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TORELLI THEOREMS FOR KÄHLER K3 SURFACES

Eduard Looijenga and Chris Peters

Introduction

A K3 *surface* is by definition a compact connected non-singular complex-analytic surface which is regular and has vanishing first Chern class. Examples of K3 surfaces are furnished by twofold coverings of $\mathbf{P}^2$ branched along a nonsingular curve of degree 6, nonsingular surfaces of degree 4 in $\mathbf{P}^2$, nonsingular complete intersections of bidegree (2, 3) in $\mathbf{P}^4$ and nonsingular complete intersections of three quadrics in $\mathbf{P}^5$. Another interesting class of K3 surfaces can be constructed out of two-dimensional tori: the canonical involution of a two-dimensional complex torus T has 16 distinct fixed points (the points of order two) and each of them determines an ordinary double point on the orbit space $T/(-id)$. A quadratic transformation at each of these points desingularizes the orbit space and the resulting non-singular surface can proved to be a K3 surface. The K3 surfaces thus obtained are usually called *Kummer surfaces.*

The second integral cohomology group of a K3 surface is a free $\mathbf{Z}$-module of rank 22. Endowed with the bilinear symmetric form $<,>$ coming from the cup product, it becomes a unimodular lattice of signature (3, 19).

It is not difficult to show that a K3 surface X has trivial canonical class. So up to a scalar multiple, there is a unique holomorphic 2-form ω_X on X. The cohomology class $[\omega_X]$ of X spans the subspace $H^{2,0}(X, \mathbf{C})$ of $H^2(X, \mathbf{C})$ and satisfies the well-known relations $\langle[\omega_X], [\omega_X]\rangle = 0$ and $\langle[\omega_X], [\bar{\omega}_X]\rangle > 0$. We can now formulate the

WEAK TORELLI THEOREM FOR KÄHLERIAN[1] K3 SURFACES. Two

[1]By a *Kähler* surface we mean a surface which carries a Kähler metric; a *kählerian* surface is a surface which can be endowed with such a metric.

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kählerian K3 surfaces X, X' are isomorphic if and only if there exists an isometry from $H^2(X, \mathbf{Z})$ to $H^2(X', \mathbf{Z})$ which sends $H^{2,0}(X, \mathbf{C})$ to $H^{2,0}(X', \mathbf{C})$.

This is an analogue of the classical Torelli theorem which asserts that two nonsingular compact curves are isomorphic if and only if their jacobians are. The weak Torelli theorem is actually derived from the strong Torelli theorem, whose formulation requires some more preparation.

Let X be a kählerian K3 surface and put $H^{1,1}(X, \mathbf{R}) := \{c \in H^2(X, \mathbf{R}) : \langle c, \omega_X \rangle = 0\}$ (we identify $H^2(X, \mathbf{R})$ with the real part of $H^2(X, \mathbf{C})$). The form $\langle , \rangle$ is of signature (1,19) on $H^{1,1}(X, \mathbf{R})$ and so the set $\{c \in H^{1,1}(X, \mathbf{R}) : \langle c, c \rangle > 0\}$ has two connected components. Exactly one of these components contains Kähler classes (for two such classes have positive inner product) and is called the *positive cone*. The strong form of the Torelli theorem, due to Rapoport and Burns [4], then reads as follows.

STRONG TORELLI THEOREM FOR KÄHLERIAN K3 SURFACES. Let X and X' be kählerian K3 surfaces and suppose we are given an isometry $\phi^* : H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$ which

(i) sends $H^{2,0}(X', \mathbf{C})$ to $H^{2,0}(X, \mathbf{C})$
(ii) sends the positive cone of X' to the positive cone of X
(iii) sends the cohomology class of any positive divisor on X' to the class of a positive divisor on X.

Then ϕ^* is induced by a *unique* isomorphism $\phi : X \to X'$.

When X and X' are both *algebraic* K3 surfaces, then the conditions (ii) and (iii) are equivalent to the condition that ϕ^* maps the class of some ample divisor on X' to the class of an ample divisor on X. In this form, the Torelli theorem was proved earlier by Piatetski-Shapiro and Shafarevič [20] and is for Burns-Rapoport the point of departure.

The main goal of the present paper is to give a complete account of the proof of these two Torelli theorems. In order to relate it to the work cited earlier, we briefly outline how the strong Torelli theorem is proved.

Certainly a key rôle is played by the period mapping, which we shall now define. Let fix (once and for all) an abstract lattice L which is isometric to the second cohomology lattice of a K3 surface and introduce the *period space*

$$\Omega := \{\omega \in L_{\mathbf{C}} : \langle \omega, \omega \rangle = 0, \quad \langle \omega, \bar{\omega} \rangle > 0\}/\mathbf{C}^*,$$

which is an open piece of a nonsingular projective quadric of dimension 20. A *marking* of a K3 surface X is by definition an isometry $\alpha : H^2(X, \mathbf{Z}) \to L$. Then $\alpha(\omega_X)$ determines an element of Ω, called the *period point* of (X, α). More generally, a marking of a family $p : \mathscr{X} \to S$ of K3 surfaces is a trivialization $\alpha : R^2 p_*(\mathbf{Z}) \to L$ of local systems and determines a *period mapping* $\tau : S \to \Omega$, which assigns to $s \in S$ the period point of $(X, \alpha(s))$.

Small deformations of the complex-analytic structure of a K3 surface X can be effectively parametrized by a family of K3 surfaces $p : \mathscr{X} \to S$ whose base space S is a neighbourhood of the origin in $\mathbf{C}^{20}$ (with $X \cong X_0$). If S is simply connected, we can choose a marking $\alpha : R^2 p_*(\mathbf{Z}) \to L$ and then according to Tyurin and Kodaira the associated period mapping $\tau : S \to \Omega$ is a local isomorphism at $s \in$ S. It is this rather unusual situation which explains why the Torelli problem for K3 surfaces has received so much attention. Besides the period mapping, the proof has four main ingredients.

First Piatetski-Shapiro and Shafarevič consider the case when one of the surfaces, say X, is a *special Kummer surface.* This means that X is a Kummer surface originating from a reducible abelian surface. Such surfaces have a rich geometric structure, which is reflected by a correspondingly rich arithmetic structure in their cohomology lattice. This enables them to give a direct proof of the strong Torelli theorem in this special case.

The next step is to show that the period points of marked special Kummer surfaces lie dense in Ω.

The proof then proceeds as follows. Given X and X' as in the theorem, embed both in marked universal families $(p : \mathscr{X} \to S, \alpha)$ resp. $(p' : \mathscr{X}' \to S', \alpha')$ such that ϕ corresponds with $\alpha(0)^{-1} \circ \alpha(0)$. By Kodaira's result $\tau : S \to \Omega$ and $\tau' : S' \to \Omega$ are local isomorphisms at the origin and condition (i) implies that $\tau(0) = \tau'(0)$. So for suitable S and S', the map $\psi := \tau^{-1} \circ \tau' : S' \to S$ is an isomorphism. The third step consists in verifying that the hypotheses of the theorem are also satisfied by the triples

$$(X_{\psi(s')}, X'_{s'}, \phi^*(s') := \alpha(\psi(s'))^{-1} \circ \alpha'(s') : H^2(X'_{s'}, \mathbf{Z}) \to H^2(X_{\psi(s')}, \mathbf{Z}))$$

for s' sufficiently close to $0 \in S'$. The two previous steps then imply that we can find a sequence $\{s'_i \in S'\}$ converging to $0 \in S'$

such that each X'_{s_i} is a special Kummer surface and a sequence of isomorphisms $\{\phi_i : X_{\psi(s'_i)} \to X'_{s'_i}\}$ inducing $\phi^*(s'_i)$.

The final step is the "main lemma" of Burns-Rapoport which roughly asserts that under these circumstances the sequence $\{\phi_i\}$ converges (pointwise) to an isomorphism $\phi : X_0 \to X'_0$, inducing $\phi^*(0) = \phi^*$.

Unfortunately, the paper of Piatetski-Shapiro and Shafarevič contains several gaps and errors, some of which seem to be quite essential. Perhaps the most striking one is a claimed Torelli theorem for abelian surfaces (needed for a proof of the corresponding result for special Kummer surfaces) which in the way it is stated is certainly false. This was first observed by Shioda, who gave a correct treatment of the theorem in question in [23]. One of our aims is to fill these gaps and to give a detailed exposition of the resulting proof. This includes a few simplifications. For instance, we give a 'direct' proof of the Torelli theorem for projective Kummer surfaces which enables us to omit the intricate analysis in [20] of the Picard group of a special Kummer surface. Another example is our proof of the "main lemma", which avoids the tedious case-by-case checking of the original proof.

Finally, we mention that a complete proof of the Torelli theorem for algebraic K3 surfaces was also given by T. Shioda in a seminar held in Bonn in 1977–78. We have been told that M. Rapoport also has a proof (apparently unpublished).

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§1. Cohomology and Picard group of a K3-surface

By a *surface* we shall always mean a nonsingular compact connected complex-analytic surface, unless the contrary is explicitly stated.

(1.1.) A surface is called a *K*3-*surface* if it is regular and has vanishing first Chern class. In this section we prove some 'well-known' properties of such surfaces. Consider on a K3-surface X the exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1.$$

Since X is regular, the differential $\delta : H^1(X, \mathcal{O}_X^*) \to H^2(X, Z)$ in the associated cohomology sequence is a monomorphism. The term

$H^1(X, \mathcal{O}_X^*)$ represents the set of isomorphism classes of analytic line bundles over X and δ is given by the (first) Chern class. So the injectivity of δ means that an analytic line bundle over X is, up to isomorphism, completely determined by its Chern class. In particular, the canonical bundle of X is trivial. If a line bundle comes from a divisor D on X, then its Chern class $[D]$ is geometrically given by intersecting 2-cycles on X with D. It follows that two divisors on X are linearly equivalent if and only if they are cohomologous.

The following proposition describes the integral cohomology of a K3-surface.

(1.2.) PROPOSITION: *Let X be a K3-surface. Then $H_1(X, \mathbf{Z})$ and $H_3(X, \mathbf{Z})$ are trivial, while $H^2(X, \mathbf{Z})$ is a free $\mathbf{Z}$-module of rank 22. Endowed with the quadratic form $\langle , \rangle$ induced by the cup product, $H^2(X, \mathbf{Z})$ is isometric to the lattice $-E_8 \oplus -E_8 \oplus H \oplus H \oplus H$, where E_8 denotes the Cartan matrix of the corresponding root system and H stands for the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; in particular, the signature of this form is of type* $(3, 19)$.

PROOF: Since X is regular, its first betti number $b_1(X)$ vanishes. Now, Serre duality implies that $h^2(\mathcal{O}_X) = h^0(\Omega_X^2) = 1$ and hence the arithmetic genus $\chi(X)$ equals 2. By the Noether formula, we then have $c_1(X)^2 + c_2(X) = 24$. Since $c_1 = 0$, this gives $c_2 = 24$. As $b_3(X) = b_1(X) = 0$ by Poincaré duality, it follows that $b_2(X) = 22$.

Since $b_1(X) = 0$, $H_1(X, \mathbf{Z})$ is of finite order k say, and this determines an unramified covering $\tilde{X} \to X$ of degree k. Clearly, $\tilde{X}$ is also a K3-surface, so by what we have already proved, $c_2(\tilde{X}) = 24$. On the other hand, $c_2(\tilde{X})$, being the Euler characteristic of $\tilde{X}$, must be equal to $24k$. So $k = 1$ and $H_1(X, \mathbf{Z}) = 0$. The universal coefficient formula and Poincaré duality then imply that $H_2(X, \mathbf{Z})$ and $H^2(X, \mathbf{Z})$ are torsion free and that $H_3(X, \mathbf{Z}) = 0$.

The index of X is given by $\frac{1}{3}(c_1(X)^2 - 2c_2(X)) = -16$ and so the signature of $\langle , \rangle$ is indeed of type (3,19). By the Wu formula [17] we have for any $x \in H^2(X, \mathbf{Z})$, $\langle x, x \rangle \equiv \langle x, c_1(X) \rangle \equiv 0 \pmod 2$. Since $\langle , \rangle$ is also unimodular (by Poincaré duality), the theory of quadratic forms then asserts that $\langle , \rangle$ is of the claimed type [21].

(1.3.) Next we make a general observation about the Hodge decomposition of the cohomology of surfaces. For any surface X, $H^{1,1}(X, \mathbf{R})$ is, by definition, the set of real classes orthogonal to $H^{2,0}(X, \mathbf{C})$. Let $H^{1,1}(X, \mathbf{R})^{\perp}$ denote the orthogonal complement of

$H^{1,1}(X, \mathbf{R})$ in $H^2(X, \mathbf{R})$. Then (more or less by the definitions), the inner product defines a perfect pairing between $H^{1,1}(X, \mathbf{R})^\perp$ and the underlying real vector space of $H^{2,0}(X, \mathbf{C})$. If the latter is endowed with its natural hermitian form, then this is actually a pairing of inner product spaces. It follows that $H^{1,1}(X, \mathbf{R})^\perp$ possesses a natural complex structure, which turns it into a hermitian vector space. Since $\langle \omega, \bar{\omega} \rangle > 0$ for all nonzero $\omega \in H^{2,0}(X, \mathbf{C}), H^{1,1}(X, \mathbf{R})^\perp$ is positive definite.

For a K3-surface and a complex torus, we have $h^0(\Omega_X^2) = 1$, so then $H^{1,1}(X, \mathbf{R})^\perp$ is a positive definite space of real dimension 2. In this case a complex structure on $H^{1,1}(X, \mathbf{R})^\perp$ making the inner product hermitian amounts to nothing more than an orientation of $H^{1,1}(X, \mathbf{R})^\perp$. It also follows that in the case of a K3-surface the signature of $\langle , \rangle$ on $H^{1,1}(X, \mathbf{R})$ is of type (1, 19).

(1.4.) The Riemann-Roch theorem for an analytic line bundle over a surface X asserts that

$$h^0(X, \mathscr{L}) - h^1(X, \mathscr{L}) + h^2(X, \mathscr{L}) = \frac{1}{12}\Big[c_1(X)^2 + c_2(X)\Big] + \frac{1}{2}\Big[c_1(\mathscr{L})^2 + c_1(\mathscr{L})c_1(X)\Big].$$

When X is a K3 surface, we have $h^2(X, \mathscr{L}) = h^0(X, \mathscr{L}^{-1} \otimes K_X) = h^0(X, \mathscr{L}^{-1})$ by Serre duality, and so using prop. (1.1) we find

$$h^0(X, \mathscr{L}) + h^0(X, \mathscr{L}^{-1}) \geq +\frac{1}{2}(c_1(\mathscr{L}))^2. \tag{1.4.)$_1$}$$

It follows that if $c_1(\mathscr{L})^2 \geq -2$, then $\mathscr{L}$ or $\mathscr{L}^{-1}$ admits a section, in other words, $\mathscr{L}$ or $\mathscr{L}^{-1}$ comes from an effective divisor.

If C is an irreducible curve on a K3 surface X, then it follows from the adjunction formula that its arithmetic genus $p_a(C)$ equals $\frac{1}{2}\langle [C], [C] \rangle + 1$.

It follows that C is a smooth rational curve if and only if $\langle [C], [C] \rangle = -2$. Such a curve will be called *nodal*, since it can be blown down to yield an ordinary double point (a node in the more classical terminology). We use the same adjective for the cohomology class $[C]$ of a nodal curve C.

(1.5.) Given a surface X, then the image of the composite map $H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{R})$ is called the *algebraic lattice* in $H^2(X, \mathbf{R})$ and denoted by S_X. Elements of S_X are called algebraic classes. It can be shown (with the help of Thm. 3 of [12] which ensures the degeneration of the spectral sequence $H^p(X, \Omega_X^q) \Rightarrow H^{p+q}(X)$ for $p+q=2$) that S_X is just the set of integral points in $H^{1,1}(X, \mathbf{R})$. Following Kodaira [12], X is projective if and only if there exists a $d \in S_X$ with $\langle d, d\rangle > 0$.

Among the algebraic classes we distinguish those coming from divisors, resp. effective divisors, resp. irreducible divisors, which will be called *divisorial*, resp. *effective*, resp. *irreducible*. We say that an effective class is *indecomposable* if it is not the sum of two nonzero effective classes.

Note that on a surface X which is regular or kählerian, the cohomology class $[D]$ of an effective nonzero divisor D is never zero: in case X is regular, this is immediate from the fact that $\delta: H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z})$ is injective; if X is kählerian, let $\kappa \in H^{1,1}(X, \mathbf{R})$ be the cohomology class of a Kähler form (we call such a κ a *Kähler class*). Then $\langle \kappa, [D]\rangle = \frac{1}{\sqrt{2}} \operatorname{vol}(D) > 0$ and hence $[D] \neq 0$. So on such a surface an indecomposable class is always irreducible. The converse need not hold, but if C is an irreducible curve with $\langle [C], [C]\rangle < 0$, then $[C]$ is indecomposable. For if $[C] = [D]$, where D is an effective divisor, then $\langle [C], [D]\rangle = \langle [C], [C]\rangle < 0$ and C is a component of D. Hence $D - C$ is effective. Since $[D - C] = 0$, it follows the preceding discussion that $D = C$.

This applies in particular to nodal classes on K3 surfaces: and algebraic class c with $\langle c, c\rangle = -2$ is nodal if and only if it is indecomposable.

If X is a kählerian surface and $\kappa \in H^{1,1}(X, \mathbf{R})$ a Kähler class, then we have just seen that $\langle \kappa, [D]\rangle > 0$ for any nonzero effective divisor D. Also, κ has positive inner product with any other Kähler class. Since the cup product on $H^{1,1}(X, \mathbf{R})$ is of hyperbolic type, the set $\{x \in H^{1,1}(X, \mathbf{R}) : \langle x, x\rangle > 0\}$ consists of two disjoint cones. Only one of them contains Kähler classes; we call this component the *positive cone*, and denote it by C_X^+.

(1.6.) LEMMA: *The semi-group of effective classes of a kählerian K3 surface X is generated by the nodal classes and $\bar{C}_X^+ \cap H^2(X, \mathbf{Z})$.*

PROOF: by the discussion in (1.4), an irreducible class $c \in H^2(H, \mathbf{Z})$ is either nodal or satisfies $\langle c, c\rangle \geq 0$. In the latter case, c must lie on $\bar{C}_X^+$, since c has positive inner product with any Kähler class. Conversely, if c is a nonzero integral point of $\bar{C}_X^+$, then c is algebraic and by the Riemann-Roch inequality $(1.4)_1$, c or $-c$ is effective. Since c has positive inner product with any Kähler class, only c is effective.

(1.7.) In the remainder of this section, X denotes a kählerian $K3$ surface. We define the *Kähler cone* of X as the set of elements in the positive cone which have positive inner product with any nonzero effective class. Clearly, any Kähler class must lie in the Kähler cone. Observe that any element κ of the Kähler cone enables us to recover the semi-group of effective classes (and hence the Kähler cone itself). Indeed, using (1.6.) it is enough to observe that an algebraic class c with $\langle c, c\rangle \geq -2$ is effective if and only if $\langle c, \kappa\rangle > 0$. If κ happens to be integral, then κ is algebraic, and hence, by the Nakai criterion, κ is the class of an ample divisor. Conversely, the class of an ample divisor has positive inner product with nonzero effective class and must therefore belong to the Kähler cone. For future reference we sum up:

(1.8.) Let X and X' be projective K3 surfaces and let $\phi^*: H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$ be an isometry which respects the Hodge decomposition (such an isometry will be briefly called a Hodge isometry). The following statements are equivalent

(i) ϕ^* preserves effective classes
(ii) ϕ^* preserves ample classes
(iii) ϕ^* maps the Kähler cone of X' onto the Kähler cone of X
(iv) ϕ^* maps an element of the Kähler cone of X' to the Kähler cone of X.

A Hodge isometry which possesses one of the equivalent properties of (1.8) is called *effective*.

Any class $\delta \in H^2(X, \mathbf{Z})$ with $\langle \delta, \delta\rangle = -2$ determines an automorphism s_δ of $(H^2(X, \mathbf{Z}), \langle,\rangle)$ by $s_\delta(c) = c + \langle c, \delta\rangle\delta$. Note that $s_\delta(\delta) = -\delta$ and that s_δ is just the reflection orthogonal to δ. We are mainly interested in the case when s_δ preserves the Hodge decomposition. This is so if and only if δ is an algebraic class. We then refer to s_δ as a *Picard-Lefschetz reflection.*

(1.9.) Proposition: *The Picard-Lefschetz reflections of a kählerian $K3$ surface X leave the positive cone C_X^+ invariant and the group W_X generated by them acts on C_X^+ in a properly discontinuous fashion. The closure of the Kähler cone in C_X^+ is a fundamental domain for W_X in the sense that any W_X-orbit in C_X^+ meets it in precisely one point.*

PROOF: Let $H_\delta \subset H^{1,1}(X, \mathbf{R})$ denote the fixed point hyperplane of a Picard-Lefschetz reflection s_δ acting on $H^{1,1}(X, \mathbf{R})$ and since $\langle \delta, \delta \rangle = -2$, it follows that $\langle , \rangle$ has signature (1, 18) on H_δ. So H_δ meets C_X^+. Since s_δ preserves $\langle , \rangle$, this implies that s_δ leaves C_X^+ invariant.

The group of automorphisms of $(H^{1,1}(X, \mathbf{R}), \langle , \rangle)$ which preserve C_X^+ realize the set of half lines in C_X^+ as a Lobatchevski space and hence this group acts properly on C_X^+. Since W_X is a discrete subgroup of this automorphism group, the action of W_X on C_X^+ is properly discontinuous. The last assertion is a general fact about reflection groups operating on spaces of constant curvature [27].

§2. Unimodular lattices

The purpose of this section is to collect a few (more or less well-known) results about lattices, which will be used in the sequel. It is independent of the rest of the paper.

A *lattice* is a free finitely generated **Z**-module endowed with an integral symmetric bilinear form (which we usually denote by $\langle , \rangle$). A homomorphism $L \to M$ between two lattices is called a lattice-homomorphism if it preserves the bilinear forms. Thus the lattices form the objects of a category. A lattice-isomorphism in this category is also called an *isometry*; the lattices in question are then said to be *isometric.*

The *quadratic function* associated to a lattice L is the function $x \in L \mapsto \langle x, x \rangle$. If it takes only even values the lattice is called *even*, otherwise it is called *odd.* If the quadratic function is strictly positive (resp. negative) on $L - \{0\}$ then we say that the lattice is *positive* (resp. *negative*) definite. In either case, we call the lattice *definite*; a lattice which is not definite, is said to be *indefinite.* If p (resp. n) stands for the rank of a positive (resp. negative) sublattice of L of maximal rank, then the pair (p, n) is called the *signature* of L.

If $(e_1, \ldots, e_n)$ is an ordered basis of the lattice L, then the determinant of the matrix $(\langle e_i, e_j \rangle)$ is independent of the choice of this basis. Slightly deviating from general practice we call the absolute value of this determinant the *discriminant* of L and denote it by $\delta(l)$. If $\delta(L) \neq 0$ (resp. $= 1$), L is called *nondegenerate* (resp. *unimodular*).

Two important examples of even unimodular lattices are

(i) *The hyperbolic plane H.* As a **Z**-module, H is equal to $\mathbf{Z} \oplus \mathbf{Z}$. If e,f is the canonical basis of H, then $\langle e, e \rangle = \langle f, f \rangle = 0$, while $\langle e, f \rangle = 1$. The signature of H is (1, 1).

(ii) *The root lattice* E_8. As a $\mathbf{Z}$-module $E_8 = \mathbf{Z}^8$ and on the canonical basis $\langle , \rangle$ is the Cartan matrix of the root system E_8. This lattice is positive definite.

Indefinite unimodular lattices admit a nice and simple classification:

(2.1.) THEOREM: *(Serre* [21].*) An indefinite unimodular lattice is up to isometry determined by its signature and parity.*

(2.2.) EXAMPLE: An even unimodular lattice of signature (3,19) is isometric to $(\bigoplus^3 H) \oplus (\bigoplus^2 - E_8)$. (Here $\oplus$ denotes orthogonal direct sum. If L is any lattice, then λL has the same underlying $\mathbf{Z}$-module, but the bilinear form has been multiplied with λ).

A sublattice M of a lattice L is called *primitive* if L/M is torsion free. An equivalent condition is that any basis of M can be supplemented to a basis of L. If M is generated by one element m, then we call m primitive if M is. This means that relative some (or any) basis of L, the gcd of the coefficients of m equals one. It is also equivalent to the condition that there exists linear form on L taking the value *one* on m.

If L is a nondegenerate lattice, then we define its *dual module* as the set $L^* := \{x \in L_{\mathbf{Q}} : \langle x, y \rangle \in \mathbf{Z} \text{ for all } y \in L\}$. It contains L as a subgroup of finite index. The square of this index is equal to $\delta(L)$.

(2.3.) LEMMA: *Let L be a unimodular lattice and let $M \subset L$ be a primitive sublattice. Let $M^\perp \subset L$ denote the orthogonal complement of M in L. Then $\delta(M) = \delta(M^\perp)$. If moreover, M is unimodular, then $L = M \oplus M^\perp$.*

PROOF: If M is degenerate, we clearly have $\delta(M) = \delta(M^\perp)$. We therefore assume that M is nondegenerate. We define a natural group-isomorphism $M^*/M \to M^{\perp *}/M^\perp$ as follows. Let $x^* \in M^*$. Since L is unimodular, we can find a $z \in L$ such that $\langle x^*, x \rangle = \langle z, x \rangle$ for all $x \in M$. By definition, there is a unique $y^* \in M^{\perp *}$ such that $\langle y^*, y \rangle = \langle z, y \rangle$ for all $y \in M^\perp$. The assignment $x^* \mapsto y^*$ induces a homomorphism $M^*/M \to M^{\perp *}/M^\perp$ of which it is easy to see that it is independent of the choices involved. Clearly $M^\perp$ is also primitive. If we interchange the rôles of M and $M^\perp$, we find a two-sided inverse of this homomorphism. In particular, M^*/M and $M^{\perp *}/M^\perp$ have the same order and so $\delta(M) = \delta(M^\perp)$. Hence the index of $M \oplus M^\perp$ is equal to $\delta(M)^2$. It follows that if M is unimodular, $L = M \oplus M^\perp$.

(2.4.) THEOREM: *Let L be an even unimodular lattice which contains a sublattice isometric to a k-fold direct sum of hyperbolic planes. If Γ is an even lattice and*

(i) *$rk(\Gamma) \leq k$, then there exists a primitive embedding $i: \Gamma \to L$ (i.e. i is a lattice-monomorphism and $i(\Gamma)$ is primitive in L).*
(ii) *$rk(\Gamma) \leq k-1$, then for any two primitive embeddings $i, j: \Gamma \to L$ there exists an isometry ϕ of L such that $j = \phi \cdot i$.*

The proof of (i) is fairly simple: Let $e_1, f_1, \ldots, e_k, f_k \in L$ be such that $\langle e_\kappa, e_\lambda \rangle = \langle f_\kappa, f_\lambda \rangle = 0$ and $\langle e_\kappa, f_\lambda \rangle = \delta_{\kappa\lambda}$. If $c_1, \ldots, c_l$ $(l \leq k)$ is a basis of Γ, then we put

$$i(c_\kappa) := e_\kappa + \tfrac{1}{2}\langle c_\kappa, c_\kappa \rangle f_\kappa + \sum_{\lambda < \kappa} \langle c_\kappa, c_\lambda \rangle f_\lambda.$$

It is easily checked that i preserves the quadratic forms. Since the matrix $(\langle i(c_\kappa), f_\lambda \rangle)$ is the identity, i must be a primitive monomorphism. For the proof of (ii), we need a lemma.

(2.5.) LEMMA: *Let E be a lattice with basis e_1, f_1, e_2, f_2 such that $\langle e_\kappa, e_\lambda \rangle = \langle f_\kappa, f_\lambda \rangle = 0$ and $\langle e_\kappa, f_\lambda \rangle = \delta_{\kappa\lambda}$ (so $E \cong H \oplus H$). Then for any vector z in E, there is an isometry of E mapping z to an element of the form $\alpha e_1 + \beta f_1$ (we will briefly say that z is* isometric *to $\alpha e_1 + \beta f_1$) with $\alpha | \beta$.*

PROOF: We map E isomorphically onto $\mathrm{End}_2(\mathbf{Z})$ by

$$x^1 e_1 + y^1 f_1 + x^2 e_2 + y^2 f_2 \mapsto \begin{pmatrix} x^1 & x^2 \\ -y^2 & y^1 \end{pmatrix}$$

Then twice the determinant function pulls back to the quadratic function of E. So we may replace E by $\mathrm{End}_2(\mathbf{Z})$ endowed with the unique symmetric bilinear form whose quadratic function is twice the determinant. Clearly the right-left action of $SL_2(\mathbf{Z}) \times SL_2(\mathbf{Z})$ on $\mathrm{End}_2(\mathbf{Z})$ preserves this bilinear form. By the theorem on elementary divisors, each element of the orbit set $SL_2(\mathbf{Z})/\mathrm{End}_2(\mathbf{Z})SL_2(\mathbf{Z})$ is (uniquely) represented by a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ with $\alpha | \beta$. Whence the result.

If L is an even lattice and $e, f \in L$ are such that $\langle e, e \rangle = \langle e, f \rangle = 0$, then there is a unique isometry $\psi_{e,f}$ of L which on the orthogonal complement of e is equal to the 'unipotent translation' $x \mapsto x + \langle x, f\rangle e$. It is given by

$$x \mapsto x + \langle x, f\rangle e - \langle x, e\rangle f - \langle x, e\rangle f - \tfrac{1}{2}\langle f, f\rangle\langle x, e\rangle e.$$

We refer to $\psi_{e,f}$ as the *elementary transformation* associated to the pair (e, f).

We are now ready for the proof of (2.4)-ii.

STEP 1: Proof of (2.4)-ii in case $k = 2$.

Let $z \in L$ be a primitive vector. If $E \subset L$ denotes the lattice spanned by e_1, f_1, e_2, f_2, then it follows from (2.3) that we can write $z = z_1 + z_2$ with $z_1 \in E$ and $z_2 \in E^\perp$. Our first aim is to make z_1 primitive. Lemma (2.5) implies that there is a $\tilde{z}$ isometric to z (via an isometry acting trivially on $E^\perp$) with $\langle \tilde{z}, e_1\rangle | \langle \tilde{z}, f_1\rangle$ (interchange e_1 and f_1); we therefore assume that this is already satisfied by z. Since z is primitive there exists a $v \in L$ with $\langle v, z\rangle = 1$. Put $\tilde{v} := v - \langle v, e_1\rangle f_1$. Then $\langle \tilde{v}, e_1\rangle = 0$ and hence $\psi_{e_1,\tilde{v}}$ is defined. If $z' := \psi_{e_1,\tilde{v}}(z)$, then $\langle z', e_1\rangle = \langle z, e_1\rangle$ and

$$\begin{aligned}\langle z', f_1\rangle &= \langle z, f_1\rangle + \langle z, \tilde{v}\rangle - \langle z, e_1\rangle\langle \tilde{v}, f_1\rangle - \tfrac{1}{2}\langle \tilde{v}, \tilde{v}\rangle\langle z, e_1\rangle \\ &\equiv \langle z, \tilde{v}\rangle \bmod \langle z, e_1\rangle \\ &\equiv \langle z, v\rangle \bmod \langle z, e_1\rangle \\ &\equiv 1 \bmod \langle z', e_1\rangle\end{aligned}$$

So if we replace z by z', then we may (and do) assume that z_1 is primitive. If we apply lemma (2.3.) as before, then it follows that z is isometric to a vector of the form $e_1 + \beta f_1 + \zeta$ with $\zeta \in E^\perp$. Now $\psi_{f_1,\zeta}$ maps the latter to a vector of the form $e_1 + \gamma f_1$. The square length of $e_1 + \gamma f_1$ must equal $\langle z, z\rangle$, so $\gamma = \tfrac{1}{2}\langle z, z\rangle$ indeed.

STEP 2: Proof of (2.4)-ii in general.

We abbreviate $x_\kappa := i(c_\kappa)$, $y_\kappa := j(c_\kappa)$ and proceed by induction on k. Therefore we may assume that $rk(\Gamma) = k - 1$ (≥ 2) and that $y_\kappa = x_\kappa$ for $\kappa \leq k - 2$. We must find an isometry of L which leaves $x_1, \ldots, x_{k-2}$ fixed and maps y_{k-1} to x_{k-1}.

We begin as in step 1: Let $E \subset L$ denote the lattice generated by $e_{k-1}, f_{k-1}, e_k, f_k$. Then $L = E \oplus E^\perp$, by lemma (2.3.). Applying an automorphism of L (acting trivially on $E^\perp$), we may assume that $\langle y_{k-1}, e_{k-1}\rangle | \langle y_{k-1}, f_{k-1}\rangle$. Since the lattice spanned by $y_1, \ldots, y_{k-1}$ is primitive and L is unimodular, there exists a $v \in L$ orthogonal to

$y_1, \ldots, y_{k-2}$ and with $\langle v, y_{k-1}\rangle = 1$. If we put $\tilde{v} := v - \langle v, e_{k-1}\rangle f_{k-1}$, then the automorphism $\psi_{e_{k-1},\tilde{v}}$ leaves $y_1, \ldots, y_{k-2}$ pointwise fixed and it follows as in step 1 that y_{k-1} is mapped to an element whose orthogonal projection to E is primitive. We therefore suppose that this holds for y_{k-1} itself.

Let $M \subset L$ denote the lattice spanned by $e_1, f_1, \ldots, e_{k-2}, f_{k-2}$. Then $L = M \oplus M^\perp$ (by lemma (2.3.)) and if we write $y_{k-1} = y'_{k-1} + y''_{k-1}$ accordingly, the fact that $E \subset M^\perp$, implies that y''_{k-1} is primitive. Applying step 1 to y''_{k-1}, yields an isometry of L (acting trivially on $M^\perp$, hence on $y_1, \ldots, y_{k-2}$) which maps y_{k-1} to a vector of the form $y := \eta + e_{k-1} + \beta f_{k-1}$ with $\beta \in \mathbf{Z}$ and $\eta \in M$.

Since $\{x_1, \ldots, x_{k-2}, f_1, \ldots, f_{k-2}\}$ is a basis for the unimodular lattice M, there exists a $w \in M$ such that $\langle w, x_\kappa\rangle = 0$ $(\kappa = 1, \ldots, k-2)$ and $\langle w, f_\kappa\rangle = \langle y, f_\kappa\rangle$ $(\kappa = 1, \ldots, k-2)$. As $\langle f_{k-1}, w\rangle = 0$, the elementary transformation $\psi_{f_{k-1},w}$ is defined. It leaves $x_1, \ldots, x_{k-2}$ pointwise fixed and if we set $\tilde{y} := \psi_{f_{k-1},w}(y)$, then

$$\langle \tilde{y}, f_\kappa\rangle = \langle y, f_\kappa\rangle - \langle y, f_{k-1}\rangle\langle w, f_\kappa\rangle = 0 \quad (\kappa = 1, \ldots, k-2)$$
$$\langle \tilde{y}, x_\kappa\rangle = \langle y_{k-1}, x_\kappa\rangle = \langle x_{k-1}, x_\kappa\rangle \qquad (\kappa = 1, \ldots, k-2).$$

These equalities are also satisfied by x_{k-1}, so x_{k-1} and $\tilde{y}$ must have the same component in M. Since x_{k-1} and $\tilde{y}$ have the same length and both their $M^\perp$ component of the form $e_{k-1} + \alpha f_{k-1}$, we actually have $\tilde{y} = x_{k-1}$.

§3. The Picard group of a Kummer surface

This section introduces the notion of a Kummer surface. The main goal is to prove a (relatively weak) theorem of Torelli type for such surfaces. This will be derived from a corresponding result for a complex-analytic tori.

For the following lemma, we observe that if Γ is a free $\mathbf{Z}$-module of rank 4 which has been oriented by means of an isomorphism $\det: \Lambda^4\Gamma \to \mathbf{Z}$, then $\Lambda^2\Gamma$ is canonically endowed with a symmetric bilinear form $\langle , \rangle$ defined by $\langle u, v\rangle = \det(u \wedge v)$.

(3.1.) LEMMA: *Let Γ and Γ' oriented free $\mathbf{Z}$-modules of rank* 4 *and let $\phi: \Lambda^2\Gamma \to \Lambda^2\Gamma'$ be an isometry (with respect to the canonical bilinear forms defined above) for which there exists an isomorphism $\psi_2: \Gamma/2\Gamma \to \Gamma'/2\Gamma'$ such that $\psi_2 \wedge \psi_2$ equals the reduction of ϕ modulo* *two. Then there exists an isomorphism* $\psi : \Gamma \to \Gamma'$ *such that* $\phi = \pm\psi \wedge \psi$.

PROOF: Let $\mathbf{k}$ be any field and put $\Gamma_{\mathbf{k}} = \Gamma \oplus_{\mathbf{Z}} \mathbf{k}$, $\phi_{\mathbf{k}} = \phi \oplus id_{\mathbf{k}}$ etc. An isotropic line in $\Lambda^2\Gamma_{\mathbf{k}}$ is always of the form $a \wedge b$, where a and b are independent elements of Γ. This sets up a bijective correspondence between the isotropic lines in $\Lambda^2\Gamma_{\mathbf{k}}$ and the planes in $\Gamma_{\mathbf{k}}$. Two isotropic lines in $\Lambda^2\Gamma_{\mathbf{k}}$ span an isotropic subspace if and only if the corresponding planes in $\Gamma_{\mathbf{k}}$ have at least a line in common. So $\phi_{\mathbf{k}}$ can be understood as a mapping from the set of 2-planes in $\Gamma_{\mathbf{k}}$, to the set of 2-planes in $\Gamma'_{\mathbf{k}}$ which preserves the incidence relation. In particular, $\phi_{\mathbf{k}}$ sends a pencil of planes to a pencil of planes. The pencils of planes in a 4-dimensional vector space come in two types: those containing a fixed line and those lying in a fixed hyperplane. We claim that ϕ preserves the type of a pencil. For it suffices to verify that its reduction modulo two does so and this follows immediately from our hypothesis that $\phi_{\mathbf{F}_2}$ is induced by a linear isomorphism $\Gamma_{\mathbf{F}_2} \to \Gamma'_{\mathbf{F}_2}$.

In particular $\phi_{\mathbf{Q}}$ maps a pencil through a line to a pencil through a line and thus determines a map between the projective spaces associated to $\Gamma_{\mathbf{Q}}$ and $\Gamma'_{\mathbf{Q}}$. This map preserves the incidence relation (i.e. respects projective lines) and is by the fundamental theorem of projective geometry then projective. Hence there exists an isomorphism $\psi : \Gamma_Q \to \Gamma'_Q$ such that $\lambda\phi = \psi \wedge \psi$ for some $\lambda \in \mathbf{Q}$. After multiplying by a scalar we may (and do) assume that $\Psi(\Gamma) \subset \Gamma'$ and that $\psi(\Gamma)$ contains a primitive vector of Γ'.

Choose $e_1 \in \Gamma$ such that $\psi(e_1)$ is primitive. Then e_1 is also primitive and hence can be supplemented to a basis e_1, e_2, e_3, e_4 of Γ. Then $\psi(e_1) \wedge \psi(e_i) = \lambda\phi(e_1 \wedge e_i)$ for $i = 2, 3, 4$. As ϕ is an isometry, $\phi(e_1 \wedge e_i)$ is primitive and hence $\lambda \in \mathbf{Z}$. By adding a suitable integral multiple of e_1 to e_i, we may suppose that $\psi(e_i) \in \lambda\Gamma'$. Then $\lambda\phi(e_2 \wedge e_3) = \psi(e_2) \wedge \psi(e_3) \in \lambda^2\Gamma$. Since $\phi(e_2 \wedge e_3)$ is primitive, it follows that $\lambda = \pm 1$.

(3.2.) COROLLARY: *Let A and A' be two-dimensional complex-analytic tori and let* $\phi^* : H^2(A', \mathbf{Z}) \to H^2(A, \mathbf{Z})$ *be an isometry which preserves the Hodge decomposition. Suppose there exists an isomorphism* $\psi_2^* : H^1(A', \mathbf{F}_2) \to H^1(A, \mathbf{F}_2)$ *such that* $\psi_2^* \wedge \psi_2^*$ *equals the reduction of ϕ modulo two. Then* $\pm\phi^*$ *is induced by an isomorphism* $\phi : A \to A'$.

PROOF: We put $\Gamma := H_1(A, \mathbf{Z})$ and $\Gamma' := H_1(A', \mathbf{Z})$ and identify $H^2(A, \mathbf{Z})$ with $\Lambda^2\mathrm{Hom}(\Gamma, \mathbf{Z})$ and similarly $H^2(A', \mathbf{Z})$ with

$\Lambda^2\mathrm{Hom}(\Gamma', \mathbf{Z})$. By the previous lemma there exists an isomorphism $\psi: \Gamma \to \Gamma'$ such that $\pm\phi^* = (\psi \wedge \psi)^*$. Now, choose a nonzero $\omega' \in H^{2,0}(A', \mathbf{C})$. Since $\omega' \wedge \omega' = 0$, there exist $\omega_1', \omega_2' \in \mathrm{Hom}(\Gamma', \mathbf{C})$ such that $\omega' = \omega_1' \wedge \omega_2'$. Let Ω' denote the map $\Gamma' \to \mathbf{C}^2$ whose co-ordinates are ω_1' and ω_2', and put $\Omega := \Omega \circ \psi: \Gamma \to \mathbf{C}^2$. Then A' may be identified with $\mathbf{C}^2/\Omega'(\Gamma')$ and as $\psi^*(\omega_1') \wedge \psi^*(\omega_2') = \pm\phi^*(\omega') \in H^{2,0}(A, \mathbf{C})$ we also have an identification of A with $\mathbf{C}^2/\Omega(\Gamma)$. Clearly, ψ induces an isomorphism from $\mathbf{C}^2/\Omega(\Gamma)$ onto $\mathbf{C}^2/\Omega'(\Gamma')$ and hence also one from A to A'. This isomorphism induces ψ on $H_1(-, \mathbf{Z})$ and $\pm\phi^*$ on $H^2(-, \mathbf{Z})$.

(3.3.) Let A be a complex-analytic torus of dimension two and let $i: A \to A$ be an involution which induces minus the identity on $H_1(A, \mathbf{Z})$. We show that i has 16 fixed points and that i is "minus the identity" with respect to any of its fixed points. Since we have a canonical isomorphism $H_p(A, \mathbf{Z}) \cong \Lambda^p H_1(A, \mathbf{Z})$, i induces $(-id)^p$ on $H_p(A, \mathbf{Z})$ and so the Lefschetz number of i equals $\Sigma_{p=0}^4 \dim H_p(A) = 16$. In particular, i has a fixed point $a \in A$. Let $i_a: A \to A$ denote "minus the identity" with respect to a. We claim that $i = i_a$. As the composite $i_a i$ acts trivially on $H_1(A, \mathbf{Z})$ and leaves a fixed, $i_a i$ acts trivially on the Albanese of (A, a). Since the Albanese of (A, a) is canonically isomorphic to (A, a), $i_a i$ must be the identity.

It is now clear that the fixed point set V of i consists of 16 distinct points and is in a natural way an affine space over $\mathbf{F}_2$ of dimension 4. The translation group of V is canonically isomorphic to $H_1(A, \mathbf{F}_2)$.

Let $p: \tilde{A} \to A$ be the map which blows up each point of V. Clearly, the involution i lifts to an involution $\tilde{i}$ of $\tilde{A}$. We denote by $X := \tilde{A}/\tilde{i}$ the orbit space of $\tilde{i}$ and let $\rho: \tilde{A} \to X$ be the canonical map. As i acts as $-id$ on the tangent space at each of its fixed points, the fixed point locus of $\tilde{i}$ is just the exceptional divisor $p^{-1}(V)$ of p. This implies that X is a non-singular surface, for $p^{-1}(V)$ is everywhere of codimension one in $\tilde{A}$. The surfaces thus obtained are called *Kummer surfaces.*

Each $v \in V$ determines a nodal curve $\rho(p^{-1}(v))$ on X; we denote its class in $H^2(X, \mathbf{Z})$ by e_v and put $\mathcal{V} := \{e_v : v \in V\}$. These nodal classes are mutually perpendicular, so the lattice they generate in $H^2(X, \mathbf{Z})$ may be identified with $\mathbf{Z}^{\mathcal{V}}$ (or $\mathbf{Z}^V$).

(3.4.) PROPOSITION: *The Kummer surface X is a K3 surface. There is a natural monomorphism $\alpha: H^2(A, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ which maps onto a submodule of finite index of the orthogonal complement of $\mathbf{Z}^{\mathcal{V}}$ and multiplies the intersection form with two. Moreover, $\alpha_{\mathbf{C}}$ maps $H^{2,0}(A, \mathbf{C})$ isomorphically onto $H^{2,0}(X, \mathbf{C})$.*

PROOF: Since V is of real codimension 4 in A, the inclusion $A - V \subset A$ induces an isomorphism on $H_j(-, \mathbf{Z})$ for $j = 1, 2$. Consider for $j = 1, 2$ the composite

$$\beta_j : H_j(A) \xleftarrow{\approx} H_j(A - V) \xleftarrow{\approx} H_j(\tilde{A} - p^{-1}(V)) \xrightarrow{\gamma_j} H_j(\tilde{A}) \xrightarrow{\rho_j} H_j(X)$$

(with integral coefficients). As $p^{-1}(V)$ is of real codimension 2 in $\tilde{A}$, γ_1 is an epimorphism. In particular, the i-invariant part of $H_1(A, \mathbf{Q})$ (which is trivial, of course) maps surjectively to the $\tilde{i}$-invariant part of $H_1(\tilde{A}, \mathbf{Q})$ (which is therefore also trivial). As ρ_1 maps the latter isomorphically onto $H_1(X, \mathbf{Q})$, it follows that $H_1(X, \mathbf{Q}) = 0$, in particular that X is regular.

Next we verify that X has trivial canonical class. Let ω be a nowhere vanishing homomorphic 2-form on A. Then the divisor of $p^*(\omega)$ coincides with the reduced exceptional divisor $p^{-1}(V)$ of p. Since ω is invariant under i, $p^*(\omega)$ is invariant under $\tilde{i}$ and hence drops to a nowhere vanishing holomorphic 2-form ω_X on X. So X is a K3 surface.

We define $\alpha : H^2(A, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ as the Poincaré dual of β_2. Since ρ is of degree 2, β_2 (and hence α) multiplies the intersection form with 2. In particular, α is a monomorphism and its image is a nondegenerate sublattice of $H^2(X, \mathbf{Z})$. It is geometrically clear that the dual of β_2, $\beta^* : H^2(X) \to H^2(A)$ maps $\mathcal{V}$ to zero. Therefore, α maps into the orthogonal complement of $\mathbf{Z}^{\mathcal{V}}$.

Since $rk\mathbf{Z}^{\mathcal{V}} + rkH^2(A, \mathbf{Z}) = 16 + 6 = rkH^2(X, \mathbf{Z})$, it follows that $H^2(X, \mathbf{Q})$ decomposes isometrically into a direct sum $\mathrm{Im}(\alpha_{\mathbf{Q}}) \oplus \mathbf{Q}^{\mathcal{V}}$. So $\mathrm{Im}(\alpha)$ is of finite index in the orthogonal complement of $\mathbf{Z}^{\mathcal{V}}$. Because $\mathcal{V}$ is contained in the algebraic lattice, it also follows that $H^{2,0}(X, \mathbf{C}) \subset \mathrm{Im}(\alpha_{\mathbf{C}})$. From the definition of ω_X it is clear that $\beta^*_{2,\mathbf{C}}$ maps the class of ω_X to the class of ω and hence $H^{2,0}(X, \mathbf{C})$ to $H^{2,0}(A, \mathbf{C})$. It follows that $\alpha_{\mathbf{C}}$ maps $H^{2,0}(A, \mathbf{C})$ isomorphically onto $H^{2,0}(X, \mathbf{C})$.

We shall investigate the position of $\mathcal{V}$ in $H^2(X, \mathbf{Z})$ in more detail. An important result will be that the pair $(H^2(X, \mathbf{Z}), \mathcal{V})$ enables us to recover $H^1(A, \mathbf{F}_2)$.

(3.5.) PROPOSITION: *With the above notations, the orthogonal complement of* $\mathbf{Z}^{\mathcal{V}}$ *is precisely the image of* α.

At the same time we will prove

(3.6.) PROPOSITION: *If E denotes the (unique) primitive sublattice of* $H^2(X, \mathbf{Z})$ *which contains* $\mathbf{Z}^{\mathcal{V}}$ *as a submodule of finite index, then*

$E \subset E^* \subset \frac{1}{2}\mathbf{Z}^V$ and the image of E resp. E^* in $\frac{1}{2}\mathbf{Z}^V/\mathbf{Z}^V \cong F_2^V \cong \mathbf{F}_2^V$ consists of the space of polynomial functions $V \to \mathbf{F}_2$ of degree ≤ 1 resp. ≤ 2.

For their proof we need two lemmas.

(3.7.) LEMMA: *Let $\tau: V \to V$ be an affine transformation. Then there exists an isometry $\sigma: H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ such that $\sigma(e_v) = e_{\tau(v)}$ $(v \in V)$.*

PROOF: There is real affine-linear transformation t of A which is orientation preserving and whose restriction to V is just τ. Then t commutes with the involution i. By perturbing t in a small neighbourhood of V (with a possible loss of its affine-linear character) we can arrange that t is an i-equivariant homeomorphism which is complex-analytic near V. Then t lifts to an $\tilde{i}$-equivariant homeomorphism of $\tilde{A}$ which maps $p^{-1}(v)$ in an orientation-preserving manner to $p^{-1}(\tau(v))$. This homeomorphism drops to a selfhomeomorphism of X whose action on $H^2(X, \mathbf{Z})$ is as required.

(3.8.) LEMMA: *Let V be an affine space of finite dimension ≥ 2 over $\mathbf{F}_2$ and let $\mathscr{E}$ be a linear subspace of the space $\mathbf{F}_2^V$ of $\mathbf{F}_2$-valued functions on V. Suppose that $\mathscr{E}$ is invariant under the group of affine transformations of V. Then either $\mathscr{E}$ consists of the constant functions (the polynomial functions of degree 0) or $\mathscr{E}$ consists of the affine-linear functions (the polynomial functions of degree ≤ 1) or else $\mathscr{E}$ contains for any affine subspace $W \subset V$ of codimension two its characteristic function χ_W (these generate the polynomial functions of degree ≤ 2).*

PROOF: If $\mathscr{E}$ contains a non-constant affine-linear function then $\mathscr{E}$ contains all of them, as they form an orbit for the affine group. As the constant function 1 is the sum of two non-constant affine-linear ones, it follows that then $\mathscr{E}$ contains all affine-linear functions.

Now suppose that $\mathscr{E}$ contains a function ϕ which is not affine-linear. We prove by induction on $n := \dim V \geq 2$ that $\mathscr{E}$ then contains the characteristic functions of all codimension two subspaces. As the case $n = 2$ is trivial, we assume that $n \geq 3$. Choose an affine hyperplane $V' \subset V$ such that $\phi | V'$ is not affine-linear, and apply the induction hypothesis to $\mathscr{E}' := \{f | V' : f \in \mathscr{E}\}$. It follows that there exists a $\phi' \in \mathscr{E}$ such that $\phi' | V'$ is the characteristic function of a $(n-3)$-dimensional subspace W of V'. Choose $a \in W$ and $b \in V' - V$. Since

$V - V'$ is an affine hyperplane of V (we are working over $\mathbf{F}_2$!), there exists a (unique) affine transformation τ of V which leaves $V - V'$ pointwise fixed and maps a to b. Then τ induces a translation over $b - a$ in V' which carries W to an $(n-3)$-dimensional subspace $\tau(W)$ of V' which is disjoint with W. So $\psi := \phi' \circ \tau - f' \in \mathscr{E}$ is the characteristic function of $W \cup \tau(W)$. As $W \cup \tau(W)$ is the $(n-2)$-dimensional subspace of V generated by W and b, $\mathscr{E}$ contains at least the characteristic function of one codimension two subspace of V. The affine group acts transitively on these functions, so the lemma follows.

Remark: Using elementary representation theory, one can prove that any Aut(V)-invariant subspace of $\mathbf{F}_2^V$ is the space $\mathscr{E}_k$ of polynomial functions of degree k for some $k \in \{0, 1, \ldots, \dim V\}$. The quotient $\mathscr{E}_k/\mathscr{E}_{k-1}$ is naturally isomorphic to $\Lambda^k(\mathscr{E}_1/\mathscr{E}_0)$.

We are now ready for the proofs of (3.5.) and (3.6.). For notational convenience, we put $\Gamma := H^2(X, \mathbf{Z})$ and denote by $E^\perp$ the orthogonal complement of E or equivalently $\mathbf{Z}^\mathscr{V}$ in $H^2(X, \mathbf{Z})$. By (3.4.), $E^\perp$ contains the image of α as a submodule of finite index.

Step 1: We have inclusions $E \subset E^* \subset \frac{1}{2}\mathbf{Z}^\mathscr{V}$ and $\delta(E) \leq 2^6$ (see §2 for the definition of * and δ).

Proof: The inclusions are immediate from the fact that $\mathbf{Z}^\mathscr{V} \subset E$ and hence $E \subset E^* \subset (\mathbf{Z}^\mathscr{V})^* = \frac{1}{2}\mathbf{Z}^\mathscr{V}$. As α multiplies the intersection form with 2, we have $2^6 = \delta(\mathrm{Im}(\alpha)) \geq \delta(E^\perp) = \delta(E)$ by(2.3.).

Step 2: The image $\mathscr{E}$ of E in $\frac{1}{2}\mathbf{Z}^\mathscr{V}/\mathbf{Z}^\mathscr{V} \cong \mathbf{F}_2^V$ consists of the affine-linear functions and $\delta(E) = 2^6$. Furthermore, $\mathrm{Im}(\alpha) = E^\perp$.

Proof: It is clear that the power of $\mathscr{E}$ is just the index of E in $\mathbf{Z}^\mathscr{V}$. So $|\mathscr{E}|^2 = \delta(\mathbf{Z}^\mathscr{V})\delta(E)^{-1} \geq 2^{16} \cdot 2^{-6} = 2^{10}$ by step 1. By lemma (3.7.) $\mathscr{E}$ is invariant under the affine group. If $\mathscr{E}$ contains a function which is not affine-linear, then according to (3.8.), $\mathscr{E}$ contains the characteristic functions of all codimension two subspaces of V. Choose two of those subspaces W, W' such that $|W \cap W'| = 1$. So $\Sigma_{v \in W} \frac{1}{2} e_v$ and $\Sigma_{v \in W'} e_v$ both belong to E. But their inner product equals $\frac{1}{2} \notin \mathbf{Z}$, which is a contradiction. So by (3.8.) again, either $\mathscr{E}$ consists of the affine linear functions or $\mathscr{E}$ consists of the two constant functions. The latter is excluded since $|\mathscr{E}| \geq 2^5$.

The space of affine-linear functions on V is of dimension 5, so

$|\mathscr{E}| = 2^5$. Hence $\delta(E) = \delta(Z^{\mathscr{V}}) \cdot |\mathscr{E}|^{-2} = 2^{16} \cdot 2^{-10} = 2^6$. From this, it follows that $|E^\perp/\mathrm{Im}(\alpha)| = \delta(E^\perp)/\delta(\mathrm{Im}(\alpha)) = \delta(E)\, 2^{-6} = 1$, i.e. $E^\perp = \mathrm{Im}(\alpha)$.

STEP 3: The image $\mathscr{E}^*$ of E in $\mathbf{F}_2^{\mathscr{V}}$ consists of the space of polynomial functions of degree ≤ 2.

PROOF: We have by the previous step,

$$|\mathscr{E}^*| = |\mathscr{E}| \cdot |\mathscr{E}^*/\mathscr{E}| = |\mathscr{E}|\delta(E) = 2^5 \cdot 2^6 = 2^{11}.$$

In particular $\mathscr{E}^* \underset{\neq}{\supset} \mathscr{E}$ and so $\mathscr{E}^*$ contains by lemma (3.8.) the space of polynomial functions of degree ≤ 2. As the latter has cardinality 2^{11}, it follows that we actually have a quality.

(3.9.) COROLLARY (to (3.5.) and (3.6.)): *Let L be a lattice and $\mathscr{V} \subset L$ a subset such that there exists a Kummer surface X and an isometry of L onto $H^2(X, \mathbf{Z})$ which maps $\mathscr{V}$ onto the set of 16 distinguished nodal classes arising from the construction of X as a Kummer surface. Let $E \subset L$ denote the smallest primitive sublattice which contains $\mathscr{V}$. Then $E \subset \frac{1}{2}\mathbf{Z}^{\mathscr{V}}$ and $\mathscr{V}$ admits in a unique way the structure of an affine space over $\mathbf{F}_2$ such that the image $\mathscr{E}$ of E in $\frac{1}{2}\mathbf{Z}^{\mathscr{V}}/\mathbf{Z}^{\mathscr{V}} \cong \mathbf{F}_2^{\mathscr{V}}$ is the set of affine-linear functions relative this affine structure. Moreover, if T denotes the vector space of translations of $\mathscr{V}$, then we have a canonical isomorphism between $\Lambda^2(\mathrm{Hom}(T, \mathbf{F}_2)$ and $E^\perp/2E^\perp$.*

PROOF: As the first clause is immediate from (3.6.), let us prove the last one. It follows from (3.5.) that $(E^\perp)^* = \frac{1}{2}E^\perp$. Hence $E^\perp/2E^\perp \cong \frac{1}{2}E^\perp/E^\perp = (E^\perp)^*/E^\perp \cong E^*/E$. Now by (3.6.), E^* is contained in $\frac{1}{2}\mathbf{Z}^{\mathscr{V}}$ and the image of E^* in $\mathbf{F}_2^{\mathscr{V}}$ consists of the space $\mathscr{E}^*$ of polynomial functions of degree ≤ 2 on $\mathscr{V}$. So $E^*/E \cong \mathscr{E}^*/\mathscr{E}$. It is not hard to see that natural isomorphism from $\mathscr{E}^*/\mathscr{E}$ to $\Lambda^2\mathrm{Hom}(T, \mathbf{F}_2)$ is given by assigning to $f \in \mathscr{E}^*$ the symplectic form $s, t \in T \to f(v+s+t) + f(v) - f(v+s) - f(v+t)$ where $v \in V$ is arbitrary.

After all these preparations we are now able to prove an (albeit weak) result of Torelli type for Kummer surfaces.

(3.10.) PROPOSITION: *Let A (resp. A') be a two-dimensional complex torus with involution i (resp. i') acting as $-id$ on the first homology group. Let X (resp. X') denote the corresponding Kummer surface and let $\mathscr{V} \subset H^2(X, \mathbf{Z})$ (resp. $\mathscr{V}' \subset H^2(X', \mathbf{Z})$) be the associated* *set of* 16 *distinguished nodal classes. Suppose that A' has a nonzero algebraic class.*

Then any isometry $\Phi^*: H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$ which respects the Hodge decomposition, maps $\mathcal{V}'$ to $\mathcal{V}$ and positive classes to positive classes, is induced by an isomorphism $\Phi: X \to X'$.

PROOF: Let $\alpha: H^2(A, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ and $\alpha': H^2(A', \mathbf{Z}) \to H^2(X', \mathbf{Z})$ be the canonical monomorphism introduced in (3.4.). By (3.5.) α (resp. α') multiplies the intersection with two and its image is just the orthogonal complement of $\mathcal{V}$ (resp. $\mathcal{V}'$). Hence Φ^* determines an isometry $\phi^*: H^2(A', \mathbf{Z}) \to H^2(A, \mathbf{Z})$..The last clause of (3.4.) implies that ϕ^* respects the Hodge decomposition.

Following (3.9.) $\mathcal{V}$ (resp. $\mathcal{V}'$) is in a canonical way an affine space over $\mathbf{F}_2$ and if we denote by T (resp. T') its vector space of translations we have natural isomorphisms

$$H^2(A, \mathbf{F}_2) \cong \mathrm{Im}(\alpha)/2\mathrm{Im}(\alpha) \cong \Lambda^2\mathrm{Hom}(T, \mathbf{F}_2)$$
$$(\text{resp. } H^2(A', \mathbf{F}_2) \cong \Lambda^2\mathrm{Hom}(T', \mathbf{F}_2)).$$

Hence Φ^* restricts to an affine-linear isomorphism $\mathcal{V}' \to \mathcal{V}$. This isomorphism induces an isomorphism $\chi: T' \to T$ and the functorial character of the above isomorphisms guarantees that $\chi^* \wedge \chi^*: \Lambda^2\mathrm{Hom}(T, \mathbf{F}_2) \to \Lambda^2\mathrm{Hom}(T', \mathbf{F}_2)$ is just the reduction of $(\phi^*)^{-1}$ modulo two. If we identify T with $H_1(A, \mathbf{F}_2)$ and T' with $H_1(A', \mathbf{F}_2)$, then it follows from (3.2.) (with $\psi_2 = (\chi^*)^{-1}$ that ϕ^* or $-\phi^*$ is induced by an isomorphism $\phi: A \to A'$. Since Φ^* takes positive classes to positive classes, so does ϕ^*. As A' possesses nonzero algebraic classes, it follows that ϕ^* is induced by ϕ. By composing ϕ with a translation we may suppose that $\phi \circ i = i' \circ \phi$ and that for at least one $v' \in \mathcal{V}'$ we have $\Phi^*(e_{v'}) = e_{\phi^{-1}(v')}$. Then ϕ determines an isomorphism $\Phi: X \to X'$ whose induced map Φ^{**} on $H^2(\cdot, \mathbf{Z})$ coincides with Φ^* on $\mathrm{Im}(\alpha')$ and $e_{v'}$. Clearly, Φ^{**} restricts to an affine-linear isomorphism $\mathcal{V}' \to \mathcal{V}$. The induced map $\Lambda^2\mathrm{Hom}(T, \mathbf{F}_2) \to \Lambda^2\mathrm{Hom}(T, \mathbf{F}_2)$ coincides with the one induced by Φ^*, so Φ^* and Φ^{**} differ on $\mathcal{V}'$ by a translation at most. As $\Phi^*(e_{e'}) = \Phi^{**}(e_{v'})$, it then follows that Φ^* and Φ^{**} coincide on $\mathcal{V}'$ and hence on all of $H^2(X', \mathbf{Z})$.

§4. The Torelli theorem for Kummer surfaces

The weakness of (3.10.) lies in the fact that one of its assumptions, namely that the set 16 distinguished nodal classes be preserved, is far too strong. In this section we shall show how to remove this hypothesis for projective Kummer surfaces. In contrast to [20] we need not work with special Kummer surfaces.

(4.1.) LEMMA: *Let X be a $K3$ surface which contains* 16 *disjoint nodal curves $C_1, \ldots, C_{16}$ such that $\Sigma_{i=1}^{16} C$ is* 2*-divisible in* Pic(X). *Then X has the structure of a Kummer surface with $C_1, \ldots, C_{16}$ as its set of distinguished nodal curves.*

PROOF: Since $\Sigma_{i=1}^{16} C_i$ is 2-divisible in Pic(X), there exists a double covering $\rho : \tilde{X} \to X$ whose branch locus is the union of the C_i's. If we put $\tilde{C}_i := \rho^{-1}(C_i)$, then $\tilde{C}_i$ is an exceptional curve of the first kind and $\Sigma_{i=1}^{16} \tilde{C}_i$ is a canonical divisor of $\tilde{X}$. So if we contract each of the $\tilde{C}_i$'s, we get a smooth surface A with trivial canonical bundle. Clearly, the involution $\tilde{i}$ of $\tilde{X}$, which interchanges the sheets of $\tilde{X}$ drops to an involution i of A. The euler number $c_2(A)$ equals $c_2(\tilde{X}) - 16 = 2(c_2(X) - 16) - 16 = 0$. Then A is a complex torus by [12, §6]. Since the i-invariant part of $H^1(A, \mathbf{Q})$ is canonically isomorphic to $H^1(X, \mathbf{Q}) = \{0\}$, i acts as $-id$ on $H^1(A, \mathbf{Q})$ and so the lemma follows.

(4.2.) TORELLI THEOREM FOR PROJECTIVE KUMMER SURFACES: *Let X be any $K3$ surface and let X' be a projective Kummer surface. Suppose we are given a Hodge-isometry $\phi^* : H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$, which preserves the effective classes. Then ϕ^* is induced by an isomorphism $\phi : X \to X'$.*

PROOF: Let $\{c_1', \ldots, c_{16}'\}$ be the set of distinguished nodal classes on X'. Since ϕ^* preserves effective classes, it also preserves indecomposable classes. Hence, by the discussion in (1.4.), ϕ preserves the nodal classes. Put $c_i = \phi^*(c_i')$ and let C_i denote the unique nodal curve representing c_i. Since $\Sigma_{i=1}^{16} C_i$ is two-divisible (for $\Sigma_{i=1}^{16} C_i'$ is), lemma (4.1.) applies and we may conclude that X is a Kummer surface having $\{C_1, \ldots, C_{16}\}$ as its set of distinguished nodal curves. Since X' is projective, it contains a nonzero effective class orthogonal to $\{C_1', \ldots, C_{16}'\}$. Such a class corresponds to a nonzero effective class of the torus from which X' has been obtained and hence all the hypotheses of (3.10.) are satisfied. The theorem follows.

(4.3.) COROLLARY (*weak form of the Torelli theorem for projective Kummer surfaces*): *Let X be a $K3$ surface and let X' be a projective Kummer surface. Suppose that there exists a Hodge-isometry from $H^2(X', \mathbf{Z})$ onto $H^2(X, \mathbf{Z})$. Then X and X' are isomorphic.*

PROOF: It follows from the assumptions that $S_{X'}$ contains elements with positive length, so X' must be projective by Kodaira's result [12, Thm. 8]. By (1.9.) we may compose the Hodge isometry with an element in $\{\pm id\}$. W_X such that the resulting Hodge isometry ϕ^* maps the Kähler cone of X' onto the Kähler cone of X. Following (1.8,) this means that ϕ^* preserves effective classes. So X and X′ are isomorphic, by (4.2.).

§5. The local Torelli theorem for K3-surfaces

(5.1.) In order to prove the local Torelli theorem for K3-surfaces, it is necessary to recall some of the general theory of deformations.

A *deformation* of a compact connected analytic manifold X_0 consists of a cartesian diagram of (germs of) analytic spaces

$$\begin{array}{ccc} X_0 & \overset{i}{\hookrightarrow} & (\mathscr{X}, x_{s_0}) \\ \downarrow & \square & \downarrow p \\ \{s_0\} & \subset & (S, s_0) \end{array}$$

where (some representative of) p is a proper and flat morphism. When no confusion can arise, we identify X_0 with its image under i and then speak of $p:(\mathscr{X}, X_0)\to(S, s_0)$ as a deformation of X_0.

The deformation is called *smooth* when (S, s_0) (and hence $(\mathscr{X}, X_0)$) is. The deformations of a fixed X_0 actually form the objects of a category whose morphisms are pairs of analytic map-germs (Ψ, ψ) such that the diagram below commutes

$$\begin{array}{ccc} & X_0 & \\ {}^{i'}\swarrow & & \searrow{}^{i} \\ (\mathscr{X}', X'_{s_0}) & \xrightarrow{\Psi} & (\mathscr{X}, X_{s_0}) \\ \downarrow p' & & \downarrow p \\ (S', s'_0) & \xrightarrow{\Psi} & (S, s_0) \end{array}$$

and the square is cartesian. A deformation p of X_0 is called *complete* if for any other deformation p' of X_0 there exists a morphism from p' to p. Notice that we do not insist that this morphism be unique. If this happens to be the case, then we call the deformation *universal.* A universal deformation is a final object of our category and hence two such are canonically isomorphic.

Kuranishi [15] has shown that complete deformations exist. In order to state his result in a more precise manner we need another notion.

If $p:(\mathscr{X}, x_0)\to(S, s_0)$ is deformation of X_0, then over X_0 we have an exact sequence of tangent sheaves

$$0\to\Theta_{X_0}\to\Theta_{\mathscr{X}|X_0}\to T_{S,s_0}(\mathcal{O}_{X_0})\to 0,$$

where the last term should be understood as the sheaf of local sections in the (trivial) pull-back of the Zariski-tangent space T_{S,s_0} under $p|X_0:X_0\to\{s_0\}$. In the associated cohomology sequence we find a differential

$$\rho_p: T_{S,s_0}=H^0(X_0, T_{S,s_0}(\mathcal{O}_{X_0}))\to H^1(X_0, \Theta_{X_0})$$

which in a sense measures up to first order how the complex structure on the underlying $\mathscr{C}^\infty$-manifold is deformed. This homomorphism is called the *Kodaira-Spencer map*. Note that the Kodaira-Spencer map is compatible with base-change, i.e. if (Ψ, ψ) is a morphism from p' to p, then $\rho_{p'}=\rho_p\circ d\psi(s_0')$. According to Kuranishi [15], complete deformations exist (and if $H^2(X_0, \Theta_{X_0})=0$, then there even exists a smooth deformation) for which the Kodaira-Spencer map is an isomorphism (such a deformation is necessarily complete). Note that this last criterion is met by K3-surfaces, for Serre duality implies $h^2(X_0, \Theta_{X_0})=h^0(X_0, \Omega^1_{X_0})=0$.

(5.2.) Before we are going to discuss the Torelli theorem for K3-surfaces, it is convenient to introduce the notion of a marked deformation. Fix once and for all a lattice L which is isometric the second cohomology lattice of a K3-surface. If $p:\mathscr{X}\to S$ is a family of K3-surfaces, then a *marking* of p is an isomorphism of $R^2p_*(\mathbf{Z})$ onto the constant local system L over S which induces over each base point an isometry. Clearly, the family $p:\mathscr{X}\to S$ admits a marking if and only if the local system $R^2p_*(\mathbf{Z})$ is trivial. This is for instance the case when S is simply connected. Likewise, one defines the notion of a marked deformation of a K3-surface. Such a marking is entirely determined by a marking of the surface X_0 which is subject to deformation.

(5.3.) If X_0 is a K3-surface and $\omega\in H^{2,0}(X_0, C)$ a generator, then by elementary Hodge-de Rham theory, we have $\langle\omega,\omega\rangle=0$ and $\langle\omega,\bar{\omega}\rangle>0$.

This leads us to introduce the period space

$$\Omega := \{\omega \in L_{\mathbf{C}} : \langle \omega, \omega \rangle = 0, \quad \langle \omega, \bar{\omega} \rangle > 0\}/\mathbf{C} \subset \mathbf{P}(L_C).$$

Notice, that Ω is an open subset of a nonsingular quadric in $\mathbf{P}(L_{\mathbf{C}})$. In particular, Ω is an analytic manifold of dimension 20. Observe also, that Ω is an orbit in $\mathbf{P}(L_{\mathbf{C}})$ of the Lie group $\mathrm{Aut}(L_{\mathbf{R}})$. As a homogeneous manifold Ω is then isomorphic to $O(3,19)/(SO(2) \times O(1,19))$, so this action is not proper.

If $\omega \in \Omega$ is represented by the complex line $\ell \subset L_{\mathbf{C}}$, then the tangent space of Ω at ω is naturally isomorphic to $\mathrm{Hom}(\ell, \ell^{\perp}/\ell)$, where $\ell^{\perp} \subset L_{\mathbf{C}}$ denotes the orthogonal (non-hermitian) complement of ℓ.

(5.4.) To any marked family $(p : \mathscr{X} \to S, \alpha : R^2p^*(\mathbf{Z}) \to \mathrm{L})$ of K3-surfaces we associate a *period mapping* $\tau : S \to \Omega$ which assigns to $s \in S$ the complex line $\alpha(H^{2,0}(X_s, \mathbf{C})) \subset L_{\mathbf{C}}$. Since $p^*K_{\mathscr{X}/S}$ is locally free of rank 1 it is locally generated by a nowhere zero section. Here we let $K_{\mathscr{X}/S}$ be the sheaf of relative holomorphic 2-forms on over S. This section gives a holomorphically varying nonzero holomorphic 2-form ω_s on X_s. Following Kodaira-Spencer [13] the period map τ is holomorphic.

In the case of a marked deformation $p : (\mathscr{X}, X_0) \to (S, s_0)$ of a K3-surface we of course get an analytic germ $\tau : (S, s_0) \to \Omega$. The differential of τ at s_0 can be viewed as a homomorphism $\mathrm{d}\tau(s_0) : T_{s_0} \to \mathrm{Hom}(H^0(X_0, \Omega^2_{X_0}), H^1(X_0, \Omega^1_{X_0}))$. Indeed for any tangent vector $\frac{\partial}{\partial t} \in T_{s_0}(S)$ the image $\tau(s_0)_*\left(\frac{\partial}{\partial t}\right)$ is an element of $\mathrm{Hom}(H^{2,0}(X_0), H^{2,0}(X_0)^{\perp}/H^{2,0}(X_0))$ under the identification of L with $H^2(X_0, \mathbf{C})$ via the marking, and there is a natural identification of $H^{2,0}(X_0)^{\perp}/H^{2,0}(X^0)$ with $H^1(X_0, \Omega^1_{X_0})$.

(5.5.) Lemma: *The differential* $\mathrm{d}\tau(s_0)$ *is the composite of the Kodaira-Spencer map and the homomorphism*

$$\nabla : H^1(X_0, \Theta_{X_0}) \to \mathrm{Hom}(H^{2,0}(X_0), H^{1,1}(X_0))$$

obtained by the cup product

$$\mathrm{cup} : H^1(X_0, \Theta_{X_0}) \otimes H^0(X_0, \Omega^2_{X_0}) \to H^1(X_0, \Omega^1_{X_0}).$$

Proof: $\tau(s_0)_*\left(\frac{\partial}{\partial t}\right)$ associates to any holomorphic 2-form $\omega(s)$ on X_s

depending differentiably on s the derivative $\omega'(s_0) = \frac{d}{dt}\omega(s_0)$ in the t-direction modulo (2,0)-forms. Now choose $\omega(s)$ to be holomorphically varying (cf. (5.4.) above). Let (z_1, z_2) be local coordinates on X_{s_0} and $(w_1(z,t), w_2(z,t))$ local coordinates on X_t depending holomorphically on t with boundary conditions $(w_1(z,0), w_2(z,0)) = (z_1, z_2)$. If the local expression for $\omega(s)$ equals $f(z,s)\ dw_1 \wedge dw_2$ with $f(z,s)$ holomorphic in s we find; $\omega'(s_0) \equiv f(z,s_0)\bar\partial(w_1'(z,0)) \wedge dz_2 + \bar\partial(w_2'(z,0)) \wedge dz_1]$ modulo (2,0) forms. But $\left[\bar\partial(w'(z,0)) \wedge \frac{\partial}{\partial z_1} + \bar\partial(w_1'(z,0)) \wedge \frac{\partial}{\partial z_2}\right]$ is a Dolbeault-representative for the Kodaira-Spencer class $\rho_p\left(\frac{\partial}{\partial t}\right)$ and the right hand side is a Dolbeault representative for the cup product $\mathrm{cup}\left(\rho_p\left(\frac{\partial}{\partial t}\right) \otimes \omega(s_0)\right)$: it is obtained by contracting representatives for both classes.

(5.6.) LEMMA: *The map 'cup' (and hence ∇) is an isomorphism.*

PROOF: Choose a generator $\omega \in H^0(X_0, \Omega^2_{X_0})$. Then ω is a nowhere vanishing 2-form and the contraction homomorphism $\Theta_{X_0} \to \Omega^1_{X_0}$ obtained by contracting with ω is an isomorphism. Since $\sigma \in H^1(X, \Theta_X) \mapsto \mathrm{cup}(\sigma \otimes \omega) \in H^1(X, \Omega^1_X)$ is induced by this coefficient-isomorphism, the lemma follows.

(5.7.) COROLLARY. *Any K3-surface X_0 admits a smooth universal deformation. A marked deformation is universal if and only if the associated period map-germ is a local isomorphism.*

PROOF: As observed in (5.2.), Kuranishi's criterion implies the existence of a smooth complete deformation $p:(\mathscr{X}, X_0) \to (S, s_0)$ such that the Kodaira-Spencer map is an isomorphism. Choose a marking for X. This determines a marking for p. Then the differential of the associated period map-germ $d\tau(s_0): T_{s_0} \to T_{\tau(s_0)}\Omega$, being the composite of the Kodaira-Spencer map and ∇, is an isomorphism. So $\tau:(S, s_0) \to \Omega$ is a local isomorphism.

We prove that p is universal. Let $p':(\mathscr{X}', X_0) \to (S', s_0)$ is any deformation of X_0 and let $\tau':(S, s_0) \to \Omega$ denote the period map-germ, determined by the marking of X_0. Since p is complete, there is a morphism (Ψ, ψ) of deformations from p' to p. Then $\tau \circ \psi = \tau'$. Since τ is a local isomorphism it follows that ψ is unique.

So any morphism from p' to p is of the form (Ψ', ψ). In that case, there is a (unique) automorphism $\Phi : (\mathscr{X}', X_0) \to (\mathscr{X}', X_0)$ which commutes with p' and satisfies $\Phi | X_0 = id_{X_0}$ and $\Psi' = \Phi \circ \Psi$. Suppose $\Psi' \neq \Psi$. Then $\Phi \neq id$, so there is a largest $k \in \mathbf{N}$ with the property that the homomorphism $\Phi^* - id : \mathcal{O}_{\mathscr{X}'} \to \mathcal{O}_{\mathscr{X}'}$ maps into $\mathcal{O}_{\mathscr{X}'}(-kX_0)$. Then the composite

$$\mathcal{O}_{\mathscr{X}'} \xrightarrow{\Phi^*-id} \mathcal{O}_{\mathscr{X}'}(-kX_0) \longrightarrow \mathcal{O}_{\mathscr{X}'}(-kX_0)/\mathcal{O}_{\mathscr{X}'}(-(k+1)X_0)$$

is nontrivial. As the kernel of this map contains $\mathcal{O}_{\mathscr{X}'}(-X_0)$ as well as the constants, it factorizes over a (nontrivial) homomorphism

$$\Omega^1_{X_0} \to \mathcal{O}_{\mathscr{X}}(-kX_0)/\mathcal{O}_{\mathscr{X}'}(-(k+1)X_0).$$

If we identify the right hand side with $\mathcal{O}_{\mathscr{X}_0'} \otimes_{\mathbf{C}} m^k_{S',s_0'}/m^{k+1}_{S',s_0'}$, we may also view this homomorphism as a section of $\Theta_{X_0} \otimes_{\mathbf{C}} m^k_{S',s_0'}/m^{k+1}_{S',s_0'}$. But this contradicts the fact that X_0 does not possess nontrivial vector fields. So $\Psi = \Psi'$, indeed. If τ' were also a local isomorphism, then ψ (and hence Ψ) would be an isomorphism, thus implying that p' is universal.

§6. The density theorem

If X is a surface and $S_X \subset H^2(X, \mathbf{R})$ its algebraic lattice, then the set of integral points in $H^2(X, \mathbf{R})$ orthogonal to S_X is a lattice which we denote by T_X and refer to as the transcendental lattice. The subspace spanned by T_X in $H^2(X, \mathbf{C})$ can be characterized as the smallest rationally defined subspace of $H^2(X, \mathbf{C})$ which contains $H^{2,0}(X, \mathbf{C})$. We call X *exceptional* if rank $S_X = \dim H^{1,1}(X, \mathbf{R})$, or equivalently if $H^{2,0}(X) \oplus \bar{H}^{2,0}(X)$ is defined over $\mathbf{Q}$. By (1.3.), $T_X \otimes \mathbf{R}$ is then naturally endowed with a complex structure, in particular, T_X is naturally oriented.

If X is a Kummer surface arising from a complex torus A, then following (3.4.) there is a canonical isomorphism (defined over $\mathbf{Q}$) from $H^{2,0}(A)$ onto $H^{2,0}(X)$ and so X is exceptional if and only if A is. We shall prove that the period points of exceptional Kummer surfaces form a dense subset of Ω.

(6.1.) Proposition: *Let T be a primitive oriented sublattice of L of rank two such that $\langle x, x\rangle \in 4\mathbf{Z}$ for all $x \in T$. Then there exists a marked exceptional Kummer surface $(X, \alpha : H^2(X, \mathbf{Z}) \to L)$ such that α maps T_X isomorphically and orientation preserving onto T.*

PROOF: We prove this proposition in three steps. Let T' denote the oriented **Z**-module T endowed with the even integral form $\langle,\rangle' := \frac{1}{2}\langle,\rangle$.

STEP 1. There exists a two-dimensional complex torus A whose transcendental lattice is isometric to T'.

PROOF: As at the beginning of §3, let Γ be an oriented free Z-module of rank 4 and endow $\Lambda^2\Gamma$ with the symmetric bilinear form $\langle,\rangle$ coming from the orientation. Since $\Lambda^2\Gamma$ is isometric to $H \oplus H \oplus H$, there exists by (2.4.) an isometric primitive embedding $j: T' \to \Lambda^2\Gamma$. Let $\omega: \Lambda^2\Gamma \to \mathbf{C}$ be a linear form which vanishes on the orthogonal complement of $j(T')$ and maps $j(T')$ isometrically into $\mathbf{C}$ such that $\omega \circ j$ is orientation preserving. Choose an orthonormal basis $\{e_1, e_2\}$ of $j(T')_{\mathbf{R}}$.

Then $\langle\omega, \omega\rangle = \omega(e_1)^2 + \omega(e_2)^2 = \omega(e_1)^2 + (i\omega(e_1))^2 = 0$ and $\langle\omega, \bar{\omega}\rangle = |\omega(e_1)|^2 + |i\omega(e_1)|^2 = 2|\omega(e_1)|^2 > 0$ and so $\omega \wedge \omega = 0$ and $\omega \wedge \bar{\omega}$ is a positive multiple of det (both are viewed as elements of $\mathrm{Hom}(\Lambda^4\Gamma, \mathbf{C})$). Hence we can write $\omega = \omega_1 \wedge \omega_2$ with $\omega_i \in \mathrm{Hom}(\Gamma, \mathbf{C})$. Define $\Omega = (\omega_1, \omega_2): \Gamma \to \mathbf{C}^2$. Then $\Omega(\Gamma)$ is a lattice in $\mathbf{C}^2$, since $\omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 \neq 0$. Let A denote the torus $\mathbf{C}^2/\Omega(\Gamma)$. Then we have a natural orientation preserving isomorphism $\Gamma \cong H_1(A, \mathbf{Z})$. The corresponding isometry between $\mathrm{Hom}(\Lambda^2\Gamma, \mathbf{Z})$ and $H^2(A, \mathbf{Z})$ maps the line $\mathbf{C} \cdot \omega$ onto $H^{2,0}(A, \mathbf{C})$ and maps therefore $\mathrm{Ker}(\omega)$ onto S_A and $j(T')$ onto T_A.

STEP 2.: There exists an exceptional Kummer surface X whose transcendental lattice is isometric to T.

PROOF: Let X be the Kummer surface obtained from a canonical involution of A. The injection $\alpha: H^2(A, \mathbf{Z}) \to H^2(X, \mathbf{Z})$, defined in (3.4.), multiplies the intersection form with two and maps $H^{2,0}(A, \mathbf{C})$ isomorphically onto $H^{2,0}(X, \mathbf{C})$ and so $T_X = \alpha(T_A)$ and $T_X \cong T$.

STEP 3

PROOF OF THE PROPOSITION: Choose an isometry $\phi': H^2(X, \mathbf{Z}) \to L$. Following (2.4.) there exists an automorphism of $(L, \langle,\rangle)$ which maps $\phi'(T_X)$ orientation preserving onto T. Composing ϕ' with this automorphism yields an isometry $\phi: H^2(X, \mathbf{Z}) \to L$ with $\phi(T_X) = L$.

Next we show that there are enough sublattices of L satisfying the conditions of prop. (6.1.).

(6.2.) PROPOSITION: *The set of rationally defined 2-planes $P \subset L_{\mathbf{R}}$ satisfying $\langle x, x\rangle \in 4\,\mathbf{Z}$ for all $x \in P \cap L$ is dense in the Grassmannian $G_2(L_{\mathbf{R}})$.*

We prove this with the help of the following:

(6.3) LEMMA: *Let $m, n \in \mathbf{N}$ and let M be a lattice containing a primitive vector e_0 with $\langle e_0, e_0 \rangle = m (\mathrm{mod}\, n)$. Then the set of lines $l \subset M_{\mathbf{R}}$ spanned by a primitive vector e with $\langle e, e \rangle \equiv m (\mathrm{mod}\, n)$ is dense in the projective space $\mathbf{P}(M_{\mathbf{R}})$.*

PROOF: Let U be an open nonvoid subset of $\mathbf{P}(M_{\mathbf{R}})$. Then U contains a rationally defined element l, spanned by a primitive vector $e \in M$. If $e \neq \pm e_0$, then e and e_0 are linearly independent. Choose $f \in (\mathbf{R}e + \mathbf{R}e_0) \cap M$ such that $\{e, f\}$ is a basis of $(\mathbf{R}e + \mathbf{R}e_0) \cap M$ and write $e_0 = ae + bf$ with $a, b \in \mathbf{Z}$. Since e_0 is primitive, $gcd(a, b) = 1$. Then for any $N \in \mathbf{Z}$, $e_N := e_0 + Nbe = (a + Nb)e + be$ is also primitive since $gcd(a + Nb, b) = gcd(a, b) = 1$. Taking $N \in n\mathbf{Z}$, we have $\langle e_N, e_N \rangle \equiv \langle e_0, e_0 \rangle \equiv m (\mathrm{mod}\, n)$. For N sufficiently large we also have that $\mathbf{R}e_N = \mathbf{R}\left(e + \frac{1}{Nb} e_0\right) \in U$. So U contains an element with the required properties.

PROOF OF THE PROPOSITION: Let U be a nonvoid open subset of $G_2(L_{\mathbf{R}})$. Since the hyperbolic plane contains a primitive vector of square length 4, so does L. It follows from the lemma that there exists a $P' \in U$ which contains a primitive vector e_1 with $\langle e_1, e_1 \rangle \equiv 4 (\mathrm{mod}\, 8)$. Let M denote the orthogonal complement of e_1 in L. By (2.4.), M contains a hyperbolic plane as a direct summand and so M contains a primitive vector of square length 64. Our lemma implies that M contains a primitive vector e_2 with $\langle e_2, e_2 \rangle \equiv 0 (\mathrm{mod}\, 64)$, such that $P := \mathbf{R}e_1 + \mathbf{R}e_2 \in U$.

We claim that any $f \in P \cap L$ satisfies $\langle f, f \rangle \in 4\mathbf{Z}$. Since $\langle e_1, e_1 \rangle f - \langle e_1, f \rangle e_1$ is an element of $P \cap L$ orthogonal to e_1, it is an integral multiple ae_2 of e_2. Taking the square length of both sides of $\langle e_1, e_1 \rangle f = \langle e_1, f \rangle e_1 + ae_2$ yields

$$\langle e_1, e_1 \rangle^2 \langle f_1, f_1 \rangle = \langle e_1, f \rangle^2 \langle e_1, e_1 \rangle + a^2 \langle e_2, e_2 \rangle.$$

Letting v_2 denote the 2-adic valuation on $\mathbf{Z}$, then $v_2(\langle e_1, f \rangle^2 \langle e_1, e_1 \rangle) = 2v_2(\langle e_1, f \rangle) + 2$ and $v_2(a^2 \langle e_2, e_2 \rangle) \geq 6$. So $v_2(\langle e_1, e_1 \rangle^2 \langle f_1, f_1 \rangle)$ is even or ≥ 6. This implies that $v_2(\langle f, f \rangle) \geq 2$, i.e. that $\langle f, f \rangle \in 4\mathbf{Z}$.

(6.4.) COROLLARY: *The period points of marked projective Kummer surfaces lie dense in Ω.*

PROOF: By (6.2.) the set of $\omega \in \Omega$ with the property that $\{\mathrm{Re}(\omega), \mathrm{Im}(\omega)\}$ spans a rationally defined 2-plane $P \subset L_{\mathbf{R}}$ such that $\langle x, x\rangle \in 4\mathbf{Z}$ for all $x \in P \cap L$ is dense in Ω. Following (6.1.) any such ω is the image under τ of a marked exceptional Kummer surface X. Since S_X contains elements with positive length, X is projective.

(6.5.) REMARK: The proof of (6.1.) depends in an essential way on the fact that L contains $H \oplus H \oplus H$ as a direct summand. If we were only considering K3 surfaces with a fixed polarization (as is done in[20]), then the arguments used in (6.1) would no longer be valid. For this reason the proof given in [20, §6] is incomplete.

§7. The main lemma of Burns-Rapoport

By a *family of analytic manifolds* we mean a proper and flat morphism $p:\mathscr{X}\to S$ between analytic spaces such that each closed fibre is smooth. The family is called *smooth* if S is smooth and p is everywhere of maximal rank. The main result of this section is the following proposition, which is essentially due to Burns-Rapoport [4].

(7.1.) PROPOSITION: *Let S be an analytic manifold which serves as a common base for two smooth families $p:\mathscr{X}\to S$ and $p':\mathscr{X}'\to S$ of Kählerian K3-surfaces*[1] *and let $\Phi^*: R^2p'_*(\mathbf{Z})\to R^2p_*(\mathbf{Z})$ be an isomorphism of local systems which induces over each base point an isometry. Let further be given a sequence $\{s_i\in S\}_{i=1}^{\infty}$ converging to $s_0\in S$ and a sequence of isomorphisms $\{\phi_i: X_{s_i}\to X'_{s_i}\}_{i=1}^{\infty}$ such that $\phi_i^* = \Phi^*(s_i)$. Then X'_{s_0} and X_{s_0} are isomorphic. If moreover $\Phi^*(s_0)$ sends effective classes to effective classes, then there is a subsequence of $\{\phi_i\}_{i=1}^{\infty}$ which converges uniformly to an isomorphism $\phi_0: X_{s_0}\to X'_{s_0}$ such that $\phi_0^* = \Phi^*(s_0)$.*

We divide the proof over three separate lemmas.

Let $\Gamma_i \subset X_{s_i}\times X'_{s_i}\subset X \times_S X$ denote the graph of ϕ_i. We view Γ_i as a positive integral analytic current of dimension 4 on $X\times_S X$.

(7.2.) LEMMA: *A subsequence of $\{\Gamma_i\}$ converges (in the space of functionals on the space of continuous 4-forms) to a positive integral analytic current Γ_0 with support in $X_{s_0}\times X'_{s_0}$.*

[1]The hypothesis that we are dealing with K3-surfaces is only used in a modest way. Indeed it may be replaced by the condition that the surfaces are minimal non-ruled.

PROOF: By a result of Bishop [2], it suffices to show that the 4-volumes of cycles Γ_i (in terms of any hermitian metric on $\mathscr{X} \times_S \mathscr{X}'$) are bounded. Since the cocycles associated to Γ_i are cohomologous for i sufficiently large, it is convenient to choose such a hermitian metric in a particular way.

We choose an open neighbourhood V of s_0 in S and Kähler metrics η_s on X_s and η'_s on X'_s depending continuously on $s \in S$ and we compute volumes with respect to the product metric θ_s on $X_s \times X'_s$. We let $[\Gamma_n] \in H^4(X_s \times X'_s, \mathbf{R}) \simeq H^4(\mathscr{X} \times_S \mathscr{X}', \mathbf{R})$ be the class of θ_s and finally $[\Phi^*]$ the class of Φ^* in $H^4(\mathscr{X} \times_S \mathscr{X}', \mathbf{R})$. We find:

$$\operatorname{Vol}(\Gamma_n) = \tfrac{1}{2}[\Gamma_n] \cup \kappa_{s_n}^2 = \tfrac{1}{2}[\Phi^*] \cup \kappa_{s_n}^2,$$

which is the value af s_n of the continuous function $f(s) = \tfrac{1}{2}[\Phi^*] \cup \kappa_s^2$, hence bounded on any compact neighbourhood of 0 in S.

(7.3.) LEMMA: *The limit current Γ_0 is of the form $\Delta + \Sigma_{i=1}^N C_i \times C'_i$ where Δ is the graph of an isomorphism and $\{C_i\}$ resp. $\{C'_i\}$ are effective divisors on X_{s_0} resp. X'_{s_0}.*

PROOF: The cohomology class $[\Gamma_i] \subset H^4(X_{s_j} \times X'_s, \mathbf{Z})$ defines homomorphism from $H_j(X_{s_i}, \mathbf{Z})$ to $H_j(X'_{s_j}, \mathbf{Z})$ (which we also denote by $[\Gamma_i]$) as follows: if z is any j-cycle on X_{s_i}, choose a j-cycle $\tilde{z}$ on X_{s_i} homologous to z, such that $\tilde{z} \times X'_{s_i}$ is in general position with respect to Γ_i and assign to z the j-cycle on X' obtained by projecting $(\tilde{z} \times X'_{s_i}) \cap \Gamma$ onto X'_{s_i}. This assignment induces a map on homology which is, of course, nothing but the one induced by ϕ_i. Hence the dual $[\Gamma_i]^*$ of $[\Gamma_i]$ equals $\phi^*(s_i)$ in dimension 2 and is the obvious isomorphism in the other dimensions. Since $X \times_S X' \to S$ is locally trivial in the $\mathscr{C}^\infty$-sence, this also holds 'in the limit', that is, $[\Gamma_0]^*$, defined in a similar way as $[\Gamma_i]^*$, equals $\phi^*(s_0)$ in dimension 2 and is the obvious isomorphism in the other dimensions.

Now if d' denotes the degree of the projection of Γ_0 onto X'_{s_0}, the map induced by $[\Gamma_0]$ on 4-dimensional homology is multiplication with d'. So $d' = 1$, in other words, we can write $\Gamma_0 = \Delta + \Gamma'_0$ where Δ projects birationally onto X'_{s_0} and Γ'_0 projects onto a lower dimensional subvariety on X'. The same argument applied to 0-dimensional homology shows that either Δ projects birationally onto X_{s_0} or that the projection $\Gamma'_0 \to X_{s_0}$ is of degree one. We claim that the former holds. If not, then Δ projects onto a subvariety of lower dimension in X_{s_0} and hence the image of $\phi^*(s_0) = [\Gamma]^* = [\Delta]^* + [\Gamma'_0]^* : H^2(X'_{s_0}, \mathbf{Z}) \to H^2(X_{s_0}, \mathbf{Z})$ would consist of algebraic classes only. Then $\phi^*(s_0)$ would map into a proper sublattice of $H^2(X_{s_0}, \mathbf{Z})$, which contradicts the assumption that $\phi^*(s_0)$ is an isomorphism. Hence Δ is the graph of a birational map $\phi_0': X_{s_0} \to X'_{s_0}$ and Γ_0' projects onto a lower dimensional subvariety in both X_{s_0} and X'_{s_0}. Since X_{s_0} is absolutely minimal ϕ_0' must be an isomorphism. Finally, Γ_0' being a purely 2-dimensional non-negative integral current is necessarily a finite sum of products of effective divisors on X_{s_0} and X'_{s_0}.

The previous lemma already implies the first clause of (7.1.). We therefore assume from now on that the last condition, $\phi^*(s_0)$ maps effective classes to effective classes, is also fulfilled. We further identify X_{s_0} with X'_{s_0} by means of the isomorphism ϕ_0' and we write X_0 instead of X_{s_0}. So $\Phi^*(s_0)$ is an automorphism of the lattice $H^2(X_0, \mathbf{Z})$ which respects the set of positive classes. According to the previous lemma, $\Phi^*(s_0)$ is of the form $\Phi(s_0)(x) = x + \Sigma_{i=1}^N \langle c_i', x\rangle\, c_i$ where c_i and c_i' are classes of effective divisors. By leaving out the terms for which $c_i = 0$, we may (and do) suppose that each c_i is effective. The following lemma then completes the proof of our proposition.

(7.4.) LEMMA: *For all i, $c_i' = 0$, in particular $\Phi^*(s_0)$ is the identity.*

PROOF: Let ψ denote the restriction of $\Phi^*(s_0)$ to $H^{1,1}(X_0, \mathbf{R})$. Then ψ leaves the Kähler cone V in $H^{1,1}(X_0, \mathbf{R})$ invariant. The Brouwer fixed point theorem, applied to the set of half-lines in the closure V of V, implies that ψ stabilizes a half-line in V, i.e. V contains an eigen vector ξ with positive eigen value λ. So $\langle \xi, \xi\rangle \geq 0$ and $\langle \xi, c\rangle \geq 0$ for any class c of an effective divisor. Let $H \subset H^{1,1}(X_0, \mathbf{R})$ denote the orthogonal complement of ξ in $H^{1,1}(X_0, \mathbf{R})$. Since $\langle \xi, \xi\rangle \geq 0$, the restriction of $\langle,\rangle$ to H is negative, with nullity at most one. The remainder of the proof is split up into four steps.

STEP 1: ψ acts with finite order on the subspace H' of H spanned by irreducible classes in H.

PROOF: First of all $\langle,\rangle$ is negative semi-definite on H' and ψ preserves a lattice inside of H'. So, if the nullspace is $\{0\}$, ψ has finite order. In case there is a 1-dimensional nullspace, (spanned by ξ) we observe that ψ has finite order mod ξ, i.e. for some $k \in \mathbf{N}$ we have $\psi^k(x) = x + \alpha(x)\xi$, for all $x \in H'$.

Let D_i $(i = 1, \ldots, r)$ be irreducible curves on X_0, whose classes d_i belong to H. Assume that $\bigcup_{i=1}^r D_i$ is connected. Then $\langle d_i, d_j\rangle \geq 0$ $(i \neq j)$, the matrix $(-\langle d_i, d_j\rangle)$ is negative semi-definite and there is not a proper subset I of $\{1, \ldots, r\}$ such that $i \in \dot{I}$ and $j \in \{1, \ldots, n\} - \dot{I}$ imply that $\langle d_i, d_j \rangle = 0$. Hence the matrix $(-\langle d_i, d_j \rangle)$ satisfies the assumptions of ([3], §3 Lemma 4) and hence the null space of $\langle , \rangle | H_0$ is $\{0\}$ or 1-dimensional with basis $\Sigma_{i=0}^{r} a_i d_i = \xi$ $(a_i \in \mathbf{N})$—here $H_0 = \operatorname{span}\{d_1, \ldots, d_r\}$. Now put $\psi_0 := \psi^k | H_0$ and choose a Kähler class κ. Since ψ_0 preserves effective classes $\alpha(d_i) \in \mathbf{N} \cup \{0\}$. Since ψ_0 preserves Kähler classes we have $\langle \psi_0^{-1}(d_i), \kappa \rangle = \langle d_i, \psi(\kappa) \rangle > 0$ and one checks that this would contradict $\alpha(d_i) > 0$. Hence $\alpha(d_i) = 0$ and $\psi_0 = id$. Similarly $\psi^k | H' = id$.

STEP 2: Any effective class in. H is a linear combination of irreducible classes in H. Moreover $\psi(\xi) = \xi$.

PROOF: Let $c \in H$ be positive and write $c = \Sigma_j n_j d_j$ where each d_j is irreducible and $n_j \in \mathbf{N}$. Then $0 = \langle c, \xi \rangle = \Sigma_j n_j \langle d_j, \xi \rangle$. Since $\langle d_j, \xi \rangle \geq 0$ it follows that $\langle d_j, \xi \rangle = 0$, that is $d_j \in H$ for all j.

Taking the inner product of the equality $\psi(\xi) = \lambda \xi = \xi + \Sigma_i \langle c_i', \xi \rangle c_i$ with ξ yields $0 = \Sigma_i \langle c_i', \xi \rangle \langle c_i, \xi \rangle$. In each term of the right hand side the factors are ≥ 0, so for any i we have $\langle c_i', \xi \rangle = 0$ or $\langle c_i, \xi \rangle = 0$. Hence $(\lambda - 1)\xi = \Sigma_{c_i \in H} \langle c_i', \xi \rangle c_i$. By step 1 and the already proven clause of step 2, the right hand side of the last equality is invariant under some power of ψ. So if $\lambda \neq 1$, then it must be a root of unity. But this leads to a contradiction, as a positive root of unity necessarily equals 1.

STEP 3: ψ is of finite order.

PROOF: Choose any $\kappa \in V$ and consider the equality

$$0 = \langle \psi(\xi), \kappa \rangle - \langle \xi, \kappa \rangle = \sum_i \langle c_i', \xi \rangle \langle c_i, \kappa \rangle.$$

Since $\langle c_i, \kappa \rangle > 0$ (for c_i is positive) and $\langle c_i', \xi \rangle \geq 0$ for all i, it follows that $\langle c_i', \xi \rangle = 0$. This proves that the subspace P generated by $\{c_1', \ldots, c_N'\}$ is contained in H. It follows from the two previous steps that ψ acts with finite order on P. Let Q denote the orthogonal complement of P in H. Since $\langle , \rangle$ is negative on H, P and Q span all of H. Clearly ψ acts as the identity on Q. Hence ψ acts with finite order on H. Now step 3 follows from the (readily verified) fact that any orientation preserving automorphism of a vector space leaving a nondegenerate quadratic form invariant and a hyperplane pointwise fixed must be the identity.

STEP 4: For all i, $c_i' = 0$.

PROOF: Choose any element κ' in the Kähler cone V and put $\kappa := \kappa' + \psi(\kappa') + \cdots + \psi^{k-1}(\kappa')$, where k denotes the order of ψ. Since V is a ψ-invariant convex cone we have that $\kappa \in V$. Moreover $\psi(\kappa) = \kappa$. So

$$0 = \langle \psi(\kappa), \kappa \rangle - \langle \kappa, \kappa \rangle = \textstyle\sum_i \langle c_i', \kappa \rangle \langle c_i, \kappa \rangle.$$

Since $\langle c_i', \kappa \rangle \geq 0$ and $\langle c_i, \kappa \rangle > 0$ it follows that $\langle c_i', \kappa \rangle = 0$ for all i. This implies that $c_i' = 0$ for all i.

The following proposition shows that in the conclusion of (7.1.) we may delete 'a subsequence of'.

(7.5.) PROPOSITION: *Any automorphism of a kählerian* K3-*surface which acts trivially on its second cohomology group is the identity.*

PROOF: Let X be a kählerian K3-surface and denote by G the group of its automorphisms which act trivially on $H^2(X, \mathbf{Z})$.

STEP 1: G is finite.

PROOF: Let $g \in G$ and apply (7.1.) to the trivial family with $\phi_i = g^i$. It follows that a subsequence of $\{g^i\}_{i=1}^{\infty}$ converges to some $g_0 \in G$. This proves that G is compact. Since a K3-surface does not possess nontrivial holomorphic vector fields, G is of dimension zero. Hence G is finite.

STEP 2: Let $g \in G$ be of prime order p (>1). Then the fixed point set of g consists of 24 distinct points, at each of which the action of g is given in local coordinates (u, v) by $(u, v) \mapsto (\xi u, \xi^{-1} v)$ with ξ a pth root of unity $\neq 1$.

PROOF: Since X is a K3-surface, it possesses a nowhere vanishing holomorphic 2-form ω_X. As $H^0(X, \Omega_X^2)$ is a direct summand of $H^2(X, \mathbf{C})$, ω_X is left invariant by g. So the jacobian of g at a fixed point has determinant 1. Since, at a fixed point, g is in local coordinates (u, v) given by $(u, v) \mapsto (\xi u, \xi^j v)$ for some $j \in \mathbf{Z}/p$, we must have $j = -1$. In particular, the fixed points of g are all isolated. Since g acts trivially on the homology of X, the Lefschetz number of g equals the euler number of X, that is, equals 24. The Lefschetz fixed point formula then implies that there are 24 fixed points.

STEP 3: G reduces to the identity.

PROOF: Since G is finite it suffices to show that G does not contain elements of prime order. Suppose not and let $g \in G$ be of prime order p. Denote by $\hat{X}$ the g-orbit space of X. Following step 2, $\hat{X}$ has 24 singular points, each of which is locally isomorphic to the orbit space of the $\mathbf{Z}/p$-action described there. As is well known (see for instance [6]) such a singularity admits a (minimal) resolution of which the exceptional divisor consists of a string of $p-1$ nodal curves. Let $r:\tilde{X} \to \hat{X}$ denote the resolution of $\hat{X}$, thus obtained.

We claim that $\tilde{X}$ is a regular surface with trivial canonical bundle (and hence a K3-surface). The regularity follows from the fact that r induces an isomorphism $H_1(\tilde{X}, \mathbf{R}) \to H_1(\hat{X}, \mathbf{R})$ and the identification $H_1(\hat{X}, \mathbf{R}) \cong H_1(X, \mathbf{R})^g = \{0\}$. The 2-form ω_X on X is g-invariant and thus determines a nowhere vanishing 2-form $\omega_{\hat{X}}$ on the smooth part of $\hat{X}$. It is not difficult to check that $r^*\omega_{\hat{X}}$ extends to a holomorphic nowhere vanishing 2-form on $\tilde{X}$.

Then $\tilde{X}$, being a K3-surface, has second betti number 22. This yields a contradiction, for by choosing a nodal curve over each singular point of X, we obtain 24 mutually orthogonal nodal classes.

(7.6.) REMARK: From (7.5.) together with the density theorem (6.4.) it readily follows that in (7.5.) the assumption that X be kählerian is superfluous.

Compare [4], Proposition 1.1.

§8. Openness of the Kähler cone

In this section we prove the following:

(8.1.) PROPOSITION: *Let $p:\mathscr{X} \to S$ and $p':\mathscr{X}' \to S$ be two families of kählerian* K3 *surfaces over a common base S and let $\phi^*:R^2p'_*(\mathbf{Z}) \to R^2p_*(\mathbf{Z})$ be an isomorphism of sheaves which induces over each base point $s \in S$ a Hodge isometry. Then the set of $s \in S$ with the property that $\phi^*(s)$ maps the Kähler cone of X'_s onto the Kähler cone of X_s, is open in S.*

For this purpose we consider the space Ω' consisting of pairs $(\omega, \kappa) \in \Omega \times L_{\mathbf{R}}$ satisfying $\langle \kappa, \kappa \rangle > 0$ and $\langle \tilde{\omega}, \kappa \rangle = 0$ for some representative $\tilde{\omega} \in L_{\mathbf{C}} - \{0\}$ of ω. In this case, the vectors $\mathrm{Re}(\tilde{\omega})$, $\mathrm{Im}(\tilde{\omega})$ and κ are mutually orthogonal and have all three positive length. So they span a 3-dimensional positive definite subspace of $L_{\mathbf{R}}$, of which it is easy to see that it only depends on the pair $(\omega, \kappa) \in \Omega'$. Clearly, the

Lie group $\mathrm{Aut}(L_{\mathbf{R}})$ acts on Ω and the map just defined from Ω' to the space of 3-dimensional positive definite subspaces is a $\mathrm{Aut}(L_{\mathbf{R}})$-equivariant fibre bundle. Since $\mathrm{Aut}(L_{\mathbf{R}})$ acts in a proper fashion on the base space of this bundle (it is the symmetric space associated to $\mathrm{Aut}(L_{\mathbf{R}})$), the action of $\mathrm{Aut}(L_{\mathbf{R}})$ on Ω' is also proper.

Now recall that any $\delta \in L$ of length -2 determines a reflection s_δ, given by $x \mapsto x + \langle \delta, x\rangle\delta$, which belongs to $\mathrm{Aut}(L)$. Since $\mathrm{Aut}(L)$ is a discrete subgroup of $\mathrm{Aut}(L_{\mathbf{R}})$, it follows that the union of the fixed point loci of these reflections is closed in Ω'. This is of course equivalent to saying that the set $\{(\omega, \kappa) \in \Omega' : \langle \bar{\omega}, \delta\rangle = 0$ and $\langle \kappa, \delta\rangle = 0, \langle \delta, \delta\rangle = -2\}$ is closed in Ω'. This observation will be used to prove the following result.

(8.2.) LEMMA: *Let $p: \mathscr{X} \to S$ be a family of kählerian $K3$-surfaces. Then the union of vector spaces $\{H^{1,1}(X_s, \mathbf{R})\}_{s\in S}$ makes up a real analytic subbundle of $R^2p_*(\mathbf{R})$ in which the union of Kähler cones is open.*

PROOF: Without loss of generality we may (and do) suppose that S is simply connected. Then the family admits a marking $\alpha: R^2p_*(\mathbf{Z}) \to L$. We denote the associated period mapping by $\tau: S \to \Omega$. Since τ is holomorphic, the subspaces $\alpha(H^{1,1}(X_s, \mathbf{R})$ (= real part of the orthogonal complement of $\tau(s)$) vary in a real-analytic way with $s \in S$ and hence $\cup\{H^{1,1}(X_s, \mathbf{R}) : s \in S\}$ defines a real-analytic subbundle $H^{1,1}(\mathscr{X}/S, \mathbf{R})$ of $R^2p_*(\mathbf{R})$, as asserted.

Now, let $\kappa \in H^{1,1}(X_s, \mathbf{R})$ be contained in the Kähler cone of X_s, Since the Kähler cone of X_s is convex (in particular connected) and contains a Kähler class, there exists a compact connected neighbourhood K of κ in the Kähler cone of X_s which contains a Kähler class in its interior. More or less by definition, there is no vector of length -2 in L which is simultaneously orthogonal to $\tau(s)$ and an element of K. It follows from the discussion preceding this proposition, that we can find a neighbourhood V of K in $H^{1,1}(\mathscr{X}/S, \mathbf{R})$ such that for any $\kappa' \in V \cap H^2(X_{s'}, \mathbf{R})$, no vector of length -2 in $H^2(X_s, \mathbf{Z}) \cap H^{1,1}(X_s, \mathbf{R})$ is orthogonal to κ'. By shrinking V we can arrange that $\langle \kappa', \kappa'\rangle > 0$ for any $\kappa' \in V$ and that $V \cap H^2(X_{s'}, \mathbf{R})$ is connected for any $s' \in S$. According to Kodaira-Spencer [14], the set of Kähler classes is open in $H^{1,1}(\mathscr{X}/S, \mathbf{R})$. Since K contains a Kähler class, there exists a neighbourhood V of s in S such that $V \cap H^{1,1}(X_{s'}, \mathbf{R})$ contains a Kähler class for any $s' \in V$. But then any such intersection must be contained in the Kähler cone of $X_{s'}$. The lemma follows.

PROOF OF (8.1.): The conditions imposed on ϕ^* imply that ϕ^* induces a bundle isomorphism $H^{1,1}(\mathscr{X}'/S, \mathbf{R}) \to H^{1,1}(\mathscr{X}/S, \mathbf{R})$. It is clear from (1.7.) that if ϕ^* maps an element of the Kähler cone of X'_s into the Kähler cone of X_s, then ϕ^* maps the whole Kähler cone of X'_s onto the Kähler cone of X_s. The theorem now follows from the last proposition.

§9. The Torelli theorem for $K3$-surfaces

We are now able to prove the Torelli theorem for kählerian K3-surfaces as announced in the introduction.

(9.1.) THE TORELLI THEOREM FOR KÄHLERIAN K3-SURFACES: *Let X and X' be kählerian $K3$-surfaces and suppose there exists a Hodge isometry $\phi^*: H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$ which respects the Kähler cones (or equivalently: sends the positive cone of X' to the positive cone of X and nodal classes to nodal classes). Then ϕ^* is induced by a unique isomorphism $\phi: X \to X'$.*

PROOF: We embed the two surfaces in locally universal families:

$$\begin{array}{ccc} X & \hookrightarrow & \mathscr{X} \\ \downarrow & \square & \downarrow{\scriptstyle p} \\ \{s_0\} & \subset & S \end{array} \qquad\qquad \begin{array}{ccc} X' & \hookrightarrow & \mathscr{X}' \\ \downarrow & \square & \downarrow \\ \{s'_0\} & \subset & S' \end{array}$$

having both a simply-connected base. Following Kodaira-Spencer [14], we may assume that both are families of *kählerian* K3-surfaces. Let $\alpha: R^2p_*(\mathbf{Z}) \to L$ be a marking of the first family, let $\Phi^*: R^2p'_*(\mathbf{Z}) \to R^2p_*(\mathbf{Z})$ be the unique isomorphism extending ϕ^* and put $\alpha' := \alpha \circ \Phi^*$. We denote the associated period mapping by $\tau: S \to \Omega$ and $\tau': S' \to \Omega$.

By the local Torelli theorem (5.7.) τ and τ' are local isomorphisms. Since $\phi^* = \Phi^*(s'_0)$ is a Hodge isometry, $\tau(s_0) = \tau(s'_0)$. Hence, after shrinking S and S' if necessary, there is a unique isomorphism $\Psi: S \to S'$ such that $\tau = \tau' \circ \Psi$. The family which p' induces over S (via Ψ) is, of course still locally universal and so we may as well assume that $S = S'$, $\Psi = id$ and (hence) $\tau = \tau'$. Now Φ^* induces over each $s \in S$ a Hodge isometry and $\Phi^*(s_0) = \phi^*$ respects the Kähler cones. Then following (8.1.), there is an open neighbourhood V of s_0 in S such that $\Phi^*(s)$ respects the Kähler cones for any $s \in V$.

By the Density Theorem (6.4.), the period points of projective Kummer surfaces lie dense in Ω, and so we can find a sequence $\{s_i \in U\}_{i\in \mathbf{N}}$ converging to s_0 such that $\tau(s_i)$ $(= \tau'(s_i))$ is the period point of a projective Kummer surface. Then the special Torelli theorem (weak form) asserts that X_{s_i} and X'_{s_i} are both projective Kummer surfaces. Since $\Phi^*(s_i)$ is a Hodge isometry which preserves the Kähler cones, it is by the strong form of the special Torelli theorem induced by an isomorphism $\Phi(s_i): X_{s_i} \to X'_{s_i}$. It now follows from proposition (7.1.) that $\phi^* = \Phi^*(s_0)$ is induced by an isomorphism $\phi: X \to X'$. The uniqueness of ϕ is implied by (7.5.).

§10. Applications

Our first application concerns the diffeomorphism type of K3 surfaces.

(10.1) PROPOSITION: *Any nonsingular quartic surface in* $\mathbf{P}^3$ *is a* K3-*surface. All* K3-*surfaces are mutually diffeomorphic and each* K3-*surface is simply connected.*

PROOF: A nonsingular quartic X in $\mathbf{P}^3$ has by adjunction formula trivial canonical bundle. The Lefschetz theorem implies that such a surface is simply connected, hence regular. By definition, X is then a K3-surface.

It remains to show that all K3-surfaces are mutually diffeomorphic. We first observe that all Kummer surfaces are diffeomorphic, since any two such can be connected by a family of Kummer surfaces. Now, if X_0 is an arbitrary K3-surface, realize X_0 as a fibre $p^{-1}(s_0)$ in a locally universal family $p: \mathscr{X} \to S$. Arguing as in (9.1) we see that the set of $s \in S$ for which X_s is a Kummer surface is dense in S. It follows that $X_0 \cong X_{s_0}$ is diffeomorphic to a Kummer surface.

Next we discuss an alternative formulation of the global Torelli theorem in terms of a moduli space for unpolarized Hodge structures, introduced by Burns-Rapoport [4].

Consider the class of marked families $(p: \mathscr{X} \to S,\ \Phi: R^2p_*(\mathbf{Z}) \to L)$ of kählerian K3-surfaces. A *morphism* (Ψ, ψ) from one such family $(p: \mathscr{X} \to S, \Phi)$ to another $(p': \mathscr{X}' \to S', \Phi)$ defined in the obvious way: the diagram

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\Psi} & \mathscr{X}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ S & \xrightarrow[\psi]{} & S' \end{array}$$

should be cartesian and the composition of the map $R^2p'_*(\mathbf{Z})\to R^2p_*(\mathbf{Z})$ induced by the pair (Ψ, ψ) with Φ should equal Φ'. We thus obtain the *category of marked kählerian K3 surfaces.* Our aim is to construct a final object $(p_M:\mathscr{X}_M\to M,\Phi_M)$ for this category. Usually, one calls such an object a *fine moduli space* of marked kählerian K3-surfaces.

(10.2) PROPOSITION: *There exists a fine moduli space $(p_M:\mathscr{X}_M\to M, \Phi_M)$ of marked kählerain* K3-*surfaces. The associated period mapping $\tau_M: M\to \Omega$ exhibits M as an analytic space étale over Ω.*

PROOF: Let $(p_{M'}:\mathscr{X}_{M'}, \Phi_{M'})$ denote the disjoint union of the class of locally universal families of kählerian K3 surfaces. Let $\mathscr{R}_1$ resp. $\mathscr{R}_2$ denote the equivalence relation on $\mathscr{X}_{M'}$ resp. M' generated by the mappings Ψ resp. ψ coming from the morphisms (Ψ, ψ) in our category. Clearly, $\mathscr{R}_1$ is an open equivalence relation. As morphisms in the full subcategory of locally universal families are unique, it follows that $\mathscr{R}_2$ is also open and that each $\mathscr{R}_2$-equivalence class meets any fibre of p'_M in at most one point. So the quotient spaces $\mathscr{X}_M := \mathscr{X}_{M'}/\mathscr{R}_1$ and $M := M'/\mathscr{R}_2$ are analytic manifolds and the quotient $p_M:\mathscr{X}_M\to M$ of $p_{M'}$ is in a natural way a family of kählerian K3-surfaces. Clearly, $\Phi_{M'}$ induces a marking Φ_M of p. It is not hard to see that this family is a final object of our category.

The fact that τ_M is étale is immediate from (5.7).

(10.3) The following example, due to Atiyah [1], shows that the space M doesn't satisfy the Hausdorff axiom.

Consider the family of quartic surfaces $\{X_t\}$ in $\mathbf{P}^3$ which, in affine coordinates is given by

$$x^2(x^2-2)+y^2(y^2-2)+z^2(z^2-2)=2t^2.$$

Letting t run over the open unit disk $\mathbf{D}\subset\mathbf{C}$, we get a family $p:\mathscr{X}\to\mathbf{D}$ smooth over $\mathbf{D}-\{0\}$, while X_0 has the origin x_0 as its unique singular point (an ordinary double point). This is also the unique singular point of the total space $\mathscr{X}$. According to (10.1.) the nonsingular fibres of p are K3 surfaces. The tangent cone of $\mathscr{X}$ at x_0 is given by $x^2+y^2+z^2+t^2=0$, so the blow up at x_o, $\pi:\tilde{\mathscr{X}}\to\mathscr{X}$, desingularizes $\mathscr{X}$ and has a nonsingular quadric surface $E\subset\tilde{\mathscr{X}}$ as exceptional divisor. The proper transform $\tilde{Y}_0$ of X_0 is nonsingular and it is not hard to verify that it is a K3-surface. The surface E, being a nonsingular quadric, is isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$ and meets $\tilde{Y}_0$ with normal crossing along the curve

$E \cap \tilde{Y}_0$ of bidegree (1, 1). Each of the two possible rulings of E defines a contraction of E onto $E \cap \tilde{Y}_0$. We thus obtain two smooth families $p_i : \mathscr{X}_i \to \mathbf{D}(i = 1, 2)$ of algebraic K3-surfaces which are naturally isomorphic over $\mathbf{D} - \{0\}$. We claim that this isomorphism doesn't extend over $\mathbf{D}$. For otherwise we would have an automorphism of $\mathscr{X}$ acting trivially on $\mathscr{X} - X_0$ but nontrivially on the tangent cone of x_0, which is clearly impossible.

Now choose a marking $\Phi_1 : Rp_{1*}(\mathbf{Z}) \to L$ of p_1 (this possible since $\mathbf{D}$ is simply connected). The isomorphism over $\mathbf{D} - \{0\}$ induces a marking $\Phi_2 : Rp_{2*}(\mathbf{Z}) \to L$. These markings determine "classifying maps" $\psi_i : \mathbf{D} \to M$ which coincide on $\mathbf{D} - \{0\}$ but are nevertheless distinct. This can only happen if M is non-Hausdorff.

Recall the definition of Ω': it consists of the pairs $(\omega, \kappa) \in \Omega \times L_{\mathbf{R}}$ with $\langle \tilde{\omega}, \kappa \rangle = 0$ for some representative $\tilde{\omega} \in L_{\mathbf{C}}$ of ω and $\langle \kappa, \kappa \rangle > 0$. To each $\delta \in L$ with $\langle \delta, \delta \rangle = -2$ there is associated a Picard-Lefschetz reflection s_δ in L which also acts on Ω'. Let $\Omega'' \subset \Omega'$ denote the complement of the set of fixed points of Picard-Lefschetz reflections. Following the discussion in §8, Ω'' is open in Ω'. We define an equivalence relation $\sim$ on Ω' by letting $(\omega, \kappa) \sim (\omega', \kappa')$ if and only if $\omega = \omega'$ and κ, κ' belong to the same connected component $\Omega' \cap (\{\omega\} \times L_{\mathbf{R}})$. In other words, an equivalence class is an "abstract" Kähler cone. Let $\tilde{\Omega} := \Omega''/\sim$ denote the quotient space. It is provided with a canonical projection $\pi : \tilde{\Omega} \to \Omega$.

(10.4) Lemma: *The map $\pi : \tilde{\Omega} \to \Omega$ is a topological sheaf over Ω and thus $\tilde{\Omega}$ receives the structure of an analytic space, étale over Ω. For any $\omega \in \Omega$ the group $\{\pm id\} \times W_\omega$, where W_ω denotes the group generated by the Picard-Lefschetz reflections which fix ω, acts in a simple transitive manner on the fibre $\pi^{-1}(\omega)$.*

Proof: Since Ω'' is open in Ω', there exists for any $(\omega, \kappa) \in \Omega''$ an open neighbourhood U of ω in Ω and a continuous local section $\sigma : U \to \Omega''$ of the projection $\Omega'' \to \Omega$. The induced section $\tilde{\sigma} : U \to \tilde{\Omega}$ of π is clearly open, hence $\sigma(U)$ is a neighbourhood of the equivalence class of (ω, κ) in Ω''. The assertion concerning the group $\{\pm id\} \times W_\omega$ follows from the fact that this group acts in a simple transitive manner on the connected components of $\Omega'' \cap (\{\omega\} \times L_{\mathbf{R}})$.

The period mapping $\tau_M : M \to \Omega$ lifts in a natural way to a holomorphic mapping $\tilde{\tau}_M : M \to \tilde{\Omega}$ by assigning to $m \in M$ the equivalence class of ($\tau(m)$, Kähler cone of X_m). We then have the following reformulation of the Torelli theorem.

(10.5) THE TORELLI THEOREM (alternative version): *The diagram*

$$\begin{array}{ccc} M & \xrightarrow{\tilde\tau_M} & \tilde\Omega \\ & {\scriptstyle\tau_M}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \Omega & \end{array}$$

is commutative and identifies τ_M with the restriction of π to an dense-open subset of Ω. In particular, $\tilde\tau_M$ maps M isomorphically onto its image.

PROOF: The injectivity of $\tilde\tau_M$ is nothing but the Torelli theorem as stated in (9.1.). The map $\tilde\tau_M$ is a local isomorphism, since τ_M and π are. For any $\omega \in \Omega$, $\tilde\tau_M$ defines a $\{\pm id\} \times W_\omega$-equivariant injection from $\tau_M^{-1}(\omega)$. Since $\{\pm id\} \times W_\omega$ acts transitively in the latter $\tau_M^{-1}(\omega) \to \tilde\tau_M^{-1}(\omega)$ is an isomorphism if $\tau_M^{-1}(\omega) \neq \phi$.

What is clearly lacking in the statement of (10.5.) is a description of the image of τ_M. We know already that it is open and dense. A natural question to ask is whether τ_M is surjective.

In order to discuss the recent developments concerning this problem, we need the notion of an almost-polarized K3 surface: if X is a kählerian K3 surface, then a primitive algebraic class $d \in S_X$ is called an *almost-polarization* of X if d is in the closure of the Kähler cone and $\langle d, d\rangle > 0$. The existence of such a d implies, of course, that X is projective. The pair (X, d) is called an almost-polarized K3 surface and $\langle d, d\rangle$ is called its *degree*. An almost-polarization d of X is the class of an almost-ample divisor D on X, i.e. a multiple of D defines a birational morphism of X onto a projective surface whose singular locus consists of rational double points. Kulikov claims in [14] to prove a surjectivity property of τ_M for algebraic K3 surfaces: $(\omega, d) \in \Omega \times L$ is such that d is orthogonal to ω and $\langle d, d\rangle > 0$, then there exists a marked algebraic K3-surface (X, α) with $\tau(X, \alpha) = \omega$. As pointed out to us by D. Morrison this result can be sharpened as follows: there exists moreover an almost polarisation d_X with $\alpha(d_X) = d$. Kulikov's ingeneous proof contains some important ideas, but unfortunately there is a gap. This gap has recently been filled in by U. Persson and H. Pinkham. For almost-polarised K3-surfaces of degree two and four, resp. two this result has been obtained earlier by J.

Shah [22], resp. E. Horikawa [8]. Then A. Todorov [26] using the sharpening due to D. Morrison of the Kulikow-Persson-Pinkham result and S.T. Yau's proof of the Calabi conjecture proved that τ_M is surjective. Recently E. Looijenga obtained a proof that only uses the refined density theorem as given §6 as well as S.T. Yau's solution of the Calabi conjecture.

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