle r\rangle = \{x \in M \mid \tau(x) = r^2\} = M[r] - M(r). \tag{4.2}$$

Define $M_* = M - M[0]$ and $\Delta = \sup \sqrt{\tau}$. Then τ is said to be an *exhaustion* with *maximal radius* Δ if and only if $\sqrt{\tau} < \Delta$ on M and if $M[r]$ is compact for every $r \in \mathbf{R}$ with $0 \leq r < \Delta$. Here we call $M[0]$ the *center* of τ. Also $M[r]$ and $M(r)$ are called the *closed* and *open pseudoballs* of radius r of τ and $M\langle r\rangle$ is the *pseudosphere* of radius r of τ.

Let M be an irreducible complex space of dimension m. Then (M, τ) is called *a strictly parabolic space of dimension m and maximal radius* Δ and τ is called a *strictly parabolic exhaustion of maximal radius* Δ if and only if τ is a strictly parabolic function and an exhaustion of M with maximal radius Δ.

Initially, only slightly weaker *assumptions* are needed:

(B1) *Let M be an irreducible complex space of dimension m.*

(B2) *Let τ be an exhaustion of maximal radius Δ and of class C^∞ on M.*

(B3) *Let τ be strictly parabolic on M_*.*

(B4) *Let f, F and Y be the vector fields of class C^∞ on M_* defined by* (3.15) *and* (3.16).

(B5) *Abbreviate* $\delta = \sqrt{\tau} : M \to \mathbf{R}_+$.

For each $p \in M_*$ there exists one and only one maximal integral curve

$$\psi_p : \mathbf{R}(\alpha_p, \beta_p) \to M_* \tag{4.3}$$

of Y where

$$\alpha_p < \delta(p) < \beta_p \qquad \psi_p(\delta(p)) = p. \tag{4.4}$$

LEMMA 4.1: *Assume* (B1)–(B4). *Then* $\alpha_p = 0$ *and* $\beta_p = \Delta$ *for all* $p \in M_*$ *and*

$$\tau(\psi_p(t)) = t^2 \quad \textit{for all } t \in \mathbf{R}(0, \Delta). \tag{4.5}$$

PROOF: First (4.5) shall be proved. Take $t_0 \in \mathbf{R}(\alpha_p, \beta_p)$. Then there exists an embedded chart $\mathfrak{p} : U \to G$ at $\psi_p(t_0)$ where G is an open subset of $\mathbf{C}^n$, and where there exists a strictly parabolic extension $\theta > 0$ of τ on G. Let $\tilde{f}$, $\tilde{F}$ and $\tilde{Y}$ be the associated extensions of the vector fields f, F and Y. There are numbers α, β with $\alpha_p \leq \alpha < t_0 < \beta \leq \beta_p$ such that $\psi_p(\mathbf{R}(\alpha, \beta)) \subseteq U$. On $\mathbf{R}(\alpha, \beta)$ we have

$$\begin{aligned}\frac{d}{dt}(\delta \circ \psi_p) = d\delta(\psi_p, \dot{\psi}_p) &= (1/(2\delta \circ \psi_p))d\theta(\psi_p, \dot{\psi}_p)\\ &= (1/2\delta \circ \psi_p)((\theta_j \circ \psi_p)\dot{\psi}_p^j + (\theta_{\bar{j}} \circ \psi_p)\dot{\bar{\psi}}_p^j)\\ &= (1/2\theta \circ \psi_p)((\theta_j \circ \psi_p)(f^j \circ \psi_p) + (\theta_{\bar{j}} \circ \psi_p)(\bar{f}^j \circ \psi_p))\\ &= (1/2\theta \circ \psi_p)(\theta \circ \psi_p + \theta \circ \psi_p) = 1\end{aligned}$$

on $\mathbf{R}(\alpha, \beta)$. Consequently, $d/dt\,(\delta \circ \psi_p) = 1$. A constant c exists such that $\delta(\psi_p(t)) = t + c$ for all $t \in \mathbf{R}(\alpha_p, \beta_p)$. Since $\delta(p) \in \mathbf{R}(\alpha_p, \beta_p)$, we have

$$\delta(p) + c = \delta(\psi_p(\delta(p))) = \delta(p).$$

Hence $c = 0$. Therefore $\delta(\psi_p(t)) = t$ and $\tau(\psi_p(t)) = t^2$. Since $0 < \sqrt{\tau} = \delta < \Delta$ we obtain $0 \le \alpha_p < \beta_p \le \Delta$.

Now, we shall prove that $\alpha_p = 0$. Assume that $\alpha_p > 0$. Then $K = M[\delta(p)] - M(\alpha_p)$ is compact. For all $t \in \mathbf{R}(\alpha_p, \delta(p)]$ we have $\psi_p(t) \in K$. Therefore a decreasing sequence $\{t_\lambda\}_{\lambda \in \mathbf{N}}$ exists such that $\alpha_p < t_\lambda < \delta(p)$ for all $\lambda \in \mathbf{N}$, such that $t_\lambda \to \alpha_p$ and $\psi_p(t_\lambda) \to q \in K$ for $\lambda \to \infty$. Then $t_\lambda = \delta(\psi_p(t_\lambda)) \to \delta(q)$ for $\lambda \to \infty$. Hence $\delta(q) = \alpha_p$. A local one parameter group

$$\phi : \mathbf{R}(-\epsilon, \epsilon) \times U \to M_*$$

of diffeomorphisms associated to Y exists with $q \in U \subseteq M_*$. Let U_0 and V be open neighborhoods of q such that $\bar{U}_0$ and $\bar{V}$ are compact with

$$q \in V \subset \bar{V} \subset U_0 \subset \bar{U}_0 \subset U.$$

A number $\epsilon_0 \in \mathbf{R}(0, \epsilon)$ exists such that

$$\bar{V} \subset U_t = \phi(t, U_0) \subset \bar{U}_t = \phi(t, \bar{U}_0) \subset U \quad \text{for all } t \in \mathbf{R}(-\epsilon_0, \epsilon_0).$$

Take $\lambda \in \mathbf{N}$ such that $0 < t_\lambda - \alpha_p = r_\lambda < \epsilon_0$ and such that $\psi_p(t_\lambda) \in V \subset U_{r_\lambda}$. Hence $q_\lambda \in U_0$ exists such that $\psi_p(t_\lambda) = \phi(r_\lambda, q_\lambda) = \phi(t_\lambda - \alpha_p, q_\lambda)$. Because $\psi_p(t)$ and $\phi(t - \alpha_p, q_\lambda)$ for $t \in \mathbf{R}(\alpha_p, t_\lambda]$ are integral curves of Y, we have $\psi_p(t) = \phi(t - \alpha_p, q_\lambda)$ for all $t \in \mathbf{R}(\alpha_p, t_\lambda]$. An integral curve $\chi : \mathbf{R}(\alpha_p - \epsilon_0, \beta_p) \to M_*$ of Y is defined by $\chi(t) = \phi(t - \alpha_p, q_\lambda)$ for $t \in \mathbf{R}(\alpha_p - \epsilon, \alpha_p + \epsilon)$ and by $\chi(t) = \psi_p(t)$ if $t \in \mathbf{R}(\alpha_p, \beta_p)$. Then $\chi(\delta(p)) = \psi_p(\delta(p)) = p$. By maximality, we have $\alpha_p \le \alpha_p - \epsilon_0$ which contradicts $\epsilon_0 > 0$. Therefore $\alpha_p = 0$. Now, $\beta_p = \Delta$ is proved by the same method. Q.E.D.

A map

$$\phi : \mathbf{R}(0, \Delta) \times M_* \to M_* \tag{4.6}$$

is defined by $\psi(t, p) = \psi_p(t)$ for $t \in \mathbf{R}(0, \Delta)$ and $p \in M_*$.

LEMMA 4.2: *Assume (B1)–(B5). Then* $\psi : \mathbf{R}(0, \Delta) \times M_* \to M_*$ *is of class* C^∞.

PROOF: The set N of all points in $\mathbf{R}(0, \Delta) \times M_*$ at which ψ is of class C^∞ is open.

1. CLAIM: $(\delta(p_0), p_0) \in N$ for each $p_0 \in M_*$.

PROOF OF THE 1. CLAIM: Let $\phi : \mathbf{R}(-\epsilon, \epsilon) \times U \to M_*$ be a local one parameter group of diffeomorphisms associated to Y at $p_0 \in U$ such that $0 < \delta(p_0) - \epsilon < \delta(p_0) + \epsilon < \Delta$. Take an open neighborhood V of p_0 in U such that $0 < \delta(p) - \epsilon < \delta(p) + \epsilon < \Delta$ for all $p \in V$. Then

$$W = \{(t, p) \in \mathbf{R} \times V \mid |t - \delta(p)| < \epsilon\}$$

is an open neighborhood of $(\delta(p_0), p_0)$ in $\mathbf{R}(0, \Delta) \times M_*$. If $p \in V$, then $\phi(\square, p)$ and $\psi(\square + \delta(p), p)$ are integral curves of Y on $\mathbf{R}(-\epsilon, \epsilon)$ with $\phi(0, p) = p = \psi(\delta(p), p)$. Therefore $\psi(t, p) = \phi(t - \delta(p), p)$ for all $(t, p) \in W$. Hence ψ is of class C^∞ on W and $(\delta(p_0), p_0) \in N$. The 1. Claim is proved.

Take $p_0 \in M_*$. Define

$$S = \{t \in \mathbf{R}(0, \delta(p_0)) \mid \mathbf{R}(t, \delta(p_0)) \times \{p_0\} \subset N\}$$
$$T = \{t \in \mathbf{R}(0, \delta(p_0)) \mid \mathbf{R}[\delta(p_0), t) \times \{p_0\} \subset N\}.$$

According to the 1. Claim $S \neq \emptyset \neq T$. We have

$$0 \leq s_0 = \inf S < \delta(p_0) < \sup T = t_0 \leq \Delta \leq \infty.$$

2. CLAIM: $t_0 = \Delta$.

PROOF OF THE 2. CLAIM: Assume that $t_0 < \Delta$. Then $q_0 = \psi(t_0, p_0) \in M_*$ is defined. There exists a local one parameter group of diffeomorphisms $\phi : \mathbf{R}(-\epsilon, \epsilon) \times U \to M_*$ at $q_0 \in U$ associated to Y such that $\delta(p_0) < t_0 - \epsilon < t_0 + \epsilon < \Delta$. Take open neighborhoods U_0 and X of q_0 with compact closures such that $q_0 \in X \subset \bar{X} \subset U_0 \subset \bar{U}_0 \subset U$. A number $\epsilon_0 \in \mathbf{R}(0, \epsilon)$ exists such that

$$\bar{X} \subset U_t = \phi(t, U_0) \subset \bar{U}_t = \phi(t, \bar{U}_0) \subset U \quad \text{for all } t \in \mathbf{R}(-\epsilon_0, \epsilon_0).$$

Take $t_1 \in \mathbf{R}(t_0 - \epsilon_0, t_0)$ such that $\psi(t_1, p_0) \in X$. Then $(t_1, p_0) \in N$. An open neighborhood V of p_0 in M_* exists such that $\psi(t_1, p) \in X$ for all $p \in V$ and such that $\delta(p) < t_0 - \epsilon$ for all $p \in V$. Define $r = t_1 - t_0 \in \mathbf{R}(-\epsilon_0, \epsilon_0)$. Then $\chi = \phi(r, \square): U_0 \to U_r$ is a diffeomorphism of class C^∞. Hence $\rho = \chi^{-1} \circ \psi(t_1, \square): V \to U_0$ is a map of class C^∞ with $\chi \circ \rho = \psi(t_1, \square)$ on V. Therefore $\psi(t_1, p) = \phi(t_1 - t_0, \rho(p))$ for all $p \in V$. Since $\psi(\square, p)$ and $\phi(\square - t_0, \rho(p))$ are integral curves of Y on the interval $\mathbf{R}(t_0 - \epsilon_0, t_0 + \epsilon_0)$ which contains t_1, we obtain $\psi(t, p) = \phi(t - t_0, \rho(p))$ for all $t \in \mathbf{R}(t_0 - \epsilon_0, t_0 + \epsilon_0)$ and $p \in V$. Hence ψ is of class C^∞ on $\mathbf{R}(t_0 - \epsilon_0, t_0 \in \epsilon_0) \times V$. In particular, we see that $\mathbf{R}[\delta(p_0), t_0 + \epsilon_0) \times \{p_0\} \subseteq N$. Hence $t_0 + \epsilon_0 \in T$ which implies $t_0 + \epsilon_0 \leq \sup T = t_0$. Contradiction! Therefore $t_0 = \Delta$. The 2. Claim is proved.

3. Claim: $s_0 = 0$.

Proof of the 3. Claim: Assume that $s_0 > 0$. Then $q_0 = \psi(s_0, p_0) \in M_*$ is defined. There exists a local one parameter group of diffeomorphisms $\phi: \mathbf{R}(-\epsilon, \epsilon) \times U \to M_*$ at $q_0 \in U$ associated to Y such that $0 < s_0 - \epsilon < s_0 + \epsilon < \delta(p_0)$. Take open neighborhoods U_0 and X of q_0 with compact closures such that $q_0 \in X \subset \bar{X} \subset U_0 \subset \bar{U}_0 \subset U$. A number $\epsilon_0 \in \mathbf{R}(0, \epsilon)$ exists such that

$$\bar{X} \subset U_t = \phi(t, U_0) \subset \phi(t, \bar{U}_0) \subset \bar{U}_t = \phi(t, \bar{U}_0) \subset U$$
$$\text{for all } t \in \mathbf{R}(-\epsilon_0, \epsilon_0).$$

Take $t_1 \in \mathbf{R}(s_0, s_0 + \epsilon_0)$ such that $\psi(t_1, p_0) \in X$. Then $(t_1, p_0) \in N$. An open neighborhood V of p_0 in M_* exists such that $\psi(t_1, p) \in X$ for all $p \in V$ and such that $s_0 + \epsilon < \delta(p)$ for all $p \in V$. Define $r = t_1 - s_0 \in \mathbf{R}(-\epsilon_0, \epsilon_0)$. Then $\chi = \phi(r, \square): U_0 \to U_r$ is a diffeomorphism of class C^∞. Hence $\rho = \chi^{-1} \circ \psi(t_1, \square): V \to U_0$ is a map of class C^∞ with $\chi \circ \rho = \psi(t_1, \square)$ on V. Therefore $\psi(t_1, p) = \phi(t_1 - s_0, \rho(p))$ for all $p \in V$. Since $\psi(\square, p)$ and $\phi(\square - s_0, \rho(p))$ are integral curves of Y on $\mathbf{R}(s_0 - \epsilon_0, s_0 + \epsilon_0)$ which contains t_1, we obtain $\psi(t, p) = \phi(t - s_0, \rho(p))$ for all $t \in \mathbf{R}(s_0 - \epsilon_0, s_0 + \epsilon_0)$ and $p \in V$. Hence ψ is of class C^∞ on $\mathbf{R}(s_0 - \epsilon, s_0 + \epsilon) \times V$ which implies $\mathbf{R}(s_0 - \epsilon_0, \delta(p_0)] \times \{p_0\} \subset N$. Hence $s_0 - \epsilon_0 \in S$ which implies $s_0 - \epsilon_0 \geq \inf S = s_0$. Contradiction! Therefore $s_0 = 0$. The 3. Claim is proved.

Consequently, $N = \mathbf{R}(0, \Delta) \times M_*$. Q.E.D.

Let X and Y be complex spaces. Let $A \neq \emptyset$ be a subset of X and let $B \neq \emptyset$ be a subset of Y. A map $h: A \to B$ is said to be of class C^∞ if for every point $a \in A$ there exists an open neighborhood U of a in X and a map $H: U \to Y$ of class C^∞ such that $H \mid U \cap A = h \mid U \cap A$. The map $h: A \to B$ is said to be a diffeomorphism of class C^∞, if and only if h is bijective and if h and h^{-1} are of class C^∞.

THEOREM 4.3: *Assume (B1)–(B5). Then there exists one and only one map*

$$\psi : \mathbf{R}(0, \Delta) \times M_* \to M_* \tag{4.6}$$

of class C^∞ called the gradient flow on M_* such that

(1) *For each $p \in M_*$, the curve $\psi(\square, p): \mathbf{R}(0, \Delta) \to M_*$ is an integral curve of Y.*

(2) *For each $p \in M_*$, we have $\psi(\sqrt{\tau(p)}, p) = p$.*

(3) *If $p \in M_*$ and $t \in \mathbf{R}(0, \Delta)$, then $\tau(\psi(t, p)) = t^2$ which means $\psi(t, p) \in M\langle t\rangle$.*

(4) *If $p \in M_*$, if $t \in \mathbf{R}(0, \Delta)$ and if $r \in \mathbf{R}(0, \Delta)$, then $\psi(t, \psi(r, p)) = \psi(t, p)$.*

(5) *If t, r and s belong to $\mathbf{R}(0, \Delta)$ a diffeomorphism $\psi_{rs}: M\langle s\rangle \to M\langle r\rangle$ is defined by $\psi_{rs}(p) = \psi(r, p)$ for all $p \in M\langle s\rangle$. Then ψ_{rr} is the identity and $\psi_{sr} = \psi_{rs}^{-1}$ and $\psi_{tr} \circ \psi_{rs} = \psi_{ts}$.*

PROOF: The existence of ψ with the properties (1), (2) and (3) has already been shown. Also (1) and (2) define ψ uniquely. Only (4) and (5) remain to be proved. Take $p \in M_*$ and $r \in \mathbf{R}(0, \Delta)$. Then $\psi(\square, \psi(r, p))$ and $\psi(\square, p)$ are integral curves of Y on $\mathbf{R}(0, \Delta)$ with $\psi(r, \psi(r, p)) = \psi(\delta(\psi(r, p)), \psi(r, p)) = \psi(r, p)$. Therefore $\psi(t, \psi(r, p)) = \psi(t, p)$ for all $t \in \mathbf{R}(0, \Delta)$ which proves (4). Clearly ψ_{rs} maps $M\langle s\rangle$ into $M\langle r\rangle$ by (3). Also (4) implies $\psi_{tr} \circ \psi_{rs} = \psi_{ts}$. We have $\psi_{ss}(p) = \psi(s, p) = \psi(\delta(p), p) = p$ for $p \in M\langle s\rangle$. Hence ψ_{ss} is the identity. Hence $\psi_{sr} \circ \psi_{rs}$ and $\psi_{rs} \circ \psi_{sr}$ are identities. Hence $\psi_{rs}: M\langle s\rangle \to M\langle r\rangle$ is a diffeomorphism of class C^∞ with $(\psi_{rs})^{-1} = \psi_{sr}$. Q.E.D.

THEOREM 4.4: *Let (M, τ) be a strictly parabolic space of dimension m and of maximal radius Δ. Then the center $M[0]$ consists of one and only one point.*

PROOF: Take $p \in M_*$ and $r \in \mathbf{R}(0, \Delta)$. Then $\psi(t, p) \in M[r]$ for all $t \in \mathbf{R}(0, r]$. Since $M[r]$ is compact, there exists a sequence $\{t_\lambda\}_{\lambda \in \mathbf{N}}$ with $t_\lambda \in \mathbf{R}(0, r]$ such that $t_\lambda \to 0$ and $\psi(t_\lambda, p) \to q \in M[r]$ for $\lambda \to \infty$. Then $t_\lambda = \delta(\psi(t_\lambda, p)) \to \delta(q)$ for $\lambda \to \infty$. Hence $\delta(q) = 0$ and $q \in M[0]$. Therefore $M[0] \neq 0$.

By Lemma 3.7, the compact set $M[0]$ consists of isolated points only. Hence $M[0]$ is finite. For every $a \in M[0]$ take an open neighborhood U_a of a such that $U_a \cap U_b = \emptyset$ if $a \neq b$ and $a \in M[0]$ and $b \in M[0]$. Then

$$U = \bigcup_{a \in M[0]} U_a \tag{4.7}$$

is an open neighborhood of $M[0]$. Since $\delta : M \to \mathbf{R}[0, \Delta)$ is proper with $M[0] = \delta^{-1}(0)$, a number $t_0 > 0$ exists such that $M[t_0] \subset U$. Take any $p \in M_*$. Then $\psi(t, p) \in M[t_0] \subset U$ for all $t \in \mathbf{R}[0, t_0]$. Since the union (4.7) is disjoint, one and only one point $\alpha(p) \in M[0]$ exists such that $\psi(t, p) \in U_{\alpha(p)}$ for all $t \in \mathbf{R}(0, t_0)$. Take $a \in M[0]$. Take $p \in M_* \cap U_a \cap M[t_0]$. Then $0 < t_1 = \delta(p) \leq t_0$ and $p = \psi(t_1, p) \in U_a \cap U_{\alpha(p)} \cap M[t_0]$. Since the union (4.7) is disjoint, we conclude that $a = \alpha(p)$. The map $\alpha : M_* \to M[0]$ is surjective. Take $p \in M_*$. Then $\psi(t_0, p) \in U_{\alpha(p)}$. An open neighborhood V of p exists such that $\psi(t_0, q) \in U_{\alpha(p)}$ for all $q \in V$. Then $\psi(t_0, q) \in U_{\alpha(p)} \cap U_{\alpha(q)}$ which implies $\alpha(q) = \alpha(p)$ for all $q \in V$. The map $\alpha : M_* \to M[0]$ is locally constant. Since M is irreducible, M_* is connected. Therefore $\alpha : M_* \to M[0]$ is constant. Since $\alpha : M_* \to M[0]$ is surjective, $M[0]$ consists of one and only point. Q.E.D.

The single point of $M[0]$ is denoted by O_M and is called the *center point* of M. The map $\psi : \mathbf{R}(0, \Delta) \times M_* \to M_*$ is extended to $\psi : \mathbf{R}[0, \Delta) \times M_* \to M$ by setting $\psi(0, p) = O_M$ for all $p \in M_*$.

LEMMA 4.5: *Let (M, τ) be a strictly parabolic space of dimension m and of maximal radius Δ. Then $\psi : \mathbf{R}[0, \Delta) \times M_* \to M$ is continuous.*

PROOF: *Take* $p_0 \in M_*$. Take any open neighborhood U of O_M. A number $t_0 \in \mathbf{R}(0, \Delta)$ exists such that $M[t_0] \subset U$. Then $\psi(t, p) \in M[t_0] \subset U$ for all $t \in \mathbf{R}[0, t_0]$ and $p \in M_*$. Hence ψ is continuous at $(0, p_0)$. Q.E.D.

In fact, $\psi : \mathbf{R}[0, \Delta) \times M_* \to M$ is of class C^∞ as will be shown. We make the following construction which is possible by Kobayashi–Nomizu [10] pp. 149–151 and 165–166 and by Whithead [21].

(C1) *Let (M, τ) be a strictly parabolic space of dimension m and maximal radius Δ.*

(C2) *Let f, F and Y be the vector fields of class C^∞ on M respectively M_* defined by* (3.15) *and* (3.16).

(C3) *Abbreviate* $\delta = \sqrt{\tau} : M \to \mathbf{R}_+$. *Define* ψ *by* (4.6) *and by* $\psi(0, p) = O_M$ *for all* $p \in M_*$.

(C4) *Let* $\mathfrak{p} : U \to G$ *be an embedded chart of* M *at* O_M, *where* G *is an open neighborhood of* $O_M = 0 \in \mathbf{C}^n$. *Let* $w^1, \ldots, w^n$ *be the coordinate functions on* $\mathbf{C}^n$.

(C5) *Let* θ *be a strictly parabolic extension of* τ *on* G *and let* $\tilde{f}$, $\tilde{F}$ *and* $\tilde{Y}$ *be the associated extensions of the vector fields* f, F *and* Y.

(C6) *Take the base of* $\mathbf{C}^n$ *such that* $\theta_{j\bar{k}}(0)$ *if* $1 \leq j < k \leq n$ *and* $\theta_{j\bar{j}}(0) = 1$ *for all* $j \in \mathbf{N}[1, n]$.

(C7) *Let* κ *be the Kaehler metric on* G *defined by* $dd^c\theta > 0$.

(C8) *Let* G_0 *be an open neighborhood of* $O_M = 0$ *in* G *such that* G_0 *is convex in respect to* κ. *Define* $U_0 = G_0 \cap U$.

(C9) *If* $p \in G_0$ *and* $q \in G_0$, *one and only one geodesic* $\alpha(\square, p, q) : \mathbf{R}[0, 1] \to G_0$ *exists with* $\alpha(0, p, q) = p$ *and* $\alpha(1, p, q) = q$. *The map*

$$\alpha : \mathbf{R}[0, 1] \times G_0 \times G_0 \to G_0 \tag{4.8}$$

is of class C^∞.

(C10) *For* $p \in G_0$ *let* T_p *be the real tangent space of* G_0 *at* p *endowed with the euclidean metric defined by* κ. *For* $r > 0$ *define*

$$T_p[r] = \{X \in T_p \mid \|X\| \leq r\} \qquad T_p(r) = \{X \in T_p \mid \|X\| < r\} \tag{4.9}$$

$$T_p\langle r\rangle = \{X \in T_p \mid \|X\| = r\} = T_p[r] - T_p(r). \tag{4.10}$$

(C11) *For* $p \in G_0$, *there exists a number* $s(p) > 0$ *and an open neighborhood* H_p *of* p *in* G *such that* $G_0 \subseteq H_p \subseteq G$ *and such that* $\exp_p : T_p(s(p)) \to H_p$ *is a diffeomorphism of class* C^∞.

(C12) *If* $p \in G_0$, *if* $X \in T_p(s(p))$ *and if* $\exp_p X \in G_0$, *then*

$$\alpha(t, p, \exp_p X) = \exp_p tX \quad \text{for all } t \in \mathbf{R}[0, 1]. \tag{4.11}$$

(C13) *There exists a number* $r_0 \in \mathbf{R}(0, s(0))$ *such that* $\exp_0 : T_0(r_0) \to G_0$ *is a diffeomorphism. Moreover if* $t \in \mathbf{R}[0, 1]$ *and* $X \in T_0(r_0)$, *then*

$$\alpha(t, 0, \exp_0 X) = \exp_0 tX. \tag{4.12}$$

(C14) *Take* $t_0 \in \mathbf{R}(0, \Delta)$ *with* $0 < t_0 < r_0$ *such that* $M[t_0] \subset G_0$.

PROPOSITION 4.6: *Assume* (C1)–(C14). *Take* $p \in M_*$ *and* $t \in \mathbf{R}[0, t_0]$. *Then*

$$\psi(t, p) = \alpha\left(\frac{t}{t_0}, 0, \psi(t_0, p)\right). \tag{4.13}$$

If $p \in M\langle t_0\rangle$ and $t \in \mathbf{R}[0, t_0]$, then

$$\psi(t, p) = \alpha\left(\frac{t}{t_0}, 0, p\right). \tag{4.14}$$

The map $\psi: \mathbf{R}[0, \Delta) \times M_* \to M$ is of class C^∞.

PROOF: Take a sequence $\{t_\nu\}_{\nu\in\mathbf{N}}$ with $t_\nu \in \mathbf{R}(0, t_0)$ such that $t_\nu \to 0$ for $\nu \to \infty$. By Proposition 3.15 a geodesic $\rho_\nu: \mathbf{R}[0, 1] \to G_0$ is defined by

$$\rho_\nu(t) = \psi(t(t_0 - t_\nu) + t_\nu, p) \in M[t_0] \subset G_0 \quad \text{for all } t \in \mathbf{R}[0, 1]$$

where $\rho_\nu(0) = \psi(t_\nu, p)$ and $\rho_\nu(1) = \psi(t_0, p)$. Therefore

$$\rho_\nu(t) = \alpha(t, \psi(t_\nu, p), \psi(t_0, p)) \quad \text{for all } t \in \mathbf{R}[0, 1].$$

Now $\nu \to \infty$ implies

$$\psi(tt_0, p) = \alpha(t, 0, \psi(t_0, p)) \qquad \text{for all } t \in \mathbf{R}[0, 1]$$

$$\psi(t, p) = \alpha\left(\frac{t}{t_0}, 0, \psi(t_0, p)\right) \quad \text{for all } t \in \mathbf{R}[0, t_0].$$

Consequently, $\psi: \mathbf{R}[0, t_0) \times M_* \to M$ is of class C^∞. Since ψ is of class C^∞ on $\mathbf{R}(0, \Delta) \times M_*$, we see that $\psi: \mathbf{R}[0, \Delta) \times M_* \to M$ is of class C^∞. If $p \in M\langle t_0\rangle$, then $\psi(t_0, p) = p$ which implies (4.14). Q.E.D.

Theorem 4.3(4) shows that the gradient lines are over-parameterized. A bijective parameterization shall be introduced. If $p \in G_0$, the tangent space T_P of G_0 is $\mathbf{C}^n$ but the Kaehler metric κ may not coincide with the standard euclidean metric on $\mathbf{C}^n$. However, if $p = O_M = 0 \in G$, this is the case by (C6). Then the standard euclidean exhaustion function τ_0 on $T_0 = \mathbf{C}^n$ is defined by

$$\tau_0(w) = \|w\|^2 = |w_1|^2 + \cdots + |w_n|^2 \quad \text{if } w = (w_1, \ldots, w_n) \in \mathbf{C}^n. \tag{4.15}$$

For $A \subseteq \mathbf{C}^n$ and $r \geq 0$ define

$$(4.16)\qquad \begin{aligned} A[r]_0 &= \{w \in A \mid \tau_0(w) \leq r^2\} \\ A(r)_0 &= \{w \in A \mid \tau_0(w) < r^2\} \end{aligned}$$

$$(4.17)\qquad A\langle r\rangle_0 = A[r]_0 - A(r)_0 = \{w \in A \mid \tau_0(w) = r^2\}.$$

Now we will make the additional construction.

(C15) *Let K be the Whitney tangent cone of M at $O_M = 0$ embedded into $T_0 = \mathbf{C}^n$.*

Then $K[r]_0$, $K(r)_0$ and $K\langle r\rangle_0$ are defined for $r \geq 0$ and τ_0 is a strictly parabolic exhaustion of K with maximal radius ∞. For $q \in G$, we identify $T_q(G) = \mathbf{C}^n$. If $p \in M$ and $t \in \mathbf{R}[0, t_0]$, then $q = \psi(t, p) \in U_0 \subseteq G_0$ and $\dot{\psi}(t, p) \in T_q(G) = \mathbf{C}^n$.

LEMMA 4.7: *Assume (C1)–(C15). Then $\dot{\psi}(0, p) \in K\langle 1\rangle_0$ for all $p \in M_*$.*

PROOF: Define $\xi = \dot{\psi}(0, p)$. Regard $\psi: \mathbf{R}[0, t_0] \times M_* \to M[t_0] \subset G_0$ as a map into G_0. Then $\psi(0, p) = 0$. A vector function $\psi_0: \mathbf{R}[0, t_0] \to \mathbf{C}^n$ of class C^∞ exists such that $\psi(t, p) = t^2\psi_0(t)$ for all $t \in \mathbf{R}[0, t_0]$. Here $\psi(t, p) \in U$. Hence

$$\xi = \lim_{0<t\to 0} \frac{\psi(t, p) - \psi(0, p)}{t} \in K.$$

Define $\psi = (\psi^1, \ldots, \psi^n)$ and $b_{jk} = \theta_{jk}(0)$. Observe $\theta_{j\bar{k}}(0) = 0$ if $j \neq k$ and $\theta_{j\bar{j}}(0) = 1$. Now (3.13) implies

$$\begin{aligned} t^2 &= \tau(\psi(t, p)) = \theta(\psi(t, p)) \\ &= \operatorname{Re} b_{jk}\psi^j(t, p)\psi^k(t, p) + \|\psi(t, p)\|^2 + 0(t^3). \end{aligned}$$

Division by t^2 and the limit $t \to 0$ implies

$$1 = \operatorname{Re} b_{jk}\xi^j\xi^k + \|\xi\|^2.$$

By (3.14) we have $b_{jk}\xi^j = 0$. Hence $\|\xi\|^2 = 1$. Q.E.D.

LEMMA 4.8: *Assume (C1)–(C15). Take $\xi \in K\langle 1\rangle_0$. Then there exists one and only one $\mathfrak{q}(\xi) \in M\langle t_0\rangle$ such that*

$$(4.18)\qquad \dot{\psi}(0, \mathfrak{q}(\xi)) = \xi.$$

PROOF: Since α has class C^∞, since $\alpha(0,0,q)=0$ for all $q \in G_0$, there exists a vector function $\alpha_0 : \mathbf{R}[0,1] \times G_0 \to \mathbf{C}^n$ of class C^∞ such that

$$\alpha(t,0,q) = t\dot{\alpha}(0,0,q) + t^2\alpha_0(t,q)$$

where $\dot{\alpha}$ denotes the derivative of α in respect to the first variable t, and where $t \in \mathbf{R}[0,1]$ and $q \in G_0$. Since the geodesic $\alpha(\square, 0, q)$ from 0 to q is not constant $\ell(q) = \dot{\alpha}(0,0,q) \neq 0$ if $0 \neq q \in G_0$.

Take a sequence $\{p_\nu\}_{\nu \in \mathbf{N}}$ of points $O_M \neq p_\nu \in M(t_0)$ such that $p_\nu \to O_M = 0$ and $p_\nu/\|p_\nu\| \to \xi$ for $\nu \to \infty$. Observe $\psi(t_0, p_\nu) \in M\langle t_0 \rangle$.

Since $M\langle t_0\rangle$ is compact, we can assume that $q_\nu = \psi(t_0, p_\nu) \to q \in M\langle t_0 \rangle$ for $\nu \to \infty$. Define $t_\nu = \delta(p_\nu)$. Then $0 < t_\nu < t_0$ and $t_\nu \to 0$ for $\nu \to \infty$. By (4.13) we have

$$p_\nu = \psi(t_\nu, p_\nu) = \alpha\left(\frac{t_\nu}{t_0}, 0, \psi(t_0, p_\nu)\right)$$

$$= \frac{t_\nu}{t_0}\ell(q_\nu) + \left(\frac{t_\nu}{t_0}\right)^2 \alpha_0\left(\frac{t_\nu}{t_0}, q_\nu\right)$$

which implies

$$\frac{p_\nu}{\|p_\nu\|} = \frac{t_0\ell(q_\nu) + t_\nu\alpha_0(t_\nu/t_0, q_\nu)}{\|t_0\ell(q_\nu) + t_\nu\alpha_0(t_\nu/t_0, q_\nu)\|}.$$

Since $\ell(q) \neq 0$, the limit $\nu \to \infty$ implies

$$\xi = \frac{\ell(q)}{\|\ell\ell(q)\|}.$$

For $t \in \mathbf{R}[0, t_0]$ we have $\psi(t,q) = \alpha(t/t_0, 0, q)$. Hence

$$\dot{\psi}(0,q) = \frac{1}{t_0}\dot{\alpha}(0,0,q) = \frac{\ell(q)}{t_0}.$$

Lemma 4.7 implies $\|\ell(q)\| = t_0\|\dot{\psi}(0,q)\| = t_0$. Hence $\xi = \dot{\psi}(0,q)$. If $q \in M\langle t_0\rangle$ and $p \in M\langle t_0 \rangle$ are given such that $\dot{\psi}(0,p) = \xi = \dot{\psi}(0,q)$, then $\psi(t,p) = \psi(t,q)$ for all $t \in \mathbf{R}[0,t_0]$ since $\psi(\square, p)$ and $\psi(\square, q)$ are geodesics of κ with $\psi(0,p) = O_M = \psi(0,q)$. Since $\delta(p) = t_0 = \delta(q)$ we have

$$p = \psi(t_0, p) = \psi(t_0, q) = q.$$

Hence $\mathfrak{q}(\xi) = q$ is uniquely defined such that (4.18) holds. Q.E.D.

A map $\mathfrak{q}: K\langle 1\rangle_0 \to M\langle t_0\rangle$ is defined by (4.18).

THEOREM 4.9: *Assume* $(C1)$–$(C15)$. *Then* $\mathfrak{q}: K\langle 1\rangle_0 \to M\langle t_0\rangle$ *is a diffeomorphism of class* C^∞ *such that*

$$\mathfrak{q}^{-1}(p) = \dot{\psi}(0, p) \quad \text{for all } p \in M\langle t_0\rangle \tag{4.19}$$

$$\mathfrak{q}(\xi) = \exp_0(t_0\xi) \quad \text{for all } \xi \in K\langle 1\rangle_0. \tag{4.20}$$

PROOF: By (4.18) $\mathfrak{q}$ is injective. Take $p \in M\langle t_0\rangle$. Then $\xi = \dot{\psi}(0, p) \in K\langle 1\rangle_0$. Also $\dot{\psi}(0, \mathfrak{q}(\xi)) = \xi$. By uniqueness, $p = \mathfrak{q}(\xi)$. Hence $\mathfrak{q}$ is surjective. Hence $\mathfrak{q}$ is bijective with $\mathfrak{q}^{-1}(p) = \dot{\psi}(0, p)$. Also $\mathfrak{q}^{-1}$ is of class C^∞. Take $\xi \in K\langle 1\rangle_0$. Since $0 < t_0 < r_0$, a geodesic $\rho: \mathbf{R}[0, t_0] \to G_0$ is defined by $\rho(t) = \exp_0(t\xi)$ for $t \in \mathbf{R}[0, t_0]$ where $\rho(0) = 0$ and $\dot{\rho}(0) = \xi$. Also $\psi(\square, \mathfrak{q}(\xi)): \mathbf{R}[0, t_0] \to G_0$ is a geodesic with $\psi(0, \mathfrak{q}(\xi)) = 0$ and $\dot{\psi}(0, \mathfrak{q}(\xi)) = \xi$. Hence $\psi(t, \mathfrak{q}(\xi)) = \rho(t) = \exp_0(t\xi)$ for all $t \in \mathbf{R}[0, t_0]$. Hence $\mathfrak{q}(\xi) = \psi(t_0, \mathfrak{q}(\xi)) = \exp_0(t_0\xi)$. Therefore $\mathfrak{q}$ is of class C^∞. Consequently $\mathfrak{q}: K\langle 1\rangle_0 \to M\langle t_0\rangle$ is a diffeomorphism of class C^∞. Q.E.D.

THEOREM 4.10: *Assume* $(C1)$–$(C15)$. *A map*

$$\psi: \mathbf{R}[0, \Delta) \times K\langle 1\rangle_0 \to M \tag{4.21}$$

of class C^∞ *is defined by*

$$\psi(t, \xi) = \psi(t, \mathfrak{q}(\xi)) \tag{4.22}$$

for all $t \in \mathbf{R}[0, \Delta)$ *and* $\xi \in K\langle 1\rangle_0$. *The following properties are satisfied.*

(1) *For each* $\xi \in K\langle 1\rangle_0$, *the curve* $\psi(\square, \xi): \mathbf{R}(0, \Delta) \to M_*$ *is an integral curve of* Y.

(2) *For each* $\xi \in K\langle 1\rangle_0$, *we have* $\psi(0, \xi) = O_M$ *and* $\dot{\psi}(0, \xi) = \xi$ *and* $\psi(t_0, \xi) = \mathfrak{q}(\xi)$.

(3) *If* $t \in \mathbf{R}[0, \Delta)$ *and* $\xi \in K\langle 1\rangle_0$, *we have* $\tau(\psi(t, \xi)) = t^2$, *which means* $\psi(t, \xi) \in M\langle t\rangle$.

(4) *If* $t \in \mathbf{R}[0, t_0]$ *and* $\xi \in K\langle 1\rangle_0$, *then* $\psi(t, \xi) = \exp_0(t\xi)$.

(5) *The map* $\psi: \mathbf{R}(0, \Delta) \times K\langle 1\rangle_0 \to M_*$ *is a diffeomorphism of class* C^∞ *with*

(4.23)
$$\psi^{-1}(p) = (\sqrt{\tau(p)}, \mathfrak{q}^{-1}(\psi(t_0, p))) \\ = (\sqrt{\tau(p)}, \dot{\psi}(0, \psi(t_0, p)))$$

for all $p \in M_*$.

PROOF: (1)–(3) are already established. If $t \in \mathbf{R}[0, t_0]$ and $\xi \in K\langle 1\rangle_0$, then

$$\psi(t, \xi) = \psi(t, \mathfrak{q}(\xi)) = \alpha\left(\frac{t}{t_0}, 0, \psi(t_0, \mathfrak{q}(\xi)\right) = \alpha\left(\frac{t}{t_0}, 0, \exp_0 t_0\xi\right) \\ = \exp_0\left(\frac{t}{t_0}\, t_0\xi\right) = \exp_0(t\xi)$$

which proves (4). A map $\rho: M_* \to \mathbf{R}(0, \Delta) \times K\langle 1\rangle_0$ of class C^∞ is defined by

$$\rho(p) = (\delta(p), \mathfrak{q}^{-1}(\psi(t_0, p))) \quad \text{for } p \in M_*.$$

If $p \in M_*$, then

$$\psi(\rho(p)) = \psi(\delta(p), \mathfrak{q}^{-1}(\psi(t_0, p)) = \psi(\delta(p), \psi(t_0, p)) \\ = \psi(\delta(p), p) = p.$$

If $(t, \xi) \in \mathbf{R}(0, \Delta) \times K\langle 1\rangle_0$, then

$$\rho(\psi(t, \xi) = (\delta(\psi(t, \xi)), \mathfrak{q}^{-1}(\psi(t_0, \psi(t, \xi)))) \\ = (t, \mathfrak{q}^{-1}(\psi(\delta(\mathfrak{q}(\xi)), \mathfrak{q}(\xi)))) \\ = (t, \mathfrak{q}^{-1}(\mathfrak{q}(\xi))) \\ = (t, \xi).$$

Therefore ψ is a diffeomorphism of class C^∞ with $\psi^{-1} = \rho$. Q.E.D.

Let (M, τ) be a strictly parabolic space of dimension m and maximal radius Δ. Let K be the Whitney tangent cone at O_M. We can consider the Whitney tangent cone as an analytic cone embedded in the holomorphic tangent space $\mathfrak{T}_{O_M}(M) = \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal in the ring of germs of holomorphic functions. Pick any positive definite hermitian form on $\mathfrak{T}_{O_M}(M)$ and define $\tau_0 = \|\ \|^2$ in respect to this form. Then $K(r) = K(r)_0 = \{x \in K \mid \tau_0(x) < r^2\}$ is defined for all $r > 0$. Now we introduce the construction (C1)–(C15) and identify $\mathbf{C}^n = \mathfrak{T}_{0_M}(G) = \mathfrak{T}_0(\mathbf{C}^n)$ by a complex linear isometry and identify $\mathfrak{T}_0(\mathbf{C}^n) = T_0(\mathbf{C}^n)$ by η_0. If we set $K\langle r\rangle = K\langle r\rangle_0$ in this identification, the map

$$\psi : \mathbf{R}[0, \Delta) \times K\langle 1\rangle \to M$$

becomes available. Define

$$h : K(\Delta) \to M \tag{4.24}$$

by

$$h(w) = \begin{cases} \psi\left(\|w\|, \dfrac{w}{\|w\|}\right) & \text{if } 0 \neq w \in K(\Delta) \\ O_M & \text{if } w = 0. \end{cases} \tag{4.25}$$

THEOREM 4.11: *Let (M, τ) be a strictly parabolic space of dimension m and maximal radius Δ. Then the map $h : K(\Delta) \to M$ defined in (4.24) is a diffeomorphism of class C^∞ with $\tau \circ h = \tau_0$.*

REMARK: In the language of the construction (C1)–(C15) we have

$$h(w) = \exp_0 w \quad \text{for all } w \in K(t_0)_0 = K(t_0). \tag{4.26}$$

PROOF: Define $\mathbf{C}^n_* = \mathbf{C}^n - \{0\}$ and $K_* = K - \{0\}$. A diffeomorphism

$$\rho : \mathbf{C}^n_* \to \mathbf{R}^+ \times \mathbf{C}^n\langle 1\rangle_0$$

is defined by $\rho(w) = (\|w\|, w/\|w\|)$ where $\rho^{-1}(t, \xi) = t\xi$ if $t \in \mathbf{R}^+$ and $\xi \in \mathbf{C}^n\langle 1\rangle_0$. Then ρ restricts to a diffeomorphism $\rho : K_*(\Delta)_0 \to \mathbf{R}(0, \Delta) \times K\langle 1\rangle_0$. Hence $h = \psi \circ \rho : K_*(\Delta)_0 \to M_*$ is a diffeomorphism. If $0 \neq w \in K(t_0)_0$, then

$$h(w) = \psi(\|w\|, w/\|w\|) = \exp_0\left(\|w\| \frac{w}{\|w\|}\right) = \exp_0(w).$$

If $w = 0$, then $h(0) = O_M = \exp_0(0)$. Hence h is a local diffeomorphism at 0. Since h is bijective, and a diffeomorphism on $K_*(\Delta)_0$, we see that $h : K(\Delta) \to M$ is a diffeomorphism. If $0 \neq w \in K(\Delta)$, then

$$\tau(h(w)) = \tau\left(\psi\left(\|w\|, \frac{w}{\|w\|}\right)\right) = \|w\|^2 = \tau_0(w).$$

If $w = 0$, then $\tau(h(0)) = \tau(O_M) = 0 = \tau_0(0)$. Hence $\tau \circ h = \tau_0$ on M. Q.E.D.

In fact, h is biholomorphic, but considerable effort is required to prove it. The following expansion will be needed.

LEMMA 4.12: *Assume* $(C1)$–$(C15)$. *Then there exists one and only one vector function*

$$\psi_0 : \mathbf{R}[0, t_0] \times K\langle 1\rangle_0 \to \mathbf{C}^n \tag{4.27}$$

of class C^∞ *such that*

$$\psi(t, \xi) = t\xi + t^2\psi_0(t, \xi) \tag{4.28}$$

for all $t \in \mathbf{R}[0, t_0]$ *and* $\xi \in K\langle 1\rangle_0$.

PROOF: We have $\exp_0(0) = 0$ and $d\exp_0(0, X) = X$. Hence there exists a vector function $Q : \mathbf{C}^n(r_0)_0 \times \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^n$ of class C^∞ such that $Q(X, \square, \square) : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^n$ is bilinear over $\mathbf{R}$ for each $X \in \mathbf{C}^n(r_0)_0$ and such that

$$\exp_0(X) = X + Q(X, X, X) \quad \text{for all } X \in \mathbf{C}^n(r_0)_0.$$

A vector function $\psi_0 : \mathbf{R}[0, t_0] \times K\langle 1\rangle_0 \to \mathbf{C}^n$ of class C^∞ is defined by

$$\psi_0(t, \xi) = \xi + Q(t\xi, \xi, \xi)$$

for all $t \in \mathbf{R}[0, t_0]$ and $\xi \in K\langle 1\rangle_0$. Then

$$\psi(t, \xi) = \exp_0(t\xi) = t\xi + Q(t\xi, t\xi, t\xi) = t\xi + t^2\psi_0(t, \xi)$$

if $t \in \mathbf{R}[0, t_0]$ and $\xi \in K\langle 1\rangle_0$. Q.E.D.

5. The circular flow and the complex foliation

First we assume (B1)–(B5) only. The $JF = if - i\bar{f}$ is a vector field of class C^∞ on M with $[F, JF] = 0$ (Proposition 3.14). Let $\phi : \mathbf{R}(\alpha, \beta) \to M_*$ be a maximal integral curve of JF. According to Lemma 3.16 a number $r > 0$ exists such that $\tau \circ \phi = r^2$ is constant. This means $\phi(\mathbf{R}(\alpha, \beta)) \subseteq M\langle r\rangle$ where $M\langle r\rangle$ is compact. By Proposition 2.9 the vector field JF on M_* is complete. Therefore there exists a global one parameter group

$$\sigma : \mathbf{R} \times M_* \to M_* \tag{5.1}$$

of diffeomorphisms associated to JF. The map σ is of class C^∞ and has these properties:

(1) *If $p \in M_*$, then $\sigma(\square, p) : \mathbf{R} \to M_*$ is an integral curve of JF with $\sigma(0, p) = p$.*

(2) *If $r \in \mathbf{R}(0, \Delta)$ and if $p \in M\langle r \rangle$, then $\sigma(y, p) \in M\langle r \rangle$ for all $y \in \mathbf{R}$.*

(3) *If $y \in \mathbf{R}$, then $\sigma(y, \square) : M_* \to M_*$ is a diffeomorphism of class C^∞.*

(4) *If $p \in M_*$, if $y_1 \in \mathbf{R}$ and $y_2 \in \mathbf{R}$, then $\sigma(y_1 + y_2, p) = \sigma(y_1, \sigma(y_2, p))$.*

Here σ is called the *circular flow associated* to τ.

In order to complexify the gradient flow, a change in parameter is required. Define

$$\Delta_0 = \log \Delta \leq \infty \tag{5.2}$$

$$\chi : \mathbf{R}(-\infty, \Delta_0) \times M_* \to M_* \quad \text{by} \quad \chi(x, p) = \psi(e^x, p) \tag{5.3}$$

for all $x \in \mathbf{R}(-\infty, \Delta_0)$ and $p \in M_*$. Obviously, χ is of class C^∞.

Take $p \in M_*$ and $x_0 \in \mathbf{R}(-\infty, \Delta_0)$. Let $\mathfrak{p} : U \to G$ be a chart of M_* at p such that there exists a strictly parabolic extension θ on G. Let $\tilde{F}$ be the associated extension of F. Numbers α and β exist with $\alpha < x_0 < \beta \leq \Delta_0$ such that $\chi(x, p) \in U$ for all $x \in \mathbf{R}(\alpha, \beta)$. For $x \in \mathbf{R}(\alpha, \beta)$ we have

$$\begin{aligned} \dot{\chi}(x, p) = \dot{\psi}(e^x, p) e^x &= \frac{\tilde{F}(\psi(e^x, p))}{(\tau(\psi(e^x, p)))^{1/2}} e^x \\ &= \tilde{F}(\psi(e^x, p)) = \tilde{F}(\chi(x, p)). \end{aligned}$$

Hence $\chi(\square, p) : \mathbf{R}(-\infty, \Delta_0) \to M_*$ is an integral curve of F. Theorem 4.3 implies

(1) *For each $p \in M_*$, the curve $\chi(\square, p) : \mathbf{R}(-\infty, \Delta_0) \to M_*$ is an integral curve of F.*

(2) *For each $p \in M_*$, we have $\chi(\frac{1}{2} \log \tau(p), p) = p$.*

(3) *If $p \in M_*$ and $x \in \mathbf{R}(-\infty, \Delta_0)$, then $\tau(\chi(x, p)) = e^{2x}$.*

(4) *If $p \in M_*$, if $x \in \mathbf{R}(-\infty, \Delta_0)$ and $u \in \mathbf{R}(-\infty, \Delta_0)$ then $\chi(x, \chi(u, p)) = \chi(x, p)$.*

(5) *If x, u and v belong to $\mathbf{R}(-\infty, \Delta)$ a diffeomorphism $\chi_{uv}: M\langle e^v\rangle \to M\langle e^u\rangle$ is defined by $\chi_{uv}(p) = \chi(u, p)$ for all $p \in M\langle e^v\rangle$. Then χ_{uu} is the identity and $\chi_{vu} = \chi_{uv}^{-1}$ and $\chi_{xu} \circ \chi_{uv} = \chi_{xv}$.*

THEOREM 5.1: *Assume (B1)–(B5). Take $p \in M_*$ and $x \in \mathbf{R}(-\infty, \Delta_0)$ and $y \in \mathbf{R}$. Then*

$$\chi(x, \sigma(y, p)) = \sigma(y, \chi(x, p)). \tag{5.4}$$

PROOF: Take $p_0 \in M_*$. Define $x_0 = \frac{1}{2}\log \tau(p_0)$. Observe $\chi(x_0, p_0) = p_0$.

1. CLAIM: *There exists a positive number δ_0 with $x_0 + \delta_0 < \Delta_0$ such that*

$$\chi(x, \sigma(y, p_0)) = \sigma(y, \chi(x, p_0)) \quad \text{for all } x \in \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0) \text{ and } y \in \mathbf{R}. \tag{5.5}$$

PROOF OF THE 1. CLAIM: Define $r_0 = e^{x_0} = \sqrt{\tau(p_0)}$. Then $p_0 \in M\langle r_0\rangle$ and $\sigma(y, p_0) \in M\langle r_0\rangle$ for all $y \in \mathbf{R}$.

First, δ_0 shall be constructed. Take any $q \in M\langle r_0\rangle$. Take an embedded chart $\mathfrak{p}: U_q \to G_q$ of M of q such that there exists a strictly parabolic extension θ_q of τ on G_q where G_q is an open subset of $\mathbf{C}^{n_q}$. Here U_q is an open neighborhood of q in M_*. The associated extension $\tilde{F}_q$ of F on G_q defines a local one parameter group

$$\tilde{\sigma}_q: \mathbf{R}(-\epsilon_q, \epsilon_q) \times H_q \to G_q$$

of diffeomorphisms. Here $\epsilon_j > 0$ and H_q is an open, connected neighborhood of q in $\mathbf{C}^{n_q}$ such that $\bar{H}_q$ is compact and contained in G_q. Take open neighborhoods N_q, V_q, W_q of q in H_q such that

$$q \in N_q \subset \bar{N}_q \subset W_q \subset \bar{W}_q \subset V_q \subset \bar{V}_q \subset H_q.$$

Now there are numbers $\eta_q \in \mathbf{R}(0, \epsilon_q)$ such that

$$\bar{V}_q \subset H_{qy} = \tilde{\sigma}_q(y, H_q) \qquad \bar{W}_q \subset \tilde{\sigma}_q(y, V_q) = V_{qy} \subset \bar{V}_{qy} \subset H_q$$

for all $y \in \mathbf{R}(-\eta_q, \eta_q)$. Since $\chi(x_0, p) = p$ for all $p \in M\langle r_0\rangle \cap \bar{N}_Q$, there exists a number $\lambda_q > 0$ with $x_0 + \lambda_q < \Delta_0$ such that $\chi(x, p) \in W_q$ for all $x \in \mathbf{R}(x_0 - \lambda_q, x_0 + \lambda_q)$ and all $p \in N_q \cap M\langle r_0\rangle$.

A finite subset Q of $M\langle r_0\rangle$ exists such that

$$M\langle r_0\rangle \subseteq \bigcup_{q\in Q} N_q.$$

Then $\delta_0 = \mathrm{Min}\{\lambda_q \mid q \in Q\}$ is positive with $x_0 + \delta_0 < \Delta_0$. Thus δ_0 is determined.

Let I be the set of all $y \in \mathbf{R}$ such that $\chi(x, \sigma(y, p_0)) = \sigma(y, \chi(x, p_0))$ for all $x \in \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0)$. Trivially, I is closed and $0 \in I$. Now, we shall prove that I is open.

Take $y_0 \in I$. Then $\ell_0 = \sigma(y_0, p_0) \in M\langle r_0\rangle$ and $q \in Q$ exists such that $\ell_0 \in N_q$. Take $p \in V_q \cap M_*$. Let $T_p(G_q)$ be the real tangent space of G_q at p. Take $y \in \mathbf{R}(-\eta_q, \eta_q)$. Then $\tilde{\sigma}_q(y, \tilde{\sigma}_q(-y, p)) = p$. Therefore the differential of $\tilde{\sigma}_q(y, \square)$ at the point $\tilde{\sigma}_q(-y, p)$ defines a linear map

$$d\tilde{\sigma}_q(y, \tilde{\sigma}_q(-y, p), \square) : T_{\tilde{\sigma}_q(-y,p)}(G_q) \to T_p(G_q).$$

A vector function $L : \mathbf{R}(-\eta_q, \eta_q) \to T_p(G_q)$ of class C^∞ is defined by

$$L(y) = d\tilde{\sigma}_q(y, \tilde{\sigma}_q(-y, p), \tilde{F}_q(\tilde{\sigma}_q(-y, p)))$$

for all $y \in \mathbf{R}(-\eta_q, \eta_q)$. By Kobayashi–Nomizu [10] page 16, Corollary 1.10 and Remark we have

$$L'(y) = -d\tilde{\sigma}_q(y, \tilde{\sigma}_q(-y, p), [J\tilde{F}_q, \tilde{F}_q](\tilde{\sigma}_q(-y, p)))$$

for all $y \in \mathbf{R}(-\eta_q, \eta_q)$. Since $p \in V_q \cap M_* \subseteq H_q \cap U_q$ we have

$$\tilde{\sigma}_q(-y, p) = \sigma_q(-y, p) \in U_q.$$

Therefore $[J\tilde{F}_q, \tilde{F}_q](\tilde{\sigma}_q(-y, p)) = 0$ by Proposition 3.14. Therefore $L'(y) = 0$ for all $y \in \mathbf{R}(-\eta_q, \eta_q)$. Hence $L(y) = L(0)$ for all $y \in \mathbf{R}(-\eta_q, \eta_q)$. Since $\tilde{\sigma}_q(0, h) = h$ for all $h \in H_q$ we have $d\tilde{\sigma}_q(0, p, v) = v$ for all $v \in T_p(G_q)$. Hence $L(0) = \tilde{F}_q(p)$. Therefore $L(y) = \tilde{F}_q(p)$ for all $y \in \mathbf{R}(-\eta_q, \eta_q)$.

Take $h \in W_q \cap M_*$ and $y \in \mathbf{R}(-\eta_q, \eta_q)$. Then $h \in V_{q(-y)}$. Hence $p \in V_q$ exists such that $\tilde{\sigma}_q(-y, p) = h \in H_q$. Hence $p = \tilde{\sigma}_q(y, h) \in U_q \cap H_q$. Therefore

$$d\tilde{\sigma}_q(y, h, \tilde{F}_q(h)) = \tilde{F}_q(p) = \tilde{F}_q(\tilde{\sigma}_q(y, h)).$$

Take $y \in \mathbf{R}(-\eta_q, \eta_q)$. Observe that $\ell_0 \in N_q$. If $x \in \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0)$, then $\chi(x, \ell_0) \in W_q \cap M_*$ by the construction of δ_0. Therefore a curve

$$\rho : \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0) \to V_q \cap M_*$$

of class C^∞ is defined by

$$\rho(x) = \tilde{\sigma}_q(y, \chi(x, \ell_0)) = \sigma(y, \chi(x, \ell_0))$$

for all $x \in \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0)$. Then

$$\begin{aligned}\dot{\rho}(x) &= d\tilde{\sigma}_q(y, \chi(x, \ell_0), \dot{\chi}(x, \ell_0)) = d\tilde{\sigma}_q(y, \chi(x, \ell_0), \tilde{F}(\chi(x, \ell_0))) \\ &= \tilde{F}(\tilde{\sigma}_q(y, \chi(x, \ell_0))) = \tilde{F}(\rho(x))\end{aligned}$$

for all $x \in \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0)$. Now $\ell_0 \in M\langle r_0\rangle$ implies $\chi(x_0, \ell_0) = \ell_0$ and $\sigma(y, \ell_0) \in M\langle r_0\rangle$. Hence $\chi(x_0, \sigma(y, \ell_0)) = \sigma(y, \ell_0)$. We have

$$\chi(x_0, \sigma(y, \ell_0)) = \sigma(y, \ell_0) = \sigma(y, \chi(x_0, \ell_0)) = \rho(x_0).$$

Consequently

$$\sigma(y, \chi(x, \ell_0)) = \rho(x) = \chi(x, \sigma(y, \ell_0)) \tag{5.6}$$

for all $x \in \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0)$ and $y \in \mathbf{R}(-\eta_q, \eta_q)$. Observe

$$\sigma(y, \ell_0) = \sigma(y, \sigma(y_0, p_0)) = \sigma(y + y_0, p_0) \tag{5.7}$$

for all $y \in \mathbf{R}(-\eta_q, \eta_q)$. Since $y_0 \in I$, we have

$$\chi(x, \ell_0) = \chi(x, \sigma(y_0, p_0)) = \sigma(y_0, \chi(x, p_0))$$

$$\begin{aligned}\sigma(y, \chi(x, \ell_0)) &= \sigma(y, \sigma(y_0, \chi(x, p_0))) \\ &= \sigma(y + y_0, \chi(x, p_0))\end{aligned} \tag{5.8}$$

for all $x \in \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0)$ and $y \in \mathbf{R}(-\eta_q, \eta_q)$. Now (5.6), (5.7) and (5.8) imply

$$\sigma(y + y_0, \chi(x, p_0)) = \chi(x, \sigma(y + y_0, p_0))$$

for all $x \in \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0)$ and $y \in \mathbf{R}(-\eta_q, \eta_q)$. Hence

$$\sigma(y, \chi(x, p_0)) = \chi(x, \sigma(y, p_0))$$

for all $x \in \mathbf{R}(x_0 - \delta_0, x_0 + \delta_0)$ and for each $y \in \mathbf{R}(y_0 - \eta_q, y_0 + \eta_q)$. Therefore $\mathbf{R}(y_0 - \eta_q, y_0 + \eta_q) \subseteq I$. The non-empty, closed subset I of $\mathbf{R}$ is open in $\mathbf{R}$. Therefore $I = \mathbf{R}$ and the 1. Claim is proved.

2. CLAIM: *Define*

$$K = \{x \in \mathbf{R}(-\infty, \Delta_0) \mid \sigma(y, \chi(x, p_0)) = \chi(x, \sigma(y, p_0)) \forall y \in \mathbf{R}\}.$$

Then $K = \mathbf{R}(-\infty, \Delta_0)$.

PROOF: Obviously K is closed in $\mathbf{R}(-\infty, \Delta_0)$. Also $x_0 \in K$. We shall show that K is open. Take any $x_1 \in K$. Define $p_1 = \chi(x_1, p_0)$. Then $x_1 = \frac{1}{2} \log \tau(p_1)$. If $x \in \mathbf{R}(-\infty, \Delta_0)$, then

$$\chi(x, p_1) = \chi(x, \chi(x_1, p_0)) = \chi(x, p_0). \tag{5.9}$$

According to the 1. Claim, there exists a number $\delta_1 > 0$ with $x_1 + \delta_1 < \Delta_0$ such that

$$\sigma(y, \chi(x, p_1)) = \chi(x, \sigma(y, p_1)) \tag{5.10}$$

for all $y \in \mathbf{R}$ and $x \in \mathbf{R}(x_1 - \delta_1, x_1 + \delta_1)$. Since $x_1 \in K$, we have

$$\sigma(y, p_1) = \sigma(y, \chi(x_1, p_0)) = \chi(x_1, \sigma(y, p_0))$$

$$\chi(x, \sigma(y, p_1)) = \chi(x, \chi(x_1, \sigma(y, p_0))) = \chi(x, \sigma(y, p_0)). \tag{5.11}$$

Now (5.9), (5.10) and (5.11) imply

$$\begin{aligned} \sigma(y, \chi(x, p_0)) &= \sigma(y, \chi(x, p_1)) = \chi(x, \sigma(y, p_1)) \\ &= \chi(x, \sigma(y, p_0)) \end{aligned}$$

for all $y \in \mathbf{R}$ and for all $x \in \mathbf{R}(x_1 - \delta_1, x_1 + \delta_1)$. Hence $\mathbf{R}(x_1 - \delta_1, x_1 + \delta_1) \subseteq K$. The set $K \neq \emptyset$ is open and closed in $\mathbf{R}(-\infty, \Delta_0)$. Therefore $K = \mathbf{R}(-\infty, \Delta_0)$. The 2. Claim is proved. Q.E.D.

Consider $D = \mathbf{R}(-\infty, \Delta_0) \times \mathbf{R}$ as an open subset of $\mathbf{C}$. A map

$$\mathfrak{w} : D \times M_* \to M_* \tag{5.12}$$

of class C^∞ is defined by

$$\mathfrak{w}(x+iy, p) = \chi(x, \sigma(y, p)) = \sigma(y, \chi(x, p)) \tag{5.13}$$

for all $x \in \mathbf{R}(-\infty, \Delta_0)$ and $y \in \mathbf{R}$ and $p \in M_*$.

Let $N \neq \emptyset$ be an open subset of $\mathbf{C}$. Let M be a complex manifold. Let $T(M)$ be the real tangent bundle of M. Let $T^c(M)$ be the complexified tangent bundle. Then $T^c(M) = \mathfrak{T}(M) \oplus \bar{\mathfrak{T}}(M)$ where $\mathfrak{T}(M)$ is the holomorphic tangent bundle and $\bar{\mathfrak{T}}(M)$ the conjugate holomorphic tangent bundle. Let $\eta_0: T^c(M) \to \mathfrak{T}(M)$ and $\eta_1: T^c(M) \to \bar{\mathfrak{T}}(M)$ be the projections, which restricted to $T(M)$ become $\mathbf{R}$-linear bundle isomorphisms. Let J be the almost complex structure on $T(M)$ and $T^c(M)$. Since $T(N)$ and $T^c(N)$ is trivial we can identify $T(N) = N \times \mathbf{C}$ and $T^c(N) = N \times \mathbf{C}^2$. Let $h: N \to M$ be a map of class C^1. Take $p = a + ib \in N$ where a and b are real. A number $\epsilon > 0$ exists such that $p + z \in N$ for all $z \in \mathbf{C}(\epsilon)$. Then $\phi: \mathbf{R}(-\epsilon, \epsilon) \to M$ and $\psi: \mathbf{R}(-\epsilon, \epsilon) \to M$ are curves of class C^1 defined by $\phi(t) = h(p+t)$ and $\psi(t) = h(p+it)$ for all $t \in \mathbf{R}(-\epsilon, \epsilon)$. Define

$$h_x(p) = \dot{\phi}(0) \in T_{h(p)}(M) \qquad h_y(p) = \dot{\psi}(0) \in T_{h(p)}(M). \tag{5.14}$$

Also the differential of h at p is given as a linear map

$$dh(p): T^c_p(N) \to T^c_{h(p)}(M) \quad (\text{over } \mathbf{C}) \tag{5.15}$$

$$dh(p): T_p(N) \to T_{h(p)}(M) \quad (\text{over } \mathbf{R}). \tag{5.16}$$

Then

$$h_x(p) = dh\left(p, \frac{\partial}{\partial x}(p)\right) \qquad h_y(p) = dh\left(p, \frac{\partial}{\partial y}(p)\right). \tag{5.17}$$

We define

$$h_z(p) = \eta_0\left(dh\left(p, \frac{\partial}{\partial z}(p)\right)\right) \quad h_{\bar{z}}(p) = \eta_0\left(dh\left(p, \frac{\partial}{\partial \bar{z}}(p)\right)\right) \tag{5.18}$$

$$h_z(p) = \tfrac{1}{2}\eta_0(h_x(p) - Jh_y(p)) \qquad h_{\bar{z}}(p) = \tfrac{1}{2}\eta_0(h_x(p) + Jh_y(p)). \tag{5.19}$$

The map h is holomorphic if and only if

$$dh(p, Jv) = Jdh(p, v) \quad \text{for all } p \in N \text{ and } v \in T_p(N). \tag{5.20}$$

which is the case if and only if $h_{\bar{z}}(p) = 0$ for all $p \in N$, which is the case if and only if $h_x(p) = -Jh_y(p)$ for all $p \in N$. If h is holomorphic define

$$h'(p) = h_z(p) = \eta_0(h_x(p)) = -i\eta_0(h_y(p)). \tag{5.21}$$

LEMMA 5.2: *Assume (B1)–(B5). Take $p \in M_*$. Then $\mathfrak{w}(\square, p): D \to M_*$ is a holomorphic map. If $z \in D$ with $\mathfrak{w}(z, p) \in \mathfrak{R}(M_*)$ is given, then*

$$\mathfrak{w}'(z, p) = f(\mathfrak{w}(z, p)). \tag{5.22}$$

If $\mathfrak{p}: U \to G$ is an embedded chart, if θ is a strictly parabolic extension of τ on G, if $V \neq \emptyset$ is open in D such that $\mathfrak{w}(V, p) \subseteq U$, if $\tilde{f}$ is the extension of f associated to θ, if $\mathfrak{w}(\square, p): V \to G$ is regarded as a map into G, then

$$\mathfrak{w}'(z, p) = \tilde{f}(\mathfrak{w}(z, p)). \tag{5.23}$$

PROOF: If $z = x + iy \in V$ where x and y are real, then

$$\mathfrak{w}_x(x + iy, p) = \dot{\chi}(x, \sigma(y, p)) = \tilde{F}(\chi(x, \sigma(y, p))) = \tilde{F}(\mathfrak{w}(x + iy, p))$$
$$\mathfrak{w}_y(x + iy, p) = \dot{\sigma}(y, \chi(x, p)) = J\tilde{F}(\sigma(y, \chi(x, p))) = J\tilde{F}(\mathfrak{w}(x + iy, p)).$$

Hence

$$J\mathfrak{w}_y(z, p) = JJ\tilde{F}(\mathfrak{w}(z, p)) = -\tilde{F}(\mathfrak{w}(z, p)) = -\mathfrak{w}_x(z, p).$$

Therefore $\mathfrak{w}(\square, p): V \to G$ is holomorphic, which implies that $\mathfrak{w}(\square, p): V \to M_*$ is holomorphic. Since those open subsets V cover D, we see that $\mathfrak{w}(\square, p): D \to M_*$ is holomorphic. On V we have

$$\begin{aligned}\mathfrak{w}'(z, p) &= \tfrac{1}{2}\eta_0(\mathfrak{w}_x(z, p) - J\mathfrak{w}_y(z, p)) \\ &= \eta_0(\tilde{F}(\mathfrak{w}(z, p))) = \tilde{f}(\mathfrak{w}(z, p))\end{aligned}$$

for all $z \in V$. If we take $U = G = \mathfrak{R}(M_*)$ and $V = \mathfrak{w}^{-1}(\mathfrak{R}(M_*), p)$ then (5.22) follows.

Because $\tilde{f}(\mathfrak{w}(z, p)) \neq 0$, the map $\mathfrak{w}(\square, p): D \to M_*$ is a holomorphic immersion of D into M_*. If $x \in \mathbf{R}(-\infty, \Delta_0)$ and $y \in \mathbf{R}$ and $p \in M_*$, we have

$$\tau(\mathfrak{w}(x + iy, p)) = e^{2x}. \tag{5.24}$$

Now, we shall adopt the construction (C1)–(C15). Let $\mathfrak{q}: K\langle 1\rangle_0 \to M\langle t_0\rangle$ be the diffeomorphism defined in Lemma 4.8 and Theorem 4.9. Maps of class C^∞ are defined by

(5.25) $\sigma: \mathbf{R} \times K\langle 1\rangle_0 \to M_*$ by $\sigma(y, \xi) = \sigma(y, \mathfrak{q}(\xi))$

(5.26) $\chi: \mathbf{R}(-\infty, \Delta_0) \times K\langle 1\rangle_0 \to M_*$ by $\chi(x, \xi) = \chi(x, \mathfrak{q}(\xi))$

(5.27) $\mathfrak{w}: D \times K\langle 1\rangle_0 \to M_*$ by $\mathfrak{w}(z, \xi) = \mathfrak{w}(z, \mathfrak{q}(\xi))$

where $\xi \in K\langle 1\rangle_0$, where $y \in \mathbf{R}$, where $x \in \mathbf{R}(-\infty, \Delta_0)$ and where $z \in D$. Then we obtain the following properties for these choices of ξ, y, x and z:

(5.28) $$\chi(x, \xi) = \psi(e^x, \xi) = \mathfrak{w}(x, \xi).$$

(5.29) $$\tau(\chi(x, \xi)) = e^{2x}.$$

(5.30) $$\chi: \mathbf{R}(-\infty, \Delta_0) \times K\langle 1\rangle_0 \to M_*$$

is a diffeomorphism of class C^∞.

(5.31) $$\chi(\square, \xi): \mathbf{R}(-\infty, \Delta_0) \to M_* \quad \text{is an integral curve of } F.$$

(5.32) $$\sigma(\square, \xi): \mathbf{R} \to M_* \quad \text{is an integral curve of } JF.$$

(5.33) $$\sigma(y, \xi) \in M\langle t_0\rangle \quad \text{for all } y \in \mathbf{R}.$$

(5.34) $$\sigma(0, \xi) = \mathfrak{q}(\xi) = \chi(\log t_0, \xi).$$

(5.35) $$\sigma(y, \square): K\langle 1\rangle_0 \to M\langle t_0\rangle$$

is a diffeomorphism of class C^∞.

(5.36) $$\sigma(y_1 + y_2, \xi) = \sigma(y_1, \sigma(y_2, \xi))$$

for all $y_1 \in \mathbf{R}$ and $y_2 \in \mathbf{R}$.

(5.37) $$\mathfrak{w}(x + iy, \xi) = \chi(x, \sigma(y, \xi)) = \sigma(y, \chi(x, \xi)).$$

(5.38) $$\mathfrak{w}(\square, \xi): D \to M_* \quad \text{is holomorphic.}$$

(5.39) $$\mathfrak{w}'(z, \xi) = f(\mathfrak{w}(z, \xi))$$

if $z \in D$, if $\xi \in K\langle 1\rangle_0$ and if $\mathfrak{w}(z, \xi) \in \mathfrak{R}(M_*)$.

Let $\mathfrak{p}: U \to G$ be an embedded chart of M_*. Let θ be a strictly parabolic extension of τ on G. Let $\tilde{f}$ be the associated extension of f

to θ. Let V be open in D and $\xi \in K\langle 1\rangle_0$ such that $\mathfrak{w}(V, \xi) \subseteq U$. Then

$$\mathfrak{w}'(z, \xi) = \tilde{f}(\mathfrak{w}(z, \xi)) \quad \text{for all } z \in V. \tag{5.40}$$

A map ζ of class C^∞ is defined by

$$\zeta = \mathfrak{q}^{-1} \circ \sigma : \mathbf{R} \times K\langle 1\rangle_0 \to K\langle 1\rangle_0. \tag{5.41}$$

LEMMA 5.3: *Assume* $(C1)$–$(C15)$. *Then*

$$\zeta(0, \xi) = \xi \quad \textit{for all } \xi \in K\langle 1\rangle_0 \tag{5.42}$$

$$\mathfrak{w}(x + iy, \xi) = \chi(x, \zeta(y, \xi)) = \mathfrak{w}(x, \zeta(y, \xi)) \tag{5.43}$$

for all $\xi \in K\langle 1\rangle_0$, *all* $x \in \mathbf{R}(-\infty, \Delta_0)$ *and all* $y \in \mathbf{R}$. *Moreover, if* $y_1 \in \mathbf{R}$, *if* $y_2 \in \mathbf{R}$ *and if* $\xi \in K\langle 1\rangle_0$, *then*

$$\zeta(y_1 + y_2, \xi) = \zeta(y_1, \zeta(y_2, \xi)). \tag{5.44}$$

Also if $x_j \in \mathbf{R}(-\infty, \Delta_0)$, *if* $y_j \in \mathbf{R}$ *and* $\xi_j \in K\langle 1\rangle_0$ *for* $j = 1, 2$, *then*

$$\mathfrak{w}(x_1 + iy_1, \xi_1) = \mathfrak{w}(x_2 + iy_2, \xi_2) \tag{5.45}$$

if and only if $\xi_2 = \zeta(y_1 - y_2, \xi_1)$ *and* $x_1 = x_2$.

PROOF: If $\xi \in K\langle 1\rangle_0$, then $\zeta(0, \xi) = \mathfrak{q}^{-1}(\sigma(0, \xi)) = \mathfrak{q}^{-1}(\mathfrak{q}(\xi)) = \xi$ which proves (5.42). If $x \in \mathbf{R}(-\infty, \Delta_0)$, if $y \in \mathbf{R}$ and if $\xi \in K\langle 1\rangle_0$, then

$$\begin{aligned} \mathfrak{w}(x + iy, \xi) &= \chi(x, \sigma(y, \xi)) = \chi(x, \mathfrak{q}(\zeta(y, \xi)) \\ &= \chi(x, \zeta(y, \xi)) = \mathfrak{w}(x, \zeta(y, \xi)) \end{aligned}$$

which proves (5.43). If $\xi \in K\langle 1\rangle_0$, if $y_1 \in \mathbf{R}$ and if $y_2 \in \mathbf{R}$, then

$$\begin{aligned} \zeta(y_1 + y_2, \xi) &= \mathfrak{q}^{-1}(\sigma(y_1 + y_2, \xi)) = \mathfrak{q}^{-1}(\sigma(y_1, \sigma(y_2, \xi))) \\ &= \mathfrak{q}^{-1}(\sigma(y, \mathfrak{q}(\zeta(y_2, \xi)))) = \mathfrak{q}^{-1}(\sigma(y_1, \zeta(y_2, \xi))) \\ &= \zeta(y_1, \zeta(y_2, \xi)) \end{aligned}$$

which proves (5.44). Take $x_j \in \mathbf{R}(-\infty, \Delta_0)$ and $y_j \in \mathbf{R}$ and $\xi_j \in K\langle 1\rangle_0$. Assume (5.45). Then

$$e^{2x_1} = \tau(\mathfrak{w}(x_1 + iy_1, \xi_1)) = \tau(\mathfrak{w}(x_2 + iy_2, \xi_2)) = e^{2x_2}.$$

Therefore $x_1 = x_2 = x$. Also

$$\chi(x, \zeta(y_1, \xi_1)) = \mathfrak{w}(x_1 + iy_1, \xi_1) = \mathfrak{w}(x_2 + iy_2, \xi 2)$$
$$= \chi(x, \zeta(y_2, \xi_2)).$$

Now (5.30) implies $\zeta(y_1, \xi_1) = \zeta(y_2, \xi_2)$ or

$$\xi_2 = \zeta(0, \xi_2) = \zeta(-y_2, \zeta(y_2, \xi_2))$$
$$= \zeta(-y_2, \zeta(y_1, \xi_1)) = \zeta(y_1 - y_2, \xi).$$

If $x_1 = x_2$ and $\xi_2 = \zeta(y_1 - y_2, \xi_1)$, then

$$\zeta(y_2, \xi_2) = \zeta(y_2, \zeta(y_1 - y_2, \xi_1)) = \zeta(y_1, \xi_1)$$
$$\mathfrak{w}(x_1 + iy_1, \xi_1) = \chi(x_1, \zeta(y_1, \xi_1)) = \chi(x_2, \zeta(y_2, \xi_2))$$
$$= \mathfrak{w}(x_2 + iy_2, \xi_2), \qquad \text{Q.E.D.}$$

Fortunately, the flow $\dot{\zeta}$ can be determined explicitly.

THEOREM 5.4: *Assume* $(C1)$–$(C15)$. *Take* $y \in \mathbf{R}$ *and* $\xi \in K\langle 1\rangle_0$. *Then*

$$\zeta(y, \xi) = e^{iy}\xi. \tag{5.46}$$

PROOF: Recall Lemma 4.12 with (4.26) and (4.27). Fix $\xi \in K\langle 1\rangle_0$. Maps

$$\gamma : \mathbf{R}[0, t_0] \times \mathbf{R} \to G_0 \qquad \rho : \mathbf{R}[0, t_0] \times \mathbf{R} \to \mathbf{C}^n$$

of class C^∞ are defined by

$$\gamma(t, y) = \psi(t, \zeta(y, \xi)) \tag{5.47}$$

$$\rho(t, y) = \zeta(y, \xi) + t\psi_0(t, \zeta(y, \xi)) \tag{5.48}$$

for all $y \in \mathbf{R}$ and $t \in \mathbf{R}[0, t_0]$. Then (4.28) implies

$$\gamma(t, y) = t\rho(t, y). \tag{5.49}$$

Consequently,

$$\gamma_t(t, y) = \rho(t, y) + t\rho_t(t, y)$$
$$\gamma_{ty}(t, y) = \rho_y(t, y) + t\rho_{ty}(t, y)$$
$$\gamma_{ty}(0, y) = \rho_y(0, y) = \zeta_y(y, \xi)$$
$$\gamma_{yt}(0, y) = \gamma_{ty}(0, y) = \zeta_y(y, \xi) \quad \text{for all } y \in \mathbf{R}. \tag{5.50}$$

If we identify $T(G) = \mathfrak{T}(G)$ by η_0, then we have

$$\gamma(t, y) = \psi(t, \zeta(y, \xi)) = \chi(\log t, \zeta(y, \xi)) = \mathfrak{w}(\log t + iy, \xi)$$
$$\gamma_y(t, y) = i\mathfrak{w}'(\log t + iy, \xi) = i\tilde{f}(\mathfrak{w}(\log t + iy, \xi))$$
$$= i\tilde{f}(\gamma(t, y)) = i\tilde{f}(t\rho(t, y)).$$

Consider $\tilde{f}: G \to \mathbf{C}^n$ as a vector function. Now, Lemma 3.10 implies

$$\gamma_y(t, y) = it\mathfrak{b}(\rho(t, y)) + it\rho(t, y) + it^2 R(t, \rho(t, y)).$$

Hence

$$\gamma_{yt}(0, y) = i\mathfrak{b}(\rho(0, y)) + i\rho(0, y) = i\mathfrak{b}(\zeta(y, \xi)) + i\zeta(y, \xi).$$

Since $\zeta(y, \xi) \in K$, Lemma 3.10 yields $\mathfrak{b}(\zeta(y, \xi)) = 0$. Therefore

$$\gamma_{yt}(0, y) = i\zeta(y, \xi) \quad \text{for all } y \in \mathbf{R}. \tag{5.51}$$

From (5.50), (5.51) and (5.42) we obtain

$$\zeta_y(y, \xi) = i\zeta(y, \xi) \quad \text{with} \quad \zeta(0, \xi) = \xi \tag{5.52}$$
$$\frac{d}{dy}(e^{-iy}\zeta(y, \xi)) = e^{-iy}(\zeta_y(y, \xi) - i\zeta(y, \xi)) = 0$$

for all $y \in \mathbf{R}$. Hence $e^{-iy}\zeta(y, \xi) = \zeta(0, \xi) = \xi$ or $\zeta(y, \xi) = e^{iy}\xi$ for all $y \in \mathbf{R}$. Q.E.D.

The pull back of the circular flow to the intersection of the Whitney tangent cone with the unit sphere is the restriction of the Hopf fibration of the unit sphere to this intersection. Now (5.43) reads

$$\mathfrak{w}(x + iy, \xi) = \chi(x, e^{iy}\xi) = \mathfrak{w}(x, e^{iy}\xi) \tag{5.53}$$

for all $x \in \mathbf{R}(-\infty, \Delta_0)$ and all $y \in \mathbf{R}$ and all $\xi \in K\langle 1\rangle_0$. Moreover, if $x_j \in \mathbf{R}(-\infty, \Delta_0)$, if $y_j \in \mathbf{R}$ and $\xi_j \in K\langle 1\rangle_0$ for $j = 1, 2$, then (5.45) holds if and only if $x_1 = x_2$ and

$$\xi_2 = e^{i(y_1 - y_2)}\xi_1 \tag{5.54}$$

if $\xi_1 = \xi_2$, this is the case if and only if $y_2 = y_1 + 2\pi p$ for some integer $p \in \mathbf{Z}$. Hence $\mathfrak{w}(z_1, \xi) = \mathfrak{w}(z_2, \xi)$ if and only if $z_2 = z_1 + 2\pi i p$ where $p \in \mathbf{Z}$.

6. The biholomorphic isometry

Now, we want to show that the affine algebraic cones are – up to a biholomorphic isometry – the only affine algebraic cones.

Let V be a complex vector space of dimension n. Let K be an irreducible analytic subset of V such that $z \in K$ implies $\mathbf{C}z \subseteq K$. Then K is said to be a *complex cone.* Obviously, K is affine algebraic. Hence K is also called an *affine algebraic cone.* Let $(\square \mid \square)$ be a positive definite hermitian form on V and define $\|v\| = \sqrt{(v \mid v)}$ as the norm of v. A strictly parabolic exhaustion θ_0 of V is defined by $\theta_0(v) = \|v\|^2$ for all $v \in V$. For each $r \geq 0$ and $A \subseteq V$ define

$$A[r] = \{v \in A \mid \|v\| \leq r\} \qquad A(r) = \{v \in A \mid \|v\| < r\} \tag{6.1}$$

$$A\langle r\rangle = \{v \in A \mid \|v\| = r\} \qquad A_* = A - \{0\}. \tag{6.2}$$

Define $\tau_0 = \theta_0 \mid K$ and let m be the dimension of K. Let $\mathfrak{p}: K \to V$ be the inclusion.

THEOREM 6.1: *Take $0 < \Delta \leq +\infty$. Then $(K(\Delta), \tau_0)$ is a strictly parabolic space of dimension m and maximal radius Δ. Also $\mathfrak{p}: K(\Delta) \to V(\Delta)$ is an embedded chart and θ_0 is a strictly parabolic extension of τ_0 onto $V(\Delta)$*

PROOF: Since $dd^c\theta_0 > 0$ and $dd^c \log \theta_0 \geq 0$ we have $dd^c\tau_0 > 0$ on $\Re(K)$ and $dd^c \log \tau_0 \geq 0$ on $\Re(K_*)$. Let $\mathbf{P}: V_* \to \mathbf{P}(V)$ be the projection. Then $K' = \mathbf{P}(K_*)$ is an irreducible analytic set of dimension $m-1$. Let Ω be the exterior form of the Fubini-Study Kaehler metric defined by θ_0 on $\mathbf{P}(V)$. Then $\mathbf{P}^*(\Omega) = dd^c \log \theta_0$. Let $j: K' \to \mathbf{P}(V)$ be the inclusion. Then

$$dd^c \log \tau_0 = \mathfrak{p}^*(dd^c \log \theta_0) = \mathfrak{p}^*\mathbf{P}^*(\Omega) = \mathbf{P}^*(j(\Omega)) \geq 0$$
$$(dd^c \log \tau_0)^m = \mathbf{P}^*(j^*(\Omega)^m) = \mathbf{P}^*(0) = 0.$$

Hence τ_0 is strictly parabolic on $\mathfrak{R}(K)$. Also θ_0 is a strictly parabolic extension to V. Therefore τ_0 is strictly parabolic on K. Trivially $\tau_0 \mid K(\Delta)$ is an exhaustion of maximal radius Δ of $K(\Delta)$. Therefore $(K(\Delta), \tau_0)$ is a strictly parabolic space of dimension m and maximal radius Δ. Q.E.D.

Let (M, τ) be a strictly parabolic space of dimension m and maximal radius Δ. Then the center $M[0]$ consists of one and only one point O_M called the center point. Let K be the Whitney tangent cone at the center point O_M. The center cone is an affine algebraic cone embedded into the holomorphic complex tangent space $\mathfrak{T} = \mathfrak{T}_{O_M}(M)$ of M at O_M. Take any positive definite hermitian form on $\mathfrak{T}$ and define the strictly parabolic exhaustions θ_0 of $\mathfrak{T}$ and $\tau_0 = \theta_0 \mid K$ of K as above. These are the assumptions to be made for the rest of the paper. Now we carry out the construction (C1)–(C15). By a linear isometry we can identify $\mathfrak{T} = \mathbf{C}^n$ such that $\theta_0(w) = \Sigma_{j=1}^n |w_j|^2 = \|w\|^2$ for all $w = (w_1, \ldots, w_n) \in \mathbf{C}^n$. In accordance with the conventions (6.1) and (6.2) we shall also write $K[r]$, $K(r)$ and $K\langle r\rangle$ instead of $K[r]_0$, $K(r)_0$ and $K\langle r\rangle_0$ if this does not cause any confusion. Under these assumptions we have the following parameterization.

PROPOSITION 6.2: *There exists one and only one map*

$$\mathfrak{b} : \mathbf{C}(\Delta) \times K\langle 1\rangle \to M \tag{6.3}$$

of class C^∞ such that

(6.4) $\mathfrak{b}(e^z, \xi) = \mathfrak{w}(z, \xi)$ *for all $z \in D$ and all $\xi \in K\langle 1\rangle$.*

The map $\mathfrak{b}$ is proper and surjective. Moreover

$$\mathfrak{b}(0, \xi) = O_M \quad \textit{for all } \xi \in K\langle 1\rangle \tag{6.5}$$

$$\tau(\mathfrak{b}(z, \xi)) = |z|^2 \quad \textit{for all } z \in D \textit{ and } \xi \in K\langle 1\rangle \tag{6.6}$$

$$\mathfrak{b}(t, \xi) = \psi(t, \xi) \quad \textit{for all } t \in \mathbf{R}[0, \Delta) \textit{ and } \xi \in K\langle 1\rangle \tag{6.7}$$

$$\mathfrak{b}(z, e^{i\alpha}\xi) = \mathfrak{b}(ze^{i\alpha}, \xi) \tag{6.8}$$

for all $z \in \mathbf{C}(\Delta)$, all $\alpha \in \mathbf{R}$ and $\xi \in K\langle 1\rangle$.

If $\xi \in K\langle 1\rangle$, then $\mathfrak{b}(\square, \xi): \mathbf{C}(\Delta) \to M$ is proper, injective and holomorphic with

$$\mathfrak{b}'(0, \xi) = \xi. \tag{6.9}$$

If $\xi \in K\langle 1\rangle$, if $z \in \mathbf{C}(\Delta)$ and if $\mathfrak{b}(z, \xi) \in \mathfrak{R}(M)$, then

$$z\mathfrak{b}'(z, \xi) = f(\mathfrak{b}(z, \xi)). \tag{6.10}$$

Let $\mathfrak{p}: \hat{U} \to \hat{G}$ be an embedded chart of M. Let θ be a strictly parabolic extension of τ onto $\hat{G}$. Let $\hat{f}$ be the associated extension of f. Take $\xi \in K\langle 1\rangle$ and $z \in \mathbf{C}(\Delta)$ such that $\mathfrak{b}(z, \xi) \in \hat{U}$, then

$$z\mathfrak{b}'(z, \xi) = \hat{f}(\mathfrak{b}(z, \xi)). \tag{6.11}$$

PROOF: Observe that $\exp: D \to \mathbf{C}(\Delta) - \{0\}$ is the universal covering. Since $\mathfrak{b}(z_1, \xi) = \mathfrak{b}(z_2, \xi)$ if and only if $z_2 = z_1 + 2\pi i p$ where $p \in \mathbf{Z}$, a map $\mathfrak{b}: (\mathbf{C}(\Delta) - \{0\}) \times K\langle 1\rangle \to M_*$ is uniquely defined by (6.4). If $\xi \in K\langle 1\rangle$, then $\mathfrak{b}(\square, \xi): \mathbf{C}(\Delta) - \{0\} \to M_*$ is injective and holomorphic. Define $\mathfrak{b}(0, \xi) = O_M$. Then $\tau(\mathfrak{b}(0, \xi)) = \tau(O_M) = 0$. If $0 \neq z \in \mathbf{C}(\Delta)$ and $\xi \in K\langle 1\rangle$, then $z = e^{x+iy}$ with $x \in \mathbf{R}(-\infty, \Delta_0)$ and $y \in \mathbf{R}$. Then (5.24) implies

$$\tau(\mathfrak{b}(z, \xi)) = \tau(\mathfrak{b}(e^{x+iy}, \xi)) = \tau(\mathfrak{w}(x + iy, \xi)) = e^{2x} = |z|^2.$$

Therefore (6.5) is established for all $z \in \mathbf{C}(\Delta)$ and $\xi \in K\langle 1\rangle$. Let N be any open neighborhood of O_M. Then $t_1 \in \mathbf{R}(0, \Delta)$ exists such that $M(t_1) \subset N$. If $z \in \mathbf{C}(t_0)$ and $\xi \in K\langle 1\rangle$, then (6.5) implies $\mathfrak{b}(z, \xi) \in M(t_1) \subseteq N$. Hence $\mathfrak{b}$ is continuous on $\mathbf{C}(\Delta) \times K\langle 1\rangle$. By Riemann's extension theorem $\mathfrak{b}(\square, \xi): \mathbf{C}(\Delta) \to M$ is holomorphic for each fixed $\xi \in K\langle 1\rangle$. Take the number t_0 of (C14). Then the map $\mathfrak{b}: \mathbf{C}(t_0) \times K\langle 1\rangle \to M(t_0)$ can be viewed as a map into $\mathbf{C}^n$. Take $s \in \mathbf{R}(0, t_0)$. If $z \in \mathbf{C}(s)$ and $\xi \in K\langle 1\rangle$, then

$$\mathfrak{b}(z, \xi) = \frac{1}{2\pi i} \int_{\mathbf{C}\langle s\rangle} \frac{\mathfrak{b}(\zeta, \xi)}{\zeta - z}\, d\zeta.$$

Therefore $\mathfrak{b}$ is of class C^∞ on $\mathbf{C}(s) \times K\langle 1\rangle$, which implies that $\mathfrak{b}$ is of class C^∞ on $\mathbf{C}(\Delta) \times K\langle 1\rangle$.

Let P be a compact subset of M. A number $r \in \mathbf{R}(0, \Delta)$ exists such that $P \subseteq M[r]$. Then $\mathfrak{b}^{-1}(P) \subseteq \mathbf{C}[r] \times K\langle 1\rangle$. Hence $\mathfrak{b}^{-1}(P)$ is compact. The map p is proper. If $t \in \mathbf{R}(0, \Delta)$ and $\xi \in K\langle 1\rangle$, then

$$\mathfrak{b}(t, \xi) = \mathfrak{w}(\log t, \xi) = \psi(t, \xi).$$

If $t = 0$, then $\mathfrak{b}(0, \xi) = 0_M = \psi(0, \xi)$. Therefore (6.7) is proved. By Theorem 4.10, (2) and (5), ψ is surjective. Hence $\mathfrak{b}$ is surjective. Take $z \in \mathbf{C}(\Delta)$ and $\xi \in K\langle 1\rangle$ and $\alpha \in \mathbf{R}$. If $z \neq 0$, then $x \in \mathbf{R}(-\infty, \Delta_0)$ and $y \in \mathbf{R}$ exist such that $z = e^{x+iy}$. Therefore

$$\begin{aligned}\mathfrak{b}(z, e^{i\alpha}\xi) &= \mathfrak{b}(e^{x+iy}, e^{i\alpha}\xi) = \mathfrak{w}(x + iy, e^{i\alpha}\xi) = \mathfrak{w}(x, e^{i(\alpha+y)}\xi)\\ &= \mathfrak{w}(x + i(\alpha + y), \xi) = \mathfrak{b}(e^{x+i(\alpha+y)}, \xi) = \mathfrak{b}(ze^{i\alpha}, \xi).\end{aligned}$$

If $z = 0$, then $\mathfrak{b}(0, e^{i\alpha}\xi) = O_M = \mathfrak{b}(0, \xi) = \mathfrak{b}(0e^{i\alpha}, \xi)$. Hence (6.8) is proved.

Take $\xi \in K\langle 1\rangle$. Since $\mathfrak{b}: \mathbf{C}(\Delta) \times K\langle 1\rangle \to M$ is proper, $\mathfrak{b}(\square, \xi): \mathbf{C}(\Delta) \to M$ is proper also, and as seen, $\mathfrak{b}(\square, \xi): \mathbf{C}(\Delta) \to M$ is injective and holomorphic. Also

$$\mathfrak{b}'(0, \xi) = \dot{\psi}(0, \xi) = \xi$$

which proves (6.9). Take $\xi \in K\langle 1\rangle$ and $z \in \mathbf{C}(\Delta)$. Assume that $\mathfrak{b}(z, \xi) \in \mathfrak{R}(M)$. If $z \neq 0$, then $u \in D$ exists such that $z = e^u$. Then

$$\mathfrak{w}'(u, \xi) = \frac{d}{du}\, \mathfrak{b}(e^u, \xi) = e^u \mathfrak{b}'(e^u, \xi)$$

$$z\mathfrak{b}'(z, \xi) = \mathfrak{w}'(u, \xi) = f(\mathfrak{w}(u, \xi)) = f(\mathfrak{b}(z, \xi)).$$

If $z = 0$, then $z\mathfrak{b}'(z, \xi) = 0 = f(O_M) = f(\mathfrak{b}(0, \xi)) = f(\mathfrak{b}(z, \xi))$. Hence (6.10) is proved.

Let $\mathfrak{p}: \hat{U} \to \hat{G}$ be an embedded chart of M. Let $\hat{\theta}$ be a strictly parabolic extension of τ onto $\hat{G}$. Let $\hat{\mathfrak{f}}$ be the associated extension of f. Let V be an open subset of $\mathbf{C}(\Delta)$ and $\xi \in K\langle 1\rangle$ such that $\mathfrak{b}(V, \xi) \subset \hat{U}$. Consider $\mathfrak{b}(\square, \xi): V \to \hat{U} \subseteq \hat{G}$ as a map into $\hat{G}$. Take $z \in V$. If $z \neq 0$, then $u \in D$ exists such that $e^u = z$. Then

$$z\mathfrak{b}'(z, \xi) = \mathfrak{w}'(u, \xi) = \hat{f}(\mathfrak{w}(u, \xi)) = \hat{f}(\mathfrak{b}(z, \xi)).$$

If $z = 0$, then

$$z\mathfrak{b}'(z, \xi) = 0 = \hat{f}(O_M) = \hat{f}(\mathfrak{b}(0, \xi)) = \hat{f}(\mathfrak{b}(z, \xi))$$

which proves (6.11). Q.E.D.

Recall the diffeomorphism $h: K(\Delta) \to M$ of Theorem 4.11 defined by (4.25). If $\xi \in \mathbf{C}^n_*$, a smooth injective holomorphic map $j_\xi: \mathbf{C} \to \mathbf{C}^n$ is defined by $j_\xi(z) = z\xi$ for all $z \in \mathbf{C}$. If $r > 0$ and $\|\xi\| = 1$, then $j_\xi: \mathbf{C}(r) \to \mathbf{C}^n(r)$ is proper. If $\xi \in K\langle 1\rangle$, then $j_\xi: \mathbf{C}(r) \to K(r)$.

LEMMA 6.3: *Take $\xi \in K\langle 1\rangle$ and $z \in \mathbf{C}(\Delta)$. Then*

$$h(z\xi) = \mathfrak{b}(z, \xi). \tag{6.12}$$

The map $h \circ j_\xi: \mathbf{C}(\Delta) \to M$ is holomorphic.

PROOF: If $z = 0$, then $h(z\xi) = h(0) = O_M = \mathfrak{b}(0, \xi)$. If $z \neq 0$, then

$$\begin{aligned} h(z\xi) &= \psi\left(\|z\xi\|, \frac{z\xi}{\|z\xi\|}\right) = \psi\left(|z|, \frac{z}{|z|}\xi\right) \\ &= \mathfrak{b}\left(|z|, \frac{z}{|z|}\xi\right) = \mathfrak{b}\left(|z|\frac{z}{|z|}, \xi\right) = \mathfrak{b}(z, \xi) \end{aligned}$$

Q.E.D.

We shall show that h is holomorphic. Two lemmata are needed.

LEMMA 6.4: *Let A be an affine algebraic cone with vertex 0 in $\mathbf{C}^n$. Take $0 < r \leq \infty$. Define $A(r) = \{w \in A \mid \|w\| < r\}$. Take $p \in \mathbf{N}$. Let $H: A(r) \to \mathbf{C}$ be a function of class C^p such that $H(zw) = z^p H(w)$ for all $w \in A(r)$ and for all $z \in \mathbf{C}(1)$. Then there exists a holomorphic homogeneous polynomial P of degree p such that $P \mid A(r) = H$. In particular H is holomorphic.*

PROOF: For every point $a \in \mathbf{C}^n(r)$, there exists an open neighborhood $U(a)$ of a in $\mathbf{C}^n(r)$ and a function $H_a: U(a) \to \mathbf{C}$ of class C^p such that $H_a \mid U(a) \cap A(r) = H \mid U(a) \cap A(r)$ if $U(a) \cap A(r) \neq \emptyset$. If $U(a) \cap A(r) = \emptyset$, then $H_a = 0$ can be assumed. Let $\{U(a_\lambda)\}_{\lambda \in \Lambda}$ be a locally finite covering of $\mathbf{C}^n(r)$ and let $\{g_\lambda\}_{\lambda \in \Lambda}$ be a partition of unity associated to this covering. Then $g_\lambda: \mathbf{C}^n(r) \to \mathbf{R}$ is of class C^∞ with compact support in $U(a_\lambda)$. Also

$$\sum_{\lambda\in\Lambda} g_\lambda = 1$$

on $\mathbf{C}^n(r)$. Define $\tilde{H}_\lambda = g_\lambda H_{a_\lambda}$ on $U(a_\lambda)$ and $\tilde{H}_\lambda = 0$ on $\mathbf{C}^n(r) - U(a_\lambda)$. Then $\tilde{H}_\lambda : \mathbf{C}^n(r) \to \mathbf{C}$ is a function of class C^p with compact support in $U(a_\lambda)$. Therefore

$$\tilde{H} = \sum_{\lambda\in\Lambda} \tilde{H}_\lambda : \mathbf{C}^n(r) \to \mathbf{C}$$

is a function of class C^p. If $w \in \mathbf{C}^n(r)$, then $\Lambda(w) = \{\lambda \in \Lambda \mid g_\lambda(w) > 0\}$ is finite. For $w \in A(r)$ we have

$$\begin{aligned}\tilde{H}(w) &= \sum_{\lambda\in\Lambda(w)} \tilde{H}_\lambda(w) = \sum_{\lambda\in\Lambda(w)} g_\lambda(w) H_{a_\lambda}(w)\\ &= \sum_{\lambda\in\Lambda(w)} g_\lambda(w) H(w) = \sum_{\lambda\in\Lambda} g_\lambda(w) H(w) = H(w).\end{aligned}$$

Hence $\tilde{H} \mid A(r) = H$. Let $w^1, \ldots, w^n$ be the coordinate functions of $\mathbf{C}^n$. Denote by $\tilde{H}_{\mu_1\ldots\mu_q}$ the partial derivative for $w^{\mu_1}, \ldots, w^{\mu_q}$. Take $w \in A(r)$ and $z \in \mathbf{C}(1)$, then

$$\tilde{H}(zw) = H(z, w) = z^p H(w) = z^p \tilde{H}(w).$$

Differentiation for z implies

$$\sum_{\mu=1}^{n} \tilde{H}_\mu(zw) w^\mu = p z^{p-1} \tilde{H}(w).$$

By induction, we obtain

$$\sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_p=1}^{n} \tilde{H}_\mu(zw) w^{\mu_1} \cdots w^{\mu_p} = p!\tilde{H}(w) = p!H(w).$$

Put $z = 0$ and define a holomorphic, homogeneous polynomial $P : \mathbf{C}^n \to \mathbf{C}$ of degree p by

$$P(w) = \frac{1}{p!} \sum_{\mu_1=1}^{n} \sum_{\mu_2=1}^{n} \cdots \sum_{\mu_p=1}^{n} H_{\mu_1\ldots\mu_p}(0) w^{\mu_1} \cdots w^{\mu_p}.$$

Then $P(w) = H(w)$ for all $w \in A(r)$. Q.E.D.

LEMMA 6.5: *Let A be an affine algebraic cone with vertex 0 in $\mathbf{C}^n$. Take $0 < r \leq +\infty$. Let $H : A(r) \to \mathbf{C}^k$ be a vector function of class C^∞. Assume that for each $\xi \in A\langle 1\rangle$, the vector function $H \circ j_\xi : \mathbf{C}(r) \to \mathbf{C}^k$ is holomorphic. Then $H : A(r) \to \mathbf{C}^k$ is holomorphic.*

PROOF: Without loss of generality, we can assume that $k = 1$ and that H extends to a function $H : \mathbf{C}^n(r) \to \mathbf{C}$ of class C^∞. For each non-negative integer p define a function $H_p : \mathbf{C}^n(r) \to \mathbf{C}$ of class C^∞ by

$$H_p(w) = \frac{1}{2\pi i} \int_{\mathbf{C}\langle 1\rangle} \frac{H(zw)}{z^{p+1}}\, dz.$$

Take $w \in A(r)$. Then the function $H \circ j_w : \mathbf{C}(1) \to \mathbf{C}$ is holomorphic. Therefore

$$H(zw) = \sum_{p=0}^{\infty} H_p(w) z^p) \quad \text{for all } z \in \mathbf{C}(1).$$

If $z \in \mathbf{C}(1)$ and $u \in \mathbf{C}(1)$, then $zu \in \mathbf{C}(1)$. We have

$$\sum_{p=0}^{\infty} H_p(w) z^p u^p = H(zuw) = \sum_{p=0}^{\infty} H_p(zw) u^p.$$

Therefore

$$H_p(zw) = z^p H_p(w).$$

By Lemma 6.4 $H_p \mid A(r)$ is holomorphic. Take $0 < s < r$. Take $\eta > 1$ with $s\eta < r$. A constant $C > 0$ exists such that $|H(w)| \leq C$ for all $w \in \mathbf{C}^n[s\eta]$. If $w \in A[s]$, then $|H(z\eta w)| \leq C$ for all $z \in \mathbf{C}[1]$. Hence $|H_p(\eta w)| \leq C$ for all integers $p \geq 0$ which implies $|H_p(w)| \leq C\eta^{-p}$. Therefore

$$H(w) = \sum_{p=0}^{\infty} H_p(w)$$

converges uniformly on $A[s]$ for every $s \in \mathbf{R}(0, r)$. Since $H_p \mid A(r)$ is holomorphic for each integer $p \geq 0$, the function $H : A(r) \to \mathbf{C}$ is holomorphic. Q.E.D.

Let M be a pure dimensional complex space. Malgrange [11] showed that a function $f : M \to \mathbf{C}$ of class C^∞ which is holomorphic on $\mathfrak{R}(M)$ is holomorphic on M. Obviously, the theorem extends to vector functions.

PROPOSITION 6.6: *Let M and N be complex spaces of pure dimension m. Let $g: M \to N$ be a holomorphic diffeomorphism of class C^∞. Then $g: M \to N$ is biholomorphic.*

PROOF: Let $S = \mathfrak{S}(M)$ and $T = \mathfrak{S}(N)$ be the sets of singular points of M respectively N. Then $T_0 = g^{-1}(T)$ is analytic and nowhere dense in M. Hence $E = T_0 \cup S$ is analytic and nowhere dense in M. Also $M_0 = M - E$ is open and dense in M and M_0 is a complex manifold of pure dimension m. Since $g: M \to N$ is proper, $S_0 = g(S)$ is analytic and nowhere dense in N. Hence $F = T \cup S_0$ is analytic and nowhere dense in N. Also $N_0 = N - F$ is open and dense in N and N_0 is a complex manifold of pure dimension m. Obviously, $g: M_0 \to N_0$ is a holomorphic diffeomorphism between complex manifolds. Therefore $g: M_0 \to N_0$ is biholomorphic.

Define $h = g^{-1}: N \to M$ as the inverse map. The map h is of class C^∞ and $h \mid N_0$ is holomorphic. Take any point $a \in N$. Define $b = h(a)$. Then there exists chart $\mathfrak{p}: U \to G$ of M at b where G is an open subset of $\mathbf{C}^n$. An open neighborhood V of a in N exists such that $h(V) \subseteq U$. Then $\mathfrak{p} \circ h: V \to G$ is a vector function of class C^∞ which is holomorphic on $V \cap N_0$. By the Riemann extension theorem on manifolds, $\mathfrak{p} \circ h$ is holomorphic on $V - T$. By the Theorem of Malgrange [11], $\mathfrak{p} \circ h$ is holomorphic on V. Observe that $U' = \mathfrak{p}(U)$ is an analytic subset of G and that $\mathfrak{p}: U \to U'$ is biholomorphic. We have $\mathfrak{p}(h(V)) \subseteq \mathfrak{p}(U) = U'$. Hence $\mathfrak{p} \circ h$ is holomorphic as a map into U' and $h \mid V = \mathfrak{p}^{-1} \circ \mathfrak{p} \circ h: V \to U$ is holomorphic. Therefore $h: N \to M$ is holomorphic and $g: M \to N$ is biholomorphic. Q.E.D.

THEOREM 6.7: *Let (M, τ) be a strictly parabolic space of dimension m and with maximal radius Δ. Let K be the Whitney tangent cone of M at the center point O_M. Assume that K is embedded into the holomorphic complex tangent space $\mathfrak{T} = \mathfrak{T}_{O_M}(M)$ of M at O_M. Let $(\square \mid \square)$ be a positive definite hermitian form on $\mathfrak{T}$ and define $\tau_0(z) = (z \mid z) = \|z\|^2$ for all $z \in \mathfrak{T}$. Define $K(\Delta) = \{z \in K \mid \tau_0(z) < \Delta^2\}$. Then there exists a biholomorphic map*

$$h: K(\Delta) \to M \tag{6.13}$$

such that $\tau \circ h = \tau_0$. Moreover such a map is given by (4.24) *and* (4.25).

PROOF: Let h be given by (4.25). Then $h: K(\Delta) \to M$ is a diffeomorphism of class C^∞ with $\tau \circ h = \tau_0$ (Theorem 4.11). Take any $r \in \mathbf{R}(0, \Delta)$. Then $h: K(\Delta) \to M$ restricts to a diffeomorphism

$h: K(r) \to M(r)$. Here τ is a strictly pseudoconvex exhaustion of $M(r)$ of maximal radius r. Therefore $M(r)$ is a Stein space. Since $M[r]$ is compact, the embedding dimension is bounded on $M(r)$. Therefore there exists a number $k \in \mathbf{N}$ an analytic subset N of pure dimension m in $\mathbf{C}^k$ and a biholomorphic map $\phi: M(r) \to N$. Let $\rho: N \to \mathbf{C}^k$ be the inclusion map. Then $\rho \circ \phi \circ h: K(r) \to \mathbf{C}^k$ is a map of class C^∞. For each $\xi \in K\langle 1 \rangle$, the map $\rho \circ \phi \circ h \circ j_\xi : \mathbf{C}(r) \to \mathbf{C}^k$ is holomorphic. Then $\phi \circ h: K(r) \to N$ is holomorphic. Therefore $h = \phi^{-1} \circ \phi \circ h$ is holomorphic on $K(r)$ for every $r \in \mathbf{R}(0, \Delta)$. Hence $h: K(\Delta) \to M$ is holomorphic. By Theorem 6.6 the holomorphic, C^∞-diffeomorphism $h: K(\Delta) \to M$ is a biholomorphic map. Q.E.D.

Now, the question of uniqueness can be easily settled. Let

$$(6.14) \qquad h: K(\Delta) \to M \qquad \tilde{h}: K(\Delta) \to M$$

biholomorphic maps with $\tau \circ h = \tau_0 = \tau \circ \tilde{h}$. Then

$$\ell = h^{-1} \circ \tilde{h}: K(\Delta) \to K(\Delta)$$

is a biholomorphic map with

$$(6.15) \qquad \tilde{h} = h \circ \ell \qquad \tau_0 \circ \ell = \tau_0.$$

PROPOSITION 6.8: *A linear isomorphism $L: \mathfrak{X} \to \mathfrak{X}$ exists such that $L \mid K(\Delta) = \ell$.*

PROOF: Take $w \in K(\Delta)$. Define $g: \mathbf{C}(1) \to K(\Delta)$ by $g(z) = \ell(zw)$. Let $\|g(z)\| = \|\ell(zw)\| = \|zw\| = |z|\|w\|$. In particular $g(0) = 0$.

A holomorphic vector function $u: \mathbf{C}(1) \to \mathfrak{X}$ exists such that $g(z) = zu(z)$. Then $|z|\|u(z)\| = \|g(z)\| = |z|\|w\|$. Hence $\|u(z)\| = \|w\|$ for all $z \in \mathbf{C}(1)$. Then $0 = dd^c\|u\|^2 = \|u'\|^2(i/2\pi)dz \wedge d\bar{z}$. Hence $u' = 0$ on $\mathbf{C}(1)$. The function u is constant. Hence $\ell(zw) = z\ell(tw)/t$ for all $0 < t < 1$. Now $t \to 1$ implies $\ell(zw) = zl(w)$ for all $z \in \mathbf{C}(1)$ and $w \in K(\Delta)$.

Let V be the linear hull of A in $\mathfrak{X}$. By Lemma 6.4 there exists a linear map $P: V \to V$ such that $P \mid A(r) = \ell$. Similarly there exists a linear map $Q: V \to V$ such that $Q \mid A(r) = \ell^{-1}$. Then $P \circ Q \mid A(r) = \mathrm{Id} \mid A(r)$. Hence $P \circ Q - \mathrm{Id} \mid A(r) = 0$. Since A is a cone, $P \circ Q - \mathrm{Id} \mid A = 0$. Since V is the linear hull of A, we have $P \circ Q - \mathrm{Id} = 0$ or $P \circ Q = \mathrm{Id}$. Therefore the linear map $P: V \to V$ is an isomorphism. Let $W = V^\perp$. Then $V \oplus W = \mathfrak{X}$. Let $\chi: V \to \mathfrak{X}$ and $\iota: W \to \mathfrak{X}$ be the inclusions. Let $\lambda : \mathfrak{X} \to V$ and $\pi : \mathfrak{X} \to W$ be the projections. Then

$$L = \chi \circ P \circ \lambda + \iota \circ \pi : \mathfrak{X} \to \mathfrak{X}$$

is a linear map. Since $P : V \to V$ is an isomorphism, $L : \mathfrak{X} \to \mathfrak{X}$ is an isomorphism. If $v \in A(r)$, then

$$\begin{aligned} L(\chi(v)) &= \chi(P(\lambda(\chi(v))) + (\pi(\chi(v)) \\ &= \chi(P(v)) = \chi(\ell(v)). \end{aligned}$$

Hence

$$L \mid A(r) = \ell \qquad \text{Q.E.D.}$$

Therefore h is unique up to a linear isometry of $K(\Delta) \to K(\Delta)$.

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