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YUJIRO KAWAMATA
ECKART VIEHWEG

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ON A CHARACTERIZATION OF AN ABELIAN VARIETY IN THE CLASSIFICATION THEORY OF ALGEBRAIC VARIETIES

Yujiro Kawamata and Eckart Viehweg

In this paper we shall prove the following theorem which was conjectured by S. Iitaka (B_n in p. 131 in [1]) and proven by K. Ueno for $n = 3$ [2]. In this paper everything is defined over the complex number field $\mathbb{C}$.

MAIN THEOREM: *Let X be an algebraic variety and let $f: X \to A$ be a dominant generically finite morphism to an abelian variety. If the Kodaira dimension $\kappa(X) = 0$, then f is birationally equivalent to an étale morphism and X is birationally equivalent to an abelian variety.*

To prove the main theorem we shall reduce it to the following theorem 1.

THEOREM 1: *Let A be abelian variety of dimension n, let X be a reduced irreducible divisor on A and let $\bar{X}$ be a resolution of X. If X is an algebraic variety of general type, then $q_k(\bar{X}) \equiv \dim H^0(\bar{X}, \Omega^k_{\bar{X}}) \geq \binom{n}{k}$, for $k = 1, \ldots, n-1$. Moreover, if $p_g(\bar{X}) \equiv q_{n-1}(\bar{X}) = n$, then $q_k(\bar{X}) = \binom{n}{k}$, for $k = 1, \ldots, n-2$, and in particular $|\mathbf{X}(O_{\bar{X}})| = 1$.*

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The following lemma is just a special case of a theorem of Ueno (3.3 of [2]).

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LEMMA 2: *Let the notations and assumptions be as in the Main Theorem. Then* $\dim(H^0(X, \Omega_X^{n-1})) \leq n$.

PROOF: We want to show that $f^*(dz_1 \wedge \cdots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \cdots \wedge dz_n) = \omega_i$ are generators of $H^0(X, \Omega_X^{n-1})$, for a global coordinate system $(z_1, \ldots, z_n)$ of A. Take $\omega \in H^0(X, \Omega_X^{n-1})$ and $a_i \in \mathbb{C}$, such that $\omega \wedge f^*(dz_i) = a_i \cdot f^*(dz_1 \wedge \cdots \wedge dz_n)$. This is always possible, since $H^0(X, \Omega_X^n)$ is generated by $f^*(dz_1 \wedge \cdots \wedge dz_n)$. Replacing ω by $\omega - \sum_{i=1}^{n} (-1)^{n-1} a_i \omega_i$ we may assume that $a_i = 0$ for $i = 1, \ldots, n$. Choose a small open subset $U \subseteq X$, such that $f|_U$ is an embedding. $(z_1, \ldots, z_n)$ is a local coordinate system of U. Since $\omega \wedge dz_i = 0$ for $i = 1, \ldots, n$, ω must be 0 on U and hence on X.

LEMMA 3: *Let the notations and assumptions be as in the main theorem. Let* $f_0: X_0 \to A$ *be the normalisation of* A *in* $\mathbb{C}(X)$. *Let* $D_1, \ldots, D_m$ *be the irreducible components of the discriminant* $\Delta(X_0/A)$ *and let* $\bar{D}_1, \ldots, \bar{D}_m$ *be their desingularisations. Then*

$$\sum_{i=1}^{m} p_g(\bar{D}_i) \leq \dim(A).$$

PROOF: Choose Δ_i to be one irreducible component of $f^{-1}(D_i)$, such that Δ_i is ramified over A. We may assume, that X is projective and that $\Delta_1 \cup \cdots \cup \Delta_m$ is a regular subvariety of X. Let $\omega_X = \Omega_X^n$. Then $\omega_X \otimes_{O_X} O_X \left(\sum_{i=1}^{m} \Delta_i\right) \subseteq \omega_X^2$ and, since $H^0(X, \omega_X) = H^0(X, \omega_X^2) = \mathbb{C}$, we know that $\bigoplus_{i=1}^{m} H^0(\Delta_i, \omega_{\Delta_i})$ is a subspace of $H^1(X, \omega_X) = H^{n-1}(X, O_X) = H^0(X, \Omega_X^{n-1})$. However, $H^0(\bar{D}_i, \omega_{\bar{D}_i}) \subseteq H^0(\Delta_i, \omega_{\Delta_i})$.

Now we recall the following Theorem of Ueno (p. 120 in [1]):

THEOREM 4: *Let* B *be a subvariety of an abelian variety* A. *Then there exist an abelian subvariety* A_1 *of* A *and an algebraic variety* W *which is a subvariety of an abelian variety such that*

(1) *B is an analytic fibre bundle over W whose fibre is A_1,*

(2) $\kappa(W) = \dim W = \kappa(B)$.

A_1 is characterized as the maximal connected subgroup of A such that $A_1 + B \subseteq B$.

PROOF OF "THEOREM 1 $\Rightarrow$ MAIN THEOREM": Let $\eta: A' \to A$ be any étale covering and $X_\eta = X \times_A A'$. Then $X_\eta \to A'$ also satisfies the conditions of the main theorem. Let $X_{\eta,0}$ be the normalisation of A' in $\mathbb{C}(X_\eta)$. Then $\Delta(X_{\eta,0}/A')$ is the pullback of $\Delta(X_0/A)$ by η. Suppose $\Delta(X_0/A)$ is not empty. Any abelian variety has étale coverings of arbitrary high degree (for example "multiplication with $r \gg 0$"). Every subvariety of an abelian variety has $p_g > 0$. Hence, replacing A by some étale covering we may assume, that for every étale covering $\eta: A' \to A$ the number of irreducible components of $\Delta(X_0/A)$ and $\Delta(X_{\eta,0}/A')$ is the same (Lemma 3).

Put $B = \{x \in A; x + \Delta(X_0/A) \subseteq \Delta(X_0/A)\}^0$. Again replacing A by an étale covering, we may assume that $A = B' \times B$. Let Y_0 be the Stein factorisation of $X_0 \to A \to B'$ and Y any desingularisation of Y_0. Since X_0 is a finite covering of $Y_0 \times B$ we have $\kappa(X) = 0 \geq \kappa(Y) + \kappa(B) \geq 0$ and $\kappa(Y) = 0$.

Assume that $\Delta(Y_0/B') = \emptyset$. Since $\Delta(X_0/A) \simeq \Delta' \times B$ for some positive divisor $\Delta' \subseteq B'$, the ramification divisor of $X_0 \to Y_0 \times B$ must be a (rational) multiple of the pullback of some divisor $\tilde{\Delta}$ of Y_0. Then $\kappa(X) \geq \kappa(Y_0, O(\tilde{\Delta})) > 0$, in contradiction to our assumptions. Therefore $\Delta(Y_0/B') \neq \emptyset$ and, repeating this step if necessary, we may assume $B = 0$.

Let $B_i = \{x \in A; x + D_i \subseteq D_i\}^0$ for $i = 1, \ldots, m$. We have $\cap \mathrm{B_i} = 0$. By Theorem 4 each D_i is a fibre bundle over a certain E_i with fibre B_i, for $i = 1, \ldots, m$. We have $p_g(\bar{D}_i) \geq p_g(\bar{E}_i)$ for a desingularisation $\bar{E}_i$ of E_i and by Theorem 1 $p_g(\bar{E}_i) \geq \operatorname{codim}_A B_i$.
Since the equalities must be true by Lemma 3, we have $|\mathbf{X}(O_{\bar{E}_i})| = 1$ by theorem 1, for $i = 1, \ldots, m$.

Let r be a natural number, with $r \geq 2$, and let $r: A \to A$ be the multiplication with r. Using the notation introduced above, $\Delta(X_{r,0}/A)$ must have components $D_{r,i}$, $i = 1, \ldots, m$ such that the corresponding base space $E_{r,i}$ satisfies $|\mathbf{X}(O_{\bar{E}_{r,i}})| = \operatorname{degree}(r) \cdot |\mathbf{X}(O_{\bar{E}_i})| \geq 2$. This is a contradiction.

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Proof of Theorem 1: Let $\{x_1, \ldots, x_n\}$ be a global coordinate system on A such that the set $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ gives a basis of 1-forms on A. Let $\alpha: \bar{X} \to A$ be the canonical map and let $\omega_i = \alpha^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n)$ for $i = 1, \ldots, n$. We shall prove first that these are linearly independent $(n-1)$-forms on $\bar{X}$. Suppose the contrary: $\sum_{i=1}^{n} a_i\omega_i = 0$ for $a_i \in \mathbb{C}$. Pick a smooth point p on X.

Suppose X is defined in A near p by an equation $x_n = F(x_1, \ldots, x_{n-1})$, where F is a certain holomorphic function. Then $\omega_i = (-1)^{n-i-1} \dfrac{\partial F}{\partial x_i} \omega_n$ for $i = 1, \ldots, n-1$. Therefore, $\sum_{i=1}^{n-1} (-1)^{n-i-1} a_i \dfrac{\partial F}{\partial x_i} + a_n = 0$, which means that there is a non-zero subgroup B of A such that $B + X \subseteq X$, which is a contradiction.

Put $\omega_I = \alpha^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k})$ for each set I of k-distinct integers $1 \leq i_1 < \cdots < i_k \leq n$. Since ω_i are linearly independent, $\{\omega_I\}_I$ gives a linearly independent system of k-forms on $\bar{X}$. Thus, $q_k(\bar{X}) \geq \binom{n}{k}$.

Before we prove the second part of theorem 1, we shall prove the following theorem, due to the first author.

THEOREM 5: *Let A and X be as in theorem* 1. *Let $f: X \to \mathbb{P}^{n-1}$ be the rational map defined by the system $\{\omega_1, \ldots, \omega_n\}$. If X is an algebraic variety of general type, then f is dominant.*

PROOF: Assume the contrary. Let Y be the image variety of f and let q be a smooth point of Y such that $f^{-1}(q)$ is also smooth near some smooth point $p \in f^{-1}(q)$ of X. Our assumption means that $\dim f^{-1}(q) \geq 1$. Consider everything in the universal cover $\mathbb{C}^n$ of A. Let H be the tangent plane of X at p, which we assume is defined by an equation $x_n = 0$. Then, X is defined near p by an equation $x_n = F(x_1, \ldots, x_{n-1})$, where $\{x_1, \ldots, x_n\}$ is a global coordinate system centered at p and $\dfrac{\partial F}{\partial x_i}(0) = 0$ for $i = 1, \ldots, n-1$. $f^{-1}(q)$ is defined near p by the equations $\dfrac{\partial F}{\partial x_i} = 0$ for $i = 1, \ldots, n-1$. Y is contained near q in a smooth divisor D of $\mathbb{P}^{n-1}$ (near q). After a suitable linear transformation of $x_1, \ldots, x_{n-1}$, the equation of D can be written as $\dfrac{\partial F}{\partial x_1} = G\left(\dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_{n-1}}\right)$, where G is a holomorphic function of degree ≥ 2. By the rule of derivation of products, we have on $f^{-1}(q)$ $\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial F}{\partial x_i}\right) = \dfrac{\partial G}{\partial x_i} = 0$ for $i = 1, \ldots, n-1$. Thus, $f^{-1}(q)$ is invariant under translations in the direction of x_1 and hence contains a translation of an abelian subvariety of A generated by the line $x_2 = \cdots = x_n = 0$. Let B be the maximal abelian subvariety of A such that $p + B$ is contained in X. We have proved that $B \neq 0$. Since there are only countably many abelian subvarieties, B does not depend on p. Thus, $B + X \subseteq X$, a contradiction. Q.E.D.

PROOF OF THEOREM 1 CONTINUED: Suppose $p_g(\bar{X}) = n$. Let p be a smooth point of X and let $x_1, \ldots, x_n$ be as in the proof of theorem 5. Let ω be an arbitrary k-form on $\bar{X}$. Write near p $\omega = \sum_{n \notin I} g_I(x_1, \ldots, x_{n-1})\omega_I$. Put $I^c = \{1, \ldots, n-1\} - I$. Then $\omega \wedge \omega_{I^c} = \epsilon(I, I^c)g_I\omega_n$, where ϵ is the sign of permutations. Therefore, we have

$$g_I = g_I(0) + \sum_{i=1}^{n-1} a_{Ii} \frac{\partial F}{\partial x_i}$$

for some $a_{Ii} \in \mathbb{C}$. Let J be a subset of $\{1, \ldots, n-1\}$ such that Card $J = n - k - 2$. Since

$$\omega \wedge \omega_J \wedge \mathrm{d}x_n = \sum_{\{i\} \cup I = J^c} \epsilon(I, J, i) g_I \frac{\partial F}{\partial x_i} \omega_n,$$

we have

$$\sum_{\{i\} \cup I = J^c} \epsilon(I, J, i)\left(g_I(0) + \sum_{j=1}^{n-1} a_{Ij} \frac{\partial F}{\partial x_j}\right)\frac{\partial F}{\partial x_i} = \sum_{i=1}^{n-1} b_{Ji} \frac{\partial F}{\partial x_i}$$

for some $b_{Ji} \in \mathbb{C}$. Since $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n-1}}$ are algebraically independent, we can compare the coefficients and we get (1) $a_{Ij} = 0$ for $j \in J$, (2) $a_{Ii} = 0$ for $I \cup \{i\} = J^c$, and (3) for each $K = \{i_1, \ldots, i_{k-1}\}$ such that $i_1 < \cdots < i_{k-1}$ and $K \cup \{i\} \cup \{j\} = J^c$,
$\epsilon(K \cup \{i\}, J, j)a_{K \cup \{i\}, i} + \epsilon(K \cup \{j\}, J, i)a_{K \cup \{j\}, j} = 0$, that is,
$\epsilon(K, i)a_{K \cup \{i\}, i} = \epsilon(K, j)a_{K \cup \{j\}, j}$. Put $a_K = \epsilon(K, i)a_{K \cup \{i\}, i}$.
Then, $\omega = \sum_{n \notin I} q_I(0)\omega_I + \sum_K a_K\omega_K \wedge \mathrm{d}x_n$. Q.E.D.

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Yujiro Kawamata
Department of Mathematics
University of Tokyo
Hongo, Bunkyo, Tokyo, Japan

Eckart Viehweg
Institut für Mathematik
und Informatik
A5, Seminargebäude
D-68 Mannheim