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Introduction

The purpose of the following is to introduce a new method for analyzing terms of finite type by means of ordinals and to apply it to the case of bar recursion of type zero (BR_0).

The method may be described briefly as follows. A term H is said to be semi-closed if all of its free variables have type level not exceeding 1. Suppose, in addition, that H has level not exceeding 2. We say that H has measure h if H has a computation tree of length 2^h. Let $F(X)$ be a term of type 0 with no free variables of level exceeding 2 and just one free variable X of level 2. For a suitable family or ordinal functions ξ_f parametrized by ordinals f, we say that $F(X)$ has measure f if $F(H)$ has measure $\xi_f(h)$ for all semi-closed H of measure h. Proceeding essentially in this way we extend the concept of measure to all finite type levels. The basic results about measures are derived in §2.

Let $\mathcal{T} + BR_0$ denote the theory obtained by adding BR_0 to Gödel's free variable theory $\mathcal{T}$ of primitive recursive functionals of finite type. To analyze $\mathcal{T} + BR_0$ we make use of the reformulation $\mathcal{U}$ of bar recursion of type zero given in [6]. Our results about measures yield the ordinal analysis of $\mathcal{U}$. This is carried out in §3. Let Δ denote the Bachmann ordinal $\varphi\epsilon_{\Omega+1}0$. The main result of the present paper is that every semi-closed term of type 0 in $\mathcal{U}$ has a computation tree of length less than Δ. This has the following consequences. Let $\mathcal{A}$ be elementary intuitionistic analysis plus axioms of choice plus Brouwer's bar theorem (bar induction of type zero). Let $\mathcal{S}(d)$ be

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Skolem (free variable) arithmetic of lowest type extended by ordinal recursion and transfinite induction over the ordinals less than d. Then

(i) The consistency of $\mathscr{A}$ is provable in $\mathscr{S}(\Delta)$.

(ii) Every provably recursive function of $\mathscr{A}$ is definable by ordinal recursion over the ordinals less than some ordinal less than Δ.

(iii) Suppose in $\mathscr{A}$ there is a proof that a decidable tree τ is well-founded. Then τ has length less than Δ.

These results are discussed in §4. By [2] the upper bound Δ is best possible. Thus Δ is the ordinal of bar induction of type zero.

Let $\mathscr{T}_{jk}$ denote the theory $\mathscr{T} + BR_0$ restricted to the case in which the bar recursion and primitive recursion functors have type levels not exceeding j and k, respectively. For $j \geq 4$, Vogel [7] has given a lower bound for the ordinal of $\mathscr{T}_{jk}$. In §4 we show that for $j \leq k$ the lower bounds given by Vogel are indeed upper bounds. Thus we have a detailed correspondence between ordinals and subsystems $\mathscr{T}_{jk}$ of $\mathscr{T} + BR_0$.

Let $\mathscr{H}(d)$ denote elementary intuitionistic analysis plus the axiom of choice plus transfinite induction over the ordinals less than d. The natural system for carrying out our proof of each of the results above is $\mathscr{H}(d)$ for the appropriate ordinal d. As we point out in §4, the proofs of the final results can be carried out in $\mathscr{S}(d)$.

1. Preliminaries

It is necessary for the reader to be acquainted with the following three items: the system $\mathscr{T} + BR_0$, the system $\mathscr{U}$, and computation trees. The definitions of these were given in [5] and [6], but in the interest of convenience and clarification we give the relevant features below. (A small difference from [5] and [6] is that we omit the operator δ and we simplify the definition of length).

Notation

The notation is as in [6]. In particular, n, m and c always denote numerals; c is the numeral corresponding to the sequence $\langle c_0, \ldots, c_{k-1}\rangle$ of numerals $c_0, \ldots, c_{k-1}$, where $k = l(c)$; t denotes a term of type 0; α denotes a variable of type 1, to be thought of as varying over free choice sequences; and FH denotes $FH_1 \ldots H_k$.

The *level* of a term is the level of its type symbol. The type symbol 0 has level zero. A term of type $\sigma \to \tau$ has level equal to the maximum of level $(\sigma) + 1$ and level (τ).

The system $\mathscr{T} + BR_0$. Only terms of type zero will be contracted.

Besides the primitive recursion functors R we have bar recursion functors Φ. The *deterministic* contractions are:

$$(1.1) \qquad (\lambda X F(X))G\boldsymbol{H} \quad \text{contr} \quad F(G)\boldsymbol{H}$$

$$(1.2) \qquad RFG0\boldsymbol{H} \quad \text{contr} \quad G\boldsymbol{H}$$

$$(1.3) \qquad RFG(n+1)\boldsymbol{H} \quad \text{contr} \quad Fn(RFGn)\boldsymbol{H}$$

$$(1.4) \qquad \Phi AFGc\boldsymbol{H} \quad \text{contr} \quad R(\lambda x \lambda y L_1)L_2(l(c) \dot{-} A[c]),$$

where L_1 and L_2 denote $Gc\boldsymbol{H}$ and $Fc(\lambda u \Phi AFG(c * u))\boldsymbol{H}$, respectively. The nondeterministic contractions are:

$$(1.5) \qquad x \quad \text{contr} \quad r$$

$$(1.6) \qquad Xm_1 \dots m_k \quad \text{contr} \quad \mathrm{n}$$

$$(1.7) \qquad \alpha m \quad \text{contr} \quad n$$

for all numerals $m_1, \dots, m_k$ and where x and X (and of course α) are free variables.

The system $\mathcal{U}$ is obtained by extending $\mathcal{T} + BR_0$ by adding a new rule of term formation: from α, c and t, form $\{\alpha, c, t\}$ of the same type as ΦA. For computation in $\mathcal{U}$ we replace the contraction (1.4) by the following three contractions.

$$(1.8) \qquad \Phi AFGc\boldsymbol{H} \quad \text{contr} \quad \{\alpha, c, A\alpha\}FGc\boldsymbol{H},$$

where α is chosen so as not to be free in A,

$$(1.9) \qquad \{\alpha, c, t\}FGc\boldsymbol{H} \quad \text{contr} \quad R(\lambda x \lambda y M_1)M_2(l(c) \dot{-} t_c),$$

where M_1 and M_2 denote $Gc\boldsymbol{H}$ and $Fc(\lambda u\{a, c, t\}FG(c * u))\boldsymbol{H}$, and t_c denotes the result of substituting $[c]$ for α in t, and

$$(1.10) \qquad \{\alpha, c, t\}FG(c * n)\boldsymbol{H} \quad \text{contr} \quad \{\alpha, c * n, t\}FG(c * n)\boldsymbol{H}.$$

We require that a subterm αm of $\{\alpha, c, t\}$ be contracted only when $m < l(c)$, in which case αm must be contracted to c_m. It was shown in §1 and §5 of [H4] that it is sufficient to give an ordinal analysis of $\mathcal{U}$.

A *computation tree* of a term K is a tree τ such that: K is attached to the initial node; if E is attached to a node, then the successors of E are obtained by contracting a fixed subterm of E according to (1.1)–(1.3) and (1.5)–(1.10); and each final term of τ is a numeral. It is understood that when a subterm t of E is contracted by a nondeterministic contraction, E has the successors $E_0, E_1, \ldots, E_n, \ldots$, where E_n is the result of replacing t by n in E.

At first thought it would seem that we ought to require a consistency condition; namely, if t has been contracted to a numeral at a node $\mathcal{M}$, and if $\mathcal{N}$ is a descendant of $\mathcal{M}$, then t must not be contracted to a different numeral at $\mathcal{N}$. There is no difficulty in adding the consistency condition but in the present paper we will not require it.

A computation tree is said to have ***length b*** if there is a function f from nodes $\mathcal{M}$ to ordinals $f(\mathcal{M})$ greater than zero such that if $\mathcal{M}$ is the initial node, then $f(\mathcal{M}) \leq b$, and if $\mathcal{M}$ has a successor $\mathcal{N}$, then $f(\mathcal{M}) > f(\mathcal{N})$. A term E of type 0 is said to have ***computation size b*** if E is the initial term of a computation tree of length b. If E has type level 1 or 2, then there are variables $X_1, \ldots, X_k$ such that $EX_1 \ldots X_k$ has type 0; and a computation size for $EX_1 \ldots X_k$ is understood to be a computation size for E.
If a term E has no free variables of level greater than 1, then E is said to be *semi-closed*. Whenever we speak of a computation size for E it is understood that E is semi-closed and of level not exceeding 2.

Ordinals

In this paper $d + e$ denotes the *natural sum* (Hessenberg sum) of the ordinals d and e; and de denotes the *natural product*. The natural product can be characterized as follows: it is commutative and associative, it distributes over the natural sum, and $2^d 2^e = 2^{d+e}$.

As usual, Ω denotes the first uncountable ordinal. For notational convenience let ψ denote the function $\bar{\varphi}$ treated in [1], page 215, the related function φ being based on the starting function $\varphi 0 b = \epsilon_b$. In the present paper, 'ordinal' means: ordinal less than or equal to $\epsilon_{\Omega+1}$. Hence φe and ψe are well-defined. Essentially as on page 367 of [4] let θe denote $\psi e 0$ and let $d \prec e$ denote the relation: $d < e$ and $\theta d < \theta e$. It is useful to keep in mind that $d \prec e$ is equivalent to: $d < e$ and $(|\forall| x < \Omega)(\psi dx < \psi ex)$. A number of properties which are useful for carrying out the proofs in §2 are listed on page 367 of [4]. We denote the Bachmann ordinal $\varphi \epsilon_{\Omega+1} 0$ by Δ.
To formalize the metamathematics, the ordinals are represented by notations as in [2] and then the notations are numbered.

2. The notion of measure and derive the basic results about measures

Degree

Let F be a term and $\boldsymbol{X}$ be a list of variables such that $F\boldsymbol{X}$ has type

0. We say that F has degree $j+1$ if all free variables of FX have level less than $j+1$ and at least one of the variables has level j. A term of level 0 with no free variables is said to have degree 0. We say that a list of terms $D_1, \ldots, D_k$ has level (resp. degree) j if each D_r has level (resp. degree) j.

Preliminary remarks

Suppose $F(\boldsymbol{X})$ has level 0 and degree $j+1$, where $\boldsymbol{X}$ is a list of the free variables of level j in $F(\boldsymbol{X})$. Corresponding to the list $\boldsymbol{X}$, let $\boldsymbol{D}$ be a list of terms of level and degree j. Then $F(\boldsymbol{D})$ has degree not exceeding j. Thus $F(\boldsymbol{X})$ has associated with it a mapping of lists $\boldsymbol{D}$ of level and degree j into terms $F(\boldsymbol{D})$ of degree not exceeding j. The definition of the measure of $F(\boldsymbol{D})$ is based on this mapping.

The notion of measure of a term F will be defined relative to a number called the *height*. The height is always at least 2 and at least the degree of F. Heuristically: to attach height p to F, where p is greater than the degree of F, means that we are considering F to contain free variables of level $p-1$ vacuously.

We will define three versions of the notion of measure. Each version is based on a family of functions $\xi_j(b, d)$ of ordinals b and d, where j ranges over degrees not less than 3. We write $\xi_j bd$ for $\xi_j(b, d)$. It is understood that a measure is an ordinal greater than 0.

First Version. We take $\xi_j bd$ to be $2^b d$ for all $j \geq 3$.

Second Version. We take $\xi_3 bd$ to be ψbd, where ψ is as in §1; $\xi_4 bd = b + d$, and $\xi_j bd = 2^b d$ for $j \geq 5$.

Third Version. We take $\xi_3 bd$ to be $b + d$, and $\xi_j bd = 2^b d$ for $j \geq 4$.

Say that a list of terms $D_1, \ldots, D_k$ has measure b for height p if each D_r has measure b for height p.

Definition of measure

(i) First and Second Versions: if a term F of degree not exceeding 2 has computation size 2^b, then F has measure b for height 2. Third Version: replace 2^b by ϵ_b.

(ii) For F not of type 0 and a variable X not free in F: if FX has measure b for height p, then so does F.

(iii) Suppose the notion of measure has been defined for all terms of degree j, and let $F(\boldsymbol{X})$ be a term of level 0 and degree $j+1$, where $\boldsymbol{X}$ is a list of the free variables of level j in $F(\boldsymbol{X})$. If $F(\boldsymbol{D})$ has measure $\xi_{j+1} bd$ for height j, for all $\boldsymbol{D}$ of degree j and measure d, then $F(\boldsymbol{X})$ has measure b for height $j+1$.

(iv) If a term F of level 0 has measure $\xi_{j+1} b1$ for height j, then F

has measure b for height $j+1$ except in the following cases. Second Version: if F has measure b for height 3, then F has measure b for height 4. Third Version: if F has measure b for height 2, then F has measure b for height 3.

Mainly, in the present paper, we will be concerned with the Second Version of the measure. This is the version appropriate to the ordinal analysis of BR_0 when the functor Φ has level not less than 4.

REMARK 2.1: It is easy to prove that if F has measure b for height j, and if $b < e$, then F has measure e for height j.

For the Second Version it is not hard to verify the following relations (where b, d, e, f, g are greater than 0).

(2.1) $$\xi_j bd + \xi_j ed \leqslant \xi_j(b+e)d \text{ except when } j = 4$$

(2.2) $$\xi_j fg \leqslant \xi_j(f+g)1 \text{ except when } j = 4$$

(2.3) $$d \leqslant \xi_j bd$$

(2.4) $$\xi_j f(\xi_j gd) \leqslant \xi_j(f+g)d$$

(2.5) $$\xi_j gd + 1 \leqslant \xi_j(g+1)d$$

(2.6) $$f < g \to \xi_j fd < \xi_j gd.$$

Substitution

The notation $F^{\boldsymbol{D}}$ indicates the result of substituting various members D_r of the list $\boldsymbol{D}$ for free variables of F. Some of the free variables may occur vacuously.

REMARK 2.2 Suppose G has measure g for height $j+1$ and let $\boldsymbol{D}$ be a list of degree j and measure d for height j. It is easy to see that if $G^{\boldsymbol{D}}$ has degree not exceeding j, then $G^{\boldsymbol{D}}$ has measure $\xi_{j+1}gd$ for height j.

In Lemmas 2.1–2.11 we will use the Second Version, but the proofs of Lemmas 2.1–2.8 go through (with appropriate modifications) for the First and Third Versions also.

LEMMA 2.1: *Suppose $F(X)$ and G have measures f and g for height $j+1$. Then $F(G)$ has measure $f+g$ for height $j+1$.*

PROOF: If $F(X)$ has level greater than 0, we can replace $F(X)$ by $F(X)\boldsymbol{Y}$ of level 0 and same degree as $F(X)$. Hence it suffices to consider the case in which $F(X)$ has level 0. Also note that the lemma is obviously true if the variable X occurs vacuously in $F(X)$. Hence we can assume that X occurs nonvacuously in $F(X)$. We now proceed by induction on j.

Suppose $j = 1$. Then $F(X)$ and F are terms of degree not exceeding 2 with computation sizes 2^f and 2^g, respectively. Hence $F(G)$ has computation size $2^f 2^g \leq 2^{f+g}$ by Theorem 2.1 of [6].

For the inductive step, assume the lemma is true for j replaced by $j-1$, where $j \geq 2$.

Case 1: $F(X)$ and G have degree less than $j+1$. Suppose $j \neq 3$. Then $F(X)$ and G have measures $\xi_{j+1}f1$ and $\xi_{j+1}g1$, respectively, for height j. Hence, by induction hypothesis and (2.2), $F(G)$ has measure $\xi_{j+1}(f+g)1$ for height j. Thus $F(G)$ has measure $f+g$ for height $j+1$. For the case $j = 3$, replace $\xi_{j+1}f1$, $\xi_{j+1}g1$ and $\xi_{j+1}(f+g)1$ by f, g and $f+g$ in the argument just given.

Case 2: At least one of $F(X)$, G has degree $j+1$.

Case 2.1: All free variables of $F(G)$ have level less than j.

Case 2.1*a*: X has level less than j. Then all free variables of $F(X)$ have level less than j, so $F(X)$ has degree less than $j+1$. Also G has level less than j. But G occurs nonvacuously in $F(G)$, so all free variables of G have level less than j. Hence G has degree less than $j+1$, contradicting the requirements for Case 2. Thus Case 2.1a does not occur.

Case 2.1*b*: X has level j. Then G has level j and degree j (see Case 2.1a), and G has measure $\xi_{j+1}g1$ for height j (or simply g if $j = 3$). Hence $F(G)$ has measure $\xi_{j+1}(f+g)1$ for height j by (2.4) if $j \neq 3$. If $j = 3$, then $F(G)$ has measure $\xi_4 fg = f+g$ for height 3.

Case 2.2: $F(G)$ has free variables of level at least j. The maximum level of these free variables must be j because both $F(X)$ and G have degree not exceeding $j+1$ by hypothesis of the lemma. Let $\boldsymbol{Y}$ be a list of the free variables of level j in $F(G)$, and let $\boldsymbol{D}$ be a corresponding list of degree j and measure d for height j. We must show $F(G)^{\boldsymbol{D}}$ has measure $\xi_{j+1}(f+g)d$ for height j.

Case 2.2*a*: G has level j. Then $G^{\boldsymbol{D}}$ has degree j, and, by (2.3), the list $G^{\boldsymbol{D}}, \boldsymbol{D}$ has measure $\xi_{j+1}gd$ for height j. Substituting this list into $F(X)$ we get $F^{\boldsymbol{D}}(G^{\boldsymbol{D}})$ of measure $\xi_{j+1}f(\xi_{j+1}gd)$ for height j. Hence, by (2.4), $F(G)^{\boldsymbol{D}}$ has measure $\xi_{j+1}(f+g)d$ for height j.

Case 2.2*b*: G has level less than j. Then $F^{\boldsymbol{D}}(X)$ is the result of substituting elements of the list $\boldsymbol{d}$ for all free variables of level j in $F(X)$. Hence. by Remark 2,2, $F^{\boldsymbol{D}}(X)$ has measure $\xi_{j+1}fd$ for height j. Hence, by induction hypothesis and (2.1), $F^{\boldsymbol{D}}(G^{\boldsymbol{D}})$ has measure $\xi_{j+1}(f+g)d$ for height j so long as $j \neq 3$. It remains to discuss the case $j = 3$. In this case $G^{\boldsymbol{D}}$ has measure $g+d$ for height 3. We must show that $(F(G)^{D})^{E}$ has measure $\psi(f+g+d)e$ for height 2, for all suitable lists E of degree 2 and measure e for height 2. *Assume G has level* 2. Then the list $(G^{D})^{E}$, E has level 2, degree 2, and measure $\psi(g+d)e$ for height 2. Substituting this list into $F^{D}(X)$ we get $(F^{D})^{E}((G^{D})^{E})$ of measure $\psi(f+d)(\psi(g+d)e)$ for height 2. This is easily seen to be less than $\psi(f+g+d)e$ by use of Theorem 4.1, p. 215, of [1]. *Assume G has level less than* 2. Then $(F^{D})^{E}(X)$ has measure $\psi(f+d)e$ for height 2. Recall $((G^{D})^{E}$ has measure $\psi(g+d)e$ for height 2. Hence, by Theorem 2.1 of [6], $(F^{D})^{E}((G^{D})^{E})$ has measure $\psi(f+d)e+\psi(g+d)e$ for height 2. This is less than $\psi(f+g+d)e$ by Theorem 4.1 of [G1]. □

The proof of Lemma 2.1 goes through if we replace X and G by lists X and G of a given level.

LEMMA 2.2: *Suppose a term F of type* 0 *reduces to G in one reduction step, and G has measure g for height* $k \geq$ *degree(F). Then F has measure* $g+1$ *for height k.*

PROOF: By induction on k. If $k=2$, then G has computation size 2^{g}, so F has computation size $2^{g}+1 \leq 2^{g+1}$. For the induction step assume the lemma is true for $k-1$ in place of k.

Case 1: $k = \text{degree}(F)$. Let D be a suitable list of terms of degree and level $k-1$, where D has measure d for height $k-1$. Then F^{D} has degree not exceeding $k-1$, and G^{D} has measure $\xi_{k-1}gd$ for height $k-1$. Hence, by induction hypothesis and (2.5), F^{D} has measure $\xi_{k-1}(g+1)d$ for height $k-1$. Thus F has measure $g+1$ for height k.

Case 2: $k < \text{degree}(F)$, Replace F^{D}, G^{D} and d by F, G and 1 in Case 1. □

LEMMA 2.3: *Let the variable x be free in* $F(x)$. *Suppose* $F(n)$ *has measure* $d<e$, *for height k, for every numeral n (where d may depend on n). Then* $F(x)$ *has measure e for height k.*

PROOF: For a suitable list of variables Z the terms $F(n)Z$ and $F(x)Z$ have the same measures as $F(n)$ and $F(x)$, respectively. Hence it is sufficient to consider the case in which $F(x)$ has level 0. Now proceed by induction on k as in Lemma 2.2. □

REMARK 2.3: The proof of Lemma 2.3 uses the axiom of choice in an essential way. This occurs in the proof of the basis of the induction–namely, the case $k=2$. Specifically, we suppose that a computation tree is represented in the metamathematics by a characteristic function (see[5]) and observe that this function can be combined with the length function. Thus the assumption '$F(n)$ has measure d for every n' has the form $\forall n \exists f A(n,f)$, where f ranges over functions of type $0 \to 0$. By the axiom of choice there exists g such that $\forall n A(n, gn)$. From the existence of this g we infer that the term $F(x)$ has computation size e.

LEMMA 2.4: *Let the variable x be free in $F(x)$. Suppose $F(x)$ has measure e for height k. Then $F(n)$ has measure e for height k for every numeral n.*

PROOF: It is easy to prove that if a semi-closed term $G(x)$ of type 0 has computation size g, then so does $G(n)$. Using this, proceed by induction on the degree of $F(x)$ as in Lemmas 2.2 and 2.3. □

LEMMA 2.5: *Suppose $F(x)$ has measure e for height k, and suppose t (of type 0) has computation size b. Then $F(t)$ has measure $e+b$ for height k.*

PROOF: By induction on k as in Lemmas 2.2–2.4. For the case in which $F(x)$ is semi-closed and has level 0, it is easy to see that $F(t)$ has computation size $2^e + b \leq 2^{2+b}$. □

LEMMA 2.6: *Let R be a primitive recursion functor of level k. Then $RXYn$ has measure $2n+2$ for height k for every numeral n.*

PROOF: By induction on n. It is easy to see that the variable Y has measure 1 for height k. Also RXYOZ contracts to YZ. Hence $RXYO$ has measure 2 for height k by Lemma 2.2. For the induction step, let u and W be variables and observe that XuW has measure 1 for height k. Hence also does XnW (by Lemma 2.4). By induction hypothesis $RXYn$ has measure $2n+2$ for height k. Hence, by Lemma 2.1, $Xn(RXYn)$ has measure $1+2n+2$ for height k. Hence, by Lemma 2.2, $RXY(n+1)$ has measure $2(n+1)+2$ for height k. □

LEMMA 2.7: *Let R be a primitive recursion functor of level k. Then R has measure ω for height k.*

PROOF: Simply observe that, by Lemmas 2.3 and 2.6, $RXYu$ has measure ω for height k. □

LEMMA 2.8: *Suppose t has computation size b. Then the term $\{\alpha, c, t\}XYc$ of degree k has measure $(b\omega+2\omega+1)^2$ for height k.*

PROOF: It is assumed that we are given a computation tree τ of t. Let $ord(\alpha, c, t)$ be defined as follows. If, for computation in τ, the term to be contracted in t has the form αm with $m \geq l(c)$, say that c is m-critical in t. If t is not a numeral and c is not critical in t, define $ord(\alpha, c, t)$ to be $\omega(b+2)$. If c is m-critical in t define $ord(\alpha, c, t)$ to be $\omega(b+1)+m+1-l(c)$. If t is a numeral n, define $ord(\alpha, c, t)$ to be $\omega(b+1)+(n+1) \dot{-} l(c)$.

By transfinite induction on $ord(\alpha, c, t)$ we will show that $\{\alpha, c, t\}XYc$ has measure $(ord(\alpha, c, t)+1)^2$. In the present proof, measures are for height k.

Case 1: t is a numeral less than $l(c)$. Then $\{\alpha, c, t\}XYc\boldsymbol{Z}$ reduces, in finitely many steps, to $Yc\boldsymbol{Z}$ which has measure 1, so apply Lemma 2.2.

Case 2: t is a numeral not less than $l(c)$ or c is critical in t. Then $ord(\alpha, c*n, t) < \mathrm{ord}(\alpha, c, t)$ for every n. By induction hypothesis $\{\alpha, c*n, t\}XY(c*n)\boldsymbol{Z}$ has measure $(ord(\alpha, c*n, t)+1)^2$ for every n. Hence, by (1.10) and Lemma 2.2, $\{\alpha, c, t\}XY(c*n)\boldsymbol{Z}$ has measure $(ord(\alpha, c*n, t)+1)^2+1$ which is less than $ord(\alpha, c, t)^2+2$. Therefore $\lambda u\{\alpha, c, t\}XY(c*u)$ has measure $ord(\alpha, c, t)^2+4$ by Lemmas 2.2 and 2.3. But $XcW\boldsymbol{Z}$ has measure 1. Hence M_2–see (1.9)–has measure $ord(\alpha, c, t)^2+5$ by Lemma 2.1. Also M_1 has measure 1. For each numeral m the term $R(\lambda x\lambda yM_1)M_2m\boldsymbol{Z}$ reduces to M_1 or M_2 in no more than 2 steps and hence has measure $ord(\alpha, c, t)^2+7$. By Theorem 2.1 of [6], t_c has computation size $b\omega$. Hence $l(c) \dot{-} t_c$ has computation size less than $b\omega+\omega$. Hence, by Lemmas 2.3 and 2.5, $R(\lambda x\lambda yM_1)M_2(l(c) \dot{-} t_c)\boldsymbol{Z}$ has measure $ord(\alpha, c, t)^2+8+b\omega+\omega$ which is less than $ord(\alpha, c, t)^2+2ord(\alpha, c, t)$. But the latter term is obtained by one reduction step from $\{\alpha, c, t\}XYc\boldsymbol{Z}$ so $\{\alpha, c, t\}XYc\boldsymbol{Z}$ measure $((ord(\alpha, c, t)+1)^2$ by Lemma 2.2.

Case 3: Neither Case 1 nor Case 2. Then t has one or more successors t' in τ, and $ord(\alpha, c, t') < ord(\alpha, c, t)$. Since $\{\alpha, c, t\}XYc\boldsymbol{Z}$ has the successor or successors $\{\alpha, c, t'\}XYc\boldsymbol{Z}$ we can apply the induction hypothesis and Lemma 2.2. □

LEMMA 2.9: *For $k>4$ the BR_0 functor Φ of level k has measure Ω for height k.*

PROOF: For ordinals $b_1, \ldots, b_r, \ldots$ define $[b_1, \ldots, b_r]$ by induction on r by the equations $[b_1] = b_1$ and $[b_1, \ldots, b_{r+1}] = 2^{[b_1, \ldots, b_r]} b_{r+1}$. For suitable variables X_1, X_2, X_3, U, Z the term $\Phi X_1 X_2 X_3 uZ$ has type 0. Denote this term by L_1 and consider the sequence of terms $L_1, \ldots, L_{k-1}$ where L_{r+1} is obtained from L_r by substituting a list D^r of level and degree $k - r$ for all the variables of level $k - r$ in L_r. It is easy to see that to prove L_1 has measure Ω for height k it suffices to show that L_{k-1} has measure $\psi([\Omega, d_i, \ldots, d_{k-4}] + d_{k-3}) d_{k-2}$ for height 2, for all sequences $D^1, \ldots, D^{k-2}$ with corresponding measure $d_1, \ldots, d_{k-2}$ for heights $k - l, \ldots, 2$.

The term L_{k-2} has the form $\Phi X_1 F_1 G_1 uB$, and L_{k-1} has the form $\Phi AF_2 G_2 uC$, where A belongs to the list D^{k-2} and hence has measure d_{k-2}. Denote d_{k-2} by b and observe that $\{\alpha, c, A\alpha\} X_2 X_3 cZ$ has measure $(\omega 2^b + 2\omega + 1)^2$ for height k by Lemma 2.8. Hence $\Phi AX_2 X_3 cZ$ has measure $e + 1$ for height k by Lemma 2.2, where e denotes $(\omega 2^b + 2\omega + 1)^2$. Hence $\Phi AX_2 X_3 uZ$ has measure $e + 2$ by Lemma 2.3. But now, substituting D^1 into the latter term, then substituting D^2 into the result, and so on, we again obtain L_{k-1}. Hence L_{k-1} has measure $\psi([e + 2, d_1, \ldots, d_{k-4}] + d_{k-3}) d_{k-2}$ for height 2. To prove that this is less than $\psi([\Omega, d_1, \ldots, d_{k-4}] + d_{k-3}) d_{k-2}$ we appeal to Theorem 4.1 of [1], p. 215, which involves the notion of 'constituent' (p. 204). Observe $d_{k-2} < \Omega$. Hence $e < \Omega$, so $[e + 2, d_1, \ldots, d_{k-4}] + d_{k-3}$ is less than $[\Omega, d_1, \ldots, d_{k-4}] + d_{k-3}$. Also it is easy to see that the constituents of the former are less than the next epsilon number after the latter. The desired inequality now follows from Theorem 4.1 of [1]. □

LEMMA 2.10: *The BR_0 functors Φ of levels 3 and 4 have measure 1 for height 4.*

PROOF: This follows from Theorem 2.4 of [6] and the basic order relations for the function ψ – together with Lemma 2.2. □

LEMMA 2.11: *The primitive recursion functors R of levels 3 and 4 have measure 1 for height 4.*

PROOF: Similar to the proof of Lemma 2.10 but adapting the proof of Theorem 2.4 of [6] to the case of R rather than Φ. □

LEMMA 2.12: *For the Third Version of the measure: the BR_0 functor Φ of level 3 has measure 1 for height 3.*

PROOF: By Theorem 2.3 of [6] and Lemma 2.2. □

LEMMA 2.13: *For the Third Version of the measure: the primitive recursion functor R of levels* 2 *and* 3 *have measure* 1 *for height* 3.

PROOF: Similar to the proof of Lemma 2.12. □

3. Computation sizes

Let $\mathcal{T}_{jk}$ denote the set of all terms of $\mathcal{T} + BR_0$ in which the primitive recursion functors R and the BR_0 functors Φ have levels not exceeding j and k, respectively. We will now determine computation sizes for the semi-closed terms of $\mathcal{T}_{jk}$ (considered as terms of $\mathcal{U}$).

Elementary terms

By an elementary term is meant a term built up from the constant 0, the successor functor S and variables by means of the operations of application and λ-abstraction.

The following lemma holds for all three versions of the measure.

LEMMA 3.1: *Let F be an elementary term of degree k. Then F has finite measure for height $i \geq k$, so long as $i \geq 3$.*

PROOF: The result is easy to prove if F is 0, S or a variable. For general F, choose a fixed height $j \geq i$ such that j is greater than the level of every free or bound variable in F. In building up F, suppose we have constructed a term A of finite measure for height j and the next step is to construct λYA. From Lemma 2.2 and the fact that $(\lambda YA)YZ$ reduces to AZ in one step, it follows that λYA has finite measure for height j. On the other hand, if the next step is to construct AB, where B has finite measure for height j, observe that AB arises by substituting B for X in AX and apply Lemma 2.1. Thus F has finite measure for height j. Now lower the height to i.

Notation

(a, i, b) denotes the ordinal defined by the recursion equations $(a, 0, b) = b$ and $(a, j+1, b) = a^{(a, j, b)}$. The function φ is as in §1.

THEOREM 3.1: *For each semi-closed term E of type* 0 *in $\mathcal{T}_{jk}$ with*

$j \geq 5$ we can find $b < (\omega, k \dot{-} j, \omega)$ such that E has computation size $\varphi(\Omega, j-4, b)0$.

PROOF: Suppose $k > j$. The term E arises from an elementary term F by substituting various functors R and Φ for free variables. In the following discussion some substitutions may be vacuous. We work with the Second Version of the measure. By Lemma 3.1, F has finite measure p for height $k+1$. We now proceed by steps. The first step is to substitute all functors R of level k into F, getting a term H_1. By Lemma 2.7, H_1 has measure ωr for height k, where $r = 2^k$. The second step is to substitute all functors R of level $k-1$ into H_1, getting a term H_2 of measure $\omega^r\omega$ for height $k-1$. In $k-j$ steps we get a term H_{k-j} of measure $(\omega, k-j-1, \omega)$ for height $j+1$. Now substitute all functors Φ and R of level j into H_{k-j}, getting a term H_{k-j+1}. By Lemmas 2.7 and 2.9, H_{k-j+1} has measure Ωa for height j, where $a < (\omega, k-j, \omega)$. In $j-4$ more steps of this kind we get a term H_{k-3} with measure $(\Omega, j-4, d)$ for height 4, where $d < (\omega, k-j, \omega)$ – by possible use of Lemma 2.11. In one more step we get H_{k-2} with measure $(\Omega, j-4, e)$ for height 3, where $e < (\omega, k-j, \omega)$. Finally, in one more step we get H_{k-1} with measure $\psi(\Omega, j-4, f)1$ for height 2, where $f < (\omega, k-j, \omega)$. This measure is less than $\varphi(\Omega, j-4, b)0$, where $b = f+1$. But H_{k-1} is E. Since $\varphi(\Omega, j-4, b)0$ is epsilon number, it is a computation size for E.

The proof for the case $k \leq j$ is similar. □

Let $\mathcal{T}_j$ be the union of the sets $\mathcal{T}_{jk}$ for all k. From Theorem 3.1 we conclude the following two theorems.

THEOREM 3.2: *Each semi-closed term of type 0 in $\mathcal{T}_j$* with $j \geq 5$ *has computation size less than $\varphi(\Omega, j-4, \epsilon_0)0$.*

THEOREM 3.3: *Each semi-closed term of type 0 in $\mathcal{T}$ has computation size less than $\varphi\epsilon_{\Omega+1}0$.*

Computation sizes for the terms of $\mathcal{T}_{jk}$ for $j = 3$ and $j = 4$ were found in Theorem 3.4 of [6]. Let us observe that these results can be obtained by the present method.

THEOREM 3.4: *For each semi-closed term E of type 0 in $\mathcal{T}_{3k}$ or $\mathcal{T}_{4k}$ we can find $b < (\omega, k \dot{-} j, \omega)$, where $j = 3$ or $j = 4$, respectively, such that E has computation size ϵ_b or $\varphi b0$, respectively.*

PROOF: Similar to the proof of Theorem 3.1. For $j = 4$ use the Second Version of the measure, and Lemmas 2.10–1.11. For $j = 3$ use the Third Version and Lemmas 2.12–2.13. □

In [6] we defined operators Φ suitable for the functional interpretation of König's lemma. Using the First Version of the measure we see from Theorem 2.2 of [6] and Lemma 2.2 that the functors Φ have measure $\omega + 2$ for height 3. Hence by the methods above we find that a semi-closed term E of type 0 containing the functors Φ and primitive recursion functors R of level $k \geq 3$ has measure less than $(\omega, k-2, \omega)$ for height 2, so E has computation size less than $(\omega, k-1, \omega)$ as in Theorem 3.4 of [6]. However, for $k = 2$ the present methods yield a computation size less than $(\omega, 2, \omega)$: not quite as good as the size less than $(\omega, 1, \omega)$ obtained in [6].

REMARK 3.1: In the case of the *rule* of BR_0 we do not include the functors Φ. Rather, we introduce the following rule of term formation: for each closed term A of type 2 introduce a functor Φ_A together with the contractions obtained by replacing ΦA by Φ_A in (1.4) and (1.8). Using the First Version of the measure we see by Lemma 2.8, and the methods above, that a semi-closed term of type 0 has computation size less than ϵ_0. Thus we get an analysis of the rule of BR_0 by means of the ordinals less than ϵ_0.

4. Conclusions

We recall that (a, i, b) is defined by the recursion equations $(a, 0, b) = b$ and $(a, j+1, b) = 2^{(a,j,b)}$. Let $\mathcal{T}_{jk}$, $\mathcal{H}(d)$ and $\mathcal{S}(d)$ be as defined in the Introduction, and let $\mathcal{T}_j$ be the union, over all k, of the theories $\mathcal{T}_{jk}$. Suppose $j \geq 5$ and let $\mathcal{A}$ be a theory with a Gödel functional interpretation in $\mathcal{T}_j$. As mentioned in the Introduction, Theorem 3.1 is provable in $\mathcal{H}(\varphi(\Omega, j-4, b)0)$, where $b < (\omega, k \dot{-} j, \omega)$. Hence, by the same discussion as in §§4–5 of [6], the following theorems are provable in $\mathcal{S}(\varphi(\Omega, j-4, \epsilon_0)0)$.

THEOREM 4.1: *$\mathcal{A}$ is consistent*

THEOREM 4.2: *Each provably recursive function of $\mathcal{A}$ is definable by ordinal recursion (of lowest type) over the ordinals less than some ordinal less than $\varphi(\Omega, j-4, \epsilon_0)0$.*

THEOREM 4.3: *Suppose in $\mathscr{A}$ there is a proof that a decidable tree τ is well-founded. Then τ has length less than $\varphi(\Omega, j-4, \epsilon_0)0$.*

Similarly we have the more detailed versions of Theorems 4.1–4.3 in which $\mathscr{T}_j$ and $\varphi(\Omega, j-4, \epsilon_0)0$ are replaced by $\mathscr{T}_{jk}$ and $\varphi(\Omega, j-4(\omega, k \dot{-} j, \omega))0$, respectively, and the less detailed versions in which $\mathscr{T} + BR_0$ corresponds to $\varphi\epsilon_{\Omega+1}0$. By [7] all these ordinal bounds are the best possible. For completeness we recall that in [6] it was shown that the ordinals corresponding to $\mathscr{T}_3$ and $\mathscr{T}_4$ are ϵ_{ϵ_0} and $\varphi_{\epsilon_0}0$, respectively.

In line with the discussion in the Introduction we mention that the theorems above are of particular interest when the theory $\mathscr{A}$ is taken to be elementary analysis plus BI_0 plus axioms of choice and other axioms which have a Gödel functional interpretation in $\mathscr{T} + BR_0$ (see [3]).

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