COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 42, nº 3 (1980), p. 279-313

<http://www.numdam.org/item?id=CM_1980__42_3_279_0>

COMPOSITIO MATHEMATICA, Vol. 42, Fasc. 3, 1981, pag. 279–313

COHOMOLOGICALLY INSIGNIFICANT DEGENERATIONS OF ALGEBRAIC VARIETIES*

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Let $f: X \to D$ be a projective holomorphic map from a complex space X to the unit disk D smooth over the punctured disk $D^* = D - \{0\}$. For $t \in D$ denote $X_t = f^{-1}(t)$, the fiber X_0 is called the special fiber and can be considered as a degeneration of any fiber X_t, $t \neq 0$. Let $\beta_t : H^n(X) \to H^n(X_t)$ to be the restriction map of the cohomology spaces with real coefficients. Because X_0 is a strong deformation retract of X the map β_0 is bijective, the composite map

$$sp_t^n = \beta_t \circ \beta_0^{-1} : H^n(X_0) \to H^n(X_t), t \neq 0.$$

is called the *specialization map* and plays an important rôle in the theory of degenerations of algebraic varieties.

According to Deligne [5] for every complex algebraic variety Y the cohomology space $H^n(Y)$ has a canonical and functorial mixed Hodge structure. However in general sp_t^n is not a morphism of mixed Hodge structures. Schmid [22] and Steenbrink [25] have introduced another mixed Hodge structure on $H^n(X_t)$, the *limit Hodge structure*, such that sp_t^n becomes a morphism of these mixed Hodge structures. The precise structure of such limit Hodge structure was conjectured by Deligne (cf. [6], conjecture 9.17).

We say that X_0 is a *cohomologically n-insignificant* degeneration if sp_t^n induces an isomorphism of (p, q)-components with $pq = 0$ (this definition is independent of a choice of $t \neq 0$). We say that X_0 is *cohomologically insignificant* if it is cohomologically n-insignificant for every n. This rather obscure definition is motivated by the following facts:

* Research supported in part by the National Science Foundation.

0010-437X/81/03279-35$00.20/0

1. If the singular locus of X_0 has dimension d then X_0 is cohomologically n-insignificant for all $n < \dim X_0 - d$ and $n > \dim X_0 + d + 1$ (2.7).
2. If X_0 is a divisor with normal crossings on non-singular X then it is cohomologically insignificant (2.3).
3. If X is non-singular and $\dim X_t = 1$ then X_0 is cohomologically insignificant if and only if $X_{0,\mathrm{red}}$ (the reduced fiber) has at most nodes as its singularities and $X_0 = X_{0,\mathrm{red}}$ or X_0 is a multiple elliptic fiber of Kodaira (3.10).
4. If X is non-singular and X_0 is a surface with isolated singular points then X_0 is cohomologically insignificant if and only if its singular points are either double rational, or simple elliptic, or cusp singularities (4.13).
5. If X_0 has the same singularities as a general projection of a non-singular surface into $\mathbb{P}^3$ then X_0 is cohomologically insignificant (4.17).
6. If X_0 has only ordinary quadratic singularities and X is non-singular then X_0 is cohomologically insignificant (2.4).

The notion of cohomological insignificance is closely related to the notion of insignificant limit singularities of Mumford [17].

CONJECTURE: *Suppose that the special fibre X_0 of a family is reduced and have only insignificant limit singularities. Then the degeneration X_0 is cohomologically insignificant.*

This conjecture was checked for all known (presumably all) insignificant limit surface singularities by J. Shah [23]. He also constructed some examples that show the converse is not true.

This work was inspired by a letter of D. Mumford to me. I am grateful to him for this very much. The work of J. Shah [23] and conversations with him were very stimulating for me. I thank the referee for his critical remarks and constructive suggestions.

1. Mixed Hodge structures

(1.1) A (real) *Hodge structure* of weight n is a finite-dimensional real vector space H together with the splitting of its complexification $H_{\mathbb{C}} = H \otimes_{\mathbb{R}} \mathbb{C}$ into a direct sum of subspaces

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$$

such that $H^{p,q} = \bar{H}^{q,p}$.

A *mixed Hodge structure* is a finite-dimensional real vector space H together with a finite increasing filtration W (the weight filtration) and a finite decreasing filtration F on $H_{\mathbf{C}}$(Hodge filtration) such that each $Gr_n^W(H)$ is a Hodge structure of weight n and the filtration induced by F on $Gr_n^W(H)_{\mathbf{C}}$ is the filtration by the subspaces $\bigoplus_{p' \geq p} H^{p',q'}$. We say that H is *pure* of weight n if $Gr_i^W(H) = 0$ for $i \neq n$.

Clearly each pure mixed Hodge structure of weight n can be considered as a Hodge structure of weight n, and conversely each Hodge structure of weight n can be considered as a mixed Hodge structure which is pure of weight n.

A *morphism* $f: H \to H'$ of mixed Hodge structures is a linear map compatible with both filtrations W and F. In particular, the induced map $Gr_n^W(f): Gr_n^W(H) \to Gr_{n'}^W(H')$ maps $H^{p,q}$ into $H'^{p,q}$.

Let H be a mixed Hodge structure. For any integer m we define the *m-twisted mixed Hodge structure* $H[m]$ as follows: $H[m] = H$ as vector spaces;

$$W_i(H[m]) = W_{i+m}(H), \qquad F_i(H[m]) = F_{i+m}(H);$$

$$Gr_i^W(H[m])_{\mathbf{C}} = \bigoplus_{p+q=i} H[m]^{p,q}, \quad \text{where } H[m]^{p,q} = H^{p+m,q+m}.$$

(1.2) EXAMPLE. Let X be a compact Kähler manifold, $H^{p,q}(X)$ the space of harmonic forms of type (p,q) on X. The classical Hodge theory proves that

$$H^n(X, \mathbf{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

and $H^{p,q}(X) = \overline{H^{q,p}(X)}$. This shows that the spaces $H^n(X) = H^n(X, \mathbb{R})$ can be considered as Hodge structures of weight n.

For any complete algebraic variety X over $\mathbf{C}$ we have the similar construction:

$$H^n(X, \mathbf{C}) = \bigoplus_{p+q=n} H^{p,q}(X), \qquad H^{p,q}(X) = H^q(X, \Omega_X^p).$$

(1.3) EXAMPLE: Let X be a complete complex algebraic variety. Assume that its irreducible components X_i are nonsingular of the same dimension, all intersections $X_i \cap X_j$, $i \neq j$, are nonsingular divisors in X_i forming a divisor with normal crossings in X_i. Then $H^n(X)$ carries a canonical mixed Hodge structure which is constructed as follows (see [8, 25]):

Let

$$\tilde{X}^{(k)} = \coprod_{i_1 < \cdots < i_k} X_{i_1} \cap \cdots \cap X_{i_k}, \quad k > 0$$

and $a_k : \tilde{X}^{(k)} \to X$ the natural map, $\delta_j : \tilde{X}^{(k)} \to \tilde{X}^{(k-1)}$ the inclusion defined by components as

$$X_{i_1} \cap \cdots \cap X_{i_k} \to X_{i_1} \cap \cdots \cap X_{i_{j-1}} \cap X_{i_{j+1}} \cap \cdots \cap X_{i_k}.$$

It is easily checked that the complex

$$0 \to (a_1)_* \mathbb{R}_{\tilde{X}^{(1)}} \xrightarrow{d} (a_2)_* \mathbb{R}_{\tilde{X}^{(2)}} \xrightarrow{d} \cdots,$$

where $d = \Sigma_j (-1)^j (\delta_j)_*$, is a resolution of the constant sheaf $\mathbf{R}_X$. Thus the hypercohomology of this complex define the spectral sequence

$$E_1^{pq} = H^q(\tilde{X}^{(p+1)}, \mathbb{R}) \Rightarrow H^{p+q}(X, \mathbb{R}).$$

It is shown in [8] that this sequence degenerates at E_2. Define the weight filtration of $H^n(X)$ as

$$\begin{aligned} Gr_q^W(H^{p+q}(X)) &= E_2^{p,q} = E_\infty^{p,q} \\ &= H(H^q(\tilde{X}^{(p)}) \to H^q(\tilde{X}^{(p+1)}) \to H^q(\tilde{X}^{(p+2)})). \end{aligned}$$

It is shown in [8] that the maps $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ are morphisms of pure Hodge structures of weight q defined in (1.2). Hence they define the pure Hodge structure of weight q on $Gr_q^W(H^{p+q}(X, R)) = E_2^{p,q}$. There exists also a unique Hodge filtration F such that (H^n, W, F) is a mixed Hodge structure for all $n = 0$.

(1.4) Example: Let U be a non-singular complex algebraic variety. Then there exists a canonical mixed Hodge structure on $H^n(U) = H^n(X, \mathbb{R})$,which can be defined in the following way.

Let X be a complete non-singular algebraic variety containing U as an open piece. Assume also that the complement $Y = X - U$ is a divisor with normal crossings. This always can be done by the Hironaka theorem. Let Y_i be irreducible components of Y, $i = 1, \ldots, n$, $Y_I = \cap_{i \in I} Y_i$, $I \subset [1, n]$. Consider the Leray spectral sequence for the open immersion $j : U \to X$

$$E_2^{p,q} = H^p(X, R^q j_*(\mathbb{R})) \Rightarrow \mathrm{H}^{p+q}(\mathrm{U}, \mathbb{R}).$$

Then easy local arguments show that

$$R^p j_*(\mathbb{R}) = \bigoplus_{\#I=q} \mathbb{R}_{Y_I} = a_*(\mathbb{R}_{\tilde{Y}_{(q)}}),$$

where $\tilde{Y}^{(q)} = \coprod_{\#I=q} Y_I$, $a : \tilde{Y}^{(q)} \to Y$ the canonical projection.

Now the Leray spectral sequence for the morphism a gives

$$E_2^{p,q} = H^p(\tilde{Y}^{(q)}, \mathbb{R}), \quad \text{where } \tilde{Y}^{(0)} = X.$$

Consider $E_2^{p,q}$ as the pure Hodge structure on the cohomology of the non-singular complete variety $\tilde{Y}^{(q)}$ defined in (1.2).
Then it can be shown that the differential map d_2 defines the morphism of pure Hodge structures of weight $p+2q$

$$d_2 : E_2^{p,q}\,[-q] \to E_2^{p+2,q-1}[1-q].$$

Consequently, the terms $E_3^{p,q}$ can be endowed with the pure Hodge structure of weight $p+2q$. Now the crucial fact due to Deligne shows that the spectral sequence degenerates at E_3. Using that he constructs a Hodge filtration F on $H^n(U)$ such that $(H^n(U), F, W)$ is a mixed Hodge structure, where

$$Gr^W_{p+2q}(H^{p+q}(U)) = E_3^{p,q}.$$

In other terms we have

$$Gr^W_k(H^n(U)) = H(H^{2n-k-2}(\tilde{Y}^{(k-n+1)})[n-k-1]$$
$$\to H^{2n-k}(\tilde{Y}^{(k-n)})[n-k] \to H^{2n-k+2}(\tilde{Y}^{(k-n-1)})[n-k+1])$$

the cohomology of the complex of pure Hodge structures of weight k.

(1.5) THEOREM: (P. Deligne [5]) *Let X be an arbitrary complex algebraic variety. Then its cohomology $H^n(X) = H^n(X, \mathbb{R})$ carries a unique mixed Hodge structure such that the following properties are satisfied;*

(i) *if X is a non-singular complete variety then this structure coincides with the structure of example* (1.2);

(ii) *if X is a union of non-singular complete varieties normally intersecting each other, then it coincides with the structure of example* (1.3);

(iii) *if X is an open subset of a complete non-singular variety then this structure coincides with the structure of example* (1.4);

(iv) *for any morphism of complex algebraic varieties $f: X \to Y$ the canonical map $f^*: H^n(Y) \to H^n(X)$ is a morphism of mixed Hodge structures*;

(v) *if $Y \subset X$ is a closed subvariety* (*or open*) *then the relative cohomology $H^n(X, Y; \mathbb{R}) = H^n(X, Y)$ carries a mixed Hodge structure such that the exact sequence*

$$\cdots \to H^n(X) \to H^n(Y) \to H^{n+1}(X, Y) \to H^{n+1}(X) \to \cdots$$

is an exact sequence of the mixed Hodge structures;

(vi) *if $f: (X, Y) \to (X', Y')$ is a morphism of the pairs as above then the canonical map $f^*: H^n(X', Y') \to H^n(X, Y)$ is a morphism of mixed Hodge structures.*

(1.6) Definition: The mixed Hodge structure $H^n(X, Y)$ from theorem 1.5 is called the *Deligne mixed Hodge structure.* We denote by $H_n^{p,q}(X, Y)$ the $H^{p,q}$ spaces of $Gr_{p+q}^W(H^n(X, Y))$, and let $H_n^{p,q}(X) = H^{p,q}(X, \phi)$, $H_n^{p,q}(X) = \dim_{\mathbb{C}}(h_n^{p,q}(X))$ (the *Hodge numbers*)

(1.7) Remark: It is proven in [5] that for any complete, algebraic variety X

$$H_n^{p,q}(X) = 0 \quad \text{for } p + q > n.$$

Also, if X is non-singular (not necessary complete then

$$H_n^{p,q}(X) = 0 \quad \text{for } p + q < n.$$

(1.8) Let X be a complex algebraic variety, $p: \bar{X} \to X$ a resolution of singularities of X, $U \subset X$ the maximal open subset of X over which p is an isomorphism, $Y = X - U$, $\bar{Y} = p^{-1}(Y)$, $\bar{U} = p^{-1}(U)$. Using the previous theorem one can compute the Deligne mixed Hodge structure on $H^n(X)$ as follows. First, we have a natural commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \to & H^n(\bar{X}) & \xrightarrow{\bar{\alpha}} & H^n(\bar{Y}) & \xrightarrow{\bar{\beta}} & H^{n+1}(\bar{X}, \bar{Y}) & \xrightarrow{\bar{\delta}} & H^{n+1}(\bar{X}) & \to \cdots \\
& \uparrow{\scriptstyle\lambda} & & \uparrow{\scriptstyle\mu} & & \uparrow{\scriptstyle\nu} & & \uparrow{\scriptstyle\lambda} & \\
\cdots \to & H^n(X) & \xrightarrow{\alpha} & H^n(Y) & \xrightarrow{\beta} & H^{n+1}(X, Y) & \xrightarrow{\delta} & H^{n+1}(X) & \to \cdots
\end{array}$$

whose horizontal rows are the exact sequences of relative cohomology and the vertical arrows are induced by the map p. Secondly, we have a canonical isomorphism of mixed Hodge structures

$$\nu : H^n(X, Y) \to H^n(\bar{X}, \bar{Y}), \quad n \geq 0$$

(because $\bar{Y}$ is a strong deformation retract of one of its closed neighborhoods, see [24], Ch. 4, §8, th. 9).

Together this gives the following exact sequence of mixed Hodge structures:

$$\cdots \to H^n(X) \xrightarrow{(\alpha,\lambda)} H^n(Y) \oplus H^n(\bar{X}) \xrightarrow{\bar{\alpha}-\mu} H^n(\bar{Y}) \xrightarrow{\delta\nu^{-1}\bar{\beta}} H^{n-1}(X) \to \cdots$$

Assuming that $\bar{X}$ is non-singular, $\bar{Y}$ a divisor with normal crossings, this sequence can be used for computation of the mixed Hodge structure $H^n(X)$ by induction on dimension.

(1.9) PROPOSITION: *Let X be an algebraic variety embedded into a non-singular variety Z, Y a non-singular subvariety of X, $\pi : Z' \to Z$ the monoidal transformation centered at Y, $X' = \pi^{-1}(X)$ the total inverse transform of X. Then the morphism of the Deligne mixed Hodge structures $H^n(X) \to H^n(X')$ induced by the projection $\pi|_{X'} : X' \to X$ defines an isomorphism*

$$H_n^{p,q}(X) \simeq H_n^{p,q}(X')$$

for each pair (p, q) with $pq = 0$.

PROOF: Consider the commutative diagram of mixed Hodge structures of (1.8):

$$\begin{array}{ccccccccc}
H^n(Y') & \to & H^n(X', Y') & \to & H^n(X') & \to & H^{n+1}(Y') & \to & H^{n+1}(X', Y') \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^n(Y) & \to & H^n(X, Y) & \to & H^n(X) & \to & H^{n+1}(Y) & \to & H^{n+1}(X, Y)
\end{array}$$

where $Y' = \pi^{-1}(Y) \simeq \mathbb{P}_Y(E)$, E the normal vector bundle to Y in Z.

Now, the cohomology $H^n(Y')$ with its pure Hodge structure are easily computed in terms of the cohomology of Y. We have (see [7], p. 606)

$$H^n(Y') = \bigoplus_{i=0} H^{n-2i}(Y)\,[-i]$$

and the canonical map $H^n(Y) \to H^n(Y')$ is the isomorphism of $H^n(Y)$ onto the first summand of this sum. This obviously shows that in the diagram above the first and the fourth vertical arrows induce an isomorphism of the components $H^{p,q}$ with $pq = 0$. Since the second and the fifth arrows are isomorphisms we get the assertion from the "five homorphisms lemma".

(1.10) Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, $D^* = D - \{0\}$, $f : X \to D$ a proper surjective holomorphic map of a connected complex manifold X. We assume that all fibres $X_t = f^{-1}(t)$ are connected projective algebraic varieties, non-singular for $t \neq 0$.

Let $X^* = X - X_0 = f^{-1}(D^*)$ and $X_\infty = X^* \times_{D^*} \tilde{D}^*$, where $p : \tilde{D}^* \to D^*$ is the universal covering of D^*. Identify $\tilde{D}^*$ with the upper half plane $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and the map p with the map $H \to D^*$ given by $z \to e^{-iz}$. Then the fundamental group $\pi_1(D^*)$ is identified with the group of the transformations of H given by $z \mapsto z + 2\pi i m$, $m \in \mathbb{Z}$. The group $\pi_1(D^*)$ acts on the space X_∞ through its natural action on $\tilde{D}^*$. Let

$$T : H^n(X_\infty) \to H^n(X_\infty)$$

be the induced action of the generator $z \mapsto z + 2\pi i$ of $\pi_1(D^*)$ on the cohomology space $H^n(X_\infty) = H^n(X_\infty, \mathbb{R})$ (the monodromy transformation).

By the theorem of quasiunipotence of the monodromy ([6]) there exists a number e such that $T' = T^e$ is unipotent. Denote $N = \log(T') = \Sigma(-1)^{i+1}(T' - I)^i/i$ and let T_s be the semisimple part of T.

(1.11) THEOREM: *There exists a mixed Hodge structure on the space $H^n(X_\infty)$ such that the following properties are satisfied:*

(i) *N induces a morphism of the mixed Hodge structures*

$$H^n(X_\infty) \to H^n(X_\infty)[-1],$$

(ii) *for every $r \geq 0$ the map*

$$N^r : Gr^W_{n+r}(H^n(X_\infty)) \to Gr^W_{n-r}(H^n(X_\infty))\ [-r]$$

is an isomorphism of mixed Hodge structures;

(iii) *T_s is an isomorphism of mixed Hodge structures;*

(iv) *let $X \to X_0$ be the map which is composed of the canonical projection $X \to X^*$, the inclusion map $X^* \to X$ and the retraction* *map $X \to X_0$; then the induced map of the cohomology spaces*

$$sp_n : H^n(X_0) \to (H^n(X_\infty)$$

is a morphism of mixed Hodge structures ($H^n(X_0)$ is the Deligne mixed Hodge structure);

(v) *let $H^{p,q}(X_t)$ be the Hodge numbers of the pure Hodge structure on the cohomology of any non-singular fibre X_t, $H^{p,q}_n(X_\infty)$ be the Hodge numbers of the mixed Hodge structure $H^n(X_\infty)$, then*

$$h^{p,n-p}(X_t) = \sum_{q \geq 0} h^{p,q}_n(X_\infty).$$

(1.12) REMARKS: 1. The theorem above was conjectured by P. Deligne (see [6], conj. (9.17)) and had been proven by W. Schmid [22] and J. Steenbrink [25].

2. Since X_∞ is a smooth fibre space over a contractable base H its cohomology $H^n(X_\infty)$ are isomorphic to the cohomology of any fibre, that is to $H^n(X_t)$. However, in general, this isomorphism is not compatible with the corresponding mixed Hodge structures (defined in the theorem and the pure Deligne mixed Hodge structure).

3. The construction of the mixed Hodge structure above depends on a choice of a parameter on D. However, properties (i) – (v) determine uniquely the weight filtration and the Hodge filtration induced on each graded part (see [22], p. 255 and [25], p. 248).

(1.13) DEFINITION: The mixed Hodge structure $H^n(X_\infty)$ is called the *limit mixed Hodge structure.*

(1.14) Suppose now that the fibre X_0 from (1.10) is a divisor with normal crossings, let X_{0i} be its irreducible components. Consider any non-singular complete algebraic variety $\bar{X}$ which contains X as an open subset. Then as was explained in (1.4) there exists a canonical morphism of pure Hodge structures

$$H^{n-2}(\tilde{X}_0^{(1)})[-1] = \bigoplus_i H^{n-2}(X_{0i})[-1] \to H^n(\bar{X})$$

(in notation of (1.4) $X_0 = Y$, $\bar{X} = X = \tilde{Y}^{(0)}$). Composing this morphism with the morphism $H^n(\bar{X}) \to H^n(X_0)$ induced by the inclusion $X_0 \hookrightarrow \bar{X}$, we get the morphism of the mixed Hodge structures

$$g_n : \bigoplus_i H^{n-2}(X_{0i})[-1] \to H^n(X_0).$$

It can be shown that this morphism is independent of a choice of $\bar{X}$ ([5], 8.2.6).

The arguments similar to ones used in (I.4) show that there exists a spectral sequence

$$E_2^{p,q} = H^p(\tilde{X}^{(q)}) \Rightarrow H^{p+q}(X^*)$$

whose differential d_2 is a morphism of mixed Hodge structures $H^p(\tilde{X}_0^{(q)})[-q] \to H^{p+2}(\tilde{X}_0^{(q-1)})[-q+1]$ (we have only to check that $d_2^{p,1}$ is a morphism of mixed Hodge structures but clearly $d_2^{p,1} = g_{p+2}$).

Again it can be proved that d_i degenerate for $i \geq 3$ and $H^n(X^*)$ can be provided with a mixed Hodge structure in such a way that

$$\begin{aligned} Gr_k^W(H^n(X^*)) &= E_3^{2n-k,k-n} \\ &= H(H^{2n-k-2}(\tilde{X}_0^{(k-n+1)})[n-k-1] \to H^{2n-k}(\tilde{X}_0^{(k-n)})[n-k] \\ &\to H^{2n+2-k}(\tilde{X}_0^{(k-n-1)})[n-k+1]) \end{aligned}$$

for $k > n$ and

$$W_n(H^n(X^*)) = H^n(X_0)/\mathrm{Im}\,(g_n).$$

(1.14) THEOREM: *In the notation of* (1.14) *the sequence*

$$\bigoplus_i H^{n-2}(X_{0i})[-1] \xrightarrow{g_n} H^n(X_0) \xrightarrow{sp_n} H^n(X_\infty) \xrightarrow{N} H^n(X_\infty)[-1]$$

is an exact sequence of mixed Hodge structures.

PROOF: The projection $X_\infty \to X^*$ is a non-ramified infinite cyclic covering of X^* with the automorphism group isomorphic to $\pi_1(D^*)$. The group $\pi_1(D^*)$ acts by functoriality on the spaces $H^n(X_\infty)$, and, this action is determined by the monodromy transformation T of (1.11). Consider the standard spectral sequence associated with a covering ([14]), p. 343)

$$E_2^{p,q} H^p(\pi_1(D^*),\ H^q(X_\infty)) \to H^{p+q}(X^*).$$

Since $H^i(\mathbb{Z}, M) = 0, i > 1$ for any $\mathbb{Z}$-module M, the spectral sequence defines for each n the exact sequence

$$0 \to H^1(\pi_1(D^*),\ H^{n-1}(X_\infty)) \to H^n(X^*) \to H^0(\pi_1(D^*),\ H^n(X_\infty)) \to 0.$$

Obviously we have

$$H^0(\pi_1(D^*), H^n(X_\infty)) = \mathrm{Ker}(H^n(X_\infty) \xrightarrow{T-1} H^n(X_\infty))$$
$$= \mathrm{Ker}(H^n(X_\infty) \xrightarrow{N} H^n(X_\infty))$$

and thus the canonical map

$$H^n(X^*) \to \mathrm{Ker}(N) \overset{\mathrm{def}}{=} H^n(X_\infty)^N$$

is surjective.

Now, properties (i) and (ii) of (1.11) show that $H^n(X_\infty)^N$ is a mixed Hodge substructure of the limit Hodge structure $H^n(X_\infty)$ and

$$W_n(H^n(X_\infty)^N) = H^n(X_\infty)^N.$$

Next we use the most important fact that the map $H^n(X^*) \to H^n(X_\infty)$ is a morphism of mixed Hodge structures (where the first is defined in (1.13) and the second is the limit mixed Hodge structure) (see [25]), and hence defines the surjective map

$$W_n(H^n(X^*)) \to W_n(H^n(X_\infty)) = H^n(X_\infty)^N.$$

It remains to use that by (1.13)

$$W_n(H^n(X^*)) = H^n(X_0)/\mathrm{Im}\,(g_n).$$

(1.15) REMARK: The above exact sequence is called the *Clemens-Schmid exact sequence* and was announced first in [8]. A geometric proof of (1.14) was given by A. Todorov in his long series of talks at a seminar of Arnol'd in Moscow in 1976. A little later V. Danilov and myself (trying to understand Todorov's proof) found that a simple proof is implicitly contained in [25). That proof is given here.

2. Cohomologically insignificant degenerations

(2.1) Let $\gamma: H \to H'$ be a morphism of mixed Hodge structures. We say that γ is a *(p, q)-isomorphism* if γ induces an isomorphism $H^{p,q} \to H'^{p,q}$.

Note the following trivial properties;

(i) γ is an isomorphism if and only if γ is an (p, q)-isomorphism for each pair (p, q);

(ii) γ is an (p, q)-isomorphism if and only if γ is an (q, p)-isomorphism;
(iii) a composition of (p, q)-isomorphisms is an (p, q)-isomorphism.

(2.2) DEFINITION: Let $f{:}X \to D$ be a family of projective varieties as in (1.10). We say that the special fibre X_0 is *cohomologically n-insignificant* if the specialization map $sp_n{:}H^n(X_0) \to H^n(X_\infty)$ (defined in (1.11)) is a $(p, 0)$-isomorphism for each $p \geq 0$. We say that X_0 is *cohomologically insignificant* if it is cohomologically n-insignificant for all $n \geq 0$.

(2.3) EXAMPLE (a divisor with normal crossings): Suppose that X_0 $X_{01} + \cdots + X_{0r}$ is a divisor with normal crossings. Then the Clemens-Schmid exact sequence (1.14) shows that sp_n induces isomorphisms

$$Gr_i^W(H^n(X_0)) \to Gr_i^W(H^n(X_\infty)^N), \quad \text{for } i < n$$

and the exact sequence

$$\bigoplus_{k=1}^{r} H^{n-2}(X_{0k})[-1] \to Gr_n^W(H^n(X_0)) \to Gr_n^W(H^n(X_\infty)^N) \to 0.$$

Now, since N maps each $H^{p,q}$ into $H^{p-1,q-1}$ we get that each $H^{p,0}$ of $H^n(X_\infty)$ is contained in $H^n(X_\infty)^N$. This and the above show that X_0 is cohomologically insignificant.

(2.4) EXAMPLE (ordinary double point): Assume that the special fibre X_0 has an isolated ordinary double point x_0 and is non-singular outside x_0. Let $\sigma{:}\bar{X} \to X$ be the monoidal transformation of X centered at its non-singular point x_0, $\bar{X}_0$ be the proper inverse transform of X_0. The canonical projection $\sigma_0 = \sigma|X_0{:}\bar{X}_0 \to X_0$ is a desingularization which blows up a non-singular quadric E naturally embedded into $\sigma^{-1}(x_0) = \mathbb{P}^{d+1}$ $(d+1 = \dim X)$. Let $\bar{f} = f_0\sigma{:}\bar{X} \to D$, its special fibre $\bar{X}_0$ is equal to the union of the two divisors $D_1 = \bar{X}_0$ and $D_2 = 2D_2' = \mathbb{P}^d$. Let $\bar{D}$ be another copy of the unit disk and $\bar{D} \to D$ be the projection given by the formula $z \to z^2$. Denote by $\tilde{X}$ the normalization of $\bar{X} \underset{D}{\times} \bar{D}$ and let $\tilde{f}{:}\tilde{X} \to \bar{D}$ be the second projection. It is easily checked that the special fibre $\tilde{X}_0 = \tilde{f}^{-1}(0)$ is the union of two non-singular varieties Q_1 and Q_2 intersecting transversally. Moreover, if $\rho{:}\tilde{X} \to \bar{X}$ denotes the first projection, then its restrictions $\rho_1 = \rho|_{Q_1}{:}Q_1 \to D_1$ and $\rho_2 = \rho|_{Q_2}{:}Q_2 \to D_2'$ are an isomorphism and a double covering branched along $E = D_1 \cap D_2'$ respectively. Also it is seen that $\rho_{12} = \rho|_{Q_1 \cap Q_2}$ defines an isomorphism onto $E = D_1 \cap D_2'$.

Now we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
H^{n-1}(Q_1)\oplus H^{n-1}(Q_2) & \to & H^{n-1}(Q_1\cap Q_2) & \to & H^n(\tilde{X}_0) & \to & H^n(Q_1)\oplus H^n(Q_2) & \to & H^n(Q_1\cap Q_2)\\
\uparrow{\scriptstyle(\rho_1^*,0)} & & \uparrow{\scriptstyle\rho_{12}^*} & & \uparrow{\scriptstyle(\rho|\tilde{X}_0)^*\sigma_0^*} & & \uparrow{\scriptstyle(\rho_1^*,0)} & & \uparrow{\scriptstyle\rho_{12}^*}\\
H^{n-1}(\bar{X}_0) & \to & H^{n-1}(E) & \to & H^n(X_0) & \to & H^n(\bar{X}_0) & \to & H^n(E)
\end{array}$$

Here the first row is constructed from the resolution of $\mathbb{R}_{\tilde{X}_0}$ described in example (1.3), and the second row is the exact sequence from (1.8), where $Y=\{x_0\}$ is a point.

Now Q_2 being a double covering of $\mathbb{P}^d$ branched along a nonsingular quadric is a non-singular quadric [10] itself. Its pure Deligne Hodge structure is easily computed (see [11], exp. XII) and it turns out that all its $H_n^{p,0}$ components are zero for $n>0$. Since ρ_1^* and ρ_{12}^* are isomorphisms, passing to $H_n^{p,0}$ components and using "the five lemma" we get that $H^n(X_0)\to H^n(\tilde{X}_0)$ is a $(p,0)$-isomorphism for each p. It remains to notice that the spaces X_∞ and $\tilde{X}_\infty$ are canonically isomorphic and the composition $H^n(X_0)\to H^n(\tilde{X}_0)\xrightarrow{sp_n}H^n(\tilde{X}_\infty)$ coincides with the specialization map $H^n(X_0)\to H^n(X_\infty)$. Applying (2.3) we get that X_0 is cohomologically insignificant.

(2.5) Remark: In fact, one can prove along the same lines the following more general result. Let us assume that X_0 has a unique ordinary m-tuple point. Then X_0 is cohomologically k-insignificant for all $k\neq\dim X_0$ and cohomologically insignificant if and only if $m<\dim X_0+2$.

(2.6) Let $f:X\to D$ be a family as in (1.10), $Y\subset X_0$ a non-singular closed subvariety of its special fibre, $p:X'\to X$ the monoidal transformation of X centered at Y, $f':X'\to D$ the composition $X'\xrightarrow{p}X\xrightarrow{f}D$, $X_0'=f'^{-1}(0)$ the special fibre.

Proposition: *X_0 is a cohomologically n-insignificant degeneration of f if and only if X_0' is a cohomologically n-insignificant degeneration of f'.*

Proof: Since X and X' are isomorphic outside its special fibres the spaces X_∞ and X_∞' are canonically isomorphic. Also it is seen that the composition $H^n(X_0)\xrightarrow{p^*}H^n(X_0')\xrightarrow{sp_n}H^n(X_\infty')$ coincides with the specialization map $H^n(X_0)\to H^n(X_\infty)$. These remarks show that it suffices to prove that the map p^* is an $(p,0)$-isomorphism for every p. But this follows from proposition (1.9).

(2.7) PROPOSITION: *Let S be the singular locus of the reduced special fibre X_0 of a family $f:X \to D$, $d = \dim S$. Then the specialization morphism*

$$sp_i : H^i(X_0) \to H^i(X_\infty)$$

is injective for $i = \dim X_0 - d$, surjective for $i = \dim X_0 + d + 1$, and bijective for $i > \dim X_0 + d + 1$, $i < \dim X_0 - d$.

PROOF: Let $f(z_1, \ldots, z_{n+1})$ be a complex polynomial in $n+1$ variables, for any point $a = (a_1, \ldots. a_{n+1})$ with $f(a) = 0$ let

$$V_{\epsilon,\delta}(a) = \{z \in \mathbb{C}^{n+1} : f(z) = \epsilon, \|z - a\| < \delta\}$$

and

$$\Phi^i(a) = H^i(V_{\epsilon,\delta}(a)), \quad i \geq 0.$$

It follows from [16] that for sufficiently small ϵ and $\delta \ll \epsilon$ $V_{\epsilon,\delta}(a)$ is an open manifold whose diffeomorphy type is independent of ϵ and δ.

Let S be the set of critical points of f with the critical value 0 (that is, the set of singular points of the variety $f = 0$).

The following properties of the spaces Φ^i (called the *spaces of vanishing cohomology*) are known:

(a) $\Phi^i(a) = 0$ for $0 < i < n - \dim_a S$ ([12]),
(b) $\Phi^i(a) = 0$ for $i > n$ ([16]),
(c) the closure $\bar{S}_i$ (in the Zariski topology) of the subset

$$S_i = \{a \in S : \Phi^i(a) \neq 0\}$$

is a subvariety of dimension $\leq n - i$ (Apply the Thom isotopy theorem stated as in [10], p. 126; see also [15]).

Let $f:X \to D$ be a mapping of a complex manifold onto a disk. We may define the spaces $\Phi^i(x)$ for any $x \in X_0$ identifying some open neighborhood of x in X with an open neighborhood of 0 in $\mathbb{C}^{n+1}$ and the map f with the map $f:\mathbb{C}^{n+1} \to \mathbb{C}$ given by some polynomial.

Returing to our proof, let $\pi : X_\infty \to X$ be the canonical projection. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathbb{R}) \Rightarrow H^{p+q}(X_\infty).$$

According to Deligne ([10], exp. XIV) we have

$$(R^q\,\pi_*\mathbb{R})_x = \begin{cases} 0, & x\epsilon X - X_0 \\ \Phi^q(x), & x\epsilon X_0 \end{cases}, \quad q>0$$

$$R^0\pi_*\mathbb{R}_{X_\infty} = (\mathbb{R})_X.$$

Now it is easy to check that the specialization map sp_i coincides with the edge homorphism $d_{i,0}: E_2^{i,0} \to H^i$ composed with the retraction isomorphism $H^i(X_0) \to H^i(X)$.

Applying results (a)–(c) above we get ($n = \dim X_0$):

$$E_2^{p,q} = 0 \quad \text{for } p > 2n - 2q \quad \text{if } q \geq n - d,$$

$$E_2^{p,q} = 0 \quad \text{for } 0 < q < n - d, \quad p > 0.$$

This implies that

$$E_2^{p,q} = 0 \quad \text{for } p + q < n - d, \quad q > 0,$$

hence $d_{i,0}$ is bijective for $i < n - d$ and injective for $i = n - d$; and

$$E_2^{p,q} = 0 \quad \text{for } p + q > n + d,$$

hence $d_{i,0}$ is bijective for $i > n + d + 1$, and surjective for $i = n + d + 1$.

(2.8) Remark. An algebraic analogue of this result (not including the case $i < n - d$) was considered by Grothendieck (see [9], exp. I).

3. Families of curves

(3.1) Let X be a reduced connected complete algebraic curve, $X_1, \ldots, X_h$ its irreducible components, $\bar{X}_i$ the normalization of X_i, g the genus of $\bar{X}_i$ $(i = 1, \ldots, h)$.

Consider the normalization map

$$p : \bar{X} = \coprod_{i=1}^{h} \bar{X}_i \to X$$

and let $\bar{S} = p^{-1}(S)$, where S is the set of singular points of X. Put $s = \#S$, $\bar{s} = \#\bar{S}$.

To compute the Deligne mixed Hodge structure of X we use the exact sequence from (1.8)

$$\cdots \to H^n(X) \to H^n(S) \oplus H^n(\bar{X}) \to H^n(\bar{S}) \to H^{n+1}(X) \to \cdots$$

We easily get

$$Gr_0^W(H^1(X)) = R^{\bar{s}-s-h+1}$$

$$Gr_1^W(H^1(X)) = H^1(\bar{X}) = \bigoplus_{i=1}^{h} H^1(\bar{X}_i), \quad H^1(\bar{X}) = \mathbb{R}^{2g_i}$$

$$Gr_i^W(H^1(X)) = 0, \quad i \neq 0,1$$

$$Gr_2^W(H^2(X)) = H^2(X) = H^2(\bar{X}) = \bigoplus_{i=1}^{h} H^2(\bar{X}_i), \ H^2(\bar{X}) = \mathbb{R}.$$

(3.2) Until the end of this section we will consider the situation of (1.10), where the total space X of a family $f: X \to D$ is a complex surface. Each fibre of f is a complete algebraic curve, the special fibre X_0 considered as a positive divisor on X can be represented in the form $X_0 = m_1C_1 + \cdots + m_hC_h$, where C_i are the reduced irreducible components of X_0.

Let $DF(X)$ be the subgroup of the divisor group of X generated by the components C_i, $DF_+(X)$ be the sub-semigroup of positive divisors. As usually we write $Z \leq Z'$ for $Z, Z' \in DF(X)$ if $Z' - Z \in DF_+(X)$. Also recall that there exists a symmetric bilinear form $DF(X) \times DF(X) \to \mathbb{Z}((Z, Z') \to (Z \cdot Z'))$ with the following properties (see, for example, [10], exp. X):

Let $d = g.c.d.(m_1, \ldots, m_h)$, $\bar{m}_i = m_i/d$, $X^{(k)} = k(\bar{m}_1C_1 + \cdots + \bar{m}_hC_h)$, $k \in \mathbb{Z}$. Then

(3.2.1) for every $Z \in DF(X)$ $(Z \cdot Z) \leq 0$ and the equality takes place if and only if $Z = X^{(k)}$ for some integer k.

(3.2.2) $(Z \cdot X^{(k)}) = 0$ for every $Z \in DF(X)$ and any integer k.

(3.3) Lemma: *There exists a number k such that for any $Z \in DF_+(X)$ with $X_{0,\text{red.}} = C_1 + \cdots + C_h \leq Z \leq X^{(k)}$ we have*

$$H^0(Z, \mathcal{O}_Z) = \mathbb{C}.$$

Proof: Let

$S = \{Z \in DF(X): X_{0,\text{red.}} \leq Z \leq X_0,\ H^0(Z', \mathcal{O}_{Z'}) = \mathbb{C}$
for all Z' with $X_{0,\text{red.}} \leq Z' \leq Z\}$.
This set is non-empty because it contains $X_{0,red}$. Choose some maxi-

mal element Z from S (in sense of order $\leq$). Assume that $Z \neq X^{(k)}$ for $k = 1, \ldots, d$. Then $X_0 - Z \neq X^{(k)}$ for any $k = 1, \ldots, d$, and hence by (3.2.1) and (3.2.2) we have

$$(Z \cdot X_0 - Z) = -(X_0 - Z \cdot X_0 - Z) > 0.$$

This obviously implies that there exists a component $C_i \leq X_0 - Z$ such that $(Z \cdot C_i) > 0$. Now consider the standard exact sequence of sheaves on X:

$$0 \to \mathcal{O}_{C_i}(-Z) \to \mathcal{O}_{Z+C_i} \to \mathcal{O}_Z \to 0.$$

Since $H^0(C_i, \mathcal{O}_{C_i}(-Z)) = 0$ we get from this sequence that $H^0(Z + C_i, \mathcal{O}_{Z+C_i}) \subset H^0(Z, \mathcal{O}_Z) = \mathbb{C}$. Thus, $H^0(Z + C_i, \mathcal{O}_{Z+C_i}) = \mathbb{C}$ that contradicts to the maximality of Z.

Hence, we get that $Z = X^{(k)}$ for some k and by the definition of the set S we are done.

(3.4) Proposition (*M. Raynaud*):

$$H^0(X_0, \mathcal{O}_{X_0}) = \mathbb{C}.$$

Proof: Let $\mathcal{N} = \mathcal{O}_{X^{(1)}}(X^{(1)})$ be the normal sheaf to $X^{(1)}$ in X. Since $\mathcal{O}_X(dX^{(1)}) = \mathcal{O}_X(X_0) \simeq \mathcal{O}_X$ (X_0 has a global equation $\pi^*(z) = 0$, z a local parameter in D)

$$\mathcal{N}^{\otimes d} = \mathcal{O}_{X^{(1)}}(dX^{(1)}) = \mathcal{O}_X(dX^{(1)}) \otimes \mathcal{O}_{X^{(1)}} \simeq \mathcal{O}_{X^{(1)}}.$$

Let us show that d is the minimal positive integer with this property. Assuming the opposite we can find an integer k such that $d = sk$, $s > 1$ and $\mathcal{N}^{\otimes k} = \mathcal{O}_{X^{(1)}}$. Since $\mathcal{O}_X(kX^{(1)})^{\otimes s} = \mathcal{O}_X(X^{(k)})^{\otimes s} = \mathcal{O}_X(X^{(k)s}) = \mathcal{O}_X(X_0) = \mathcal{O}_X$, the sheaf $\mathcal{O}_X(X^{(k)})$ defines an element of the group ${}_s\mathrm{Pic}(X)$, the subgroup of the Picard group of elements killed by multiplication by s. Also the sheaf $\mathcal{O}_X(X^{(k)}) \neq \mathcal{O}_X$, otherwise the divisor $X^{(k)}$ has a global equation $\{\phi = 0\}$, where $\phi \in H^0(X, \mathcal{O}_X) = H^0(D, \mathcal{O}_D)$ and $\phi^s = z$. The latter is obviously impossible.

Consider the restriction map

$$r: \mathrm{Pic}(X) \to \mathrm{Pic}(X^{(1)}), \quad \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{(1)}}.$$

The induced map

$$r_s: {}_s\mathrm{Pic}(X) \to {}_s\mathrm{Pic}(X^{(1)})$$

is bijective (This is a standard fact which can be proven as follows. The Kummer exact sequence $0 \to \mathbb{Z}/s \to \mathcal{O}^* \to \mathcal{O}^* \to 0$ shows that ${}_s\mathrm{Pic}(X) = H^1(X, Z/s)$, ${}_s\mathrm{Pic}(X^{(1)}) = H^1(X^{(1)}, \mathbb{Z}/s)$ and r_s coincides with the natural homomorphism $H^1(X, \mathbb{Z}/s) \to H^1(X^{(1)}, \mathbb{Z}/s)$ which is bijective because X can be retracted onto X_0).

This implies that

$$r_s(\mathcal{O}_X(X^{(k)}) = \mathcal{O}_{X^{(1)}}(X^{(k)}) = N^{\otimes k} \neq \mathcal{O}_{X^{(1)}}$$

and we get a contradiction.

Let us proceed with the proof. Consider the standard exact sequences

$$0 \to \mathcal{O}_{X^{(1)}}(-iX^{(1)}) \to \mathcal{O}_{X^{(i+1)}} \to \mathcal{O}_{X^{(i)}} \to 0, \quad 1 \leq i \leq d$$

and the corresponding cohomology sequences, we get the exact sequences:

$$0 \to H^0(X^{(1)}, \mathcal{O}_{X^{(1)}}(-iX^{(1)})) \to H^0(X^{(i+1)}, \mathcal{O}_{X^{(i+1)}}) \to H^0(X^{(i)}, \mathcal{O}_{X^{(i)}}).$$

Since $\mathcal{O}_{X^{(1)}}(-iX^{(1)})$ is a non-trivial torsion element of $\mathrm{Pic}(X^{(1)})$ for $i = 1, \ldots, d-1$ and $H^0(X^{(1)}, \mathcal{O}_{X^{(1)}}) = \mathbb{C}$ (lemma (3.3)) we obtain $H^0(X^{(1)}, \mathcal{O}_{X^{(1)}}(-iX^{(1)})) = 0$, $i = 1, \ldots, d-1$. Starting from $i = k$ chosen in lemma 3.3 we obtain from the above sequence that

$$\begin{aligned} H^0(X_0, \mathcal{O}_{X_0}) = H^0(X^{(d)}, \mathcal{O}_{X^{(d)}}) &= H^0(X^{(d-1)}, \mathcal{O}_{X^{(d-1)}}) \\ &= \cdots = H^0(X^{(k)}, \mathcal{O}_{X^{(k)}})\mathbb{C}. \end{aligned}$$

(3.5) Remark: The same argument works also in an algebraic situation (D is replaced by the spectrum of a complete discrete valuation ring) over a field of characteristic $p = 0$ (or, more generally, prime to d). The assumption on characteristic is needed for proving that the map r_s is bijective.

The original proof of M. Raynaud differs from ours. It is based on the machinery of representability of the Picard functor developed by him in [18].

(3.6) Let $f: X \to D$ be a family of curves satisfying the assumptions of (1.10). We say that f is a minimal family if it cannot be factorized into $X \xrightarrow{g} X' \xrightarrow{f'} D$, where f' also satisfies the assumptions of (1.10) and g is a bimeromorphic map which is not an isomorphism. It is equivalent to requiring that there are no exceptional curves of the first kind among the components of the special fibre X_0.

(3.7) LEMMA: *Let* $C = X_{0,\mathrm{red}}$ *be the reduced special fibre. Then*
(i) $\dim \mathbb{C}H^1(X_0, \mathcal{O}_{X_0}) \geq \dim \mathbb{C}H^1(C, \mathcal{O}_C)$;
(ii) *if* $f: X \to D$ *is minimal then the equality in* (*i*) *takes place if and only if* $X_0 = C$ *or* $X_0 = dC$ *for some* $d \geq 1$ *and* $H^1(C, \mathcal{O}_C) = \mathbb{C}$.

PROOF: For each closed subscheme Z of X_0 we have a surjection $\mathcal{O}_{X_0} \to \mathcal{O}_Z$ which gives the surjection $H^1(X_0, \mathcal{O}_{X_0}) \to H^1(Z, \mathcal{O}_Z)$ because $\dim X_0 = 1$. This proves (i).

Assume that $X_0 \neq dC$ for any d. Take $X^{(k)}$ from lemma (3.3), then for any component C_i of X_0 with $m_i > 1$, we have

$$H^0(Z, \mathcal{O}_Z) = \mathbb{C}, \quad \text{where } Z = X^{(k)} - C_i.$$

Now, by (3.2.1) and (3.2.2) we get

$$(C_i \cdot Z) = (C_i \cdot X^{(k)} - C_i) = -(C_i \cdot C_i) > 0.$$

This shows that $H^0(C_i, \mathcal{O}_{C_i}(-Z)) = 0$ and the exact sequence

$$0 \to \mathcal{O}_{Ci}(-Z) \to \mathcal{O}_{X^{(k)}} \to \mathcal{O}_Z \to 0$$

gives the exact sequence of vector spaces

$$0 \to H^1(C_i, \mathcal{O}_{C_i}(-Z)) \to H^1(X^{(k)}, \mathcal{O}_{X^{(k)}})) \to H^1(Z, \mathcal{O}_Z) \to 0.$$

Suppose that $H^1(X_0, \mathcal{O}_{X_0}) = H^1(C, \mathcal{O}_C)$. Then also we have $H^1(X^{(k)}, \mathcal{O}_{X^{(k)}}) = H^1(Z, \mathcal{O}_Z)$ (since $C \leq Z \leq X^{(k)} \leq X_0$) and hence the above sequence shows that $H^1(C_i, \mathcal{O}_{C_i}(-Z)) = 0$. By Riemann-Roch

$$\dim_{\mathbb{C}} H^1(C_i, \mathcal{O}_{C_i}(-Z)) = -(C_i \cdot Z) + \dim_{\mathbb{C}} H^1(C_i, \mathcal{O}_{C_i}) - 1$$

and since $(C_i \cdot Z) = -(C_i \cdot C_i) > 0$ this dimension is zero if and only if $(C_i \cdot C_i) = -1$, $H^1(C_i, \mathcal{O}_{C_i})) = 0$. But this is exactly the characterization of exceptional curves of the first kind. The assumption of minimality of f shows that this is impossible, and, hence the equality in (i) may happen only in the case $X_0 = dC$ for some d.

Suppose that $X_0 = dC$. Certainly, if $d = 1$ we have the equality in (i). Assume that $d \neq 1$. The exact sequence

$$0 \to \mathcal{O}_C(-iC) \to \mathcal{O}_{(i+1)C} \to \mathcal{O}_{iC} \to 0$$

gives the exact sequence

$$0 \to H^1(C, \mathcal{O}_C(-iC) \to H^1((i+1)C, \mathcal{O}_{(i+1)C}) \to H^1(iC, \mathcal{O}_{iC}) \to 0$$

(because as we saw in the proof of (3.4) the sheaf $\mathcal{O}_C(-iC)$ is a non-trivial torsion element in $\mathrm{Pic}(C)$ for $i < d$, hence $H^0(C, \mathcal{O}_C(-iC)) = 0$ and starting with $i = 1$ we get $H^0(iC, \mathcal{O}_{iC}) = \mathbb{C}$ for all $i > 1$).

This implies that $H^1(C, \mathcal{O}_C) = H^1(X_0, \mathcal{O}_{X_0})$ if and only if $H^1(C, \mathcal{O}_C(-iC)) = 0$ for $i = 1, \ldots, d-1$. But by Riemann-Roch

$$\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C(-iC)) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) - 1, \quad i = 1, \ldots, d-1.$$

(Again we used that $H^0(C, \mathcal{O}_C(-iC)) = 0$ and $\deg \mathcal{O}_C(-iC) = 0$). This proves (ii).

(3.8) Remark: The arguments of the proofs of lemmas (3.3) and (3.7) are borrowed from the proof of lemma 2.6 in [2].

(3.9) Lemma: *In the notations of* (3.1) *let* $\delta = \dim_{\mathbb{C}} H^0(X, p_*\mathcal{O}_{\bar{X}}|\mathcal{O}_X)$. *Then* $\delta \geq \bar{s} - s$ *and the equality takes place if and only if* X *has at most double ordinary points as singularities.*

Proof: Let $\delta_X = \dim_{\mathbb{C}}(p_*\mathcal{O}_{\bar{X}}/\mathcal{O}_X)_X$ for a point x of X. The sheaf $p_*\mathcal{O}_{\bar{X}}/\mathcal{O}_X$ is a sky-scrapper sheaf concentrated at singular points of X and $\delta = \Sigma\delta_x$. Now, we have

$$\delta_x = \sum_i m_i(m_i - 1)/2$$

where $m_1, \ldots, m_k$ are the multiplicities of all singular points infinitesimal near to x. Since $\#p^{-1}(x)$ is the number of branches of X at x and the multiplicity is equal to the sum of the multiplicities of the branches, we get

$$\delta_x \geq \#p^{-1}(x) - 1, \quad \delta = \sum \delta_x \geq \sum(\#p^{-1}(x) - 1) = \bar{s} - s$$

and the equality takes place if and only if $k = 1$, $m_1 = 2$, $\#p^{-1}(x) = 2$ for each singular point x of X. Clearly, these conditions characterize ordinary double points.

(3.10) Theorem: *Let* $f: X \to D$ *be a minimal family of curves over a disk. Then* X_0 *is cohomologically insignificant if and only if*

(*a*) $C = X_{0,\mathrm{red.}}$ *has only ordinary double points as singularities,*
(*b*) $X_0 = X_{0,\mathrm{red.}}$ *or* $X_0 = mC$, *where* $H^1(C, \mathcal{O}_C) = \mathbb{C}$.

PROOF: First let us compute the limit Hodge structure $H^1(X_\infty)$. We have (see (1.11)):

$$Gr_2^W(H^1(X_\infty)) \simeq Gr_0^W(H^1(X_\infty)(-1), \qquad Gr_2^W(H^1(X_\infty)) = H_1^{1,1}(X_\infty),$$

$$Gr_1^W(H^1(X_\infty)) = H_1^{1,0}(X_\infty) \oplus H_1^{0,1}(X_\infty).$$

Let $h_1^{p,q}(X_\infty) = \dim \mathbb{C} H_1^{p,q}(X_\infty)$, then

$$h^{1,0}(X_t) = h_1^{1,0}(X_\infty) + h_1^{1,1}(X_\infty) = h_1^{1,0}(X_\infty) + h_1^{0,0}(X_\infty).$$

Now by (3.1) we have

$$h_1^{1,0}(X_0) = \sum_{i=1}^{h} g_i, \quad h_1^{0,0}(X_0) = \bar{s} - s - h + 1$$

(we preserve the notations of (3.1)).

Since the specialization map $H^1(X_0) \to H^1(X_\infty)$ is always injective (2.7) the inequalities

$$h_1^{1,0}(X_0) \leq h_1^{1,0}(X_\infty), \quad h_1^{0,0}(X_0) \leq h_1^{0,0}(X_\infty)$$

hold and they turn to equalities if and only if X_0 is cohomologically 1-insignificant.

Now by invariance of $\chi(X_t, \mathcal{O}_{X_t})$

$$1 - h^{1,0}(X_t) = \dim_{\mathbb{C}} H^0(X_0, \mathcal{O}_{X_0}) - \dim_{\mathbb{C}} H^1(X_0, \mathcal{O}_{X_0})$$

and by (3.4) $\dim_{\mathbb{C}} H^0(X_0, \mathcal{O}_{X_0}) = 1$. Using this and the above we get

$$\dim_{\mathbb{C}} H^1(X_0, \mathcal{O}_{X_0}) - \sum g_i + h - 1 + s - \bar{s} \geq 0$$

and the equality takes place if and only if X_0 is cohomologically 1-insignificant.

Let $p: \bar{C} \to C$ be the normalization projection of $C = X_{0,\mathrm{red.}}$. Considering the exact sequence

$$0 \to \mathcal{O}_C \to p_*(\mathcal{O}_{\bar{C}}) \to p_*(\mathcal{O}_{\bar{C}})/\mathcal{O}_C \to 0$$

we get the formula

(3.10.1) $$\dim \mathbb{C} H^1(C, \mathcal{O}_C) = \Sigma g_i + \delta - h + 1$$

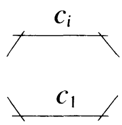
where $\delta = \dim_{\mathbb{C}} H^0(C, p_*(O_{\bar{C}})/\mathcal{O}_C)$. Plugging this formula into inequality (*) we get

$$(\dim_{\mathbb{C}} H^1(X_0, \mathcal{O}_{X_0}) - \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C)) + (\delta - \bar{s} + s) \geq 0$$

and the equality holds if and only if X_0 is cohomologically 1-insignificant. It remains to apply lemmas (3.7), (3.9) and notice that X_0 is always cohomologically 0-insignificant and 2-insignificant (2.7).

(3.11) REMARK: The curves $C = X_{0,\text{red.}}$ with $H^1(C, \mathcal{O}_C) = \mathbb{C}$ can be easily described [13]. They are of the following types:

(i) C is a nonsingular elliptic curve;
(ii) C is an irreducible rational curve with a node:
(iii) $C = C_1 + \cdots + C_h$. where C_i are non-singular rational curves which intersect each other transversally forming a cycle:



4. Families of surfaces

(4.1) Let F be a complete connected complex algebraic surface with an isolated normal singular point x_0. Let $\pi: \bar{F} \to F$ be a resolution of F at x_0 and $E = \pi^{-1}(x_0)_{\text{red.}}$ the exceptional divisor. We can always choose π with the following properties

(i) $E = E_1 + \cdots + E_h$, where all E_i are nonsingular,
(ii) E_i intersects transversally E_j for $i \neq j$ and $E_i \cap E_j \cap E_k = \emptyset$ for three distinct indices i,j,k.

If we also assume

(iii) each exceptional curve of the first kind among the E_i intersects at least three others E_j.

Then conditions (i)-(iii) determine π uniquely.

Denote by $\Gamma(E)$ the following graph:

its vertices v_i correspond to the components E_i,
its edges (v_i, v_j) correspond to the points of $E_i \cap E_j$.

Let $c(x_0) = b_1(\Gamma(E))$. It is easy to see that $c(x_0)$ is independent of choice of π with the properties (i), (ii) above. It is immediately seen (considering "the normalization" of the graph $\Gamma(E)$) that

$$c(x_0) = \dim_{\mathbb{R}} Gr_0^W(H^1(E)) \text{ computed in (3.1).}$$

(4.2) Proposition: *The Deligne mixed Hodge structure $H^i(F)$ is computed as follows:*

$$H^i(F) \simeq H^i(\bar{F}), \quad i \neq 1, 2.$$
$$H^1(F) = \ker(H^1(\bar{F}) \to H^1(E)),$$

a sequence

$$0 \to \mathrm{Coker}(H^1(\bar{F}) \to H^1(E)) \to H^2(F) \to \mathrm{Ker}(H^2(\bar{F} \to H^2(E)) \to 0$$

is exact.

Here all the maps between the cohomology spaces are induced by the inclusion of the curve E into $\bar{F}$.

Proof: Applying the exact sequence from (1.8), where $Y = \{x_0\}$, we obtain the following exact sequence:

$$0 \to H^0(F) \to H^0(\bar{F}) + H^0(\{x_0\}) \to H^0(E) \to H^1(F) \to H^1(\bar{F}) \to H^2(F)$$
$$\to H^2(\bar{F}) \to H^2(E) \to H^3(F) \to H^3(\bar{F}) \to 0 \to H^4(F) \to H^4(\bar{F}) \to 0.$$

The maps $H^0(F) \to H^0(\bar{F})$ and $H^0(\{x_0\}) \to H^0(E)$ are bijective, because F is connected and x_0 is normal respectively. Hence, we have only to show that the map $H^2(\bar{F}) \to H^2(E)$ is surjective. Restricting this map onto the subgroup generated by the cohomology classes of the components E_i we identify the obtained map $\mathbb{Z}^h \to \mathbb{Z}^h$ with the map given by the intersection form on the subgroup of $\mathrm{Pic}(\bar{F})$ formed by divisors supported in E. By Mumford' result([10], X) this form is negatively definite. This implies that the map is surjective.

(4.3) Corollary: *Let F be a normal complete surface with finitely many singular points $x_1, x_2, \ldots, x_n$. Let $\pi : \bar{F} \to F$ be a resolution of singular points of F, $E^i = \pi^{-1}(x_i)_{\mathrm{red.}}$ the exceptional curve at x_i. Assume that E^i satisfies properties (i), (ii) of (4.1) for each $i = 1, \ldots, n$. Then the Deligne mixed Hodge structure $H^i(F)$ is computed as follows:*

$$H^i(F) \simeq H^i(\bar{F}) \text{ is pure of weight } i, \quad i \neq 1, 2.$$

$$H^1(F) = \mathrm{Ker}(H^1(\bar{F}) \to Gr_1^W(H^1(E)), E = E^1 \cup \cdots \cup E^n.$$

$$Gr_O^W(H^2(F)) = Gr_0^W(H^1(E)) = \bigoplus_i Gr_O^W(H^1(E^i)) = \bigoplus \mathbb{R}^{c(x_i)},$$

$$Gr_1^W(H^2(F)) = \mathrm{Coker}(H^1(F) \to Gr_1^W(H^1(E)),$$

$$Gr_2^W(H^2(F)) = \mathrm{Ker}(H^2(\bar{F})) \to H^2(E)).$$

PROOF: Take for π the composition of a resolution $\pi_1{:}\bar{F}_1 \to F$ of F at x_1, a resolution $\pi_2{:}\bar{F}_2 \to \bar{F}_1$ of $\bar{F}_1$ at $\pi_1^{-1}(x_2), \ldots$, a resolution $\pi_n : \bar{F}_n \to \bar{F}_{n-1}$ at $(\pi_{n-1} \ldots o\pi_1)^{-1}(x_n)$. Then apply proposition (4.2) taking into account that $h^i(\bar{F})$ is a pure Hodge structure, considering first the resolution π_n, then π_{n-1} and so on.

(4.4) In the notation of (4.1) the *genus* $\delta p_a(x_0)$ of a singular point x_0 of a surface F is defined as

$$\delta p_a(x_0) = \dim_{\mathbb{C}}(R^1\pi_*\mathcal{O}_{\bar{F}})_{x_0}.$$

By Zariski's Holomorphic Function Theorem

$$(R^1\pi_*\mathcal{O}_{\bar{F}})_{x_0} = \varprojlim_Z H^1(Z, \mathcal{O}_Z)$$

where Z runs the set of all divisors supported on E and the projective limit is taken with respect to the canonical surjections $H^1(Z, \mathcal{O}_Z) \to H^1(Z', \mathcal{O}_Z)$ if $Z' \geq Z$.

The notation δp_a is explained as follows. Let $f{:}X \to D$ be a family over a disk as in (1.10) and $f = X_0$ is the special fibre. Then by the invariance of $\chi(X_t, \mathcal{O}_{X_t})$ we get $p_a(X_t) = p_a(X_0)$, where for any surface V

$$p_a(V) = -\dim_{\mathbb{C}} H^1(V, \mathcal{O}_V) + \dim_{\mathbb{C}} H^2(V, \mathcal{O}_V).$$

Now, applying the Leray spectral sequence for $\pi : \bar{X} \to X$

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{O}_{\bar{X}}) \Rightarrow H^{p+q}(\bar{X}, \mathcal{O}_{\bar{X}})$$

we obtain that $p_a(X_0) - p_a(\bar{X}_0) = \delta p_a(x_0)$. Hence

$$\delta p_a(x_0) = p_a(X_t) - p_a(\bar{X}_0). \tag{4.4.1}$$

(4.5) Let ω_F denote the Grothendieck canonical sheaf on normal surface F, that is, the sheaf of germes of differential 2-forms on F which are regular outside the point x_0. Recall (see [3]) that F is Gorenstein at X_0 iff ω_F is free in a neighborhood of x_0. Any F which is a locally complete intersection at x_0 is Gorenstein. For example, if F is embedded into a nonsingular threefold.

Let $\pi:\bar{F}\to F$ be a resolution of F at its only singular point x_0. Assume that it is *weakly minimal* in the following sense: each exceptional curve of the first kind in the exceptional divisor E intersects at least two other components of E.

(4.6) PROPOSITION: ([19]). *Suppose that F is Gorenstein at x_0. Then*

$$\omega_{\bar{F}} = \pi^*(\omega_F)\otimes \mathcal{O}_{\bar{F}}(-Z)$$

where $Z\geq 0$ and either $Z=0$ or the support Supp $(Z)=E$.

PROOF: Since ω_F is an invertible sheaf on F and $\omega_{\bar{F}}$ coincides with $\pi^*(\omega_F)$ outside the exceptional divisor E, we can always find some divisor Z supported in E with the property above. Let $\omega_{\bar{F}} = \mathcal{O}_{\bar{F}}(K_{\bar{F}})$ and and $\omega_F = \mathcal{O}_F(K_F)$ for some Cartier divisors $K_{\bar{F}}$ and K_F on $\bar{F}$ and F respectively. Since, obviously $(\pi^*(K_F)\cdot E_i)=0$ for any component E_i of E we get

$$(K_{\bar{F}}\cdot E_i) = (2\dim_{\mathbb{C}}H^1(E_i,\mathcal{O}_{E_i})-2)-(E_i\cdot E_i) = -(Z\cdot E_i).$$

As $(E_i\cdot E_i)$ is always negative, we get that $(Z\cdot E_i)\geq 0$ except the case $H^1(E_i,\mathcal{O}_{E_i})=0$ and $(E_i\cdot E_i)=-1$, that is, E_i is an exceptional curve of the first kind.

Let $Z=Z_+-Z_-$, where Z_+ and Z_- are positive divisors without common components. Suppose that $Z_-\neq 0$, then

$$(Z\cdot Z_-) = (Z_+\cdot Z_-)-(Z_-\cdot Z_-)>0.$$

This shows that for some E_i belonging to Z_- $(Z\cdot E_i)$ is positive, and hence E_i is an exceptional curve of the first kind. Thus we get the contradiction in the case where E does not contain such curves. Now, suppose that we have an exceptional curve of the 1st kind E_i among the components of E. Let $p:\bar{F}\to\bar{F}'$ be its blowing down. By induction on the number of components of the exceptional divisor we may assume that the proposition is true for the resolution $\pi:\bar{F}'\to F$. Let $\omega_{\bar{F}'} = \pi^*(\omega_F)\otimes\mathcal{O}_{\bar{F}'}(Z')$, where Z' satisfies the properties of the proposition. Taking the inverse image of the both sides under the map p

we easily get

$$-Z = -p^*(Z') + E_i.$$

By the assumption, E_i intersects at least two other components of E. This implies that $Z'_{red} = E'$ has at least a double singular point at the image of E_i under p (obviously E' is the exceptional divisor for π'). Hence, $p^*(Z') \geq 2E_i$, $Z \geq 0$ and $\mathrm{Supp}(Z) = p^{-1}(E') + E_i = E$. Thus, we see that we may assume that E does not contain exceptional curves of the first kind, and by the above $Z = Z_+ \geq 0$. Assume that $Z \neq 0$, $\mathrm{Supp}(Z) \neq E$. Since E is connected (Zariski's Connectedness Theorem), there exists a component E_j of E such that $(Z \cdot E_j) > 0$. But we saw already that this implies that E_j is an exceptional of the 1st kind. Thus, $Z = Z_+$ is supported on the whole E. The proposition is proven.

(4.7) Proposition ([19]): *Under the hypotheses of* (4.6)

(i) *the number* $p_a(Z) = \dim_{\mathbf{C}} H^1(Z, \mathcal{O}_Z)$ *is bounded on the set of all positive divisors supported on* E;
(ii) *the maximum value of* $p_a(Z)$ *coincides with the genus* $\delta p_a(x_0)$;
(iii) *there exists a unique minimal positive divisor* Z_0 *supported on* E *with*

$$p_a(Z_0) = \delta p_a(x_0) \quad \text{and} \quad p_a(Z) = p_a(Z_0) \text{ iff } Z \geq Z_0.$$

Proof: Firstly, notice that $p_a(Z') \geq p_a(Z)$ if $Z \geq Z'$ because Z, Z' are one-dimensional schemes (see (3.7)). Assuming that $p_a(Z)$ is unbounded, we get that the function $p_a(nE)$ is unbounded as a function of n. But the standard exact sequence

$$H^1(E, \mathcal{O}_E(-nE)) \to H^1((n+1)E, \mathcal{O}_{(n+1)E}) \to H^1(nE, \mathcal{O}_{nE}) \to 0$$

shows then that the left space is nonzero for some n larger than any given number N. By the duality

$$H^1(E, \mathcal{O}_E(-nE)) \simeq H^0(E, \omega_E \otimes \mathcal{O}_E(nE))$$

and (since $(E \cdot E) < 0$) for some component E_i of E we have $\deg(\omega_E \otimes \mathcal{O}_E(nE) \otimes \mathcal{O}_{E_i}) < 0$ for sufficient large n. This shows that the sheaf $\omega_E \otimes \mathcal{O}_E(nE))$ has no nontrivial sections for large n and thus $H^1(E, \mathcal{O}_F(-nE)) = 0$ for large n. Contradiction! (ii) immediately follows from Zariski's Holomorphic Function Theorem:

$$(R^1\pi_*\mathcal{O}_{\tilde{F}})_{x_0} = \varprojlim_{Z} H^1(Z, \mathcal{O}_Z),$$

where Z runs the projective system of positive divisors supported on E.

(iii) Let $Z_1 = \Sigma n_iE_i$, $Z_2 = \Sigma m_iE_i$ be two divisors such that $p_a(Z_1) = p_a(Z_2) = \delta p_a(x_0)$. Let us show that the divisor $Z_1 \cap Z_2 = \Sigma k_iE_i$ with $k_i = \min(n_i, m_i)$ also has this property. This obviously proves (iii).

Let $Z_1' = Z_1 - (Z_1 \cap Z_2)$, $Z_2' = Z_2 - (Z_1 \cap Z_2)$, $Z_1 \cup Z_2 = \Sigma k_i'E_i$, where $k_i' = \max(n_i, m_i)$. Clearly $Z_1 \cup Z_2 = Z_1 + Z_2' = Z_2 + Z_1'$. Next, consider the following commutative diagram:

$$\begin{tikzcd}
 & H^1(Z_1', \mathcal{O}_{Z_1'}(-Z_2)) \arrow[r] \arrow[d] & H^1(Z_1', \mathcal{O}_{Z_1'}(-Z_1 \cap Z_2)) \arrow[r] \arrow[d] & 0 \\
H^i(Z_2', \mathcal{O}_{Z_2'}(-Z \arrow[r] \arrow[d] & H^1(Z_1 \cup Z_2, \mathcal{O}_{Z_1\cup Z_2}) \arrow[r] \arrow[d] & H^1(Z_1, \mathcal{O}_{Z_1}) \arrow[r] \arrow[d] & 0 \\
H^1(Z_2', \mathcal{O}_{Z_2'}(-Z_1 \cap Z_2)) \arrow[r] \arrow[d] & H^1(Z_2, \mathcal{O}_{Z_2}) \arrow[r] \arrow[d] & H^1(Z_1 \cap Z_2, \mathcal{O}_{Z_1\cap Z_2}) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Here the top row and the left columns come from the exact sequence of sheaves

$$0 \to \mathcal{O}_{Z_2}(-Z_1) \to \mathcal{O}_{Z_2}(-Z_1 \cap Z_2) \to \mathcal{O}_{Z_1'}(-Z_1 \cap Z_2) \otimes \mathcal{O}_{Z_2} \to 0$$
$$0 \to \mathcal{O}_{Z_1}(-Z_2) \to \mathcal{O}_{Z_1'}(-Z_1 \cap Z_2) \to \mathcal{O}_{Z_2'}(-Z_1 \cap Z_2) \otimes \mathcal{O}_{Z_1} \to 0$$

and their surjectivity is explained by the zero-dimensionality of the supports of the third sheaves in these sequences. Now the diagram shows that we have the exact sequence

$$H^1(Z_1 \cup Z_2, \mathcal{O}_{Z_1\cup Z_2}) \to H^1(Z_2, \mathcal{O}_{Z_2}) \oplus H^1(Z_1, \mathcal{O}_{Z_1}) \to H^1(\mathcal{O}_{Z_1\cap Z_2}) \to 0$$

and hence

$$p_a(Z_1 \cap Z_2) \geq p_a(Z_1) + p_a(Z_2) - p_a(Z_1 \cup Z_2) = \delta p_a(x_0).$$

This proves (iii).

(4.8) Proposition ([19]): *Under the hypotheses of* (4.6) *the two divisors* Z_1 *from* (4.6) *and* Z_0 *from* (4.7) *coincide.*

PROOF: Let us prove first that $Z_1 \geq Z_0$. It suffices to show that for any $Z \geq Z_1$ we have $p_a(Z) \geq p_a(Z_1)$. Considering the exact sequence

$$0 \to \mathcal{O}_A(-Z_1) \to \mathcal{O}_Z \to \mathcal{O}_{Z_1} \to 0, \quad A = Z - Z_1$$

we see that it will follow from the vanishing of $H^1(A, \mathcal{O}_A(-Z_1))$. By the adjunction formula

$$\omega_A = \mathcal{O}_{\bar{F}}(A) \otimes \mathcal{O}_A \otimes \omega_{\bar{F}} = \mathcal{O}_A(-Z_1 + A)$$

and hence by the duality

$$H^1(A, \mathcal{O}_A(-Z_1)) = H^0(A, \mathcal{O}_A(A)).$$

Since $(A \cdot A) < 0$ the right space is zero if A is reduced. To show that it is zero in general case, we may argue by induction on the number of components of$A$ and use the exact sequence

$$0 \to \mathcal{O}_{A-C}(A - C) \to \mathcal{O}_A(A) \to \mathcal{O}_C(A) \to 0,$$

where C is a component with $(A \cdot C) < 0$.

Let us prove that $Z_0 \geq Z_1$. For this it suffices to show that $p_a(Z_1) > p_a(Z)$ if $Z < Z_1$. By the duality as above

$$H^1(Z, \mathcal{O}_Z) = H^0(Z, \mathcal{O}_Z(-Z + Z_1)), \qquad H^1(Z_1, \mathcal{O}_{Z_1}) \simeq H^0(Z_1, \mathcal{O}_{z_1}).$$

Now, the exact sequence

$$0 \to \mathcal{O}_Z(-Z_1 + Z) \to \mathcal{O}_{Z_1} \to \mathcal{O}_{Z_1 - Z} \to 0$$

shows that $H^0(Z, \mathcal{O}_Z(-Z_1 + Z))$ is a proper subspace of $H^0(Z_1, \mathcal{O}_{Z_1})$, because the map $H^0(Z_1, \mathcal{O}_{Z_1}) \to H^0(Z - Z_1, \mathcal{O}_{Z - Z_1})$ is non-trivial. This proves the proposition.

(4.9) The following is the list of three important classes of algebraic hypersurface isolated singularities of dimension 2.

I. *Rational double points* (or *simple*):

$$A_n : x^2 + y^2 + z^{n+1} = 0, \quad n \geq 1,$$

$$D_n : x^2 + y^2 z + z^{n-1} = 0, \quad n \geq 4,$$

$$E_6 : x^2 + y^3 + z^4 = 0,$$

$$E_7 : x^2 + y^3 z + z^3 = 0,$$
$$E_8 : x^2 + y^3 + z^5$$

II. *Simple elliptic* (*or parabolic*):

$$\tilde{E}_6 : x^3 + y^3 + z^3 + axyz = 0, \quad a^3 + 27 \neq 0,$$
$$\tilde{E}_7 : x^2 + y^4 + z^4 + ay^2z^2 = 0, \quad a^2 - 4 \neq 0,$$
$$\tilde{E}_8 : x^2 + y^3 + z^6 + ay^2z^2 = 0, \quad 4a^3 + 27 \neq 0.$$

III. *Cusp singularities* (or *hyperbolic*, or *cyclic*):

$$xyz + x^p + y^q + z^r = 0, \quad p^{-1} + q^{-1} + r^{-1} < 1.$$

They can be characterized as follows.

(4.10) PROPOSITION: *Let F be a normal surface and x_0 be its singular point. Let $E = E_1 + \cdots + E_r$ be the exceptional curve of its resolution satisfying properties* (*i*) (*iii*) *of* (4.1). *Then*

(i) *The germ (F, x_0) of F at x_0 is analytically isomorphic to a rational double point if and only if all E_i are rational curves with $(E \cdot E_i) = -2$ (the -2-curves);*
(ii) *(F, x_0) is analytically isomorphic to a simple elliptic point if and only if E is a non-singular elliptic curve with $(E \cdot E) = -1, -2$, or -3,*
(iii) *(F, x_0) is analytically isomorphic to a cusp singularity if and only if the following three properties are satisfied: (a) F is a hypersurface at x_0:(b) $E = E_1 + \cdots + E_r$, where the E_i's are non-singular rational curves, if $r > 1$, or a nodal rational curve, if $r = 1$; c) E_i transversally intersects E_{i-1} and E_{i+1} at one point ($E_{r+1} = E_1$, $E_0 = E_r$).*

PROOF: (i) I could not find a direct reference for this result. One may argue as follows: By Tjurina [28] the resolution of this form determines the singularity up to an analytic isomorphism. Now direct computation shows that any double rational point has the resolution of this form;

(ii) see Saito [21];
(iii) see Karras [11].

(4.11) COROLLARY: *Let $p : \bar{F} \to F$ be a resolution of a singular point x_0 of a normal surface F. Assume that locally at x_0 F is a hypersurface in*

$\mathbb{C}^3$. *Let* $E = p^{-1}(x_0)_{\text{red.}}$ *be the exceptional curve. Assume that all irreducible components of E are nonsingular and p is weakly minimal. Then*

$$\delta p_a(x_0) = p_a(E)$$

if and only if (F, x_0) *is either a double rational point, or a simple elliptic point, or a cusp point.*

PROOF: Let Z_1 be the divisor supported at E defined in proposition (4.6). If $Z_1 = 0$, then as we saw in its proof for each irreducible component E_i of E $(K_{\bar{F}} \cdot E_i) = 0$. Since $(E_i \cdot E_i) < 0$, we get that the only possible case is $(E_i \cdot E_i) = -2$ and $p_a(E_i) = 0$. Applying (4.10) we infer that (F_0, x_0) is a double rational point. Conversely, if (F, x_0) is a double rational point, then by Artin [1] $H^1(Z, \mathcal{O}_Z) = 0$ for any positive divisor supported in the exceptional divisor E. This shows that $Z_1 = 0$.

Now we assume that $Z_1 \neq 0$. Then by (4.8) $\delta p_a(x_0) = p_a(E)$ implies $Z_1 = Z_0 = E$. Thus,

$$p_a(E) = \frac{(-Z_1 \cdot E) + (E \cdot E)}{2} + 1 = 1.$$

Since for any positive $Z \leq E$ we have $p_a(Z) \leq p_a(E)$, that is, each component E_i of E is either a nonsingular elliptic curve or a nonsingular rational curve. Suppose that some E_i is an elliptic curve and $E \neq E_i$. Then in the exact sequence

$$0 \to \mathcal{O}_{E_i}(-E + E_i) \to \mathcal{O}_E \to \mathcal{O}_{E_i} \to 0$$

the sheaf $\mathcal{O}_{E_i}(-E + E_i) = \mathcal{O}_{E_i} \otimes \mathcal{O}_{\bar{F}}(K_{\bar{F}} + E_i) = \omega_{E_i}$ and by duality $\dim_{\mathbb{C}} H^0(E_i, \mathcal{O}_{E_i}(-E + E_i)) = p_a(E_i) = 1$. However, $H^0(E, \mathcal{O}_E) = \mathbb{C}$ and the map $H^0(E, \mathcal{O}_E) \to H^0(E_i, \mathcal{O}_{E_i})$ is non-trivial. This contradiction shows that either $E = E_i$, a nonsingular elliptic curve, or all components E_1 of E are rational. In the first case, by (4.10)(ii) we get that (F, x_0) is a simple elliptic point. In the second case we have

$$((E - E_i) \cdot E_i) = (-(K_{\bar{F}} + E_i) \cdot E_i) = 2 - 2p_a(E_i) = 2.$$

This obviously implies that E satisfies the conditions of (4.10)(iii) and hence (F, x_0) is a cusp singularity.

It remains only to check that for a simple elliptic or cusp singular point we always have $\delta p_a(x_0) = p_a(E)$.

Suppose that (F, x_0) is a simple elliptic singularity. Then obviously, $p_a(E) = 1$. From other hand, it is well known that $\delta p_a(x_0) = 1$ (see [21], [29]).

Suppose that (F, x_0) is a cusp singularity. Again it is well known that $\delta p_a(x_0) = 1$ (loc. cit.). It follows from (4.10)(iii) that for any component E_i of $E - E - E_i$ is a chain of rational curves., and $(E \cdot (E - E_i)) = 2$. Now for any reduced connected divisors Z_1, Z_2 and $Z_1 + Z_2$ we have

$$p_a(Z_1 + Z_2) = p_a(Z_1) + p_a(Z_2) + (Z_1 \cdot Z_2) - 1.$$

Applying this formula first to $E - E_i$ we easily get $p_a(E - E_i) = 0$. Then, applying it to $E = (E - E_i) + E_i$, we get $p_a(E) = 1$.

(4.12) PROPOSITION: *Let $f: X \to D$ be a family of surface over a disk as in* (1.10). *Suppose that X_0 has only isolated singular points. Then X_0 is cohomologically i-insignificant for all $i \neq 2$ and the specialization homomorphism*

$$sp_2: H^2(X_0) \to H^2(X_\infty)$$

is injective.

This is an immediate corollary of proposition (2.7) and the fact that $H^3(X_0)$ has the pure Hodge structure of $H^3(\bar{X}_0)$ (4.2).

(4.13) THEOREM: *Under the hypothesis of* (4.12) *X_0 is cohomologically insignificant if and only if each of its singular points is either a double rational point, or a simple elliptic point, or a cusp point.*

PROOF: Let $\pi: \bar{X}_0 \to X_0$ be a resolution of singular points $x_1, \ldots, x_n$ of X_0 satisfying the properties (i), (ii) of (4.1) and also assumed to be weakly minimal (4.5) at each singular point.

Since the specialization homorphism sp_1 is bijective (4.12) the cohomology $H^1(X_\infty)$ has the pure Hodge structure of $H^1(X_0)$, more precisely, it follows from (4.2) that

$$H^1(X_\infty) = H^1(X_0) = \mathrm{Ker}(H^1(\bar{X}_0) \to Gr_1^W(H^1(E)))$$

where $E = E^1 \ldots E^n$, $E^1 = \pi^{-1}(x_i)_{\mathrm{red.}}$ the exceptional curve of π at x In particular, we have (4.3)

$$h_1^{1,0}(X_\infty) = h_1^{1,0}(X_0) = \tfrac{1}{2} \dim \mathrm{Ker}(H^1(X_0) \to Gr_1^W(H^1(E))).$$

By (1.11) (v) we have

$$h_1^{1,0}(X_\infty) + h_1^{1,1}(X_\infty) = h_1^{1,0}(X_\infty) = h^{1,0}(X_t), \quad t \neq 0.$$

Thus, finally, we get

$$\dim_{\mathbf{C}} H^1(X_t, \mathcal{O}_{X_t}) = h^{1,0}(X_t) = \tfrac{1}{2} \dim \operatorname{Ker}(H^1(\bar{X}_0) \to Gr_1^W(H^1(E))).$$

Now, since sp_2 is injective, X_0 is cohomologically 2-insignificant if and only if $h_2^{p,0}(X_\infty) = h_2^{p,0}(X_0)$ for all $p \geq 0$. This is equivalent to vanishing of the number

$$a = (h_2^{00}(X_\infty) - h_2^{00}(X_0)) - (h_2^{1,0}(X_\infty) - (h_2^{1,0}(X_0)) + (h_2^{2,0}(X_\infty) - h_2^{2,0}(X_0)).$$

Let us compute the number a. It follows from (4.3) that

$$h_2^{0,0}(X_0) = \dim Gr_0^W(H^1(E)) = \sum \dim Gr_0^W(H^1(E^i)) = \sum c(x_i))$$

$$h_2^{1,0}(X_0) = \tfrac{1}{2} \dim \operatorname{Coker}(H^1(\bar{X}_0) \to Gr_1^W(H^1(E)))$$

$$h_2^{2,0}(X_0) = h^{2,0}(\bar{X}_0) = \dim H^2(\bar{X}_0, \mathcal{O}_{\bar{X}_0}).$$

Applying (1.11) (i) and (v) we get

$$\begin{aligned} h_2^{0,0}(X_\infty) + h_2^{1,0}(X_\infty) + h_2^{2,0}(X_\infty) &= h_2^{2,2}(X_\infty) + h_2^{2,1}(X_\infty) + h_2^{2,0}(X_\infty) \\ &= h^{2,0}(X_t) = \dim H^2(X_t, O_{X_t}). \end{aligned}$$

Taking into account (*) we can rewrite the number a in the form

$$\begin{aligned} a &= h^{2,0}(X_t) - h^{2,0}(\bar{X}_0) - \sum c(x_i) - \tfrac{1}{2} \dim \operatorname{Coker}(H^1(\bar{X}_0) \to Gr_1^W(H^1(E))) \\ &= h^{2,0}(X_t) - h^{2,0}(\bar{X}_0) - \sum c(x_i) - \tfrac{1}{2} \dim \operatorname{Coker}(H^1(\bar{X}_0) \to Gr_1^W(H^1(E))) \\ &\quad + \tfrac{1}{2} \dim \operatorname{Ker}(H^1(\bar{X}_0) \to Gr_1^W(H^1(E))) - h^{1,0}(X_t) \\ &= h^{2,0}(X_t) - h^{1,0}(X_t) - h^{2,0}(\bar{X}) + h^{1,0}(\bar{X}_0) - h_1^{1,0}(E) - \sum c(x_i). \end{aligned}$$

Now, by (3.1)

$$h_1^{1,0}(E) = h^{1,0}(E^i) = \sum_i \left(\sum_j g_j^i \right), \quad \text{where}$$

$E^i = E^i_1 + \cdots + E^i_{r_i}$. the decomposition into irreducible components of E^i, g^i_j is the genus of E^i_j. Also, since E^i have only ordinary double points as its singularities

$$c(x_i) + \sum_j g^i_j = p_a(E^i), \quad i = 1, \ldots, n$$

(this follows from the formula for the $c(x_i)$ in (3.1), (3.10.1) and lemma (3.9)).

Next, by the invariance of the Euler-Poincare characteristic $\chi(X_t, \mathcal{O}_{X_t}) = 1 - \dim H^1(X_t, \mathcal{O}_{X_t}) + \dim H^2(X_t, \mathcal{O}_{X_t})$ we have

$$\begin{aligned} &h^{2,0}(X_t) - h^{1,0}(X_t) - h^{2,0}(\bar{X}_0) + h^{1,0}(\bar{X}_0) \\ &= \chi(X_0, \mathcal{O}_{X_0}) - \chi(\bar{X}_0, \mathcal{O}_{\bar{X}_0}) = \sum_i \delta p_a(x_i) \end{aligned}$$

(see (4.4.1)). Taking into account all these relations we finally get that

$$a = \sum_i (\delta p_a(x_i) - p_a(E^i)).$$

Since $p_a(x_i) \geq p_a(E^i)$ (4.7) we get

$$a = 0 \text{ iff } p_a(x_i) - p_a(E^i) \quad \text{for } i = 1, \ldots, n.$$

It remains to apply (4.11).

(4.14) Remark: According to J. Steenbrink [26] there exists an exact sequence of mixed Hodge structures

$$0 \to H^2(X_0) \xrightarrow{sp_2} H^2(X_\infty) \to \bigoplus_{i=1}^n H^2(V_i) \to H^3(X_0) \xrightarrow{sp_3} H^3(X_\infty) \to 0$$

where $H^2(V_i)$ is the mixed Hodge structure on the vanishing cohomology for a singular point x_i.

Applying theorem (4.13) we easily obtain the classification of isolated hypersurface singularities of dimension 2 whose vanishing cohomology does not contain nonzero $H^{p,0}$-components in its mixed Hodge structure. They are either double rational points, or simple elliptic, or cusp singularities.

(4.15) Let F be an irreducible projective surface, $\bar{F}$ its normalization $p : \bar{F} \to F$ the canonical projection. Let $\mathscr{C}$ be the Conductor of

$p_*\mathcal{O}_{\bar{F}}$ in $\mathcal{O}_F$. By definition, $\mathscr{C}$ is the largest sheaf of ideals in $\mathcal{O}_F$ which annihilates the $\mathcal{O}_F$-Module $F = p_*\mathcal{O}_{\bar{F}}/\mathcal{O}_F$. Let Δ be the closed subscheme of F defined by this Ideal, since $\mathscr{C}$ can be also considered as an Ideal of $\mathcal{O}_{\bar{F}}$ it defines the closed subscheme $\bar{\Delta}$ of $\bar{F}$. Denote by p_Δ the restriction of the projection p onto Δ.

(4.16) We say that F is a generic projection if the following conditions are satisfied:

(i) $\bar{F}$ is nonsingular;
(ii) Δ is a reduced subscheme whose singular points are triple points $t \in \Delta$ such that $\hat{\mathcal{O}}_{\Delta,t} = \mathbb{C}[[x, y, z]]/(xy, yz, xz)$;
(iii) $\bar{\Delta}$ is a reduced subscheme whose singular points are nodes;
(iv) the map p_Δ is generically a non-trivial two-sheeted covering such that three distinct nodes of $\bar{\Delta}$ are mapped to each triple point of Δ.

It is well known that any non-singular projective surface can be projected into $\mathbb{P}^3$ with the image satisfying the above conditions (see [20]).

(4.17) Theorem: Let *$f: X \to D$ be a family of surfaces. Assume that the special fibre X_0 is a general projection surface. Then X_0 is cohomologically insignificant degeneration.*

Proof. See [27].

(4.18) Remark: The first version of the present paper contained a rather long and clumsy proof of this result under an additional assumption that $H^1(X_0, \mathcal{O}_{X_0}) = 0$. Comparing it with the short proof of [27] we have decided to omit it.

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(Oblatum 17-IV-1979 & 18-III-1980)

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