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SOME APPLICATIONS OF STRONG LUSIN SETS

Jan Brzuchowski, Jacek Cichoń and Bogdan Węglorz

Introduction

A main tool of this paper are strong Lusin sets. It turns out that the use of strong Lusin sets permits to construct some special σ-fields of subsets of the real line $\mathbf{R}$. Namely, we consider the following situation: $\mathcal{B}$ is the σ-field of Borel subsets of $\mathbf{R}$ and $\mathcal{I}$ is a σ-ideal on $\mathbf{R}$ with a Borel basis. We wish to extend the σ-field $\mathcal{B}(\mathcal{I})$ to a field which has some combinatorial properties. If A is a strong Lusin set for $\mathcal{B}(\mathcal{I})$ then we can test any $X \in \mathcal{B}(\mathcal{I})$ whether it belongs to $\mathcal{I}$, namely it suffices to look whether the cardinality of $X \cap A$ is less than $\mathfrak{c}$. Consequently if A is a strong Lusin set for $\mathcal{B}(\mathcal{I})$ then $\mathcal{I} \cap \mathcal{P}(A) \subseteq [A]^{<\mathfrak{c}}$ and $(\mathcal{B} - \mathcal{I}) \cap \mathcal{P}(A) \subseteq [A]^{\mathfrak{c}}$; thus we have a quite big freedom of extension of $\mathcal{I}$ to an ideal $\mathcal{J}$ in this manner that $\mathcal{B} - \mathcal{J} = \mathcal{B} - \mathcal{I}$. This freedom in the choice of $\mathcal{J}$ allows us to solve some problems arisen from some Ulam's problems on σ-fields on $\mathbf{R}$ (for a more detailed discussion of these problems see [4]).

The present paper is divided into two parts. In §1 (Tools), we clarify the question of the existence of strong Lusin sets and testing mappings and we construct some special strong Lusin sets like Hamel bases. In §2 (Applications), we apply our tools to give an answer to a question from [4] and to get a strengthening of a theorem from [12]. Of course in §0 we give all necessary definitions and clarify our notation.

§0. Notation and terminology

We use the standard set-theoretical notation and terminology, e.g. ordinals are sets of all smaller ordinals, cardinals are initial ordinals,

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$\omega = \{0, 1, 2, \ldots\}$ is the set of all natural numbers, $\mathbf{R}$ is the set of all reals and $\mathbf{Q}$ is the set of all rationals. The cardinality of a set A is denoted by $|A|$. In particular $|\mathbf{R}| = \mathfrak{c} = 2^\omega$ and $|\omega| = |\mathbf{Q}| = \omega$. $\mathcal{P}(X)$ denotes the set of all subsets of X. A cardinal κ is regular iff κ is not any union of fewer than κ sets of the cardinality less than κ.

We consider some ideals and fields of sets on either $\mathbf{R}$ or $\mathfrak{c}$. All ideals and fields under consideration are closed under countable unions and contain all singletons. $\mathcal{L}$ and $\mathcal{K}$ are the ideals of all subsets of $\mathbf{R}$ of Lebesgue measure zero and of all meager subsets of $\mathbf{R}$, respectively. $\mathcal{B}$ denotes the field of Borel subsets of $\mathbf{R}$ and $NS_\mathfrak{c}$ the ideal of nonstationary subsets of $\mathfrak{c}$. In the case when $\mathfrak{c}$ is regular we may use the Fodor Theorem for $NS_\mathfrak{c}$ (see e.g. [3]).

If $\mathcal{I}$ is an ideal then a family $\mathcal{A} \subseteq \mathcal{I}$ is a basis for $\mathcal{I}$ iff for any element of $\mathcal{I}$ there is an element of $\mathcal{A}$ which includes it; in particular an ideal $\mathcal{I}$ on $\mathbf{R}$ has a Borel basis iff $\mathcal{I} \cap \mathcal{B}$ is a basis for $\mathcal{I}$. For any ideal $\mathcal{I}$ on $\mathbf{R}$ we denote by $\mathcal{B}(\mathcal{I})$ the field generated by $\mathcal{B}$ and $\mathcal{I}$. It is easy to see that $\mathcal{B}(\mathcal{I}) = \{B \triangle I : B \in \mathcal{B} \text{ and } I \in \mathcal{I}\}$, where $\triangle$ denotes the symmetric difference.

Let $\mathcal{I}$ be an ideal on $\mathbf{R}$. By $\alpha(\mathcal{I})$ we denote the least cardinal κ such that any set from $\mathcal{B}(\mathcal{I}) - \mathcal{I}$ can be presented as a union of κ sets from $\mathcal{I}$. Similarly $\beta(\mathcal{I})$ denotes the least cardinality of sets from $\mathcal{P}(\mathbf{R}) - \mathcal{I}$. For a discussion of properties of $\alpha(\mathcal{I})$ and $\beta(\mathcal{I})$ see [2]. Notice that for any $\mathcal{I}$ we always have the following obvious relations

$$\omega_1 \leq \alpha(\mathcal{I}) \leq \mathfrak{c} \qquad \omega_1 \leq \beta(\mathcal{I}) \leq \mathfrak{c}.$$

We say that a boolean algebra $\mathcal{C}$ satisfies C.C.C. if any family of pairwise disjoint non-zero elements of $\mathcal{C}$ is at most countable. A boolean algebra $\mathcal{C}$ is homogeneous if for each non-zero element $a \in \mathcal{C}$ the algebras $\mathcal{C}$ and $\mathcal{C}^{(a)} = \{x \in \mathcal{C} : x \leq a\}$ are isomorphic.

We deal with the following three properties of ideals on $\mathbf{R}$ or on $\mathfrak{c}$ (for more detailed discussion of these notions see [1], [3] and [11]).

(1) An ideal $\mathcal{I}$ is a P-ideal if for each family $\{A_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{I}$ there is some $A \in \mathcal{I}$ such that for all $\alpha < \mathfrak{c}$ we have $|A_\alpha - A| < \mathfrak{c}$.

(2) An ideal $\mathcal{I}$ has the property $U(\mathfrak{c})$ if there is a family of pairwise disjoint sets $\{A_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\mathbf{R})$, each of which has the cardinality $\mathfrak{c}$, such that for each $A \in \mathcal{I}$ there is some $\alpha < \mathfrak{c}$ with $A_\alpha \cap A = \emptyset$.

(3) An ideal $\mathcal{I}$ is selective, if for each partition $\mathcal{U} \subseteq \mathcal{I}$ there is a selector S of $\mathcal{U}$ such that the complement of S is in $\mathcal{I}$.

Now we adapt the property (3) for consideration of fields on $\mathbf{R}$. We say that a field $\mathcal{S}$ on $\mathbf{R}$ is selective, if for each partition $\mathcal{U} \subseteq \mathcal{S}$ of $\mathbf{R}$ there is a selector of $\mathcal{U}$ in $\mathcal{S}$.

If μ is a countably additive measure on a field of subsets of $\mathbf{R}$ then by $\mathscr{I}_\mu$ we denote the ideal of all subsets of $\mathbf{R}$ of μ-measure zero. We say that μ is invariant under translations if for each μ-measurable set A and $x \in \mathbf{R}$ the set $A + x = \{a + x\colon a \in A\}$ is also μ-measurable and $\mu(A + x) = \mu(A)$. More general, if $\mathscr{A}$ is a family of subsets of $\mathbf{R}$ then we say that $\mathscr{A}$ is invariant under translations if for each $A \in \mathscr{A}$ and each $x \in \mathbf{R}$ we have $A + x \in \mathscr{A}$. We say that $\mathscr{A}$ is invariant if $\mathscr{A}$ is invariant under translations and for each $A \in \mathscr{A}$ and each rational $r \in \mathbf{Q}$ we have $rA = \{ra\colon a \in A\} \in \mathscr{A}$.

We treat very often $\mathbf{R}$ as a linear space over rationals. In particular we say that a set $X \subseteq \mathbf{R}$ is linearly independent if X is independent in the linear space $\mathbf{R}$ over $\mathbf{Q}$. A basis of the space $\mathbf{R}$ over $\mathbf{Q}$ is called a Hamel basis of $\mathbf{R}$. If $X \subseteq \mathbf{R}$ then by $[X]$ we denote the linear subspace of $\mathbf{R}$ spanned by X.

§1. Tools

In this section we consider only those σ-ideals on $\mathbf{R}$ which have Borel bases.

Let us recall the following two notions.

Definition: (i) A set $A \subseteq \mathbf{R}$ of the cardinality $\mathfrak{c}$ is a Lusin set for an ideal $\mathscr{I}$ on $\mathbf{R}$, if for each $I \in \mathscr{I}$ we have $|A \cap I| < \mathfrak{c}$ (see Sierpiński [8]).

(ii) A set $A \subseteq \mathbf{R}$ is a strong Lusin set for $\mathscr{B}(\mathscr{I})$, if for each $B \in \mathscr{B}(\mathscr{I})$ we have $|B \cap A| < \mathfrak{c}$ iff $B \in \mathscr{I}$ (compare McLaughlin [6]).

Lemma 1: *Suppose that $\alpha(\mathscr{I}) = \mathfrak{c}$. Then*

(i) *there exists a strong Lusin set for $\mathscr{B}(\mathscr{I})$,*

(ii) *if $\mathscr{I}$ is invariant under translations then there exists a Hamel basis which is a strong Lusin set for $\mathscr{B}(\mathscr{I})$.*

Proof: Let $\{X_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of all sets from $\mathscr{B} \cap \mathscr{I}$ and let $\{Y_\alpha\colon \alpha < \mathfrak{c}\}$ be a sequence of sets from $\mathscr{B} - \mathscr{I}$ such that each element of $\mathscr{B} - \mathscr{I}$ occurs $\mathfrak{c}$ times in this sequence.

To prove (i), pick for every $\alpha < \mathfrak{c}$ an element p_α from $Y_\alpha - (\bigcup_{\xi<\alpha} X_\xi \cup \{p_\xi\colon \xi < \alpha\})$. It is easy to see that the set $\{p_\alpha\colon \alpha < \mathfrak{c}\}$ is a strong Lusin set for $\mathscr{B}(\mathscr{I})$.

To prove (ii), fix an enumeration $\{x_\alpha\colon \alpha < \mathfrak{c}\}$ of all reals. We construct two sequences of reals: a sequence $\{p_\alpha\colon \alpha < \mathfrak{c}\}$ and a sequence $\{q_\alpha\colon \alpha < \mathfrak{c} \text{ and } \alpha \text{ is odd}\}$. We proceed as follows:

We put $N_\alpha = \bigcup_{\xi<\alpha} X_\xi \cup [\{p_\xi\colon \xi < \alpha\} \cup \{q_\xi\colon \xi < \alpha \text{ and } \xi \text{ is odd}\}]$. Now we consider two cases:

(a) α is even, i.e. $\alpha = \lambda + 2n$. Then let p_α be any element from $Y_{\lambda+n} - N_\alpha$.

(b) α is odd, i.e. $\alpha = \lambda + 2n + 1$. By assumption on $\mathcal{I}$, we have $N_\alpha \cup (N_\alpha + x_{\lambda+n}) \neq \mathbf{R}$. Thus we can choose p_α, $q_\alpha \notin N_\alpha$ such that $p_\alpha - q_\alpha = x_{\lambda+n}$.

Let $A = \{p_\xi : \xi < \mathfrak{c}\}$ and $B = \{q_\xi : \xi < \mathfrak{c}$ and ξ odd$\}$. Then A is linearly independent set which is a strong Lusin set for $\mathcal{B}(\mathcal{I})$, B is a Lusin set for $\mathcal{B}(\mathcal{I})$, and $[A \cup B] = \mathbf{R}$. Let X be any maximal linearly independent set such that $A \subseteq X$ and $X \subseteq A \cup B$. Then X is a strong Lusin set and a Hamel basis.

Remark: It is easy to see that if $cf(\mathfrak{c}) = \mathfrak{c}$, then the existence of a strong Lusin set for $\mathcal{B}(\mathcal{I})$ implies that $\alpha(\mathcal{I}) = \mathfrak{c}$.

We do not have to assume that $\alpha(\mathcal{I}) = \mathfrak{c}$ in order to produce a strong Lusin set for $\mathcal{B}(\mathcal{I})$. In fact, we can construct a strong Lusin set just from the assumption that there exists a Lusin set, provided the algebra $\mathcal{B}(\mathcal{I})/\mathcal{I}$ satisfies some extra conditions. Notice that both the ideals $\mathcal{L}$ and $\mathcal{K}$ satisfy them.

Proposition: *Suppose the algebra $\mathcal{B}(\mathcal{I})/\mathcal{I}$ is homogeneous and satisfies C.C.C. If there exists a Lusin set for $\mathcal{I}$ then there exists a strong Lusin set for $\mathcal{B}(\mathcal{I})$.*

Proof: Let A be a Lusin set for $\mathcal{B}(\mathcal{I})$. Let $\mathcal{X}$ be a maximal family of $\mathcal{I}$-almost disjoint sets from $\mathcal{B} - \mathcal{I}$ such that for each $X \in \mathcal{X}$ we have $|X \cap A| < \mathfrak{c}$. By C.C.C. we see that $|\mathcal{X}| \leq \omega$. Let $B = \mathbf{R} - \cup\mathcal{X}$. Then obviously B is a Borel set, $B \notin \mathcal{I}$ and for each $U \subseteq B$ if $U \notin \mathcal{I}$ then $|A \cap U| = \mathfrak{c}$.

By Sikorski's theorem ([7], Theorem 32.5) there exists a Borel isomorphism f from B onto $\mathbf{R}$ such that for each Borel subset X of B we have $X \in \mathcal{I}$ iff $f(X) \in \mathcal{I}$. Thus $f(A)$ is a strong Lusin set for $\mathcal{B}(\mathcal{I})$.

Remark: The assumption of homogeneity of the algebra $\mathcal{B}(\mathcal{I})/\mathcal{I}$ in the Proposition above is necessary. In fact, if we add $\aleph_2$ Cohen reals to a model for $ZFC + V = L$, then in the resulted model the ideal $\mathcal{I} = \{X \subseteq \mathbf{R}: \mathbf{R}^+ \cap X \in \mathcal{K}$ and $\mathbf{R}^- \cap X \in \mathcal{L}\}$ has a Lusin set, the algebra $\mathcal{B}(\mathcal{I})/\mathcal{I}$ satisfies C.C.C., and there is no strong Lusin set for $\mathcal{B}(\mathcal{I})$.

Our main tool is the following notion.

Definition: A mapping $f : \mathfrak{c} \to \mathbf{R}$ is a testing mapping for $\mathcal{B}(\mathcal{I})$ if for each $X \in \mathcal{B}(\mathcal{I})$ we have

$$X \in \mathcal{J} \quad \text{iff} \quad f^{-1}(X) \in NS_{\mathfrak{c}}.$$

LEMMA 2: *Let $cf(\mathfrak{c}) = \mathfrak{c}$. Then $\mathcal{B}(\mathcal{J})$ has a strong Lusin set iff it has a testing mapping.*

PROOF: Let A be a strong Lusin set for $\mathcal{B}(\mathcal{J})$. Let $\{X_\alpha: \alpha < \mathfrak{c}\}$ be an enumeration of all sets from $\mathcal{B} - \mathcal{J}$. Notice that since $\mathcal{J}$ has a Borel basis, for each $X \in \mathcal{B}(\mathcal{J}) - \mathcal{J}$ there is some $\alpha < \mathfrak{c}$ such that $X_\alpha \subseteq X$.

Consider the family $\{X_\alpha \cap A: \alpha < \mathfrak{c}\}$. Since A is a strong Lusin set for $\mathcal{B}(\mathcal{J})$, we see that, for all $\alpha < \mathfrak{c}$, we have $|X_\alpha \cap A| = \mathfrak{c}$. By Sierpiński's Refining Theorem (see [9]), there is a family $\{Y_\alpha: \alpha < \mathfrak{c}\}$ of pairwise disjoint sets such that for each $\alpha < \mathfrak{c}$, we have $Y_\alpha \subseteq X_\alpha \cap A$, $|Y_\alpha| = \mathfrak{c}$ and $\bigcup_{\alpha<\mathfrak{c}} Y_\alpha = A$. By Solovay's Partition Theorem (see [10]), there is a family of pairwise disjoint stationary sets $\{Z_\alpha: \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\mathfrak{c})$. Let f be any one-to-one function which maps $\mathfrak{c}$ onto A such that for each $\alpha < \mathfrak{c}$ we have $f(Z_\alpha) = Y_\alpha$. By our construction, if $X \in \mathcal{B}(\mathcal{J}) - \mathcal{J}$ then for some $\alpha < \mathfrak{c}$ we have $Y_\alpha \subseteq X_\alpha \subseteq X$. Consequently $Z_\alpha \subseteq \{\xi: f(\xi) \in X\}$. Thus $f^{-1}(X)$ is stationary. Similarly, if $X \in \mathcal{J}$ then $|X \cap A| < \mathfrak{c}$. Thus $f^{-1}(X)$, as a bounded subset of $\mathfrak{c}$, is nonstationary. This shows that f is a testing mapping for $\mathcal{B}(\mathcal{J})$.

Conversely, suppose that f is a testing mapping for $\mathcal{B}(\mathcal{J})$. Let $\{X_\alpha: \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{B} \cap \mathcal{J}$ and let $N_\alpha = f^{-1}(X_\alpha)$. Then $N_\alpha \in NS_{\mathfrak{c}}$ for each $\alpha < \mathfrak{c}$. Consequently the diagonal union $N = \nabla_{\alpha<\mathfrak{c}} N_\alpha$ is in $NS_{\mathfrak{c}}$. Now it is easy to check that $f(\mathfrak{c} - N)$ is a strong Lusin set for $\mathcal{B}(\mathcal{J})$.

It would be interesting to know if the assumption $cf(\mathfrak{c}) = \mathfrak{c}$ is essential in the Lemma above.

§2. Applications

Our first application of notions and methods introduced in §1 is the following theorem, which is a solution of a problem from [4].

THEOREM 1: *Suppose $\alpha(\mathcal{J}) = \mathfrak{c}$, $cf(\mathfrak{c}) = \mathfrak{c}$ and $\mathcal{B}(\mathcal{J})/\mathcal{J}$ satisfies C.C.C. Then there exists a proper ω_1-complete selective field extending $\mathcal{B}(\mathcal{J})$.*

PROOF: By Lemma 1 there is a strong Lusin set for $\mathcal{B}(\mathcal{J})$. Thus using Lemma 2 we get a testing mapping f for $\mathcal{B}(\mathcal{J})$. Define an ideal $\mathcal{J}$ on $\mathbf{R}$ by: $X \in \mathcal{J}$ if $f^{-1}(X) \in NS_{\mathfrak{c}}$. Let $\mathcal{S} = \mathcal{B}(\mathcal{J})$. Then obviously $\mathcal{S}$ is an ω_1-complete field extending $\mathcal{B}(\mathcal{J})$.

Claim 1. $\mathcal{S}$ is proper.

Suppose not. Define a mapping $\Psi:\mathcal{P}(\mathfrak{c})\to\mathcal{B}(\mathcal{J})/\mathcal{J}$ by $\Psi(X)=[f(X)]_{\mathcal{J}}$ for $X\in\mathcal{P}(\mathfrak{c})$. Then Ψ yields a one-to-one mapping $\Phi:\mathcal{P}(\mathfrak{c})/NS_{\mathfrak{c}}\to\mathcal{B}(\mathcal{J})/\mathcal{J}$. By Solovay's Partition Theorem (see [10]) we have $|\mathcal{P}(\mathfrak{c})/NS_{\mathfrak{c}}|>\mathfrak{c}$, consequently $|\mathcal{B}(\mathcal{J})/\mathcal{J}|>\mathfrak{c}$. On the other hand we have $|\mathcal{B}(\mathcal{J})/\mathcal{J}|\leq|\mathcal{B}|=\mathfrak{c}$. This contradiction proves our Claim 1.

Claim 2. $\mathcal{S}$ is selective.

First, consider the case when a partition $\mathcal{U}$ of $\mathbf{R}$ is such that $\mathcal{U}\subseteq\mathcal{J}$. Then, since $NS_{\mathfrak{c}}$ is selective, there exists a selector S of $\mathcal{U}$ which has the complement in $\mathcal{J}$, consequently $S\in\mathcal{B}(\mathcal{J})$.

So, consider the general case, i.e. let $\mathcal{U}\subseteq\mathcal{B}(\mathcal{J})$ be any partition of $\mathbf{R}$. Since for each $A\in\mathcal{B}$ we have $A\in\mathcal{I}$ iff $A\in\mathcal{J}$, we see that $\mathcal{B}(\mathcal{J})/\mathcal{J}=\mathcal{B}(\mathcal{I})/\mathcal{I}$. In particular $\mathcal{B}(\mathcal{J})/\mathcal{J}$ satisfies C.C.C. Consequently at most countably many members of $\mathcal{U}$ are in $\mathcal{B}(\mathcal{J})-\mathcal{J}$. Thus $\mathcal{U}=\{Y_n:n<\omega\}\cup\{Z_\alpha:\alpha<\mathfrak{c}\}$, where $Z_\alpha\in\mathcal{J}$ for $\alpha<\mathfrak{c}$. Let $Y=\bigcup_{n<\omega}Y_n$. Then $Y\in\mathcal{B}(\mathcal{J})$. Consider the partition $\mathcal{V}=\{Z_\alpha:\alpha<\mathfrak{c}\}\cup\{\{y\}:y\in Y\}$ of $\mathbf{R}$. Then $\mathcal{V}\subseteq\mathcal{J}$. Thus, as we have noticed before, there is a selector S of $\mathcal{V}$ such that $S\in\mathcal{B}(\mathcal{J})$. Let F be a selector of $\{Y_n:n<\omega\}$. Then $F\in\mathcal{B}(\mathcal{J})$. But $(S-Y)\cup F$ is a selector of $\mathcal{U}$ which clearly belongs to $\mathcal{B}(\mathcal{J})$. This shows the selectivity of $\mathcal{S}$.

REMARKS: (1) In fact, in [4], the Authors stated the following (added in proof): "Let $\mathcal{C}$ be a σ-complete field of subsets of real line 2^ω, which contains all Lebesgue measurable sets. Suppose that for every partition $\mathcal{V}\subseteq\mathcal{C}$ of 2^ω there exists a selector of $\mathcal{V}$ in $\mathcal{C}$. Does $\mathcal{C}=\mathcal{P}(2^\omega)$? Our conjecture is NO, at least in $ZFC+CH$".

E. Grzegorek has remarked that if $cf(\mathfrak{c})=\omega_1<\mathfrak{c}$, and $\beta(\mathcal{L})=\mathfrak{c}$ then the answer is YES. Notice that the assumption that $cf(\mathfrak{c})=\mathfrak{c}$ itself does not suffice to prove Theorem 1. Indeed, if we add $\aleph_2$ Cohen reals to a model for $ZFC+CH$ then in resulted model we have $cf(\mathfrak{c})=\mathfrak{c}$ and the answer is YES.

(2) Notice that, if we apply our Theorem 1 exactly to the case of the problem mentioned above, i.e. to $\mathcal{B}(\mathcal{L})$, then on the field $\mathcal{S}$ constructed in the proof of Theorem 1, we can define a countably additive measure μ by: $\mu(B\triangle I)=m(B)$, where $B\in\mathcal{B}$, $I\in\mathcal{J}$, and m is the Lebesgue measure on $\mathbf{R}$.

A next application of the method of strong Lusin sets is a theorem which improves a theorem from [12].

THEOREM 2: *Let m be the Lebesgue measure on* $\mathbf{R}$ *and suppose that $\alpha(\mathcal{L})=cf(\mathfrak{c})=\mathfrak{c}$. Then there exists a countably additive invariant* *measure μ on* $\mathbf{R}$ *such that:*

(i) *if* $X \in \mathcal{B}(\mathcal{L})$ *then* $X \in \mathcal{B}(\mathcal{I}_\mu)$ *and* $\mu(X) = m(X)$;

(ii) $\mathcal{I}_\mu \notin U(\mathfrak{c})$;

(iii) $\mathcal{I}_\mu$ *is a* P*-ideal.*

PROOF: By our assumptions and Lemma 1 there exists a Hamel basis H which is a strong Lusin set for $\mathcal{B}(\mathcal{L})$. Let $H = \{h_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of H. By proof of Lemma 2 we can assume that for each $X \in \mathcal{B}(\mathcal{L}) - \mathcal{L}$ we have $(\alpha : h_\alpha \in X\} \notin NS_\mathfrak{c}$. Let $R_\alpha = [\{h_\beta : \beta < \alpha\}]$. Define a function $r : \mathbf{R} \to \mathfrak{c}$ by $r(x) = \min\{\alpha : x \in R_\alpha\}$. Notice that if $x \neq 0$ then $r(x) = \alpha_0$ iff there are some $s_0, s_1, \ldots, s_n \in \mathbf{Q} - \{0\}$ and $\alpha_0 > \alpha_1 > \cdots > \alpha_n$ such that $x = s_0 h_0 + \cdots + s_n h_n$.

Claim 1. If $N \in \mathcal{L}$ then $r(N) \in NS_\mathfrak{c}$.

Suppose, to get a contradiction, that there is some $N \in \mathcal{L}$ such that $r(N) \notin NS_\mathfrak{c}$. Without loss of generality, we can assume that $0 \notin N$ and that r is one-to-one on N. Then each $a \in N$ has the form $a = s_0 h_{\alpha_0} + \cdots + s_n h_{\alpha_n}$, where $s_0, \ldots, s_n \in \bar{\mathbf{Q}} = \mathbf{Q} - \{0\}$, and $r(a) = \alpha_0 > \cdots > \alpha_n$. Because we have only countably many finite sequences from $\bar{\mathbf{Q}}$, and $NS_\mathfrak{c}$ is $\mathfrak{c}$-complete, without loss of generality we can assume that there are some $s_0, \ldots, s_n \in \bar{\mathbf{Q}}$ such that for each $a \in N$ we have $a = s_0 h_{\alpha_0} + \cdots + s_n h_{\alpha_n}$, where $r(a) = \alpha_0$. Define a function $f : r(N) \to \mathfrak{c}$ by $f(\alpha_0) = \alpha_1$ if there is (by our assumption exactly one) such $a \in N$ that $a = s_0 h_{\alpha_0} + s_1 h_{\alpha_1} + \cdots + s_n h_{\alpha_n}$. Since f is regressive there exists $M_1 \subseteq N$ and $\beta_1 < \mathfrak{c}$ such that $r(M_1) \notin NS_\mathfrak{c}$ and f has a constant value β_1 on $r(M_1)$.

Repeating this argument n-times, we get a set $M \subseteq N$ and $\beta_1 > \cdots > \beta_n$ such that $r(M) \notin NS_\mathfrak{c}$, and for each $a \in M$ we have $a = s_0 h_{\alpha_0} + s_1 h_{\beta_1} + \cdots + s_n h_{\beta_n}$. But then $|M| = \mathfrak{c}$. Let $C = \frac{1}{s_0}(M - (s_1 h_{\beta_1} + \cdots + s_n h_{\beta_n}))$. Then $C \in \mathcal{L}$, $|C| = \mathfrak{c}$ and $C \subseteq H$. But this contradicts our assumption that H is a strong Lusin set for $\mathcal{B}(\mathcal{L})$. This finishes the proof of Claim 1.

Let $\mathcal{J}$ be an ideal defined by $X \in \mathcal{J}$ iff $r(X) \in NS_\mathfrak{c}$. It is easy to see that $\mathcal{J}$ is $\mathfrak{c}$-complete and, by Claim 1, $\mathcal{L} \subseteq \mathcal{J}$. Moreover for each $a \in \mathbf{R}$, each $X \subseteq \mathbf{R}$ and each $s \in \bar{\mathbf{Q}}$ we have $r(X + a) - (r(a) + 1) = r(X) - (r(a) + 1)$ and $r(sX) = r(X)$. Consequently $\mathcal{J}$ is an invariant ideal on $\mathbf{R}$. Finally, notice that for every $X \in \mathcal{B}(\mathcal{L})$ we have $X \in \mathcal{J}$ iff $X \in \mathcal{L}$. Thus we can define a measure μ on $\mathcal{B}(\mathcal{J})$ by: $\mu(B \triangle I) = m(B)$, where $B \in \mathcal{B}$ and $I \in \mathcal{J}$. It is easy to see that μ is a countably additive invariant measure on $\mathbf{R}$ and $\mathcal{I}_\mu = \mathcal{J}$.

Claim 2. $\mathcal{J} \notin U(\mathfrak{c})$.

Indeed, let $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\}$ be a partition of $\mathbf{R}$ into sets of cardinality $\mathfrak{c}$. Consider the family $\{r(U_\alpha): \alpha < \mathfrak{c}\}$. Since the coimage of any point has the cardinality less than $\mathfrak{c}$ and $cf(\mathfrak{c}) = \mathfrak{c}$, we see that the cardinality of any set from $\{r(U_\alpha): \alpha < \mathfrak{c}\}$ is also $\mathfrak{c}$. Thus, by Sierpiński's Refining Theorem, there is a family of pairwise disjoint sets $\{V_\alpha : \alpha < \mathfrak{c}\}$ such that for every $\alpha < \mathfrak{c}$ we have $|V_\alpha| = \mathfrak{c}$ and $V_\alpha \subseteq r(U_\alpha)$. Now, since $NS_\mathfrak{c} \notin U(\mathfrak{c})$, there is a selector S for $\{V_\alpha : \alpha < \mathfrak{c}\}$ which is in $NS_\mathfrak{c}$. But then, for each $\alpha < \mathfrak{c}$, $r^{-1}(S) \cap U_\alpha \neq \emptyset$ and $r^{-1}(S) \in \mathcal{J}$. Thus there is a selector of $\mathcal{U}$ in $\mathcal{J}$ which proves our Claim 2.

Finally by a similar argument like used in the proof of Claim 2, we can show that $\mathcal{J}$ is a P-ideal.

REMARK: As far as we know, extensions of the Lebesgue measure in the Kakutani–Oxtoby way (see for example [5]) have the property $U(\mathfrak{c})$.

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