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THE GROWTH OF IWASAWA INVARIANTS IN A FAMILY

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1. Introduction

Approximately twenty years ago, Iwasawa initiated the study of Z_p-extensions. If k is a number field and p is a rational prime, a Galois extension K of k is called a Z_p-extension if $G(K/k)$ is topologically isomorphic to the additive group in the ring Z_p. If K/k is a Z_p-extension, then for each integer n, there is a unique subfield k_n of K so that $[k_n : k] = p^n$. In [7] Iwasawa proved the following:

THEOREM: *If p^{e_n} denotes the power of p which divides the class number of k_n, then there are constants μ, λ, and ν, independent of n, such that for all sufficiently large n, $e_n = \mu p^n + \lambda n + \nu$.*

The constants $\mu = \mu(K/k)$ and $\lambda = \lambda(K/k)$ are non-negative integers and they are called the Iwasawa invariants of the Z_p-extension K/k.

It should be noted that if k is a number field, then k has at least one Z_p-extension. In fact, if we adjoin to k all p-power roots of unity, the resulting extension will have Galois group isomorphic to the product of a finite group with Z_p. This extension will contain a Z_p-extension of k, called the cyclotomic Z_p-extension of k. Moreover, if we let k_{Z_p} denote the composite of all Z_p-extensions of k, then k_{Z_p} is known to be a Galois extension of k such that $G(k_{Z_p}/k) \cong Z_p^d$ where $r_2 + 1 \leq d \leq [k : Q]$ (r_2 is the number of complex primes in k). It is conjectured ("Leopoldt's conjecture") that $d = r_2 + 1$, but this conjecture plays no role in what follows.

This work concerns itself with Z_p^2-extensions of number fields. If k is a number field and p is a rational prime, a Galois extension K of k

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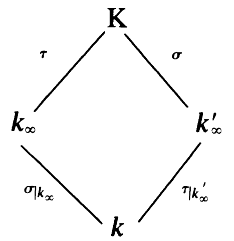
will be called a Z_p^2-extension if G(K/k) is topologically isomorphic to the additive group in $Z_p \oplus Z_p$. To insure the existence of such extensions, we will assume throughout that k has at least one complex prime. The major purpose of our investigation is to prove a theorem which can be described as follows:

Let k be a number field and let k_∞ and k'_∞ be two Z_p-extensions of k so that $k_\infty \cap k'_\infty = k$. If $\mathrm{K} = k_\infty k'_\infty$, then K is a Z_p^2-extension of k (conversely, it is not hard to see that every Z_p^2-extension of k is the composite of two Z_p-extensions of k whose intersection is precisely k). Let $\mathrm{G} = \mathrm{G}(\mathrm{K}/k)$ and choose topological generators σ and τ for G so that if $\mathrm{H} = \mathrm{G}(\mathrm{K}/k_\infty)$ and $\mathrm{H}' = \mathrm{G}(\mathrm{K}/k'_\infty)$, then H is generated topologically by τ and H′ is generated topologically by σ. Also, $\sigma|_{k_\infty}$ generates $\mathrm{G}(k_\infty/k)$ and $\tau|_{k'_\infty}$ generates $\mathrm{G}(k'_\infty/k)$. Let the subfield of k_∞ fixed by σ^{p^n} be denoted by k_n, and let k'_n denote the subfield of k'_∞ fixed by τ^{p^n}. Then if we put $\mathrm{K}_n = k'_\infty k_n$, we see that K_n is a Z_p-extension of k_n, and hence we can speak of the Iwasawa invariants $\lambda_n = \lambda(\mathrm{K}_n/k_n)$ and $\mu_n = \mu(\mathrm{K}_n/k_n)$. These invariants grow regularly with n as described by the following result:



THEOREM 1.1: *There are constants ℓ, m_0, m_1, c, and c_1, independent of n, such that for all sufficiently large n, $\lambda_n = \ell p^n + c$ and $\mu_n = m_0 p^n + m_1 n + c_1$.*

The proof of this theorem is the main concern of this paper. We will also be able to give a precise description of the invariant m_0, and to show that it depends only on K/k and not on the individual Z_p-extensions used to obtain K. We will also be able to construct examples where m_0 is arbitrarily large, and we will give necessary and sufficient conditions for m_0 to vanish.

In §2 we set up the module-theoretic machinery needed to prove Theorem 1.1, and in §3 we use these results to carry out the proof. The rest of the paper is devoted to some consequences of Theorem 1.1 and to a description of m_0.

The proof of this result forms part of my Brandeis Ph.D. thesis,

conducted under the direction of Ralph Greenberg. I would like to express my deep gratitude to Dr. Greenberg for helping me with many of the ideas in this paper and for his constant encouragement during the course of this research.

In the rest of this section we develop some notation and obtain some basic facts which will be useful in what follows.

If G is any multiplicative group isomorphic to the additive group $\mathbf{Z}_p^d$ ($d \geq 1$), and J is any subgroup of G, we let Λ_J denote $\mathbf{Z}_p[[J]]$, the complete group ring of J over $\mathbf{Z}_p$. If we choose topological generators $\{\sigma_1, \sigma_2, \ldots, \sigma_d\}$ of G, then we can identify Λ_G with the power series ring $\mathbf{Z}[[T_1, \ldots, T_d]]$ by putting $T_i = \sigma_i - 1$. If H_i is the subgroup of G generated topologically by σ_i, then under this identification, $\Lambda_{H_i} = \mathbf{Z}_p[[T_i]]$.

We will be concerned with finitely generated Λ_G-modules, and there is a structure theory for such modules which can be described as follows (for more details and proof, see [2], [9], and [10]):

A finitely generated torsion Λ_G-module is called pseudo-null if its annihilator is not contained in any prime ideal of height 1. Viewing Λ_G as a power series ring, we see that Λ_G is a unique factorization domain and that a pseudo-null Λ_G-module is annihilated by two relatively prime elements of Λ_G. Now if X and Y are finitely generated Λ_G-modules and $\phi: X \to Y$ is a Λ_G-homomorphism, we say that ϕ is a pseudo-isomorphism if both the kernel and the cokernel of ϕ are pseudo-null. If such a ϕ exists, we write $X \sim Y$. In general, $X \sim Y$ does not imply $Y \sim X$, but if X and Y are torsion Λ_G-modules, then $X \sim Y$ implies $Y \sim X$ and we simply say that X and Y are pseudo-isomorphic.

A finitely generated Λ_G-module Z is called elementary if $Z = \Lambda_G^a \oplus \frac{\Lambda_G}{\mathfrak{p}_1^{s_1}} \oplus \cdots \oplus \frac{\Lambda_G}{\mathfrak{p}_t^{s_t}}$ where each $\mathfrak{p}_i$ is a prime ideal of height 1 in Λ_G so that each $\mathfrak{p}_i$ is a principal ideal generated by an irreducible element of Λ_G. Now, the structure theorem for finitely generated Λ_G-modules says that for any such module X, there is a pseudo-isomorphism $\phi: X \to Z$ where Z is an elementary Λ_G-module. Furthermore, X is a torsion Λ_G-module if and only if $a = 0$.

We will be concerned mainly with the cases where $d = 1$ or 2 and with torsion Λ_G-modules.

If $d = 1$, then it is known that pseudo-null Λ_G-modules are finite. Also, viewing Λ_G as $\mathbf{Z}_p[[T]]$, we see that by the Weierstrass Preparation Theorem, every prime ideal of height 1 in Λ_G is either of the form (f), where f is a polynomial in $\mathbf{Z}_p[T]$ irreducible in $\mathbf{Z}_p[[T]]$, or (p), the ideal generated by the prime p.

For $d = 2$, we can view Λ_G as $\boldsymbol{Z}_p[[S, T]]$. In this case a prime ideal of height 1 is either of the form (f) where f is an irreducible power series in $\boldsymbol{Z}_p[[S, T]]$, or (p). Although pseudo-null Λ_G-modules are not necessarily finite, we can still give a fairly precise description of them. The following result is proved for the case $d = 2$, but it also has an analogous formulation for arbitrary d. This is done in [6].

Proposition A: *If* $G \cong \boldsymbol{Z}_p^2$ *and* N *is a pseudo-null* Λ_G*-module, then for all but a finite number of subgroups* J *of* G *so that* $G/J \cong \boldsymbol{Z}_p$, N *is a finitely generated torsion* Λ_J*-module.*

Proof: Let f be an element of Λ_G which annihilates N and which is prime to p. Put $\bar{\Lambda}_G = \dfrac{\Lambda_G}{p\Lambda_G}$ and if $g \in \Lambda_G$, let $\bar{g}$ denote its image in $\bar{\Lambda}_G$. Then $\bar{f} \neq 0$. Also we see that $\bar{\Lambda}_G \cong \boldsymbol{Z}/p\boldsymbol{Z}[[G]]$, the complete group ring of G over $\boldsymbol{Z}/p\boldsymbol{Z}$. If J is any subgroup of G so that $G/J \cong \boldsymbol{Z}_p$, suppose that J is generated topologically by τ_J. As above, we put $\overline{\Lambda_{G/J}} = \dfrac{\Lambda_{G/J}}{p\Lambda_{G/J}} \cong (\boldsymbol{Z}/p\boldsymbol{Z})[[G/J]]$. Now, the canonical surjection $\bar{\Lambda}_G \twoheadrightarrow \bar{\Lambda}_{G/J}$ has kernel which is generated by $\overline{\tau_J - 1}$ as an ideal in $\bar{\Lambda}_G$. Since $\overline{\Lambda_{G/J}}$ is entire, this kernel is a prime ideal in $\bar{\Lambda}_G$, and hence $\overline{\tau_J - 1}$ is irreducible. Note also that if $\tau \notin J$ then $\bar{\tau} \neq 1$ in $\bar{\Lambda}_{G/J}$ so that $\overline{\tau - 1}$ is not divisible by $\overline{\tau_J - 1}$. Now, $\bar{\Lambda}_G$ is a unique factorization domain, so the above discussion shows that for all but a finite number of choices for J, $(\bar{f}, \overline{\tau_J - 1}) = 1$. Choose J in this fashion; we claim that N is a finitely generated torsion Λ_J-module.

We first show that N is a finitely generated Λ_J-module. Choose σ_J in G so that G is generated topologically by σ_J and τ_J, and put $S_J = \sigma_J - 1$ and $T_J = \tau_J - 1$. Since $(\bar{f}, \bar{T}_J) = 1$, we see that $f \notin (T_J, p)$. Viewing Λ_G as $\boldsymbol{Z}_p[[S_J, T_J]]$ we see that f is *regular* in S_J (that is, when viewed as a power series in S_J and T_J, f contains some term of the form uS_J^n where u is a unit in $\boldsymbol{Z}_p$). By the Weierstrass Preparation Theorem, the ideal generated by f can be generated by a monic polynomial in S_J with coefficients in Λ_J. Calling this polynomial f also (it differs from our original f by a unit factor), we see that since f annihilates N, and N is finitely generated over Λ_G, there is a surjective homomorphism $(\Lambda_G/f\Lambda_G)^u \twoheadrightarrow N$ for some integer u. Now, it is not hard to see that $\Lambda_G/f\Lambda_G$ is finitely generated over Λ_J and hence N is finitely generated over Λ_J as desired.

To see that N is a torsion Λ_J-module, choose an annihilator h of N so that $(h, f) = 1$. Adding f to h if necessary, we can assume h is regular in S_J and hence that it is a monic polynomial in S_J with coefficients in Λ_J. The ideal generated by f and h is then seen to contain an element of Λ_J, giving the desired result.

This proposition will be our major tool in studying torsion Λ_G-modules when $G = \mathbf{Z}_p^2$. Roughly speaking, when we want to prove a certain property about finitely generated torsion Λ_G-modules, we will prove it for elementary torsion Λ_G-modules, and then we will use the fairly well developed theory of finitely generated Λ_J-modules (where $J \cong \mathbf{Z}_p$) to describe the difference between our original module and the elementary module pseudo-isomorphic to it (this difference is described by a pair of pseudo-null Λ_G-modules).

We will also adopt the following notation. If $G \cong \mathbf{Z}_p^2$, and we choose a pair of topological generators σ and τ for G, we let $\eta_n = \sigma^{p^n} - 1$ and $\omega_n = \tau^{p^n} - 1$. If $m > n$, we can define two elements of Λ_G by the formulae:

$$\nu_{n,m}(\sigma) = \frac{\eta_m}{\eta_n} = 1 + \sigma^{p^n} + \sigma^{2p^n} + \cdots + \sigma^{(p^{m-n}-1)p^n}$$

and

$$\nu_{n,m}(\tau) = \frac{\omega_m}{\omega_n} = 1 + \tau^{p^n} + \tau^{2p^n} + \cdots + \tau^{(p^{m-n}-1)p^n}.$$

If n_0 is a fixed integer, we let $\alpha_{n_0,n} = \nu_{n_0,n}(\sigma)$ and $\beta_{n_0,n} = \nu_{n_0,n}(\tau)$. Often we will simply write α_n and β_n for $\alpha_{n_0,n}$ and $\beta_{n_0,n}$, but the context of the discussion will always make the value of n_0 clear.

Finally, suppose Ω_p is a fixed algebraic closure of $\mathbf{Q}_p$. Let ν_p denote the p-adic exponential valuation on Ω_p, normalized so that $\nu_p(p) = 1$. If $\mathcal{W}$ denotes the multiplicative group of p-power roots of unity in Ω_p, we define a mapping $O: \mathcal{W} \to \mathbf{Z}$ by the conditions: If $\zeta \in \mathcal{W}$, then $\zeta^{p^{O(\zeta)}} = 1$ and if $0 < n < O(\zeta)$, then $\zeta^{p^n} \neq 1$. Note that if $\zeta \in \mathcal{W}$, $\nu_p(\zeta - 1) = \dfrac{1}{p^{O(\zeta)-1}(p-1)}$, and that $\prod_{O(\zeta)=n} (\zeta - 1) = p$ for $n \geq 1$.

We will also want to consider rings other than Ω_p as the domain for ν_p. For example, if F is a finite extension of $\mathbf{Q}_p$ and $\mathcal{O}$ is the ring of integers in F, we can extend ν_p to $\mathcal{O}[[S, T]]$ by putting $\nu_p\left(\sum_{i,j} \alpha_{ij} S^i T^j\right) = \inf_{i,j} \nu_p(\alpha_{i,j})$. The usual properties of exponential valuations are seen to hold for ν_p when it is extended to $\mathcal{O}[[S, T]]$ in this manner. Also if E is a finite extension of $\mathbf{Q}_p$ with $F \subset E$ and the ring of integers in E is $\mathcal{O}'$, the extension of ν_p to $\mathcal{O}'[[S, T]]$ consistent with the extension of ν_p to $\mathcal{O}[[S, T]]$.

§2. Λ_G-modules

Let G be any multiplicative group isomorphic to the additive group in $\boldsymbol{Z}_p \oplus \boldsymbol{Z}_p$ and let H be a subgroup so that $G/H \cong \boldsymbol{Z}_p$. Choose topological generators σ and τ of G so that τ generates H topologically, and identify Λ_G with $\boldsymbol{Z}_p[[S, T]]$ and Λ_H with $\boldsymbol{Z}_p[[T]]$, where $S = \sigma - 1$ and $T = \tau - 1$. If V is any finitely generated torsion Λ_H-module, then there is a unique torsion elementary Λ_H-module Z and a Λ_H pseudo-isomorphism $\phi: V \to Z$. Z can be written as

$$\Lambda_H/(f_1^{s_1}) \oplus \cdots \oplus \Lambda_H/(f_q^{s_q}) \oplus \Lambda_H/(p^{s_{q+1}}) \oplus \cdots \oplus \Lambda_H/(p^{s_r})$$

where each f_i can be taken to be an irreducible monic polynomial in Λ_H. If we put $f = f_1^{s_1} f_2^{s_2} \ldots f_q^{s_q} p^{s_{q+1}+\cdots+s_r}$, then f is called the characteristic power series of V. If we let $\lambda(V) = \deg f$ and $\mu(V) = s_{q+1} + \cdots + s_r$, then $\lambda(V)$ and $\mu(V)$ are the Iwasawa invariants for the Λ_H-module V, and $\lambda(V) = \dim_{\boldsymbol{Q}_p}(V \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p)$.

Let W be a finitely generated torsion Λ_G-module, and suppose n_0 is some fixed positive integer. For $n > n_0$, suppose that $W/\alpha_n W$ is a finitely generated torsion Λ_H-module. Then we can speak of the invariants of $W/\alpha_n W$: $\lambda(W/\alpha_n W)$ and $\mu(W/\alpha_n W)$. The following result describes how these invariants grow with n.

PROPOSITION 2.1: *With the above notation, there exist constants ℓ, m_0, m_1, c_1, and c, independent of n, such that for $n \gg 0$, $\mu(W/\alpha_n W) = m_0 p^n + m_1 n + c_1$ and $\lambda(W/\alpha_n W) = \ell p^n + c$.*

The idea of the proof is as follows: We let $\phi: W \to Z$ be the Λ_G pseudo-isomorphism which associates W to the elementary torsion Λ_G-module Z, and suppose that ϕ has kernel N and image R. We show that $Z/\alpha_n Z$ and $N + \alpha_n W/\alpha_n W$ are finitely generated torsion Λ_H-modules whose invariants can be related to those of $W/\alpha_n W$, and then we show that the invariants of $Z/\alpha_n Z$ and $N + \alpha_n W/\alpha_n W$ can be calculated for n large enough.

To this end, suppose that

$$Z = \Lambda_G/(f_1^{s_1}) \oplus \cdots \oplus \Lambda_G/(f_1^{s_q}) \oplus \Lambda_G/(p^{s_{q+1}}) \oplus \cdots \oplus \Lambda_G/(p^{s_r})$$

where each f_i is an irreducible element of Λ_G. The following observation will be useful. If $Z_i = \Lambda_G/(f_i^{s_i})$, let R_i denote the projection of R onto Z_i ($i = 1 \ldots q$). Then $R_i = H_i/(f_i^{s_i})$ where H_i is an ideal of Λ_G with $f_i^{s_i}\Lambda_G \subset H_i$. Since Z/R is pseudo-null, so too is $Z_i/R_i = \Lambda_G/H_i$ ($i =$

$1 \ldots q$). This implies that for $i = 1 \ldots q$, H_i is not contained in any principal ideal. We will need the following lemma.

LEMMA 2.2: *For $n > n_0$, $(\alpha_n, f_i) = 1$ ($i = 1 \ldots q$).*

PROOF: Suppose for some j, $f_j = \xi$ where ξ is an irreducible factor of α_n. We will show that in this case, $H_j \subset \xi\Lambda_G$, contradicting the above remark. This will prove our lemma. Let f be a nonzero annihilator of $W/\alpha_n W$ in Λ_H. Then $fR_j \subset \alpha_n R_j$ so that f annihilates $R_j/\alpha_n R_j = H_j/(\alpha_n H_j + \xi^{s_j}\Lambda_G)$ and hence $fH_j \subset \alpha_n H_j + \xi^{s_j}\Lambda_G \subset \xi\Lambda_G$. Then f annihilates $H_j + \xi\Lambda_G/\xi\Lambda_G$. This latter module is contained in $\Lambda_G/\xi\Lambda_G$ which is torsion-free over Λ_H. Hence $H_j + \xi\Lambda_G/\xi\Lambda_G = 0$, i.e., $H_j \subset \xi\Lambda_G$ as desired.

One consequence of Lemma 2.2 is that for $n > n_0$, multiplication by α_n is injective on Z. This will be useful several times.

Note that since α_n can be viewed as a monic polynomial in $\boldsymbol{Z}_p[S]$ ($\alpha_n = (S+1)^{p^n - p^{n_0}} + (S+1)^{p^n - 2p^{n_0}} + \cdots + (S+1)^{p^{n_0}} + 1$), and Λ_H can be viewed as $\boldsymbol{Z}_p[[T]]$, Λ_G/α_n is a finitely generated Λ_H-module. In fact $\Lambda_G/\alpha_n \cong (\Lambda_H)^{p^n - p^{n_0}}$ as a Λ_H-module. Also since we are assuming that $W/\alpha_n W$ is a finitely generated torsion Λ_H-module, so too is $R/\alpha_n R$. Since N is finitely generated over Λ_G and $N + \alpha_n W/\alpha_n W$ is annihilated by α_n, we see that $N + \alpha_n W/\alpha_n W$ is finitely generated over Λ_H. Since $N + \alpha_n W/\alpha_n W$ is contained in $W/\alpha_n W$, it is also a torsion Λ_H-module. As a consequence of Lemma 2.2, we also have:

COROLLARY 2.3: *$Z/\alpha_n Z$ is a finitely generated torsion Λ_H-module for all $n > n_0$.*

PROOF: Since α_n is relatively prime to the characteristic power series of Z, we see that $Z/\alpha_n Z$ is a pseudo-null Λ_G-module. Since $\alpha_n \notin (\tau - 1, p)$ the proof of Proposition A gives the desired result.

Now, since $N + \alpha_n W/\alpha_n W$ and $R/\alpha_n R$ are finitely generated torsion Λ_H-modules, we can speak of their invariants. More precisely, we have:

LEMMA 2.4: *For $n > n_0$ $\mu(W/\alpha_n W) = \mu(R/\alpha_n R) + \mu(N + \alpha_n W/\alpha_n W)$ and $\lambda(W/\alpha_n W) = \lambda(R/\alpha_n R) + \lambda(N + \alpha_n W/\alpha_n W)$.*

PROOF: We have a surjection $W/\alpha_n W \twoheadrightarrow R/\alpha_n R$ induced by ϕ. It is easy to see that the kernel of this mapping is precisely $N + \alpha_n W/\alpha_n W$, giving the desired result.

Finally, using the injectivity of α_n on Z, it is seen that the kernel of the composite of the surjective homomorphisms $Z \xrightarrow{\alpha_n} \alpha_n Z \twoheadrightarrow \alpha_n Z/\alpha_n R$ is precisely R, giving:

LEMMA 2.5: $Z/R \cong \alpha_n Z/\alpha_n R$ *as* Λ_G*-modules.*

Keeping the same notation as above, we see that to determine the invariants of $W/\alpha_n W$, we must, by Lemma 2.4, determine the invariants of $N + \alpha_n W/\alpha_n W$ and $R/\alpha_n R$. The following lemmas are directed to this end.

LEMMA 2.6: *For* $n > n_0$, $\lambda(R/\alpha_n R) = \lambda(Z/\alpha_n Z)$.

PROOF: If V is any finitely generated torsion Λ_H-module, we know that $\lambda(V) = \dim_{Q_p}(V \otimes_{Z_p} \boldsymbol{Q}_p)$, and hence the λ invariant of V depends only on its $\boldsymbol{Z}_p$ structure and not on its Λ_H structure. Now since Z/R is a pseudo-null Λ_G-module, Proposition A implies the existence of a subgroup J of G such that $J \cong \boldsymbol{Z}_p$ and Z/R is a finitely generate torsion Λ_J-module. Hence $\dim_{Q_p}(Z/R \otimes_{Z_p} \boldsymbol{Q}_p)$ is finite and so $\lambda(Z/R)$ is finite. But then

$$\lambda(Z/R) = \lambda(Z/\alpha_n Z) + \lambda(\alpha_n Z/\alpha_n R) - \lambda(R/\alpha_n R).$$

We get the desired result by applying Lemma 2.5.

LEMMA 2.7: *For* $n \gg 0$, $\mu(R/\alpha_n R) = \mu(Z/\alpha_n Z)$.

Let $H_n \cong \boldsymbol{Z}_p$ by the subgroup of G which is generated topologically by $\sigma^{p^n}\tau$. By Proposition A we see that Z/R is a finitely generated torsion Λ_{H_n}-module for n sufficiently large.

For any subgroup J of G, $J \cong \boldsymbol{Z}_p$, and any Λ_G-module Y, denote the μ-invariant of Y considered as a Λ_J-module by $\mu_J(Y)$ (provided it exists). Now, for any Λ_G-module Y, the Λ_{H_n}-structure of $Y/\alpha_n Y$ is identical with the Λ_H-structure of Y, because if $y \in Y$, then $\sigma^{p^n}\tau y - \tau y = \eta_n \tau y \equiv 0 \pmod{\alpha_n Y}$, so that $\sigma^{p^n}\tau y = \tau y$ on $Y/\alpha_n Y$. Hence, if $\mu_{H_n}(Y/\alpha_n Y)$ is defined, then so too is $\mu_H(Y/\alpha_n Y)$ and $\mu_{H_n}(Y/\alpha_n Y) = \mu_H(Y/\alpha_n Y)$.

So if n is large enough to insure that Z/R is a finitely generated torsion Λ_{H_n}-module, we see that

$$\mu_{H_n}(Z/R) = \mu_{H_n}(Z/\alpha_n Z) + \mu_{H_n}(\alpha_n Z/\alpha_n R) - \mu_{H_n}(R/\alpha_n R).$$

Applying Lemma 1.2.5 again, we see that $\mu_{H_n}(Z/\alpha_n Z) = \mu_{H_n}(R/\alpha_n R)$ and so $\mu_H(Z/\alpha_n Z) = \mu_H(R/\alpha_n R)$ as desired.

The following lemma is more general than we need here, but it will also be used in a later result.

LEMMA 2.8: *Let* V *be any finitely generated torsion* Λ_G*-module and let* N *be a pseudo-null submodule. Suppose that* $\{V_n\}_{n \in Z^+}$ *is a sequence of submodules of* V *so that for* $n > n_0$, $V_n = \alpha_n V_{n_0}$ *and* $\eta_n V \subset V_n$. *Then, for* $n > n_0$, $N + V_n/V_n$ *is a finitely generated torsion* Λ_H*-module and for* $n \gg 0$, *the invariants* $\mu(N + V_n/V_n)$ *and* $\lambda(N + V_n/V_n)$ *become constant.*

PROOF: The proof of Proposition A shows that for $n > n_0$, $N/\eta_n N$ is a finitely generated torsion Λ_H-module. Since we have a surjection $N/\eta_n N \twoheadrightarrow N + V_n/V_n$, we see that for $n > n_0$, $N + V_n/V_n$ is also a finitely generated torsion Λ_H-module. In fact, we see that $\mu(N + V_n/V_n) \leq \mu(N/\eta_n N)$ and $\lambda(N + V_n/V_n) \leq \lambda(N/\eta_n N)$. Since the sequences $\{\mu(N + V_n/V_n)\}_{n \in Z^+}$ and $\{\lambda(N + V_n/V_n)\}_{n \in Z^+}$ are increasing, it suffices to show that the invariants of $N/\eta_n N$ eventually stabilize.

For the λ invariant, note that from Proposition A, $\lambda(N) = \dim_{Q_p} N \otimes_{Z_p} Q_p$ is finite, and since $\{\lambda(N/\eta_n N)\}_{n \in Z^+}$ is an increasing sequence of integers bounded by $\lambda(N)$, we see that $\lambda(N/\eta_n N)$ must eventually stabilize.

Now consider the μ-invariant. If ${}^tN$ denotes the Z_p-torsion submodule of N, then using Proposition A, we see that $N/{}^tN$ is finitely generated over Z_p (and hence a finitely generated torsion Λ_H-module). Since $\eta_n N/\eta_n{}^tN$ is a homomorphic image of $N/{}^tN$, it too is finitely generated over Z_p,. But then, we see that for each n, $\mu(N/\eta_n N) = \mu(N/{}^tN) + \mu({}^tN/\eta_n{}^tN) - \mu(\eta_n N/\eta_n{}^tN) = \mu({}^tN/\eta_n{}^tN)$ so that we can assume that N is a Z_p-torsion module of exponent p^e for some $e \geq 1$.

Now N has an annihilator f prime to p, so, under the above assumption, there is a surjection:

$$(*) \qquad (\Lambda_G/(p^e, f, \eta_n))^u \twoheadrightarrow N/\eta_n N,$$

where u is independent of n.

For any $n > 0$, consider the module $\Lambda_G/(f, \eta_n)$. This is clearly a finitely generated Λ_H-module, and hence there is a Λ_H pseudo-isomorphism from $\Lambda_G/(\eta_n, f)$ to a Λ_H-module of form

$$\Lambda_H^a \oplus \Lambda_H/(h_1^{a_1}) \oplus \cdots \oplus \Lambda_H/(h_r^{a_r}) \oplus \Lambda_H/(p^{r_1}) \oplus \cdots \oplus \Lambda_H/(p^{r_v})$$

where $h_i \in \Lambda_H$ and h_i is an irreducible distinguished polynomial in $\tau - 1$. Now, for any integer b,

$$\Lambda_G/(f, \eta_n, p^b) \sim (\Lambda_H/p^b)^a \oplus \Lambda_H/(p^b, h_1^{a_1}) \oplus \cdots$$
$$\cdots \oplus \Lambda_H/(p^b, h_r^{a_r}) \oplus \Lambda_H/(p^{\ell_1}) \oplus \cdots \oplus \Lambda_H/(p^{\ell_v})$$

where $\sim$ denotes Λ_H pseudo-isomorphism and $\ell_i = \min(r_i, b) \leq b$. Since $\Lambda_H/(p^b, h_i^{a_i})$ is finite for $i = 1 \ldots r$, we see that

$$\mu(\Lambda_G/(p^b, \eta_n, f)) = ba + \sum_{i=1}^{v} \ell_i \leq b(a+v).$$

But then $\mu(\Lambda_G/(p^e, \eta_n, f)) \leq e(a+v) = e\mu(\Lambda_G/(p, \eta_n, f))$, and so, in view of (*), we see that

$$\mu(\mathrm{N}/\eta_n\mathrm{N}) \leq eu\mu(\Lambda_G/p, \eta_n, f).$$

Since $\{\mu(\mathrm{N}/\eta_n\mathrm{N})\}_{n\in Z^+}$ is an increasing sequence of integers, we will be done if we show that $\mu(\Lambda_G/(p, \eta_n, f))$ eventually stabilizes.

To this end, we let $\bar{\Lambda}_G = \Lambda_G/p\Lambda_G$ and, as before, if $h \in \Lambda_G$, we let $\bar{h}$ denote its image in $\bar{\Lambda}_G$, so that $\bar{\eta}_n = (\sigma-1)^{p^n}$. Suppose $\bar{f} = (\sigma-1)^k\bar{g}$ where $(\bar{g}, \sigma-1) = 1$. Choose v so large that $n \geq v$ implies $p^n > k$. Then for $n > v$

$$\begin{aligned}\mu(\Lambda_G/(p, \eta_n, f)) &= \mu(\bar{\Lambda}_G/(\sigma-1)^k\bar{g}, (\sigma-1)^{p^n}))\\ &= \mu(\bar{\Lambda}_G/((\sigma-1)^k\bar{g}, (\sigma-1)^{p^v}))\\ &\quad + \mu[((\sigma-1)^k\bar{g}, (\sigma-1)^{p^v}))/((\sigma-1)^k\bar{g}, (\sigma-1)^{p^n}))].\end{aligned}$$

Now multiplication by $(\sigma-1)^k$ induces a surjective homomorphism:

$$(\bar{g}, (\sigma-1)^{p^v-k})/(\bar{g}, (\sigma-1)^{p^n-k})$$
$$\twoheadrightarrow ((\sigma-1)^k\bar{g}, (\sigma-1)^{p^v})/((\sigma-1)^k\bar{g}, (\sigma-1)^{p^n}).$$

But $(\bar{g}, (\sigma-1)^{p^v-k})/(\bar{g}, (\sigma-1)^{p^n-k}) \subset \bar{\Lambda}_G/(\bar{g}, (\sigma-1)^{p^n-k})$ and this latter module is a pseudo-null $\bar{\Lambda}_G$-module. Since $\bar{\Lambda}_G$ is a regular local ring of dimension 2, $\bar{\Lambda}_G/(\bar{g}, (\sigma-1)^{p^n-k})$ is finite, and hence $((\sigma-1)^k\bar{g}, (\sigma-1)^{p^v})/((\sigma-1)^k\bar{g}, (\sigma-1)^{p^n})$ is finite, so it has 0 μ-invariant. That is, for $n > v$,

$$\mu(\Lambda_G/(p, \eta_n, f)) = \mu(\bar{\Lambda}_G/((\sigma-1)^k\bar{g}, (\sigma-1)^{p^v})),$$

and hence is constant, giving the desired result.

Now, returning to the previous notation, we see that the submodules $\{\alpha_n W\}_{n\in Z^+}$ satisfy the hypothesis of Lemma 2.8. Combining the results from Lemmas 2.4, 2.6, 2.7, and 2.8, we see that there are constants d, d' independent of n so that for $n \gg 0$, $\mu(W/\alpha_n W) = \mu(Z/\alpha_n Z) + d$ and $\lambda(W/\alpha_n W) = \lambda(Z/\alpha_n Z) + d'$. Proposition 2.1 will be established if we can calculate the invariants of $Z/\alpha_n Z$ for $n \gg 0$. This is what we do next.

PROPOSITION 2.9: *There are constants ℓ, m_0, m_1, c and c_1, independent of n, such that for $n \gg 0$,*

$$\lambda(Z/\alpha_n Z) = \ell p^n + c \quad \textit{and} \quad \mu(Z/\alpha_n Z) = m_0 p^n + m_1 n + c_1.$$

PROOF: In light of the structure of Z, it suffices to determine the invariants for $Z/\alpha_n Z$ when Z is a module of form $Z = \Lambda_G/(p^s)$ or $Z = \Lambda_G/(f^s)$ where f is an irreducible element of Λ_G so that $(f, \alpha_n) = 1$ and $f \neq p$.

Case 1. $Z = \Lambda_G/p^s$. Then $Z/\alpha_n Z = \Lambda_G/(\alpha_n, p^s)$. Now, viewing Λ_G as $\boldsymbol{Z}_p[[S, T]]$ where $S = \sigma - 1$, $T = \tau - 1$, we see that

$$\begin{aligned} \alpha_n &= 1 + (S+1)^{p^{n_0}} + (S+1)^{2p^{n_0}} + \cdots + (S+1)^{(p^{n-n_0}-1)p^{n_0}} \\ &= S^{p^n - p^{n_0}} + p \text{ (terms of lower degree).} \end{aligned}$$

Then we see that

$$\begin{aligned} \Lambda_G/(\alpha_n, p^s) &\cong \boldsymbol{Z}_p[[S,T]]/(S^{p^n-p^{n_0}} + \cdots, p^s) \\ &\cong (\boldsymbol{Z}/p^s\boldsymbol{Z})[[S,T]]/(S^{p^n-p^{n_0}} + \cdots) = ((\boldsymbol{Z}/p^s\boldsymbol{Z})[[T]])^{p^n-p^{n_0}} \end{aligned}$$

(as Λ_H-modules), and hence $\lambda(Z/\alpha_n Z) = 0$ and $\mu(Z/\alpha_n Z) = s(p^n - p^{n_0}) = sp^n + c_1$ for all $n > n_0$.

Case 2. $Z = \Lambda_G/(f^r)$ where $f \in \Lambda_G$, $f \neq p$, f is irreducible and $(f, \alpha_n) = 1$ for $n > n_0$. We view Λ_G as $\boldsymbol{Z}_p[[S,T]]$ and f as an irreducible power series $f(S,T)$. Put $U_n = \Lambda_G/\alpha_n\Lambda_G$. Then U_n is a free finitely generated $\boldsymbol{Z}_p[[T]]$-module on which S acts as a linear mapping, and the eigenvalues of S form the set $\{\zeta - 1 \mid \zeta \in \mathcal{W},\ n_0 < O(\zeta) \leq n\}$. Similarly, multiplication by $f(S,T)^r$ is a $\boldsymbol{Z}_p[[T]]$-linear mapping on U_n, and the eigenvalues of this mapping form the set $\{f(\zeta - 1, T)^r \mid \zeta \in \mathcal{W},\ n_0 < 0(\zeta) < n\}$. Viewing $f(S,T)^r$ as a linear mapping: $f(S,T)^r: U_n \to U_n$, we see that the cokernel of this mapping is precisely $Z/\alpha_n Z$. Letting $f(S,T)^r$ act on U_n, we can take its determinant and obtain an element $\det_n(f(S,T)^r)$ of $\boldsymbol{Z}_p[[T]]$. Now, it is proved in [2] that the ideal generated by $\det_n(f(S,T)^r)$ is the same as the ideal generated by the characteristic power series of $U_n/f(S,T)^rU_n = Z/\alpha_n Z$ in $Z_p[[T]]$. Hence, we see that $\mu(Z/\alpha_n Z) = \mu(U_n/f(S,T)^rU_n)$ is the power of p dividing $\det_n(f(S,T)^r)$, and $\lambda(Z/\alpha_n Z)$ is the reduced order of $\frac{1}{p^{\mu(Z/\alpha_n Z)}}[\det_n(f(S,T)^r)]$, i.e., the degree of the term in $\frac{1}{p^{\mu(Z/\alpha_n Z)}}[\det_n(f(S,T)^r)]$ of smallest degree with a unit coefficient. Now $\det_n(f(S,T)^r) = \prod_{n \geq O(\zeta) > n_0} f(\zeta - 1, T)^s$ where the product is over all $\zeta \in \mathcal{W}$ whose orders are in the prescribed range.

Now suppose first that $f(S,T) \notin (S,p)$. Then $f(S,T)$ is regular in T, so by the Weierstrass Preparation Theorem, we can assume $f(S,T)$ is a distinguished polynomial in $(Z_p[[S]])[T]$, and hence so is $f(S,T)^r$. That is, we can suppose, $f(S,T)^r = T^v + f_{v-1}(S)T^{v-1} + \cdots + f_0(S)$ where $f_i(S)$ is a nonunit in $Z_p[[S]]$ for $i = 0 \dots r-1$. Then if $n > n_0$, letting $f(S,T)$ act on U_n, we have

$$\det_n f(S,T)^r = \prod_{n \geq O(\zeta) > n_0} (T^v + f_{v-1}(\zeta - 1)T^{v-1} + \cdots + f_0(\zeta - 1)).$$

Now it is not hard to see that this expression gives a polynomial in T which is not divisible by p and whose first unit coefficient is in the term $T^{v(p^n - p^{n_0})}$. Hence, in this case, $\mu(Z/\alpha_n Z) = 0$ and $\lambda(Z/\alpha_n Z) = vp^n + c$ for $n > n_0$.

Next suppose $f(S,T) \in (S,p)$ so that $f(0,T) \equiv 0 \bmod p$. Then we can write $f(S,T)^r = S^a H(S,T) + p^b G(T)$ where $p^b G(T) = f(0,T)^r$, $p \nmid G(T)$ and $S \nmid H(S,T)$.

If $G(T) = 0$, then $f(0,T) = 0$, and since f is irreducible, $f(S,T) = S$. Then we see that

$$\prod_{n \geq O(\zeta) > n_0} f(\zeta - 1, T)^r = \prod_{n \geq O(\zeta) > n_0} (\zeta - 1)^r = p^{r(n - n_0)},$$

so that $\mu(Z/\alpha_n Z) = rn + c$, $\lambda(Z/\alpha_n Z) = 0$.

Next suppose $G(T) \neq 0$. Then $H(S,T) \neq 0$ (otherwise $f = p$ or 0) and $a \geq 1$. Writing $H(S,T)$ as a power series in S with coefficients in $Z_p[[T]]$, we see that:

$$f(S,T)^r = p^b G(T) + S^a(h_0(T) + h_1(T)S + \cdots),$$

where $h_i(T) \in Z_p[[T]]$. Now since $f(S,T) \neq p$, there is some index i so that $p \nmid h_i(T)$. Let t be the first index so that $h_t(T) \neq 0 \pmod p$.

We first determine the power of p which divides $\det_n(f(S,T)^r)$ for $n \gg 0$.

Choose $n_1 > n_0$ so that $O(\zeta) > n_1$ implies $\frac{a+t+1}{p^{O(\zeta)-1}(p-1)} < 1$. Suppose that $O(\zeta) > n_1$ and consider

$$f(\zeta-1,T)^r = p^b G(T) + (\zeta-1)^a(h_0(T) + h_1(T)(\zeta-1) + \cdots).$$

Now $\nu_p(p^b G(T)) = b \geq 1$. If $j < t$, then $p \mid h_j(T)$, so

$$\nu_p((\zeta-1)^{a+j}h_i(T)) \geq \frac{a+j}{p^{O(\zeta)-1}(p-1)} + 1.$$

If $j > t$, then

$$\nu_p((\zeta-1)^{a+j}h_j(T)) \geq \frac{a+j}{p^{O(\zeta)-1}(p-1)} > \frac{a+t}{p^{O(\zeta)-1}(p-1)}.$$

But then, since

$$\nu_p((\zeta-1)^{a+t}h_t(T)) = \frac{a+t}{p^{O(\zeta)-1}(p-1)} \quad \text{(because } p \nmid h_t(T)\text{)},$$

we see that

$$\nu_p(f(\zeta-1,T)^r) = \frac{a+t}{p^{O(\zeta)-1}(p-1)} = \nu_p(\zeta-1)^{a+t}.$$

Using this fact, we calculate as follows:

$$\begin{aligned}
&\nu_p\Bigg(\prod_{n\geq O(\zeta)>n_0} (f(\zeta-1,T))^r\Bigg) \\
&= \nu_p\Bigg(\prod_{n\geq O(\zeta)>n_1} f(\zeta-1,T)^r\Bigg) + \nu_p\Bigg(\prod_{n_1\geq O(\zeta)>n_0} f(\zeta-1,T)^r\Bigg) \\
&= \nu_p\Bigg(\prod_{n\geq O(\zeta)>n_1} (\zeta-1)^{a+t}\Bigg) + c \ (c \text{ is independent of } n) \\
&= (a+t)(n-n_1) + c = (a+t)n + c' \\
&\quad (c', a, t \text{ independent of } n).
\end{aligned}$$

Hence for $n \gg 0$, $\mu(Z/\alpha_n Z) = \ell n + c'$.

Finally, we have to determine the reduced order of $\frac{1}{p^{\mu(Z/\alpha_n Z)}} \det_n(f(S,T)^r)$ for $n \gg 0$.

If we denote the reduced order of a power series $g(\mathrm{T})$ by $\deg g(\mathrm{T})$, we see that:

$$\deg \det_n(f(\mathrm{S},\mathrm{T})^r) = \deg \prod_{n\geq \mathrm{O}(\zeta)>n_0} (f(\zeta-1,\mathrm{T}))^r$$

$$= \deg \prod_{n\geq \mathrm{O}(\zeta)>n_1} (f(\zeta-1,\mathrm{T})^r) + \deg \prod_{n_1\geq \mathrm{O}(\zeta)>n_0} (f(\zeta-1,\mathrm{T})^r)$$

$$= \sum_{\substack{i=n_1+1\\ \mathrm{O}(\zeta)=i}} \deg(f(\zeta-1,\mathrm{T}))^r + c \text{ (c is independent of n).}$$

Now we have seen above that for $0(\zeta) > n_1$,

$$\nu_p(f(\zeta-1,\mathrm{T})^r) = \frac{a+t}{p^{\mathrm{O}(\zeta)-1}(p-1)}.$$

Suppose that the term in $h_t(\mathrm{T})$ of least degree with unit coefficient is $u_\ell \mathrm{T}^\ell$. If we write $f(\zeta-1,\mathrm{T})^r$ as a power series in T with coefficients in $\boldsymbol{Z}_p[\zeta-1]$, an inspection of the coefficients shows that

$$f(\zeta-1,\mathrm{T})^r = (\zeta-1)^{a+t}u_\ell \mathrm{T}^\ell + \mathrm{R}_\zeta(\mathrm{T})$$

where $\mathrm{R}_\zeta(\mathrm{T})$ is such that every coefficient has p-ordinal $\geq \frac{a+t}{p^{\mathrm{O}(\zeta)-1}(p-1)}$ and the coefficient of T^j, for $j\leq \ell$ has p-ordinal $\geq \frac{a+t+1}{p^{\mathrm{O}(\zeta)-1}(p-1)}$. Hence we see that for $\mathrm{O}(\zeta)>n_1$, $\deg(f(\zeta-1,\mathrm{T}))^r = \ell$, and so

$$\sum_{\substack{i=n_1+1\\ \mathrm{O}(\zeta)=i}}^{n} \deg(f(\zeta-1,\mathrm{T})^r) = \sum_{\substack{i=n_1+1\\ \mathrm{O}(\zeta)=i}}^{n} \ell = \ell(p^n - p^{n_1}).$$

Hence for $n \gg 0$, $\deg \det_n f(\mathrm{S},\mathrm{T})^r = \ell(p^n - p^{n_0}) + c = \ell p^n + c'$ and so for $n \gg 0$, $\lambda(\mathrm{Z}/\alpha_n \mathrm{Z}) = \ell p^n + c'$.

By combining the above results, we see that if Z is an elementary torsion Λ_{G}-module whose characteristic power series is prime to α_n $(n > n_0)$, then for $n \gg 0$, the invariants of $\mathrm{Z}/\alpha_n\mathrm{Z}$ have the desired form. This completes the proof of Proposition 2.9 and hence the proof of Proposition 2.1.

REMARK: In the notation of the proof, we see that $\mathrm{W} \sim \mathrm{Z}$, so that the characteristic power series of W is the characteristic power series of Z. Now an analysis of the proof of Proposition 2.9 shows that if $\mu(W/\alpha_n W) = m_0 p^n + m_1 n + c$, then m_0 is the power of p dividing the characteristic power series of Z (i.e., of W). This will be useful in §4.

§3. Galois groups and Iwasawa invariants

The proof of Theorem 1.1 will be accomplished by obtaining a module theoretic characterization of our problem and then applying Proposition 2.1.

Recall that k_∞, k'_∞ are two Z_p-extensions of k such that $k_\infty \cap k'_\infty = k$. We have $k_\infty = \bigcup_{i=0}^{\infty} k_i$ where $[k_n : k] = p^n$, and $K_n = k_n k'_\infty$, so that K_n/k_n is a Z_p-extension. We let $\lambda_n = \lambda(K_n/k_n)$ and $\mu_n = \mu(K_n/k_n)$.

Now let $K = k_\infty k'_\infty$ so that K/k is a Z_p^2-extension. Let L (resp. L_n) denote the maximal unramified pro-p extension of K (resp. K_n), and put $X = G(L/K)$, $X_n = G(L_n/K_n)$. Now G acts on X by inner automorphisms, and hence we can make X into a Λ_G-module. It is shown in [5] that X is a finitely generated torsion Λ_G-module.

We also know that $G(K_n/k_n)$ is generated topologically by $\tau|_{K_n}$ so that we can make X_n into a Λ_H-module, and the theory of Z_p-extensions tells us that X_n is a finitely generated torsion Λ_H-module. The invariants λ_n and μ_n are, by definition, the Iwasawa invariants of the Λ_H-module X_n.

The following characterization of X_n closely follows that in [10] and is a slight generalization of the result proved in [1].

PROPOSITION 3.1: *There is an integer n_0 so that for $n > n_0$, there is a submodule Y_n of X such that $X_n = (X/Y_n) \oplus Z_p^d$. Here $d = 0$ or 1 and is independent of n. Also for $n > n_0$, $Y_n = \alpha_n Y_n$.*

PROOF: It is shown in [1] that if K/k'_∞ is unramified, $X_n = (X/\eta_n X) \oplus Z_p$ for $n \geq 0$. In this case we can take $n_0 = 0$ and $Y_n = \eta_n X$. Hence we need only consider the case where K/k'_∞ is ramified at some valuation.

If E/F is a Galois extension and v is some prime in E, we let $I_v(E/F)$ denote the inertia group of v in G(E/F).

Note that there are only finitely many primes in k over p, and, since K/k is p-ramified, only finitely many primes of k ramify in K. Now if v and w are primes on K which restrict to the same prime on k, $I_v(K/k) = I_w(K/k)$ because K/k is abelian. Hence $I_v(K/k'_\infty) = I_v(K/k) \cap G(K/k'_\infty) = I_w(K/k) \cap G(K/k'_\infty) = I_w(L/k'_\infty)$. So, although there may be an infinite number of primes in K which are ramified by K/k'_∞, the set of inertia groups for these primes is finite.

Now suppose V is the set of primes on K which are ramified by K/k'_∞. If $v \in V$, $I_v(K/k'_\infty) \subset G(K/k'_\infty)$, so $I_v(K/k'_\infty)$ is generated topologically by, say $\sigma^{p^{b_v}}$. The above discussion shows that the set $\{b_v\}_{v\in V}$ is, in fact, finite. Put $n_0 = \sup_{v\in V}\{b_v\}$, so that if $v \in V$, $b_v \leq n_0$ and hence v is totally ramified by K/K_n for all $n > n_0$.

If $v \in V$, let w_v be an extension for v to L. Since L/K is unramified, $I_{w_v}(L/k'_\infty) \cap X = 1$, and hence the restriction mapping $I_{w_v}(L/k'_\infty) \to I_v(K/k'_\infty)$ is an isomorphism. Hence there exists $\sigma_v \in I_{w_v}(L/k'_\infty)$ so that $\sigma_{v|K} = \sigma^{p^{n_0}}$. Now it is not so hard to see that $I_{w_v}(L/K_{n_0})$ is generated topologically by σ_v, and if $n > n_0$, $I_{w_v}(L/K_n)$ is generated topologically by $\sigma_v^{p^{n-n_0}}$.

Now, since KL_n/K is unramified, $L_n \subset L$. Since L/k'_∞ is Galois, if we put $G_n = G(L/K_n)$ and $J_n = G(L/L_n)$, then $X_n = G_n/J_n$.

We can describe J_n as follows: L_n is clearly the maximal unramified extension of K_n contained in L. Since the commutator subgroup of G_n is $\eta_n X$, we see that $J_n = \langle \eta_n X, \bigcup_{v\in V} I_{w_v}(L/K_n)\rangle$.

Choose some prime $v_0 \in V$. If $n > n_0$ and $v \in V$, put $a_{n,v} = \sigma_v^{p^{n-n_0}}\sigma_{v_0}^{-p^{n-n_0}}$, so that $a_{n_0,v} = \sigma_v\sigma_{v_0}^{-1}$. Since $a_{n,v|K} = 1$, $a_{n,v} \in X$. Now, for any $v \in V$,

$$\begin{aligned}\langle I_{w_v}(L/K_n), I_{w_{v_0}}(L/K_n)\rangle &= \langle \sigma_v^{p^{n-n_0}}, \sigma_{v_0}^{p^{n-n_0}}\rangle \\ &= \langle \sigma_{v_0}^{p^{n-n_0}}, \sigma_v^{p^{n-n_0}}\sigma_{v_0}^{-p^{n-n_0}}\rangle = \langle I_{w_{v_0}}(L/K_n), a_{n,v}\rangle.\end{aligned}$$

Hence we see that $J_n = \langle \eta_n X, I_{w_{v_0}}(L/K_n), \{a_{n,v}\}_{v\in V}\rangle$.

Now for $n > n_0$, $G_n = XI_{w_{v_0}}(L/K_n)$ because $I_{w_{v_0}}(L/K_n)|_K$ is generated topologically be $\sigma_{v_0|K}^{p^{n-n_0}} = \sigma^{p^n}$, so that $I_{w_{v_0}}(L/K_n)_{|K} = G(K/K_n)$. Also for $n > n_0$, we have a Λ_G-homomorphism $X \to X_n$ given by restriction, and, since $I_{w_{v_0}}(L/K_n) \cap X = 1$, we see that the kernel of this mapping is $J_n \cap X = Y_n$ where $Y_n = \langle \eta_n X, \{a_{n,v}\}_{v\in V}\rangle$. Hence Y_n is a Λ_G-submodule of X and we have an injection $X/Y_n \to X_n$. But if $g \in G_n$, we can write g as $x\gamma$ where $x \in X$, $\gamma \in I_{w_{v_0}}(L/K_n)$ and since $I_{w_{v_0}}(L/K_n)$ acts trivially on L_n, $g_{|L_n} = x_{|L_n}$. Hence the restriction mapping $X \to X_n$ is actually a surjection, and so, for $n > n_0$, $X/Y_n \cong X_n$ as Λ_G-modules.

It remains to show that for $n > n_0$, $Y_n = \alpha_n Y_n$. Since $\alpha_n\eta_{n_0} = \eta_n$, it suffices to show that for $n > n_0$, $\alpha_n a_{n_0,v} = a_{n,v}$ for $v \in V$. To this end:

$$\begin{aligned}\alpha_n a_{n_0,v} &= a_{n_0,v}^{1+\sigma^{p^{n_0}}+\cdots+\sigma^{(p^{n-n_0}-1)p^{n_0}}} \\ &= a_{n_0,v}\sigma_{v_0}a_{n_0,v}\sigma_{v_0}^{-1}\sigma_{v_0}^{2}a_{n_0,v}\sigma_{v_0}^{-2}\cdots a_{n_0,v}\sigma_{v_0}^{-(p^{n-n_0}-1)} = (a_{n_0,v}\sigma_{v_0})^{p^{n-n_0}}\sigma_{v_0}^{-p^{n-n_0}} \\ &= (\sigma_v\sigma_{v_0}^{-1}\sigma_{v_0})^{p^{n-n_0}}\sigma_{v_0}^{-p^{n-n_0}} = a_{n,v}, \quad \text{as desired.}\end{aligned}$$

Now Proposition 3.1 shows that $\mu_n = \mu(X_n) = \mu(X/Y_n)$ for $n > n_0$ and $\lambda_n = \lambda(X_n) = \lambda(X/Y_n) + d$ where $d = 0$ or 1 (depending on whether or not K/k'_∞ is ramified). Hence, we can prove Theorem 1.1 if we can compute the invariants of X/Y_n for $n > n_0$.

We have a Λ_G-pseudo-isomorphism $\phi: X \to Z$ where Z is an elementary torsion Λ_G-module. If $N = \mathrm{Ker}\,\phi$ and $R = \mathrm{Im}\,\phi$, then N and Z/R are pseudo-null Λ_G-modules. For $n > n_0$, put $W_n = \phi(Y_n)$ so for $n > n_0$, $W_n = \alpha_n W_n$. ϕ induces a surjection $X/Y_n \twoheadrightarrow R/W_n$ given by $x + Y_n \mapsto \phi(x) + W_n$. The kernel of this surjection is $N + Y_n/Y_n$. Since X_n is a finitely generated torsion Λ_H-module, we see that R/W_n and $N + Y_n/Y_n$ are also finitely generated torsion Λ_H-modules. Also $\lambda(X/Y_n) = \lambda(R/W_n) + \lambda(N + Y_n/Y_n)$ and $\mu(X/Y_n) = \mu(R/W_n) + \mu(N + Y_n/Y_n)$. Now the hypotheses of Lemma 2.8 apply to N and the family $\{Y_n\}_{n \in Z^+}$, so the invariants of $N + Y_n/Y_n$ eventually stabilize. Also we see that $\lambda(R/W_n) = \lambda(R/W_{n_0}) + \lambda(W_{n_0}/W_n) = c + \lambda(W_{n_0}/\alpha_n W_{n_0})$, and $\mu(R/W_n) = \mu(R/W_{n_0}) + \mu(W_{n_0}/W_n) = c' + \mu(W_{n_0}/\alpha_n W_{n_0})$, where c and c' are independent of n. But $W_{n_0} \subset Z$ and so W_{n_0} is a finitely generated torsion Λ_G-module, and for $n > n_0$, $W_{n_0}/\alpha_n W_{n_0} \subset R/W_n$ and hence $W_{n_0}/\alpha_n W_{n_0}$ is a finitely generated torsion Λ_H-module. Hence we apply Proposition 2.1 to conclude that for $n \gg 0$, the invariants of $W_{n_0}/\alpha_n W_{n_0}$ are of the right form. Since these invariants differ from λ_n and μ_n by constants, we have completed the proof of Theorem 1.1.

§4. The m_0-invariant

We keep the same notation as in §2 and §3, so that k_∞ and k'_∞ are two disjoint Z_p-extensions of k, $K = k_\infty k'_\infty$ and $K_n = k'_\infty k_n$. We have seen that for $n \gg 0$, $\mu_n = \mu(K_n/k_n) = m_0 p^n + m_1 n + c$ and $\lambda_n = \lambda(K_n/k_n) = \ell p^n + c'$. The integers m_0, m_1, and ℓ depend only on k, k_∞ and k'_∞, so that we can write $m_0(k_\infty, k'_\infty/k)$ etc. The m_0-invariant bears a striking similarity to Iwasawa's μ invariant and it is this similarity which we study in this section. We prove that m_0 depends only on K and k, and not on the individual Z_p-extensions used to obtain K, so that we can write $m_0(K/k)$. We give a module-theoretic description of m_0 very similar to the one for μ, and, imitating Iwasawa's technique in [8], we show how m_0 can be made arbitrarily large. We also give necessary and sufficient conditions for m_0 to vanish.

Now, returning to the notation of §3, we see that for $n \gg 0$, $\mu_n = \mu(X/Y_n)$. Now X/Y_{n_0} is annihilated by η_{n_0}, and since X/Y_{n_0} is a torsion Λ_H-module, it is also annihilated by a power series in T.

Hence X/Y_{n_0} is pseudo-null, so that $X \sim Y_{n_0}$. But $\phi(Y_{n_0}) = W_{n_0}$ so $Y_{n_0} \sim W_{n_0}$. Hence, the characteristic power series of X and W_{n_0} in Λ_G are the same. By the remark following the proof of Proposition 2.1, we see that m_0 is the power with which p divides this power series, and hence m_0 is an invariant attached to X (that is to say, K). Hence we have:

PROPOSITION 4.1: *$m_0(k_\infty, k'_\infty/k)$ depends only on K/k and not on k_∞ and k'_∞; it is the power of p which divides the characteristic power series of* X.

Now suppose p is an odd prime and $F = \boldsymbol{Q}(\zeta_p)$, where ζ_p is a primitive pth root of unity. Let $\Delta = G(F/\boldsymbol{Q})$ and denote complex conjugation by J so that $J \in \Delta$. In [3] it is shown that there are $\frac{p-1}{2}$ independent $\boldsymbol{Z}_p$-extensions L of F so that $L/\boldsymbol{Q}$ is Galois and when Δ acts on G(L/F) by conjugation, $(G(L/F))^{1+J} = 1$ (i.e. $J\gamma J^{-1} = \gamma^{-1}$ for every $\gamma \in G(L/F)$). Hence if $p \geq 5$, there are at least two of these $\boldsymbol{Z}_p$-extensions.

In [8] Iwasawa proves the following results: Suppose F is a number field, $\boldsymbol{Q}(\zeta_p) \subset F$, and $[F:\boldsymbol{Q}] = d$. Let L be a $\boldsymbol{Z}_p$-extension of F so that $L/\boldsymbol{Q}$ is Galois and $(G(L/\boldsymbol{Q}))^{1+J} = 1$. Let F_+ be the maximal real subfield of F and let $\mathfrak{p}_+$ be a prime of F_+ which is inert in F. If $\mathfrak{p}$ is a prime of F lying over $\mathfrak{p}_+$, then $\mathfrak{p}$ splits completely in L. Hence there are infinitely many primes in F which fully decompose in L. Let N be any positive integer and choose t primes in F which split completely in L where $t > N + d$. If these primes are $\mathfrak{p}_1 \dots \mathfrak{p}_t$, then choose $\alpha \in F$ so that $\nu_{\mathfrak{p}_i}(\alpha) = 1$, $i = 1 \dots t$. Let $k = F(\sqrt[p]{\alpha})$ and $k_\infty = L(\sqrt[p]{\alpha})$. Then k_∞/k is a $\boldsymbol{Z}_p$-extension and $\mu(k_\infty/k) \geq t - d > N$.

Now suppose $F = \boldsymbol{Q}(\zeta_p)$ where $p \geq 5$. Let L and L$'$ be two independent $\boldsymbol{Z}_p$-extensions of F, Galois over $\boldsymbol{Q}$, such that $(G(L/F))^{1+J} = 1$ and $(G(L'/F))^{1+J} = 1$. Choose any integer N and let t be so large that $t - d \geq N$ where $d = [F:\boldsymbol{Q}]$. Find primes $\mathfrak{p}_{+,1} \dots \mathfrak{p}_{+,t}$ in F_+, inert in F and let $\mathfrak{p}_i$ be a prime of F lying over $\mathfrak{p}_{+,i}$, so that $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ split completely in L and L$'$. Choose $\alpha \in F$ so that $\nu_{\mathfrak{p}_i}(\alpha) = 1$ $(i = 1 \dots t)$. If the intermediate fields for the $\boldsymbol{Z}_p$-extension L/F are denoted by F_n $(n \in \boldsymbol{Z}^+)$ and $\mathfrak{p}_i = \mathcal{P}^{(n)}_{i,1} \dots \mathcal{P}^{(n)}_{i,p^n}$ in F_n, then $\nu_{\mathcal{P}^{(n)}_{i,j}}(\alpha) = 1$ also. Also it is not hard to see that $\mathcal{P}^{(n)}_{i,j}$ splits completely in $L'F_n/F_n$. If we let $k = F(\sqrt[p]{\alpha})$, $k_\infty = L(\sqrt[p]{\alpha})$ and $k'_\infty = L'(\sqrt[p]{\alpha})$, then k_∞ and k'_∞ are two $\boldsymbol{Z}_p$-extensions of k. Also the intermediate fields of k_∞ are the fields k_n where $k_n = F_n(\sqrt[p]{\alpha})$. Now $k'_\infty k_n/k_n$ is a $\boldsymbol{Z}_p$-extension which, by the results in the previous paragraph, has μ invariant which is larger than

$tp^n - dp^n$ (because there are tp^n primes in F_n, namely $\mathcal{P}_{i,j}^{(n)}$, $i = 1 \ldots t$, $j = 1 \ldots p^n$, which divide α and split completely in $L'F_n$, and $[F_n : Q] = dp^n$). By Theorem 1.1, for $n \gg 0$, $\mu(k_n k'_\infty / k_n) = m_0 p^n + m_1 n + c$, and hence for $n \gg 0$,

$$m_0 p^n + m_1 n + c \geq (t-d)p^n.$$

Hence $m_0 \geq t - d > N$ and hence:

PROPOSITION 4.2: *If* $p \geq 5$ *and* $F = Q(\zeta_p)$, *then for any integer* $N > 0$, *there exists a cyclic extension* k *of* F *and a* Z_p^2*-extension* K *of* k *so that* $m_0(K/k) > N$.

We now derive necessary and sufficient conditions for m_0 to vanish.

Suppose first that $m_0 = 0$. Then if f is the characteristic power series for X, $p \nmid f$. Recall that $\phi : X \to Z$ is the pseudo-isomorphism which associates X to the elementary module Z and $\operatorname{Ker} \phi = N$, $\operatorname{Im} \phi = R$. Using Proposition A, we see that for all but a finite number of subgroups H of G, where $G/H \cong Z_p$ and H is generated topologically by τ_H, we have the following two conditions:

(a) If $f = f_1^{s_1} \ldots f_t^{s_t}$ where each f_i is an irreducible element of Λ_G, then $f_i \notin (\tau_H - 1, p)$ and

(b) N and Z/R are finitely generated torsion Λ_H-modules.

Choose such an H and suppose $\tau = \tau_H$, $T = \tau - 1$. If we choose $\sigma \in G$ so that G is generated topologically by σ and τ, and let $S = \sigma - 1$, then when viewed as a power series in S and T, each f_i is seen to be regular in S, and hence by the Weierstrass Preparation theorem, we can assume each f_i is in $(Z_p[[T]])[S]$. It is then seen that $\Lambda_G/(f_i^{s_i}, T)$ is finitely generated over Z_p, and hence Z/TZ is finitely generated over Z_p. We have proved the following lemma:

LEMMA 4.3: *If* $m_0(K/k) = 0$, *then for all but a finite number of subgroups* H *of* G *where* $G/H \cong Z_p$, Z/TZ *is a finitely generated torsion* Λ_H*-modules and* $\mu_H(Z/TZ) = 0$, *where* H *is generated topogically by* $T + 1$. [*Here* μ_H *is the* μ*-invariant of* Z/TZ *when considered as a* Λ_H*-module.*]

Now, keeping the same notation and supposing that $m_0(K/k) = 0$, we see that since $f_i \neq T$ $(i = 1 \ldots t)$, that multiplication by T is $1-1$ on Z. It then follows that $Z/R \cong TZ/TR$ and $N/TN \cong N + TX/TX$ as Λ_G-modules (the second isomorphism follows because $TX \cap N = TN$). Note that since N is finitely generated and torsion over Λ_H, so too is

$N+TX/TX$. Also we have a surjective homomorphism $R/TR \twoheadrightarrow R+TZ/TZ$. Now $R+TZ/TZ \subset Z/TZ$ and so the image of this homomorphism is a finitely generated torsion Λ_H-module. The kernel is $(TZ \cap R)/TR$ which is contained in TZ/TR, so it too is a finitely generated torsion Λ_H-module. But then R/TR is also a finitely generated torsion Λ_H-module. Finally, the map $X/TX \twoheadrightarrow R/TR$ has kernel $N+TX/TX$ so X/TX is a finitely generated torsion Λ_H-module also. Summarizing, we have:

LEMMA 4.4:
(a) N/TN, R/TR, *and* X/TX *are all finitely generated torsion* Λ_H-*modules,*
(b) $N/TN \cong N+TX/TX$, *and*
(c) $Z/R \cong TZ/TR$ (*as* Λ_G-*modules*).

Now considering all our modules as Λ_H-modules, we can apply Lemma 4.4 to obtain: $\mu_H(X/TX) = \mu_H(R/TR) + \mu_H(N/TN)$, where, as before, μ_H denotes the μ-invariant of a Λ_G-module when considered as a Λ_H-module. Since Z/R is a finitely generated torsion Λ_H-module, we also have $\mu_H(Z/R) = \mu_H(Z/TZ) + \mu_H(TZ/TR) - \mu_H(R/TR)$, and applying Lemma 4.4 we see that $\mu_H(Z/TZ) = \mu_H(R/TR)$. Combining these results with Lemma 4.3, we see that

$$\mu_H(X/TX) = \mu_H(Z/TZ) + \mu_H(N/TN) = \mu_H(N/TN).$$

Finally, since N is a finitely generated torsion Λ_H-module, it is not hard to see that N/TN is finitely generated over $\boldsymbol{Z}_p$, and hence $\mu_H(N/TN) = 0$, yielding:

LEMMA 4.5: *If* $m_0(K/k) = 0$, *then for all but a finite number of subgroups* H *of* G *where* $G/H = \boldsymbol{Z}_p$, *we have* $\mu_H(X/TX) = 0$, *where* H *is generated topologically by* $T+1$.

Now suppose $m_0(K/k) = 0$, and choose H and T as in Lemma 4.5. Let $k_{\infty,T}$ be the subfield of K fixed by $T+1=\tau$, and choose $\sigma \in G$ so that G is generated topologically by σ and τ. Let H' be generated topologically by σ so that $G(k_{\infty,T}/k) = H'$. Let L_T denote the maximal unramified pro-p extension of $k_{\infty,T}$ and let $X_T = G(L_T/k_{\infty,T})$. Then there is a surjective homomorphism $X/TX \to X_T$ so that X_T is finitely generated and torsion over Λ_H and $\mu_H(X_T) = 0$. But this implies that the kernel of the mapping $X_T \to X_T$ given by multiplication by p is finite, and hence $\mu_J(X_T) = 0$ for every subgroup J of G where $G/J \cong \boldsymbol{Z}_p$

and X_T is finitely generated and torsion over Λ_J. Now the theory of Z_p-extensions tells us that X_T is finitely generated and torsion over $\Lambda_{H'}$, so that $\mu_{H'}(X_T) = 0$. Now $\mu_{H'}(X_T)$ is, by definition, $\mu(k_{\infty,T}/k)$. Hence, we have the following result:

PROPOSITION 4.6: *If $m_0(K/k) = 0$, then for all but a finite number of Z_p-extensions $k_{\infty,*}$ of k contained in* K, *we have $\mu(k_{\infty,*}/k) = 0$.*

Next we develop a sufficient condition for m_0 to vanish, and the condition turns out to be a partial converse of Proposition 4.6.

Suppose $\mathcal{P}_1, \ldots, \mathcal{P}_t$ are the primes of k lying over p. Some of these primes, say $\mathcal{P}_{s+1}, \ldots, \mathcal{P}_t$ may split completely in K. If $k_{\infty,*}$ is a Z_p-extension of k contained in K, we will say that p is "almost finitely decomposed" if $\mathcal{P}_1, \ldots, \mathcal{P}_s$ are finitely decomposed in $k_{\infty,*}$. Keeping the same notation as in Theorem 1.1, we will prove the following proposition:

PROPOSITION 4.7: *If p is odd and p is almost finitely decomposed in k'_∞, then $\mu(k'_\infty/k) = 0 \Leftrightarrow \mu_n = \mu(k'_\infty k_n/k_n) = 0$ for all $n > 0$.*

Note: The proof of Proposition 4.7 will show that this result is also true if $p = 2$ and k is totally imaginary.

Once we prove Proposition 4.7, the following result will follow almost immediately.

COROLLARY 4.8: *If $k_{\infty,*} \subset K$ is a Z_p-extension of k [if $p = 2$ assume k totally imaginary] in which p is almost finitely decomposed and $\mu(k_{\infty,*}/k) = 0$, then $m_0(K/k) = 0$.*

Indeed, if we find $k_\infty \subset K$ so that k_∞/k is a Z_p-extension and $K = k_\infty k_{\infty,*}$, $k_\infty \cap k_{\infty,*} = k$, then Proposition 1.4.7 shows that $\mu(k_{\infty,*}k_n/k_n) = 0$ for $n > 0$. But then for $n \gg 0$, $m_0 p^n + m_1 n + c = 0$ so that $m_0 = 0$.

PROOF OF PROPOSITION 4.7: In [8] Iwasawa proves the following fact: If k'/k is a cyclic extension of degree p, A′ and A the Sylow-p subgroups of the ideal class groups of k and k', s the number of prime divisors in k which ramify in k' (this is the number of prime ideals which ramify in k' under our assumptions), and $r = \operatorname{rank} A$ (i.e., $\dim_{Z/pZ} A \otimes Z/pZ$), $r' = \operatorname{rank} A'$, then $r - 1 \leq r' \leq p(r + s)$.

Now let k_∞ and k'_∞ be as in the hypothesis of Proposition 4.7, so p is almost finitely decomposed in k'_∞, let $A_{n,m}$ be the Sylow-p subgroup of

$k_nk'_m$, $r_{n,m}$ = rank $A_{n,m}$, and suppose $s_{n,m}$ the number of primes of $k_nk'_m$, which are ramified in $k_{n+1}k'_m$.

Now only primes in k which lie over p can ramify in K and every prime in k which ramifies in K is finitely decomposed in k'_∞. Hence if $\mathcal{P}_1, \ldots, \mathcal{P}_s$ are the primes in k which ramify in K, then each $\mathcal{P}_i$ is finitely decomposed in k'_∞. Suppose the decomposition field for $\mathcal{P}_i$ in k' is k_{m_i}, and let $m_0 = \sup_{i=1,\ldots,s} m_i$. Then for $m, m' > m_0$ the number of primes $k'_{m'}$ over $\mathcal{P}_i$ is the same as the number of primes in k'_m, over $\mathcal{P}_i$ $(i = 1 \ldots s)$. That is, for $m > m_0$, the number of primes in k'_m which ramify in K is bounded by a constant s_0 which is independent of m. Since $[k_nk'_m : k_n] = p^n$, we see that if $m > m_0$, the number of primes in $k_nk'_m$ which ramify in K is bounded by s_0p^n. Hence, it is clear that for $m > m_0$, $s_{n,m} \leq s_0p^n$.

Now, Proposition 4.7 is proved as follows: In the theory of Z_p-extensions, it is known that if k_∞/k is a Z_p-extension with intermediate fields k_n, and if A_n is the Sylow-p subgroup of the ideal class group of k_n, then $\mu(k_\infty/k) = 0 \Leftrightarrow$ rank A_n is bounded independently of n. Now the above discussion shows that for $n \geq 0$, $m > m_0$, $r_{n,m} - 1 \leq r_{n+1,m} \leq p(r_{n,m} + s_{n,m}) \leq p(r_{n,m} + p^ns_0)$. Hence $r_{n,m}$ is bounded as $m \to \infty \Leftrightarrow r_{n+1,m}$ is bounded as $m \to \infty$; that is, $\mu(k_nk'_\infty/k_n) = 0 \Leftrightarrow \mu(k_{n+1}k'_\infty/k_n) = 0$, yielding the desired result.

Recently it has been shown [4] that if k/Q is abelian and k_∞ is the cyclotomic Z_p-extension of k, then $\mu(k_\infty/k) = 0$. Since p is finitely decomposed in k_∞, we have:

COROLLARY 4.9: *If k/Q is abelian and* K$/k$ *is a Z_p^2-extension containing the cyclotomic Z_p-extension of k, then $m_0(K/k) = 0$.*

Now it is not hard to see that there are only a finite number of Z_p-extensions $k_{\infty,*}$ of k contained in K in which p is not almost finitely decomposed. Combining Proposition 4.6 with Corollary 4.8, we therefore have:

COROLLARY 4.10: $m_0(K/k) = 0 \Leftrightarrow \mu(k_{\infty,*}/k) = 0$ *for almost all Z_p-extensions $k_{\infty,*}$ of k contained in* K.

The question naturally arises as to whether or not anything can be said about $\ell(k_\infty, k'_\infty/k)$. First of all, it seems unlikely that the ℓ invariant depends only in K$/k$ instead of k_∞, k'_∞ and k because it is essentially the degree of a power series $f(S, T)$ in one of the variables, and this depends on the choice of variables (which amounts to a choice of topological generators for G, and, hence, a choice of subfields k_∞ and k'_∞ of K). The search for an example which leads to a nontrivial ℓ-invariant amounts to finding a $\mathbf{Z}_p^2$-extension K so that the support of the Λ_G-module X contains a nontrivial power series distinct from p. I have been unable to find such examples. There are examples where $m_0(K/k) = 0$ and yet X is highly nontrivial (it is not finitely generated over Λ_H for some $H \subset G$ with $G/H \cong \mathbf{Z}_p$). Of course, this does not prohibit the possibility that X is pseudo-null and hence that there is no power series in the support of X.

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