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EXCESS INTERSECTION OF DIVISORS

Robert Lazarsfeld

Introduction

Our purpose is to give an elementary discussion of an appealing but generally forgotten idea of Severi's concerning the excess intersection of divisors on a smooth variety.

Consider n fixed effective divisors $H_1, \ldots, H_n$ on a nonsingular projective variety X of dimension n, and assume that each H_i moves in a base-point free linear system. Let $(H_1 \cdot \ldots \cdot H_n)$ denote their intersection class in the Chow group $A_0(X)$ of zero-cycles on X modulo rational equivalence. The degree of this class is the intersection number of the hypersurfaces H_i. We pose the following

PROBLEM: Decompose the intersection number $\deg(H_1 \cdot \ldots \cdot H_n)$ – or better still, the class $(H_1 \cdot \ldots \cdot H_n) \in A_0(X)$ – into a sum of local contributions from the intersection $\cap H_i \subseteq X$ of the given divisors.

In the event that the H_i intersect transversely, or merely properly (i.e. $\dim(\cap H_i) = 0$), the solution to this problem is classical. If the divisors meet improperly, however, the question becomes more interesting. Fulton and MacPherson have recently found a formula for a refined intersection class which provides a decomposition of the desired type [3, p. 11]. But as they emphasize, it is important from a foundational point of view to understand an excess intersection as a limit of proper intersections, and their construction lacked such a "dynamic" interpretation. In most other contemporary treatments, improper intersection appears simply as a pathological phenomenon. By contrast, Severi – whose approach to intersection theory centered around continuity arguments – was led quite naturally to deal with excess intersection. And in a brief remark [7, p. 258], Severi sug-

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gested a beautiful solution to the stated problem, at least for divisors on $\mathbb{P}^n$.

We may summarize Severi's proposed decomposition of the intersection number. Working on $\mathbb{CP}^n$, he lets the n given hypersurfaces move in families $\mathcal{H}_i = \{(\mathcal{H}_i)_t\}_{t\in S}$ parametrized by a smooth analytic curve S, with $(\mathcal{H}_i)_{t_0} = H_i$. Severi assumes that the divisors $(\mathcal{H}_1)_t$ intersect transversely for $t \neq t_0$. Thus for $t \neq t_0$, the intersection $(\mathcal{H}_i)_t \cap \cdots \cap (\mathcal{H}_n)_t$ consists of $\Pi \deg(H_i)$ points, which converge as $t \to t_0$ to a zero-cycle on $\cap H_i \subseteq \mathbb{P}^n$. Note that if the fixed hypersurfaces meet improperly, then this limiting cycle will depend on the families $\mathcal{H}_i$.

Severi's procedure, in effect, is to attach a "multiplicity" $i(Z)$ to each irreducible subvariety $Z \subseteq \cap H_i$ by considering *all* such deformations. Specifically, $i(Z)$ is defined inductively on $\dim(Z)$. For a point $P \in \cap H_i$, let $i(P)$ be the minimum, over all families $\mathcal{H}_i$ as above, of the number of intersection points $(\mathcal{H}_1)_t \cap \cdots \cap (\mathcal{H}_n)_t$ converging to P as $t \to t_0$. Evidently $i(P) = 0$ for all but a finite number of points P. Consider next an irreducible curve $C \subseteq \cap H_i$. Define $j(C)$ to be the minimum over all families of the number of points of $(\mathcal{H}_1)_t \cap \cdots \cap (\mathcal{H}_n)_t$ converging in the limit to some point on C, and set $i(C) = j(C) - \Sigma_{P\in C}\, i(P)$. Thus $i(C)$ counts the number of intersection points always approaching the curve C but not any particular point on C. Similarly, for an irreducible surface $F \subseteq \cap H_i$, $j(F)$ is the minimum number of intersection points converging to F, and $i(F) = j(F) - \Sigma_{Z \subsetneq F}\, i(Z)$. Severi's assertion was that continuing in this manner, one obtains for every irreducible $Z \subseteq \cap H_i$ a well-defined integer $i(Z) \geq 0$, and that these furnish a decomposition of the Bezout number:

$$(*) \qquad \sum_{Z \subseteq \cap H_i} i(Z) = \Pi \deg(H_i).$$

Unfortunately, Severi's claim is not true. A simple counterexample is given by the plane curves

$$H_1 = V(x_0^2 x_1), \quad H_2 = V(x_0 x_1^2).$$

The point to observe is that $i([0,0,1]) = 0$. In fact, consider the rational families defined in $\mathbb{P}^2 \times \mathbb{A}^1$ by

$$\mathcal{H}_1 = V(x_0^2 x_1 - t x_2^3), \quad \mathcal{H}_2 = V((x_0 - t^2 x_1)(x_1^2 - t^2 x_0^2));$$

the curves $(\mathcal{H}_1)_t$ and $(\mathcal{H}_2)_t$ intersect transversely for almost all $t \in \mathbb{A}^1 - \{0\}$, and none of the intersection points converge to $[0,0,1]$ as $t \to 0$. It follows by symmetry considerations that $\Sigma\, i(Z)$ is even, and

(*) cannot hold. (Compare [3, p. 10], and §3 below.) As the referee points out, one can vary this example to obtain one in which the H_i are reduced and irreducible by starting with the four hypersurfaces

$$H_1 = V(x_3), \quad H_2 = V(x_0^2x_1 + x_3(x_2^2 + x_4^2))$$
$$H_3 = V(x_4), \quad H_4 = V(x_0x_1^2 + x_4(x_2^2 + x_3^2))$$

in $\mathbb{P}^4$.

The object of this paper is to show how Severi's procedure can be modified so that it does yield a decomposition of the intersection class of the given divisors $H_i \subseteq X$. The approach we take, which is to focus on deformations generic to first order, can be described roughly as follows. Consider one-parameter rational algebraic families $\mathcal{H}_i = \{(\mathcal{H}_i)_t\}_{t\in S}$ satisfying the same conditions as before. Under a mild hypothesis, the $\mathcal{H}_i$ determine a point in a projective space $\mathbb{P}$ parametrizing first order approximations to such deformations. We prove a theorem to the effect that for any subvariety $Z \subseteq \cap H_i$, there exists a Zariski open dense set $U_Z \subseteq \mathbb{P}$ with the following property:

Suppose that $\mathcal{H}_i$ are families with first order terms corresponding to a point in U_Z. Then the intersection points $(\mathcal{H}_1)_t \cap \cdots \cap (\mathcal{H}_n)_t$ approaching Z as $t \to t_0$ converge to zero-cycle on Z whose rational equivalence class $\alpha(Z) \in A_0(Z)$ depends only on Z, and not on the families $\mathcal{H}_i$.

Using these classes $\alpha(Z)$ in place of the numbers $j(Z)$, one can mimic Severi's inductive definition and attach to each irreducible $Z \subseteq \cap H_i$ a rational equivalence class $\sigma(Z) \in A_0(Z)$. The degree of $\sigma(Z)$ counts the number of intersection points typically approaching Z, but not any fixed subvariety of Z. Letting $\phi^{Z,X}$ denote the inclusion $Z \hookrightarrow X$, it is easy to check that

$$\sum_{Z\subseteq \cap H_i} \phi_*^{Z,X}\sigma(Z) = (H_1 \cdot \ldots \cdot H_n)$$

in $A_0(X)$, and so we arrive at a solution in the spirit of Severi to the problem posed above. (In fact, although we shall not pursue the matter here, it turns out that the classes $\sigma(Z)$ coincide with those obtained by Fulton and MacPherson using a variant of the construction in [3, p. 11]. Thus one has a new interpretation of their refined intersection class.)

This paper is divided into three parts. §1 is devoted to preliminary definitions, and to the proof of a crucial lemma. The decomposition of the intersection class is given in §2. Finally, in §3 we turn by way of example to the case of two curves on a surface: here, a formula of B. Segre allows one to make explicit computations.

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Concerning notation and conventions: all schemes are assumed separated and of finite type over an algebraically close field k, and unless otherwise indicated we work exclusively with closed points. $\mathscr{Z}.(X) = \bigoplus \mathscr{Z}_d(X)$ denotes the group of cycles on a scheme X, graded by dimension. A closed subscheme $Y \subseteq X$ determines a cycle $[Y] \in \mathscr{Z}.(X)$, in which each irreducible component of Y appears with coefficient the length of the local ring of Y at the component's generic point. An algebraic family of d-cycles on X parametrized by a non-singular curve S is a $(d+1)$-cycle $\alpha = \Sigma\, n_i[V_i]$ on $X \times S$ such that each of the varieties V_i dominates S. The fibre $\alpha_t \in \mathscr{Z}_d(X)$ of α over $t \in S$ is defined as $\Sigma\, n_i[(V_i)_t]$, where $(V_i)_t$ is the scheme-theoretic fibre of V_i over t, considered as a subscheme of $X = X \times \{t\}$. We use Fulton's theory [1] of rational equivalence on possibly singular varieties.

§1. Preliminaries; main lemma

Let X be a non-singular projective variety of dimension n. Fix once and for all n effective divisors

$$H_1, \ldots, H_n \subseteq X,$$

and assume that each H_i moves in a base-point free linear system. One reduces the question of intersecting the H_i to an intersection involving only two schemes by forming the fibre square

$$\begin{array}{ccc} \cap H_i & \hookrightarrow & X \\ \downarrow & & \downarrow \delta \\ \displaystyle\mathop{\times}_{i=1}^{n} H_i & \hookrightarrow & \displaystyle\mathop{\times}_{i=1}^{n} X \end{array} \qquad \delta(x) = (x, \ldots, x).$$

So we actually consider the intersection of $\mathop{\times}_{i=1}^{n} H_i$ with the diagonal in

$\overset{n}{\underset{i=1}{\times}} X$. In the sequel, we denote the n-fold product $\times X$ by Y. Note that if the H_i meet properly, then the intersection cycle $H_1 \cdot \ldots \cdot H_n$ is just the cycle $[\cap H_i] \in \mathscr{Z}_0(X)$.

The study of certain global and infinitesimal deformations of $\times H_i$ in Y plays a central role in all that follows. While it is easy enough to describe intrinsically the deformations in question, it is simpler to deal directly – as Severi did – with spaces of divisors on X. Our discussion will be phrased, then, in these classical terms. We begin by introducing some notation.

Let M_i be the projective space $|H_i|$ of divisors on X linearly equivalent to H_i, and set

$$M = \overset{n}{\underset{i=1}{\times}} M_i.$$

We denote by $m_0 \in M$ the point corresponding to the product of the given divisors H_i. Let $F_i \subseteq X \times M_i$ be the universal family of divisors over M_i, and put $F = F_1 \times \cdots \times F_n \subseteq Y \times M$. Since each H_i moves in a base-point free linear system, the projection $F \to Y$ is smooth. Let $I \subset X \times M$ be the intersection of the F_i:

$$I = F \times_{Y \times M} X \times M,$$

$X \times M$ being embedded in $Y \times M$ via $\delta \times 1$. If $m \in M$ corresponds to divisors $H'_1, \ldots, H'_n$, then the fibre $I_m \subseteq X$ of I over m is isomorphic to the subscheme $\cap H_i \subseteq X$. Using the isomorphism $I = F \times_Y X$, one sees that I is a non-singular irreducible variety, with $\dim I = \dim M$.

We shall be concerned with the one-parameter deformations of $\overset{n}{\underset{i=1}{\times}} H_i$ in Y $\left(= \overset{n}{\underset{i=1}{\times}} X\right)$ determined by maps

$$f : (S, t_0) \to (M, m_0)\ ^1$$

where S is a non-singular curve, and t_0 is a fixed point on S. Such a mapping gives rise to n families $\mathscr{H}_i = \{(\mathscr{H}_i)_t\}_{t \in S}$ of linearly equivalent divisors on X, with $(\mathscr{H}_i)_{t_0} = H_i$. Assuming that the divisors $(\mathscr{H}_i)_t$ intersect properly for $t \neq t_0$, the *limiting cycle*

$$\lambda_f = \lim_{t \to t_0} (\mathscr{H}_1)_t \cdot \ldots \cdot (\mathscr{H}_n)_t \in \mathscr{Z}_0(\cap H_i)$$

[1] I.e. $f : S \to M$ is a morphism, and $f(t_0) = m_0$.

is defined as follows. Consider the subscheme $J = I \times_M S \subseteq X \times S$:

$$\begin{tikzcd} J \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & I \arrow[d] \\ S \arrow[r, "f"'] & M \end{tikzcd}$$

The cycle $[J] \in \mathscr{Z}_0(X \times S)$ restricts over $S^0 = S - \{t_0\}$ to an algebraic family $\mu^0 \in \mathscr{Z}_1(X \times S^0)$ of zero-cycles on X parametrized by S^0. Since J is flat over S^0, it follows from Lemma A in the appendix that for $t \neq t_0$,

$$(\mu^0)_t = (\mathscr{H}_1)_t \cdot \ldots \cdot (\mathscr{H}_n)_t.$$

S being a curve, μ^0 has a unique extension to a family $\mu \in \mathscr{Z}_1(X \times S)$ of zero-cycles parametrized by S: μ is obtained by taking the closures in $X \times S$ of the components of μ^0. We set

$$\lambda_f = (\mu)_{t_0}.$$

λ_f is a zero cycle on $\cap H_i \subset X$ which represents the intersection class $(H_1 \cdot \ldots \cdot H_n)$ in $A_0(X)$.

We henceforth restrict our attention to families $f: (S, t_0) \to (M, m_0)$ satisfying: (i) the divisors $(\mathscr{H}_i)_t$ intersect properly for $t \neq t_0$, and (ii) the differential $df: T_{t_0}S \to T_{m_0}M$ is non-zero. Denote by $\mathfrak{A}$ the class of all such, so that an element $f \in \mathfrak{A}$ consists of a mapping $f: (S, t_0) \to (M, m_0)$ for some smooth curve S and $t_0 \in S$. Each $f \in \mathfrak{A}$ determines a limiting cycle $\lambda_f \in \mathscr{Z}_0(\cap H_i)$. Moreover by virtue of condition (ii), the map on tangent spaces gives rise in the natural manner to a function

$$i: \mathfrak{A} \to \mathbb{P}(T_{m_0}M).$$

One verifies that *this function is surjective.* Although we do not attempt to put any geometric structure on the class $\mathfrak{A}$, we consider the projective space $\mathbb{P} = \mathbb{P}(T_{m_0}M)$ as an algebraic variety. One thinks of the points of $\mathbb{P}$ as representing the various first order approximations to families $f \in \mathfrak{A}$.

The following lemma is the basic fact that allows us to carry out Severi's idea. It asserts that for deformations $f \in \mathfrak{A}$ generic to first order, the limiting cycle λ_f depends only on first order data.

LEMMA 1.1: *There exists an open dense set $V \subseteq \mathbb{P}(T_{m_0}M)$ such that for all families $f \in i^{-1}(V)$, the limiting cycle $\lambda_f \in \mathscr{Z}_0(\cap H_i)$ depends only on $i(f)$. In fact, there exists a pure dimensional subscheme $T \subseteq X \times V$, flat and finite over V, such that*

$$[T_v] = \lambda_f \quad (\text{as zero-cycles on } X \times \{v\} = X)$$

for every $v \in V$ and $f \in \mathfrak{A}$ with $i(f) = v$.

REMARK: One must generally be content with a proper open subset $V \subset \mathbb{P}(T_{m_0}M)$. Consider for instance the example given in the Introduction. Define H_1, H_2, $\mathscr{H}_1$ as before, and let

$$\mathscr{H}_2(c) = V((x_0 - t^2x_1)(x_1 + c^3tx_0)(x_1 - c^3tx_0)) \subset \mathbb{P}^2 \times \mathbb{A}^1.$$

For any $c \in k$, the families $\mathscr{H}_1$ and $\mathscr{H}_2(c)$ determine an element of $\mathfrak{A}$ whose image in $\mathbb{P}(T_{m_0}M)$ is independent of c. But the corresponding limiting cycles do vary with c.

PROOF OF LEMMA 1.1: We use the notation introduced above. We may assume that the projection $g: I \to M$ is surjective; for if it is not then $\lambda_f = 0$ for every $f \in \mathfrak{A}$, in which case the Lemma is trivial. Let $\pi: \tilde{M} \to M$ and $\pi': \tilde{I} \to I$ denote respectively the blowings up of M at m_0, and of I along the fibre I_{m_0}. There is an induced map $\tilde{g}: \tilde{I} \to \tilde{M}$ which gives rise to a commutative diagram

$$\begin{array}{ccc} \tilde{I} & \xrightarrow{\pi'} & I \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ \tilde{M} & \xrightarrow{\pi} & M. \end{array}$$

Note that $\tilde{I} \subseteq X \times \tilde{M}$, and that $\tilde{g}$ is projection to the second factor. We identify the exceptional divisor of π with $\mathbb{P}(T_{m_0}M)$.

We claim that there exists an open set $U \subseteq \tilde{M}$, with $\operatorname{codim}(\tilde{M} - U, \tilde{M}) \geq 2$, such that res $\tilde{g}: \tilde{g}^{-1}(U) \to U$ is flat and finite. Indeed, $\tilde{M}$ is non-singular, $\tilde{I}$ is integral since I is, and the surjectivity of g implies that $\tilde{g}$ is likewise surjective. So the assertion follows from the fact that any proper surjective morphism $f: Z \to W$ between irreducible varieties of the same dimension, with W non-singular in codimension one, is flat and finite over the complement of a closed set of codimension at least two. Observe that g is flat and finite over the complement of $B = \{m \in M \mid \dim(g^{-1}(m)) \geq 1\}$; thus we may assume that $\pi^{-1}(M - B) \subseteq U$. Put

$$V = \mathbb{P}(T_{m_0}M) \cap U.$$

V is non-empty since $\operatorname{codim}(\tilde{M} - U, \tilde{M}) \geq 2$.

Consider an element $f:(S, t_0) \to (M, m_0)$ in the class $\mathfrak{A}$, so that $f(t) \notin B$ for $t \neq t_0$. Bearing in mind that S is a non-singular curve, one sees that f has a unique lifting to a morphism $\tilde{f}: S \to \tilde{M}$, and that moreover $\tilde{f}(t_0) \in \pi^{-1}(m_0) = \mathbb{P}(T_{m_0}M)$ is the point corresponding to the line $df(T_{t_0}) \subseteq T_{m_0}M$. Thus we may identify $i(f)$ with $\tilde{f}(t_0)$. Now suppose that $i(f) = \tilde{f}(t_0) \in V$. Then $\lim \tilde{f} \subseteq U$. Hence if one defines $\tilde{J} \subseteq X \times \tilde{M}$ to be the fibre product $\tilde{I} \times_{\tilde{M}} S$, then $\tilde{J}$ is flat over S, and it follows from the definition that $\lambda_f = [\tilde{J}]_{t_0} \in \mathfrak{Z}_0(\cap H_i)$. By Lemma A of the Appendix, we have $[\tilde{J}]_{t_0} = [\tilde{J}_{t_0}]$. But $\tilde{J}_{t_0} = \tilde{I}_{\tilde{f}(t_0)}$, which depends only on $\tilde{f}(t_0)$, i.e. on $i(f)$. This establishes the first assertion of the lemma. Taking $T = \tilde{I} \times_{\tilde{M}} V$ gives the desired finite flat family over V.

Q.E.D.

§2. Decomposition of the intersection class

We now show how Severi's idea, suitably modified, yields a decomposition of the intersection class $(H_1 \cdot \ldots \cdot H_n) \in A_0(X)$ into local contributions from the physical intersection of the divisors $H_i \subseteq X$. Our result takes the form of two theorems. Theorem 2.1 asserts that for deformations generic to first order, the rational equivalence class of the part of the limiting cycle lying on a given closed set $W \subseteq X$ is independent of the deformation. (The genericity condition is expressed in terms of the function $i: \mathfrak{A} \to \mathbb{P}(T_{m_0}M)$ defined in §1.) Theorem 2.2 states that the class on W so obtained arises as a sum of contributions from certain subvarieties of $\cap H_i$.

It is convenient to introduce the following notation. Given a scheme S and a subscheme T of S, $\phi^{T,S}$ denotes inclusion $T \hookrightarrow S$. If $\alpha = \Sigma_{Q \in S}\, n_Q[Q]$ is a zero-cycle on S, $p_T(\alpha)$ is the part of α lying on T:

$$p_T(\alpha) = \Sigma_{Q \in T}\, n_Q[Q].$$

THEOREM 2.1: *Let $W \subseteq X$ be a fixed closed set. Then there exists a dense open set $U_W \subseteq \mathbb{P}(T_{m_0}M)$ such that for all families $f \in$*

$i^{-1}(U_W) \subseteq \mathfrak{A}$, the cycles

$$p_W(\lambda_f) \in \mathscr{Z}_0(W)$$

lie in a single rational equivalence class $\alpha(W) \in A_0(W)$.

THEOREM 2.2: *There exist irreducible subvarieties*

$$Z_1, \ldots, Z_r \subseteq \cap H_i,$$

plus classes $\sigma(Z_j) \in A_0(Z_j)$, such that for every closed set $W \subseteq X$ the class $\alpha(W)$ is given by the formula

$$(*) \qquad \alpha(W) = \sum_{Z_j \subseteq W} \phi_*^{Z_j, W} \sigma(Z_j)$$

in $A_0(W)$.

By setting $\sigma(Z) = 0$ if Z is not one of the subvarieties occurring in Theorem 2.2, we assign a class $\sigma(Z) \in A_0(Z)$ to every irreducible $Z \subseteq \cap H_i$.

COROLLARY 2.3: *One has*

$$\sum_{Z \subseteq \cap H_i} \phi_*^{Z,X} \sigma(Z) = (H_1 \cdot \ldots \cdot H_n)$$

in $A_0(X)$.

PROOF: Apply Theorem 2.2 with $W = X$, recalling that for any $f \in \mathfrak{A}$ the limiting cycle λ_f represents the class $(H_1 \cdot \ldots \cdot H_n)$ in $A_0(X)$.

Q.E.D.

Note that one may obtain corresponding results for the intersection number by taking degrees in Theorems 2.1 and 2.2.

REMARK: The analogy with Severi's inductive approach is brought into relief by writing (*) in the form

$$\sigma(Z) = \alpha(Z) - \sum_{Z' \subset Z} \phi_*^{Z',Z} \sigma(Z').$$

In particular, this shows that the classes $\sigma(Z)$ are uniquely determined by the formula (*).

REMARK: If the H_i intersect properly, then Theorems 2.1 and 2.2 reduce to the classical decomposition of the intersection class. In fact, suppose that $Q \in \cap H_i$ is an isolated point at which the H_i have intersection multiplicity m_Q. It results from the definition of the limiting cycle that in this case $p_Q(\lambda_f) = m_Q[Q]$ for every $f \in \mathfrak{A}$. Hence $\sigma(Q) = m_Q[Q] \in A_0(Q)$.

Theorems 2.1 and 2.2 will be proved simultaneously. The idea is to interpret the assertions in terms of the spaces $V \subseteq \mathbb{P}(T_{m_0}M)$ and $T \subseteq X \times V$ constructed in Lemma 1.1; the argument is illustrated in Figure 1. In the course of the proof it becomes necessary to consider certain intersection cycles on a possibly singular ambient space. For the reader's convenience, we review from [2, §§1, 3], [3, §5] and [8, §4] the rudimentary facts which we shall need in this connection.

Let X be an irreducible variety, $i: Y \hookrightarrow X$ a local complete intersection of codimension d, and $X' \subseteq X$ a subscheme of pure dimension

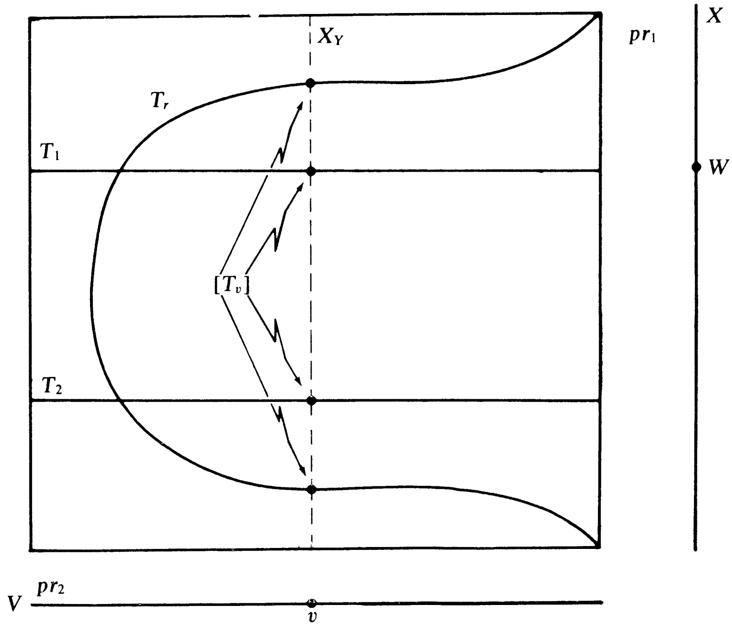

Fig. 1. $V \subseteq \mathbb{P}(T_{m_0}M)$ and $T \subseteq X \times V$ are as in Lemma 1.1: For any $f \in \mathfrak{A}$ with $i(f) = v \in V$, $\lambda_f = [T_v]$. $T_1, \ldots, T_r$ are the irreducible components of T. Given a closed set $W \subseteq \cap H_i$, for generic $v \in V$ only the components of T lying over W contribute to the part of the cycle $[T_v]$ supported on W. If T_j is such a component, then its contribution is localized to the subvariety $Z_j = \overline{pr_1(T_j)} \subseteq W$.

n. If $Y' = X' \cap Y$ has dimension $n-d$, then the intersection cycle

$$i^*([X']) = [X'] \cdot [Y] \in \mathscr{Z}_{n-d}(Y')$$

is defined. It may be computed as in Serre [6], as the alternating sum of the cycles of the sheaves $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y)$; the condition on $Y \hookrightarrow X$ ensures that the Tor_i vanish for $i > d$. If $[X'] = \Sigma\, m_j[X'_j]$, where the X'_j are the irreducible components of X', then $i^*[X'] = \Sigma\, m_j i^*[X'_j]$. If $j: Z \hookrightarrow Y$ is a local complete intersection of codimension e, and if $Z' = Y' \cap Z$ has dimension $n-(d+e)$, then

$$j^*i^*([X'])] = (i \circ j)^*([X'])$$

in $\mathscr{Z}_{n-(d+e)}(Z')$. This last fact, plus the characterization [1, Prop. 2.1] of rational equivalence, easily yields the following

LEMMA 2.4: *Let V be a smooth variety of dimension n, and let v, $v' \in V$ be points which can be joined by a non-singular rational curve $C \subseteq V$. Consider a variety Z, and a subscheme $T \subseteq Z \times V$ of dimension m such that* $\dim T \cap (Z \times \{v\}) = \dim T \cap (Z \times (v'\}) = m - n$. *Then the two cycles $[T] \cdot [Z \times \{v\}]$ and $[T] \cdot [Z \times \{v'\}]$, considered in the natural way as $(m-n)$-cycles on Z, are rationally equivalent.*
(Observe that $Z \times \{v\}$ and $Z \times \{v'\}$ are local complete intersections in $Z \times V$).

PROOFS OF THEOREMS 2.1 AND 2.2: Let $V \subseteq \mathbb{P}(T_{m_0}M)$ and $T \subseteq X \times V$ be the sets constructed in Lemma 1.1. Let $T_1, \ldots, T_r$ be the irreducible components of T (with reduced structures), and set

$$Z_j = \overline{pr_1(T_j)}.$$

Thus each Z_j is an irreducible subvariety of the intersection $\cap H_i$, and $T_j \subseteq Z_j \times V$. For $v \in V$, we will write $\langle T_j \rangle_v \in \mathscr{Z}_0(Z_j \times \{v\}) = \mathscr{Z}_0(Z_j)$ for the intersection cycle $[T_j] \cdot [Z_j \times \{v\}]$ in $Z_j \times V$. One has $\phi_*^{Z_j,X}\langle T_j \rangle_v = [T_j] \cdot [Z_j \times \{v\}]$ in $\mathscr{Z}_0(X)$, the intersection on the right being taken in $X \times V$. Denote by m_j the coefficient of $[T_j]$ in $[T]$, so that $[T] = \Sigma\, m_j[T_j]$. Observe finally that the flatness of T over V implies that $[T] \cdot [X \times \{v\}] = [T_v]$, where as usual T_v is the scheme theoretic fibre of T over $v \in V$.

Now fix a closed set $W \subseteq X$. Then for any $v \in V$ one has

$$\begin{aligned}[T_v] &= [T] \cdot [X \times \{v\}] \\ &= \sum_{Z_j \subseteq W} m_j \phi_*^{Z_j,X}\langle T_j \rangle_v + \sum_{Z_k \subsetneq W} m_k \phi_*^{Z_k,X}\langle T_k \rangle_v\end{aligned}$$

in $\mathscr{Z}_0(X)$. There exists a dense open set $U_W \subseteq V$ such that when $v \in U_W$ the cycles in the second sum on the right have support disjoint from W. For $v \in U_W$, then,

$$(*) \qquad p_W([T_v]) = \sum_{Z_j \subseteq W} m_j \phi_*^{Z_j, W} \langle T_j \rangle_v$$

in $\mathscr{Z}_0(W)$. But U_W is an open subset of a projective space, so by Lemma 2.4 the cycles $\langle T_j \rangle_v \in \mathscr{Z}_0(Z_j)$ lie in a single rational equivalence class $\tau_j \in A_0(Z_j)$ independent of $v \in U_W$. Theorem 2.1 then follows from (*), and taking $\sigma(Z_j) = m_j \tau_j \in A_0(Z_j)$ yields Theorem 2.2.

Q.E.D.

Remark: This decomposition is closely related to work of Fulton and MacPherson [2, 3], who have shown how to localize intersection classes on a non-singular variety. In the present situation, their construction yields a "refined" class

$$\{H_1 \cdot \ldots \cdot H_n\} \in \bigoplus_{Z \subseteq \cap H_i} A_0(Z),$$

which maps to the usual intersection class of the H_i under the homomorphism $\bigoplus A_0(Z) \to A_0(X)$ induced by inclusions. (The class studied in [2], which is better understood (cf. [3, §6]), is the image of $\{H_1 \cdot \ldots \cdot H_n\}$ in $A_0(\cap H_i)$.) To construct this refined class, let N be the normal bundle to $\times H_i$ in $\times X$, and C the normal cone to $\cap H_i$ in X. Denoting by $\pi : \bar{N} \to N$ (resp. $\pi' : \bar{C} \to C$) the projective completion of N (resp. C), one has the commutative diagram

$$\begin{tikzcd}
\bar{C} \arrow[r, "\pi'"] \arrow[d, hook, "p"'] & \cap H_i \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & X \arrow[d, hook, "\delta"] \\
\bar{N} \arrow[r, "\pi"'] & \times_{i=1}^{n} H_i \arrow[r, hook] & \times_{i=1}^{n} X
\end{tikzcd}$$

Let $\bar{C}_j$ be the irreducible components of $\bar{C}$ (with reduced structures), let $p_j = \bar{C}_j \hookrightarrow N$ denote the inclusion, and let $\pi'_j = \pi' \mid \bar{C}_j$. Write n_j for the coefficient of $[\bar{C}_j]$ in $[\bar{C}]$. Finally, let $\xi = \pi^*(N \oplus 1)/\mathcal{O}_{\bar{N}}(-1)$ be the universal quotient bundle of rank n on $\bar{N}$. Then the component of $\{H_1 \cdot \ldots \cdot H_n\}$ in $A_0(Z)$ is defined to be:

$$\{H_1 \cdot \ldots \cdot H_n\}_Z = \begin{cases} n_j \cdot (\pi'_j)_*(c_n(p_j^*\xi) \cap [\bar{C}_j]) \in A_0(Z) & \text{if } Z = \pi'(\bar{C}_j) \\ 0 & \text{otherwise} \end{cases}$$

(We refer to [2, 3] for details, motivation, and alternative descriptions.) Then one has the following

THEOREM: *$\sigma(Z) = \{H_1 \cdot \ldots \cdot H_N\}_Z$ in $A_0(Z)$.*

The proof will appear elsewhere. It involves intersecting sections of the normal bundle with the normal cone.

§3. The case of two curves on a surface

To illustrate the preceeding sections, we indicate how one may compute the classes $\sigma(Z)$ in examples involving two curves on a smooth surface X.

Start with effective divisors

$$H_1 = D_1 + E, \quad H_2 = D_2 + E$$

on X, where D_1 and D_2 meet properly. We choose sections $P_i \in \Gamma(X, \mathcal{O}_X(D_i))$ and $Q \in \Gamma(X, \mathcal{O}_X(E))$ such that

$$(P_i) = D_i, \quad (Q) = E.$$

(If s is a section of a line bundle, (s) denotes its divisor of zeroes.) Consider families $\mathcal{H}_i \subset X \times \mathbb{A}^1$ defined by sections of the pull-back of $\mathcal{O}_X(H_i)$ to $X \times \mathbb{A}^1$:

$$\mathcal{H}_i = (P_iQ + tS_i + t^2S'_i + \cdots) \quad i = 1, 2,$$

where $S_i, S'_i, \ldots$ are sections of $\mathcal{O}_X(H_i)$. Observe that if $m_0 \in M = |H_1| \times |H_2|$ is the point corresponding to $H_1 \times H_2$, and if $f:(\mathbb{A}^1, 0) \to (M, m_0)$ is the mapping determined by the families $\{(\mathcal{H}_i)_t\}_{t\in\mathbb{A}^1}$, then

$$df\left(\frac{d}{dt}\bigg|_0\right) = \text{image of } (S_1, S_2) \text{ in } \frac{H^0(X, \mathcal{O}_X(H_1))}{H^0(X, \mathcal{O}_X)} \oplus \frac{H^0(X, \mathcal{O}_X(H_2))}{H^0(X, \mathcal{O}_X)}.$$

We assume henceforth that the curves $(\mathcal{H}_1)_t$ and $(\mathcal{H}_2)_t$ intersect properly for almost all $t \neq 0$; deformations satisfying this condition exist in profusion provided that the $\mathcal{O}_X(H_i)$ are generated by their global sections.

In order to make concrete calculations, one needs an efficient method for determining the limiting cycles of all such families whose first order terms lie in some dense open subset of $\mathbb{P}(T_{m_0}M)$. This is provided by a formula due classically to B. Segre [5, §14]:

PROPOSITION 3.1: *Assume that the curves* $(P_1S_2 - P_2S_1)$ *and* (Q) *intersect properly. Then the limiting cycle* $\lambda = \lim_{t \to 0} (\mathcal{H}_1)_t \cdot (\mathcal{H}_2)_t$ *is given by*

$$\lambda = (P_1S_2 - P_2S_1) \cdot (Q) + (P_1) \cdot (P_2)$$

EXAMPLE: Consider again the curves

$$H_1 = V(x_0^2 x_1), \quad H_2 = V(x_0 x_1^2)$$

on $\mathbb{P}^2$. Segre's formula shows that for generic $S_1, S_2 \in \Gamma(\mathbb{P}^2, \mathcal{O}(3))$ the corresponding limiting cycle contains $[0,0,1]$ with multiplicity three. Hence $\sigma([0,0,1]) = 3 \cdot [0,0,1]$. One sees similarly that for any other point $P \in \cap H_i$, $\sigma(P) = 0$. It follows that

$$\deg \sigma(V(x_0)) = \deg \sigma(V(x_1)) = 3,$$

which determines the classes $\sigma(V(x_0))$ and $\sigma(V(x_1))$.

LEMMA 3.2: *Notation being as in* §1, *let* $Z \subseteq X \times S$ *be a closed subscheme, flat over* $S^0 = S - \{t_0\}$, *such that*

$$(*) \qquad [Z]_t = (\mathcal{H}_1)_t \cdot \ldots \cdot (\mathcal{H}_n)_t$$

for $t \neq t_0$. *Assume that* $\dim Z_{t_0} = 0$, *and suppose finally that* $\deg[Z_{t_0}] = \deg(H_1 \cdot \ldots \cdot H_n)$. *Then*

$$\lambda = [Z_{t_0}].$$

PROOF: It is enough to show that the projection $p: Z \to S$ is flat. For then Z is pure one-dimensional, in which case (*) implies that $\lambda = [Z]_{t_0}$. Moreover, Lemma A from the appendix applies to give $[Z]_{t_0} = [Z_{t_0}]$. Now it follows from (*) and the fact that $\dim Z_{t_0} = 0$ that p is quasi-finite; being proper, it is finite. Finally, the last hypothesis together with (*) shows that the function $t \to \dim_k(p_*\mathcal{O}_Z \otimes k(t)) = \deg[X_t]$ is constant, which implies the flatness of p, as desired.

Q.E.D.

PROOF OF PROPOSITION 3.1: For simplicity of notation, we will assume that $(\mathcal{H}_1)_t$ and $(\mathcal{H}_2)_t$ intersect properly for all $t \in \mathbb{A}^1 - \{0\}$. Let $Z = \mathcal{H}_1 \cap \mathcal{H}_2 \cap \Phi \subseteq X \times \mathbb{A}^1$, where $\Phi \subseteq X \times \mathbb{A}^1$ is the divisor

$$((P_1S_2 - P_2S_1) + t(P_1S_2' - P_2S_1') + \cdots).$$

Z evidently satisfies all but the last hypothesis of Lemma 3.2. The Proposition will follow if we show that $[Z_0]$ is given by the formula claimed for λ, since the latter has the correct degree.

To this end, let A be the local ring of X at a closed point, and let ϕ [resp. p_1, p_2, q] be a local equation for $P_1S_2 - P_2S_1$ [resp. P_1, P_2, Q]. It suffices to prove

$$(*) \qquad \dim_k(A/(\phi, p_1q, p_2q)) = \dim_k(A/(\phi, q)) + \dim_k(A/(p_1, p_2)).$$

But consider the exact sequence of A-modules:

$$0 \to A/(p_1, p_2) \xrightarrow{\cdot q} A/(p_1q, p_2q) \to A/q \to 0.$$

Noting that $\operatorname{Tor}_1^A(A/\phi, A/q) = 0$, (*) follows upon tensoring by A/ϕ.

Q.E.D.

Appendix

For lack of a suitable reference, we outline a proof of the following useful

LEMMA A: *Let S be a non-singular curve, X a pure one-dimensional scheme, and $X \to S$ a flat morphism. Then for $t \in S$, one has*

$$[X_t] = [X]_t.$$

PROOF: Given $x \in X_t$, let $A = \mathcal{O}_xT$, and let $u \in \mathcal{O}_tS$ be a uniformizing parameter. By flatness, u is a non zero-divisor in A. The equality at x of the two cycles in question is equivalent to

$$(*)\quad \text{length}(A/uA)=\sum_i \text{length}(A_{\mathfrak{j}_i})\cdot \text{length}(A/\mathfrak{j}_i+(u)),$$

where $\{\mathfrak{j}_i\}$ are the minimal prime ideals of A. But (*) follows from [4, IV. 21. 10. 17.7].

REFERENCES

[1] W. FULTON: Rational equivalence for singular varieties. *Publ. Math. IHES. 45* (1975) 147–167.

[2] W. FULTON and R. MACPHERSON: Intersecting cycles on an algebraic variety, in P. Holm, ed., *Real and Complex Singularities, Oslo 1976* (Sijthoff and Noordhoff) 179–197.

[3] W. FULTON and R. MACPHERSON: Defining algebraic intersections, in L. Olson, ed., *Algebraic Geometry, Lect. Notes in Math. 687* (1978) 1–30.

[4] A. GROTHENDIECK and J. DIENDONNÉ: Eléments de géométrie algébrique IV. *Publ. Math. I.H.E.S. 32* (1967).

[5] B. SEGRE: On limits of algebraic varieties. *Proc. London Math. Soc. 47* (1940) 351–403.

[6] J.-P. SERRE: Algebre locale multiplicities. *Lect. Notes in Math. 11* (1965).

[7] F. SEVERI: Il concetto generale di multiplicita della soluzioni per sistemi de equazioni algebriche e la teoria dell'eliminazione. *Annali di Math. 26* (1947) 221–270. [Reprinted in Memoire Scelte, Vol. I, where the relevant passage appears on p. 372].

[8] J.-L. VERDIER: Le théorem de Riemann-Roch pour les intersections complètes, in A. Douady and J.-L. Verdier, *Seminaire de géométrie analytique, Astérisque 36–37* (1976) Exposé IX.

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