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THE UNRAMIFIED PRINCIPAL SERIES OF p-ADIC GROUPS II THE WHITTAKER FUNCTION

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Let $\mathbf{G}$ be a connected reductive algebraic group defined over the non-archimedean local field k. We will prove in this paper an explicit formula for a certain so-called Whittaker function associated to the unramified principal series of $\mathbf{G}(k)$, under the assumption that the group $\mathbf{G}$ is itself unramified – that is to say, arises by base extension to k from a smooth reductive group over the integers $\mathcal{O}$ of k. This formula has been discovered independently by Shintani [8] when $\mathbf{G} = \mathbf{GL}_n$ and Kato [9] for Chevalley groups, and was also in fact conjectured by Langlands several years ago (in correspondence with Godement). It plays a role in establishing the functional equations of certain L-functions (see [7], for example).

As a preparation for the proof of the explicit formula, we include in the paper new proofs of results due to Rodier [6] (this is done also by Bernstein and Zelevinskii in [1]). Later on, we also include a discussion of matters partly dealt with in Jacquet's thesis [5]. Our proof of the formula itself follows along lines very similar to those pursued in [3] to prove Macdonald's formula for the zonal spherical function. (Incidentally, the technique used by Kato and Shintani is more similar to Macdonald's own proof of his formula.)

Notation

Throughout, algebraic groups will be written in boldface, the corresponding groups of k-rational points in ordinary type. Thus: $\mathbf{G}$ and G.

Let

$\mathbf{P}$ = a minimal parabolic subgroup of $\mathbf{G}$

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$\mathbf{A}$ = a maximal split torus of $\mathbf{G}$ in $\mathbf{P}$
$\mathbf{M}$ = the centralizer of $\mathbf{A}$
$\mathbf{N}$ = the unipotent radical of $\mathbf{P}$
$\mathbf{N}^-$ = the opposite of $\mathbf{N}$
Σ = roots of G with respect to $\mathbf{A}$
${}^{nd}\Sigma$ = non-divisible roots of Σ
Σ^+ = positive roots determined by the choice of P
Δ = simple roots of Σ^+
W = Weyl group of Σ
w_ℓ = longest element of W
For each $\theta \subseteq \Delta$ let
$\mathbf{P}_\theta$ = the standard parabolic subgroup corresponding to θ; similarly for $\mathbf{A}_\theta, \mathbf{M}_\theta, \mathbf{N}_\theta, W_\theta$.
δ_θ = modulus character of P_θ, taking $p = mn$ to $|\det \mathrm{Ad}_{\mathfrak{n}}(m)|$, where $\mathfrak{n}$ = Lie algebra of N

$$[W_\theta \backslash W] = \{w \in W \mid w^{-1}\theta > 0\}$$
$$A_\theta^- = \{a \in A_\theta \mid |\alpha(a)| \leq 1 \quad \text{for all } \alpha \in \Delta\}.$$

For each $\alpha \in \Sigma$ let N_α be the subgroup of N whose Lie algebra is $g_\alpha + g_{2\alpha}$. Thus N_θ is the product ΠN_α ($\alpha > 0$, not divisible, not in the linear span of θ). With a few exceptions, when we write products of expressions indexed by roots, these indices will include only elements of ${}^{nd}\Sigma$.

1. Introduction to Whittaker models

Let ψ be a smooth complex character of N, and denote as $\mathbf{C}_\psi$ the corresponding one-dimensional N-module.

If (π, V) is any smooth representation of N, define $V_{\psi,N}$ to be the Jacquet space of the twisted representation $\pi \otimes \psi^{-1}$. In other words: define $V_\psi(N)$ to be the subspace of V spanned by $\{\pi(n)v - \psi(n)v \mid n \in N, v \in V\}$ and set $V_{\psi,N} = V/V_\psi(N)$. The space $V_\psi(N)$, incidentally, is also that of $v \in V$ such that for some compact open $N_0 \subseteq N$,

$$\int_{N_0} \psi^{-1}(n)\pi(n)v\,dn = 0.$$

The group N acts on $V_{\psi,N}$ by ψ and $V_{\psi,N}$ is the largest quotient of V with this property ([2] 3.2.2):

1.1. PROPOSITION: *If V' is any space on which N acts by ψ then $V \to V_{\psi,N}$ induces*

$$\mathrm{Hom}_N(V, V') \cong (\mathrm{Hom}_{\mathbf{C}}(V_{\psi,N}, V').$$

Also, by [2] 3.2.3:

1.2. PROPOSITION: *The functor $V \to V_{\psi,N}$ is exact.*

The subgroup ΠN_α ($\alpha > 0, \alpha \notin \Delta$) is normal in N and the quotient is isomorphic to $\Pi(N_\alpha/N_{2\alpha})$ ($\alpha \in \Delta$). If ψ_α for each $\alpha \in \Delta$ is a character of $N_\alpha/N_{2\alpha}$ then $\Pi\psi_\alpha$ is a character of $\Pi(N_\alpha/N_{2\alpha})$, hence determines one of N as well. A character of N is said to be *principal* if it is of this form with no ψ_α trivial.

If ψ is principal and (π, V) an admissible representation of G, a G-embedding of V into the smooth representation $\mathrm{Ind}(\mathbf{C}_\psi \mid N, G)$ of G is called a *Whittaker model* for V. The space $\mathrm{Ind}(\mathbf{C}_\psi \mid N, G)$ (see §2.4 of [2]) is that, of all $f: G \to \mathbf{C}$ such that (1) $f(ng) = \psi(n)f(g)$ for all $n \in N$, $g \in G$ and (2) there exists an open subgroup $K \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K$, and G acts on it by the right regular representation. (The terminology arises from the analogous theory for the real group $SL_2(\boldsymbol{R})$, where such functions f are essentially classical Whittaker functions; see [5].) There is an intimate relationship between the space $V_{\psi,N}$ and Whittaker models. Let Ω be the map $\mathrm{Ind}(\mathbf{C}_\psi \mid N, G) \to \mathbf{C}$, $f \mapsto f(1)$. It is an N-morphism into $\mathbf{C}_\psi$. Frobenius reciprocity ([2] 2.4.1 (e)) and 1.1 combine to give:

1.3. PROPOSITION: *Let ψ be any smooth character of N, V a smooth representation of G. Composition with Ω induces an isomorphism*

$$\mathrm{Hom}_G(V, \mathrm{Ind}(\mathbf{C}_\psi \mid N, G)) \cong \mathrm{Hom}_{\mathbf{C}}(V_{\psi,N}, \mathbf{C}).$$

Explicitly, the map $F: V \to V_{\psi,N} \to \mathbf{C}$ corresponds to the G-morphism from V to $\mathrm{Ind}(\mathbf{C}_\psi)$ which takes v to the function $\Phi_v: g \mapsto F(gv)$.

When (π, V) is induced from a proper parabolic subgroup of G, a result of Rodier [6] reduces the problem of determining $V_{\psi,N}$ to a similar problem for the inducing representation. We give here a proof of his result along lines used in [2] to discuss V_N (which is $V_{1,N}$ in our notation).

Let (σ, U) be an admissible representation of M_θ and let $I(\sigma) = \mathrm{Ind}(\sigma \mid P_\theta, G)$, the corresponding induced representation of G. It is admissible. Because of the disjoint decomposition

$$G = UP_\theta wP \quad (w \in [W_\theta \backslash W])$$

$I(\sigma)$ is filtered by P-stable subspaces

$$I_n = \{f \in I(\sigma) \mid \mathrm{Supp}(f) \subseteq P_\theta wP(\dim P_\theta \backslash P_\theta wP \geq n)\}$$

(see [2] 6.3). The quotient I_n/I_{n+1} is isomorphic to the direct sum of spaces I_x, as x ranges over a set of representatives of elements w of $[W_\theta \backslash W]$ in $N_G(A)$ with $\dim P_\theta < P_\theta wP = n$, and where

$$I_x = \mathrm{Ind}_c(x^{-1}(\sigma\delta_\theta^{1/2}) \mid x^{-1}P_\theta x \cap P, P).$$

(This is [2] 6.3.2. The notation is that of §2 of [2], so that Ind_c is the non-normalized induced representation comprising functions of compact support modulo $x^{-1}P_\theta x \cap P$.)

Let $w_{\ell,\theta}$ be longest element of W_θ. Then $w_\theta = w_{\ell,\theta}w_\ell$ is the longest element of $[W_\theta \backslash W]$ ([2], 1.1.4(b)) and $P_\theta \backslash P_\theta w_\theta P$ is the unique open double coset in $P_\theta \backslash G$. Let d_θ be its dimension. By the above remarks, we have an injection $I_{d_\theta} \hookrightarrow I(\sigma)$.

1.4. Theorem (Rodier): *If ψ is a principal character of N then the inclusion of I_{d_θ} in $I(\sigma)$ induces an isomphorphism of $(I_{d_\theta})_{\psi,N}$ with $I(\sigma)_{\psi,N}$.*

The proof requires a preliminary result. Let Σ_θ^+ be the intersection of Σ^+ with the linear span of θ. It is, essentially, the set of positive roots of $\mathbf{M}_\theta$ with respect to $\mathbf{A}_\theta$ determined by $\mathbf{P} \cap \mathbf{M}_\theta$.

1.5. Lemma: *For $w \in [W_\theta \backslash W]$, $w \neq w_\theta$, there exists $\alpha \in \Delta$ with $w\alpha \in \Sigma^+ - \Sigma_\theta^+$.*

Proof of Lemma. If $w^{-1}\theta > 0$ then 1.1.4(b) of [2] implies that $l(w_\theta^{-1}) = l(w_\theta^{-1}w) + l(w^{-1})$. Therefore by the definition of length in W

$$\{\alpha > 0 \mid w_\theta^{-1}\alpha < 0\} = \{\alpha > 0 \mid w^{-1}\alpha < 0\} \cup w\{\alpha > 0 \mid w_\theta^{-1}w\alpha < 0\}.$$

By 1.1.4(a) of [2], the left hand side is $\Sigma^+ - \Sigma_\theta^+$. Hence

$$w^{-1}(\Sigma^+ - \Sigma_\theta^+) = w^{-1}\{\alpha > 0 \mid w^{-1}\alpha < 0\} \cup \{\alpha > 0 \mid w_\theta^{-1}w\alpha < 0\}.$$

If $w_\theta^{-1}w \neq 1$, the second term on the right must contain at least one element of Δ.

Conclusion of the proof of 1.4: by previous remarks and Proposition 1.2, it suffices to show that $(I_x)_{\psi,N} = 0$ for any x representing an element of $[W_\theta \backslash W]$ other than w_θ. By 1.1, in order to do this, it suffices to show that $\mathrm{Hom}_N(I_x, \mathbf{C}_\psi) = 0$.

An N-morphism from I_x to $\mathbf{C}_\psi$ may be considered as an element Φ in the dual of I_x which is an eigenvector for N with eigencharacter ψ^{-1}, hence as an element of the N-smooth dual of I_x. Now since $x^{-1}\theta > 0$, $x^{-1}(-\theta) < 0$, so that $x^{-1}P_\theta x \cap P = x^{-1}Px \cap P$, and as an N-space I_x is isomorphic to

$$\mathrm{Ind}_c(x^{-1}(\delta) \,|\, x^{-1}Nx \cap N, N).$$

According to [2] 2.4.2, its N-smooth dual is isomorphic to

$$\mathrm{Ind}_c(x^{-1}(\tilde{\sigma}) \,|\, x^{-1}Nx \cap N, N),$$

where $\tilde{\sigma}$ is the smooth dual of σ. Corresponding to Φ, therefore, is a function $F: N \to \tilde{U}$ (the space of $\tilde{\sigma}$) such that

$$F(n_1 n) = \tilde{\sigma}(x n_1 x^{-1}) F(n)$$

for all $n_1 \in x^{-1}Nx \cap N$, $n \in N$, and

$$F(n) = \psi^{-1}(n) F(1)$$

for all $n \in N$. In particular,

$$\tilde{\sigma}(x n_1 x^{-1}) F(1) = \psi^{-1}(n_1) F(1)$$

for all $n_1 \in x^{-1}Nx \cap N$. Since $\tilde{\sigma}$ is trivial on N_θ, in order to prove $F(1)$ and hence F null it now suffices to find $n_1 \in x^{-1}N_\theta x \cap N$ such that $\psi(n_1) \neq 1$. Since $N_\theta = \Pi N_\alpha (\alpha \in \Sigma^+ - \Sigma_\theta^+)$ and ψ is principal, such an n_1 is guaranteed by Lemma 1.5. This concludes the proof.

Now let N' be $M_\theta \cap N$, the unipotent radical of the minimal parabolic $P \cap M$ in M_θ, and for $x \in N_G(A)$ representing W_θ let ψ'_x be the principal character of N' defined by the formula

$$\psi'_x(n) = \psi(x^{-1}nx).$$

This makes sense because $w_\theta^{-1}\theta > 0$. In fact, $w_\theta^{-1}\theta = \bar{\theta}$ is a subset of Δ, the *conjugate* of θ. Because $w_{\bar\theta}^{-1}\alpha < 0$ for every $\alpha \in \Sigma^+ - \Sigma_{\bar\theta}^+$, the group $x^{-1}P_\theta x \cap N$ is equal to $M_{\bar\theta} \cap N$. Since $N = (M_{\bar\theta} \cap N)N_{\bar\theta}$, for every $f \in I_{d_\theta}$ the function $n \mapsto f(xn)$ is a function on $N_{\bar\theta}$ of compact support, and the integral

$$\Omega_\sigma(f) = \int_{N_{\bar\theta}} f(xn)\psi^{-1}(n)\,\mathrm{dn}$$

is well defined, and yields a map from $I(\sigma)_{d_\theta}$ to U.

1.6. THEOREM (*Rodier*): *The map Ω_σ induces an isomorphism of $(I_{d_\theta})_{\psi,N}$ with $U_{\psi_x',N'}$.*

PROOF. It must first be shown that if f lies in $(I_{d_\theta})_\psi(N)$ then $\Omega_\sigma(f)$ lies in $U_{\psi_x}(N')$. For this, choose compact open subgroups B_1 of $M_{\bar\theta} \cap N$ and B_2 of $N_{\bar\theta}$ such that $n \mapsto f(xn)$ vanishes outside B_2 and $f \in (I_{d_\theta})_\psi(B_1 \cdot B_2)$. Then xB_1x^{-1} is a compact open subgroup of N_θ and

$$\begin{aligned}
&\int_{xB_1x^{-1}} \sigma(n_1)\Omega_\sigma(f)\psi_x'(n_1)^{-1}\,\mathrm{dn}_1 \\
&= \int_{xB_1x^{-1}} \psi_x'(n_1)^{-1}\,dn_1 \int_{B_2} f(xn_2)\psi(n_2)^{-1}dn_2 \\
&= \int_{B_1\times B_2} f(xn_1n_2)\psi(n_1n_2)^{-1}dn_1dn_2
\end{aligned}$$

Hence Ω_σ induces a map from $(I_{d_\theta})_{\psi,N}$ to $U_{\psi_x',N'}$.

To see that this is an isomorphism, it suffices to show that the corresponding map from $\mathrm{Hom}_{N'}(U, \mathbf{C}_{\psi_x'})$ to $\mathrm{Hom}_N(I(\sigma), \mathbf{C}_\psi)$ is one. From the proof of 1.4, one sees that the N-smooth dual of I_x is isomorphic to

$$\mathrm{Ind}(x^{-1}(\tilde\sigma) \mid M_{\bar\theta} \cap N, N).$$

Following that proof a bit further one sees that the space $\mathrm{Hom}_N(I_x, \mathbf{C}_\psi)$ is isomorphic to that of $\tilde u \in \tilde U$ such that

$$\tilde\sigma(n)\tilde u = \psi^{-1}(x^{-1}(x^{-1}nx)\tilde u$$

for all $n \in M_\theta \cap N$ – i.e. to $\mathrm{Hom}_{N'}(U, \mathbf{C}_{\psi_x'})$ – and if one follows details explicitly one sees that this isomorphism is the same as that induced by Δ.

From 1.4 and 1.6 together:

1.7. COROLLARY: *If ψ is a principal character of N then $I(\sigma)_{\psi,N} \cong \sigma_{\psi'_x,N'}$, where x is any element of $N_G(A)$ representing w_θ.*

As a special case:

1.8. COROLLARY: *Assume $P_\theta = P$, σ one-dimensional, $x \in N_G(A)$ representing w_l. The functional*

$$\Omega_\sigma(f) = \int_N f(xn)\psi^{-1}(n)dn,$$

which is defined for all $f \in I(\sigma)$ with support on Pw_lP, extends uniquely to a basis element of the one-dimensional space $\mathrm{Hom}_N(I(\sigma), \mathbf{C}_\psi)$.

2. Holomorphicity

Our aim in this section is to show that the map Ω_σ defined in §1 varies holomorphically with σ. For convenience, we shall treat only the case we shall be concerned with later on, although it should be apparent that the argument can be generalized.

A representation of M is said to be *unramified* if it is trivial on the unique maximal compact subgroup M_0. Since M/M_0 is finite and free over $\mathbf{Z}$, the group $X = X_{nr}(M)$ of all such characters is isomorphic to $(\mathbf{C}^\times)^r$ for some r, and in particular has a canonical structure as a complex analytic manifold.

For $\chi \in X$, let $\mathcal{P}_\chi$ be the projection from $C_c^\infty(G)$ onto $I(\chi) = \mathrm{Ind}(\chi \mid P, G)$:

$$\mathcal{P}_\chi f(g) = \int_P \chi^{-1}\delta^{1/2}(p)f(pg)dp.$$

Here a left Haar measure on P is assumed. Let ψ be a principal character of N. Recall from §1 that Ω_χ is the unique N-morphism from $I(\chi)$ to $\mathbf{C}_\psi$ such that for $f \in C_c^\infty(Pw_lP)$ one has

$$\Omega_\chi(\mathcal{P}_\chi f) = \int_N \mathcal{P}_\chi f(xn)\psi^{-1}(n)dn$$

where x is a fixed element of $N_G(A)$ representing w_l. By the definition of $\mathcal{P}_\chi$, more explicitly:

$$\Omega_\chi(\mathcal{P}_\chi f) = \int_{Pw_lP} \Phi(y)f(y)dy$$

where

$$\Phi(n_1mxn_2) = \chi^{-1}\delta^{1/2}(m)\psi^{-1}(n_2)$$

on $Pw_lP = Pw_lN$. It is clear that for a fixed $f \in C_c^\infty(Pw_lP)$ this varies holomorphically with χ. In fact:

2.1. PROPOSITION: *For any $f \in C_c^\infty(G)$, $\Omega_\chi(\mathcal{P}_\chi f)$ is a holomorphic function of χ.*

Fix a compact open subgroup K with the property that f is bi-invariant under K. For every compact open subgroup $N_0 \subseteq N$ define a projection operator on $I(\chi)$:

$$\mathcal{P}_{\psi,N_0}\varphi(g) = (\text{meas } N_0)^{-1}\int_{N_0} \psi^{-1}(n)\varphi(gn)dn.$$

2.2. LEMMA: *There exists a compact open subgroup $N_0 \subseteq N$ such that for every $\chi \in X$ and $\varphi \in I(\chi)^K$ the function $\mathcal{P}_{\psi,N_0}\varphi$ has support in Pw_lP.*

We first point out how the lemma implies the Proposition. Since f is bi-invariant under K, $\mathcal{P}_\chi f = \varphi_\chi$ lies in $I(\chi)^K$. It depends holomorphically on χ in some obvious sense, as does $\mathcal{P}_{\psi,N_0}\varphi_\chi$. But since this latter has support in Pw_lP,

$$\Omega_\chi(\mathcal{P}_{\psi,N_0}\varphi_\chi) = \Omega_\chi(\varphi_\chi)$$

also depends holomorphically on χ.

PROOF OF THE LEMMA.

Step (1). Let G^* be the complement of Pw_lP in G; since Pw_lP is open in G, G^* is closed, and in fact it is the union of the PwP with $w \neq w_l$. For each χ, let $J(\chi)$ be the space of locally constant $\varphi: G^* \to \mathbf{C}$ such that $\varphi(nmg^*) = \chi\delta^{1/2}(m)\varphi(g^*)$ for all $n \in N, m \in M, g^* \in G^*$. Restriction is an N-morphism from $I(\chi)$ to $J(\chi)$; according to §6.1 of [2] one has an exact sequence:

$$0 \to I_{d_l} \to I(\chi) \to J(\chi) \to 0.$$

Furthermore, by 1.2 and the proof of 1.4, $J(\chi)_{\psi,N} = 0$. In other words, for each $\varphi \in J(\chi)$ there exists a compact open $N_0 \subseteq N$ such that

$\mathcal{P}_{\psi,N_0}\varphi = 0$. Since $I(\chi)^K$ is finite dimensional, one can even choose N_0 so that this vanishing holds for all $\varphi \in J(\chi)$ in the image of $I(\chi)^K$; this means in turn that *for all* $\varphi \in I(\chi)^K$ *the function* $\mathcal{P}_{\psi,N_0}\varphi$ *has support in* Pw_lP, because of the exact sequence above.

Step (2). The problem remaining is to show how one may choose this N_0 independently of χ. First of all, choose an exhaustive sequence $N_1 \subseteq N_2 \subseteq \dots$ of compact open subgroups of N. For each $n \geq 1$, let X_n be the set of all $\chi \in X$ such that for all $\varphi \in I(\chi)^K$ the function $\mathcal{P}_{\psi,N_n}\varphi$ has support in Pw_lP. By the result in Step (1), X is the union of the X_n. But then by Baire's lemma, one of the X_n contains an open subset of X. The condition $\chi \in X_n$ however, is holomorphic in χ, so that in fact X_n is all of X. This concludes the proof of 2.2 as well as that of 2.1.

For calculations, a refinement is useful:

2.3. Corollary: *Given* K,

$$\Omega_\chi(f) = \int_{N_*} \psi^{-1}(n)\varphi(xn)dn$$

for all $\varphi \in I(\chi)^K$ *and suitably large compact open subgroups* $N_* \subseteq N$.

Proof: Suppose that N_* is larger than the N_0 in 2.2 and also large enough so that $\mathcal{P}_{\psi,N_0}\varphi$ has support in Pw_lN_* for all $\varphi \in I(\chi)^K$. Then $\mathcal{P}_{\psi,N_*}\varphi$ also has support in Pw_lN_* and

$$\begin{aligned}\Omega_\chi(\varphi) &= \Omega_\chi(\mathcal{P}_{\psi,N_*}\varphi)\\ &= \int_{N_*} \psi^{-1}(n)\varphi(xn)dn.\end{aligned}$$

3. The structure of unramified groups

From now on, through §5, we shall assume the group $\mathbf{G}$ to be unramified. There are two equivalent characterizations: (1) it is obtained by base extension from a smooth reductive group scheme defined over Spec($\mathcal{O}$), where $\mathcal{O}$ is the ring of integers in k; (2) it is quasi-split over k and split over an unramified extension. We shall similarly assume that $\mathbf{P}$ and $\mathbf{M}$ also arise by base extension from subschemes of $\mathbf{G}_{\mathcal{O}}$. The group $\mathbf{P}$ is a Borel subgroup and $\mathbf{M}$ is an unramified torus.

For every ring R given with a homomorphism $\mathcal{O} \to R$, let $G(R)$ be the corresponding group of R-valued points. (We continue to write

$G(k)$ as G.) Let

$$\begin{aligned}
\mathcal{P} &= \text{prime ideal of } k \\
\mathcal{P}^* &= \mathcal{P} - \mathcal{P}^2 \\
K &= G(\mathcal{O}) \\
B &= \text{the inverse image in } G(\mathcal{O}) \text{ of } P(\mathcal{O}/\mathcal{P}) \\
M_0 &= M \cap B \\
N_0 &= N \cap B \\
N_1^- &= N^- \cap B \\
P_0 &= M_0 N_0 = P \cap B.
\end{aligned}$$

Thus K is a special maximal compact of G, B an Iwahori subgroup, M_0 the maximal compact of M. The group B has the Iwahori factorization $B = N_1^- P_0$, elements of W may be (and we shall assume them to be) represented by elements of $K \cap N_G(A)$, and K is the disjoint union of the $BwB (w \in W)$. (We do not know references for the properties of G unless **G** is split over k and semi-simple, when results are in [4]. The general case may be obtained from this one by étale descent and is implicit in various announcements by Bruhat and Tits.)

If α is a simple root, the Levi component M_α of the standard parabolic P_α has semi-simple rank one, as does the simply connected covering $\tilde{G}_\alpha$ of its derived group. The inverse image of P in $\tilde{G}_\alpha$ is a minimal parabolic of $\tilde{G}_\alpha$; the projection is an isomorphism of its unipotent radical with N_α. There are only two possible types of simply connected, semi-simple, unramified group of rank one: (1) $SL_2(F)$, where F is an unramified extension of k, considered as a group over k by restriction of scalars; (2) the special unitary group $SU_3(F)$–F as in (1) – corresponding to an unramified Hermitian form in three variables over the unramified quadratic extension E of F. These play a special role, as we have just pointed out, in the structure of general unramified groups. We shall discuss them in detail.

Let $\mathcal{O}_F$ be the integers in F, etc., and let $q = [\mathcal{O}_F : \mathcal{P}_F]$.

(1) The group $SL_2(F)$

Let

$$\begin{aligned}
P &= \text{upper triangular matrices} \\
A &= \text{diagonal matrices} \\
K &= \text{integral matrices} \\
B &= \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \mid c \in \mathcal{P} \right\} \\
w &= \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}
\end{aligned}$$

Define a special coset of $A \cap K$ in A:

$$a = \begin{pmatrix} \mathcal{P}_F^* & \\ & (\mathcal{P}_F^*)^{-1} \end{pmatrix}$$

For $m \in \mathbf{Z}$, let

$$N_m = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \,\middle|\, x \in \mathcal{P}^m \right\}$$

and similarly for N_m^-.

This equation is fundamental:

$$\text{(3.1)} \qquad \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} -x^{-1} & \\ & -x \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix}$$

for $x \neq 0$.

(2) The group $SU_3(F)$

Let $x \to \bar{x}$ be the conjugation of E/F, and let

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

Note that $J = {}^tJ = J^{-1}$. It is associated to the Hermitian form $x_1\bar{x}_3 + x_2\bar{x}_2 + x_3\bar{x}_1$. The group $SU_3(F)$ is that of all $X \in SL_3(E)$ with

$${}^tX \cdot J \cdot \bar{X} = J,$$

or

$$\bar{X} = J \cdot {}^tX^{-1} \cdot J.$$

Let

P = upper triangular matrices

M = diagonal matrices

$$w = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$$

K = integral matrices

$B = \{g \in K \mid g \equiv \text{upper triangular } (\text{mod } \mathcal{P})\}$.

Then

$$M = \left\{ \begin{pmatrix} y & & \\ & \bar{y}/y & \\ & & \bar{y}^{-1} \end{pmatrix} \;\middle|\; y \in E^{\times} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix} \;\middle|\; \text{Trace(y)} = -\text{Norm}(x) \right\}$$

and w represents the non-trivial element of the Weyl group. Define the class

$$a = \begin{pmatrix} \mathscr{P}_F^* & & \\ & 1 & \\ & & (\mathscr{P}_F^*)^{-1} \end{pmatrix}.$$

Define subgroups N_m, $m \in Z$:

$$N_{2m} = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix} \in N \;\middle|\; x \equiv 0(\mathscr{P}^m),\, y \equiv 0(\mathscr{P}^{2m}) \right\}$$

$$N_{2m} = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix}' \in N \;\middle|\; x \equiv 0(\mathscr{P}^{m+1}),\, y \equiv 0(\mathscr{P}^{2m+1}) \right\}.$$

Here the fundamental equation is:

$$\begin{pmatrix} 1 & & \\ x & 1 & 1 \\ y & -\bar{x} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\bar{x}/\bar{y} & 1/y \\ & 1 & x/y \\ & & 1 \end{pmatrix} \begin{pmatrix} 1/\bar{y} & & \\ & \bar{y}/y & \\ & & y \end{pmatrix} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} 1 & -\bar{x}/y & 1/y \\ & 1 & x/y \\ & & 1 \end{pmatrix}$$
(3.2)

for $y \neq 0$.

In either case, we have this fact;

$$N_m^- - N_{m+1}^- \subseteq N_{-m} a^{-m} w (N_{-m} - N_{-m+1})$$

and if $n \in N_m^- - N_{m+1}^-$ *is expressed accordingly as*

$$n - n_1 a^{-m} w n_*$$

the map $n \mapsto n_*$, $N^-_m - N^-_{m+1} \to N_{-m} - N_{-m+1}$ *is a bijection transforming the restriction of a Haar measure to the restriction of a Haar measure.* If one chooses measures on N, N^- such that $\mathrm{meas}(N_0) = \mathrm{meas}(N^-_0) = 1$ then, more precisely,

$$\frac{dn_*}{dn} = \begin{cases} q^{2m} & (SL_2) \\ q^{4m} & (SU_3) \end{cases}.$$

We also need to remark that, in the case of SL_2 one can find characters $\psi : N \to \mathbf{C}$ trivial on N_0 but not on N_{-1}; in the case of SU_3, one can find characters $\psi : N \to \mathbf{C}$ trivial on N_{-1} but not on N_{-2}. Such characters we call *unramified.*

Note that if δ is the modulus character of P then

$$\delta(a) = \begin{cases} q^{-2} & (SL_2) \\ q^{-4} & (SU_3) \end{cases}$$

since $aN_m a^{-1} = N_{m+2}$ in both cases.

If α is the simple root corresponding to the choice of P, define

$$q_\alpha = q$$

$$q_{\alpha/2} = \begin{cases} 1 & (SL_2) \\ q^2 & (SU_3) \end{cases}.$$

This agrees with the notation of [3].

To return to the more general case: if α is any simple root, we shall always assume an identification of $\tilde{G}_\alpha$ with one of the groups above in such a way that the inverse image of $P \cap \tilde{G}_\alpha$ is the parabolic above, the image of the above K is contained in $G(\mathcal{O})$, etc. We shall similarly refer to the groups $N_{\alpha,m}$, $N^-_{\alpha,m}$, the coset a_α of M_0 (the image of the element a in $\tilde{G}_\alpha$), and the numbers q_α, $q_{\alpha/2}$.

A remark is in order concerning the validity of our paper when G is not unramified. It seems likely that all results still hold as long as one chooses for K what might be called an absolutely special maximal compact – i.e. one descending from a special compact over any field extension. Already in SU_3, for example, there is a second type of maximal compact which is *not* absolutely special and for which our eventual formula does not hold.

We ought also to mention that globally the local groups, representations, and characters of N are unramified at almost all primes.

4. Whittaker models and intertwining operators

All characters of M will be unramified in §§4, 5.

Fix also for these sections a principal character $\psi = \Pi\psi_\alpha$ of N, which we assume to be unramified.

Assume on Pw_lP the restriction of a Haar measure with $\mathrm{meas}(P_0w_lN_0) = 1$. Fix a representative x_l of w_l. Let $\Omega_\chi : I(\chi) \to \mathbf{C}_\psi$ be the unique N-morphism which for $f \in C_c^\infty(Pw_lP)$ satisfies

$$\Omega_\chi(\mathcal{P}_\chi f) = \int_{Pw_lP} \Phi(y)f(y)dy$$

where $\Phi(n_1mx_ln_2) = \chi^{-1}\delta^{1/2}(m)\psi^{-1}(n_2)$. As we have shown in §2, Ω_χ varies holomorphically with χ.

For each χ and each $w \in W$, let $\varphi_{w,\chi} = \mathcal{P}_\chi(\mathrm{char}\ BwB)$. These form a basis of $I(\chi)^B$. The function $\varphi_{K,\chi} = \mathcal{P}_\chi(\mathrm{char}\ K)$ spans $I(\chi)^K$. (We shall often drop reference to χ in subscripts.)

For each $\alpha \in \Sigma$, let

$$\xi_\alpha(\chi) = \begin{cases} (1 - q_{\alpha/2}^{-1/2}q_\alpha^{-1}\chi(a_\alpha))(1 + q_{\alpha/2}^{-1/2}\chi(a_\alpha)) & (q_{\alpha/2} \neq 1) \\ 1 - q^{-1}\chi(a_\alpha) & (q_{\alpha/2} = 1) \end{cases}$$

$$\eta_\alpha(\chi) = \begin{cases} 1 - \chi(a_\alpha)^2 & (q_{\alpha/2} \neq 1) \\ 1 - \chi(a_\alpha) & (q_{\alpha/2} = 1) \end{cases}$$

and for each $w \in W$ let

$$\begin{aligned} \xi_w(\chi) &= \Pi\xi_\alpha(\chi) \\ \eta_w(\chi) &= \Pi\eta_\alpha(\chi) \quad (\alpha > 0,\ w\alpha < 0). \end{aligned}$$

Further let

$$\begin{aligned} c_\alpha(\chi) &= \xi_\alpha(\chi)/\eta_\alpha(\chi) \\ c_w(\chi) &= \xi_w(\chi)/\eta_w(\chi)^{\boldsymbol{\cdot}} \end{aligned}$$

In [3] it is shown that if x is a regular character – i.e. $w_\chi = \chi$ only for $w = 1$ – then there exists a unique G-morphism $T_w : I(\chi) \to I(w\chi)$ such that $T_w(\varphi_k) = c_w(\chi)\varphi_k$. The operator T_w depends holomorphically on χ; $T_{w_1w_2} = T_{1_1}T_{w_2}$ if $l(w_1w_2) = l(w_1) + l(w_2)$; and if α is a simple root, $w \in W$ such that $l(w_\alpha w) > l(w)$, then

$$T_{w_\alpha}(\varphi_{w_\alpha w} + \varphi_w) = c_\alpha(\chi)(\varphi_{w_\alpha w} + \varphi_w). \tag{4.1}$$

Let T_w^* be the transpose of T_w, from the linear dual of $I(w\chi)$ to that of $I(\chi)$. Since the space of N-morphisms from $I(\chi)$ to $\mathbf{C}_\psi$ has dimension one, $T^*_w(\Omega_{w\chi})$ is a scalar multiple of Ω_χ. The scalar will depend holomorphically on χ. Since every w is a reduced product of elementary reflections, it suffices to calculate it for $w = w_\alpha$, $\alpha \in \Delta$.

4.1. LEMMA: (a) *For any* $\alpha \in \Delta$,

$$\Omega_\chi(\varphi_{w_\alpha w_l}) = \begin{cases} -q_\alpha^{-1}\chi(a_\alpha) & (q_{\alpha/2} = 1) \\ q_{\alpha/2}^{-1/2}\chi(a_\alpha) - q_{\alpha/2}^{-1/2}q_\alpha^{-1}\chi(a_\alpha) - q_{\alpha/2}^{-1}q_\alpha^{-1}\chi(a_\alpha) \\ (q_{\alpha/2} \neq 1); \end{cases}$$

(b) $\Omega_\Phi(\varphi_{w_l}) = 1$.

PROOF: Claim (b) is trivial since $Bw_lB = P_0w_lN_0$. For claim (a): first apply 2.3 to see that we must evaluate

$$\int_{Pw_lN_* \cap Bw_\alpha w_lB} \Phi(y)dy$$

for N_* suitably large, where $\Phi(n_1ma_ln_2) = \chi^{-1}\delta^{1/2}(m)\psi^{-1}(n_2)$ as above.

Recall that $Bw_\alpha w_lB = P_0N^-_{\alpha,1}w_lwN_0$, and express $N^-_{\alpha,1}$ as $(N^-_{\alpha,1} - N^-_{\alpha,2}) \cup (N^-_{\alpha,2} - N^-_{\alpha,3}) \cup \cdots$. Recall also (from %3) that every $n \in (N^-_{\alpha,m} - N^-_{\alpha,m+1})$ may be expressed

$$n = n_1a_\alpha^{-m}m_0w_\alpha n_2$$

with $n_1 \in N_{\alpha,-m}$, $m_0 \in M_0$, $n_2 \in (N_{\alpha,-m} - N_{\alpha,-m+1})$ Therefore

$$nw_\alpha w_l = n_1a_\alpha^{-m}m_0w_l \cdot n_* \cdot n_2$$

where here

$$n_* \in (N_{\bar\alpha,-m} - N_{\bar\alpha,-m+1}) = (w_\alpha w_l)^{-1}(N_{\alpha,-m} - N_{\alpha,-m+1})(w_\alpha w_l).$$

(The element $\bar\alpha \in \Delta$ is the *conjugate* of α.)

The above integral becomes

$$\begin{aligned} &\int_{Pw_lN_* \cap P_0N^-_{\alpha,1}w_\alpha w_lN_0} \Phi(y)dy \\ &\quad = \sum_{m=1}^{\infty} \int_{Pw_lN_* \cap P_0(N^-_{\alpha,m} - N^-_{\alpha,m+1})w_\alpha w_lN_0} \Phi(y)dy \\ &\quad = \sum_{m=1}^{n} \int_{P_0(N^-_{\alpha,m} - N^-_{\alpha,m+1})w_\alpha w_lN_0} \Phi(y)dy \end{aligned}$$

which for large n

$$= \sum_{m=1}^{n} \int_{P_0} dp_0 \int_{N_0} dn_0 \int_{N^-_{\alpha,m} - N^-_{\alpha,m+1}} \Phi(p_0 n_1 a_\alpha^{-m} x_l n * n_0) dn$$

$$= \sum_{m=1}^{n} \chi^{-1}\delta^{1/2}(a_\alpha^{-m}) \int_{N^-_{\alpha,m} - N^-_{\alpha,m+1}} \psi^{-1}(n_*) dn.$$

where we have assumed measures on P_0, N_0 with total measure 1. We must in fact be careful about measures: we have assumed $\mathrm{meas}(Bw_lB) = 1$, so that since $Bw_\alpha B \cdot Bw_\alpha w_l B = Bw_l B$ is the Hecke algebra $\mathscr{H}(G, B)$, meas $Bw_\alpha w_l B = 1/q_\alpha$, and in the above integral, with the stated assumptions on P_0, N_0 we assume on N^-_α the measure with $\mathrm{meas}(N^-_{\alpha,0}) = 1$. Assuming as well the measure on N_α to be such that $\mathrm{meas}(N_{\alpha,0}) = 1$ (consistent with the assumption on N), one can check that the map from $N^-_{\alpha,m} - N^-_{\alpha,m+1}$ to $N_{\tilde{\alpha},-m} - N_{\tilde{\alpha},-m+1}$ taking n to n_* is such that

$$\frac{dn_*}{dn} = \begin{cases} q_\alpha^{2m} & (q_{\alpha/2} = 1) \\ q_\alpha^{4m} & (q_{\alpha/2} \neq 1) \end{cases}.$$

The above becomes (since $\psi = \psi_{\tilde{\alpha}}$ on $N_{\tilde{\alpha}}$)

$$\sum_{n=1}^{n} \chi^{-1}\delta^{1/2}(a_\alpha^{-m}) \left(\frac{dn}{dn_*}\right) \int_{N_{\tilde{\alpha},-m} - N_{\tilde{\alpha},-m+1}} \psi_{\tilde{\alpha}}^{-1}(n_*) dn_*$$

At this point the two cases have to be treated differently:

(1) $\tilde{G}_\alpha = SL_2$. Here $\psi_{\tilde{\alpha}}$ is trivial on $N_{\tilde{\alpha},0}$ but not on $N_{\tilde{\alpha},-1}$, so that

$$\int_{N_{\tilde{\alpha},-m}} \psi_{\tilde{\alpha}}(n) dn = \begin{cases} 1 & (m = 0) \\ 0 & (m \geq 1) \end{cases}$$

and

$$\int_{N_{\tilde{\alpha},-m} - N_{\tilde{\alpha},-m+1}} \psi^{-1}(n_*) dn_* = \begin{cases} -1 & (m = 1) \\ 0 & (m \geq 2) \end{cases}$$

Our sum only has one term and is equal to

$$-\chi^{-1}\delta^{1/2}(a_\alpha^{-1}) q_\alpha^{-2} = -q_\alpha^{-1}(a_\alpha).$$

(2) $\tilde{G}_\alpha = SU_3$. Here $\psi_{\tilde{\alpha}}$ is trivial on $N_{\tilde{\alpha},-1}$ but not on $N_{\tilde{\alpha},-2}$.

Therefore

$$\int_{N_{\tilde{\alpha},-m}-N_{\tilde{\alpha},-m+1}} \psi^{-1}(n)dn = \begin{cases} q_\alpha - 1 & (m=1) \\ -q_\alpha & (m=2) \\ 0 & (m \geq 3) \end{cases}$$

and our sum, with two terms only, equals

$$\begin{aligned} &\chi^{-1}\delta(a_\alpha^{-2})q_\alpha^{-8}(-q_\alpha) + \chi^{-1}\delta^{1/2}(a_\alpha^{-1})q_\alpha^{-4}(q-1) \\ &= -q_\alpha^{-3}\chi(a_\alpha)^2 + q_\alpha^{-1}\chi(a_\alpha) - q_\alpha^{-2}\chi(a_\alpha) \end{aligned}$$

which agrees with the Lemma since $q_{\alpha/2} = q^2$.

4.2. COROLLARY: *One has*

$$\Omega_\chi(\varphi_{w_\alpha w_l} + \varphi_{w_l}) = \zeta_\alpha(\chi).$$

Now define

$$\begin{aligned} \lambda_\alpha(\chi) &= \zeta_\alpha(\chi^{-1})/\eta_\alpha(\chi) \\ \lambda_w(\chi) &= \Pi\lambda_\alpha(\chi) \quad (\alpha > 0 \mid w\alpha < 0). \end{aligned}$$

4.3. PROPOSITION: *One has*

$$T_w^*(\Omega_{w\chi}) = \lambda_w(\chi)\Omega_\chi.$$

This follows from a comparison of 4.2 with Equation (4.1). (Note that $\lambda_w(\chi) = c_w(\chi)\zeta_w(\chi^{-1})/\zeta_w(\chi)$.)

5. The unramified Whittaker function

Corresponding to the N-morphism $\Omega_\chi : I(\chi) \to \mathbf{C}_\psi$ we have a G-morphism:
$I(\chi) \to \mathrm{Ind}(\mathbf{C}_\psi \mid N, G)$. Define the *unramified Whittaker function* corresponding to χ to be the image of φ_K under this map. It is thus defined by the formula

$$W_\chi(g) = \Omega_\chi(R_g\varphi_K)$$

and satisfies the conditions

$$W_\chi(ngk) = \psi(n)W_\chi(g)$$

for $k \in K$, $n \in N$, $g \in G$. Because $G = NAK$ the function w_χ is determined by its restriction to A. We shall obtain an explicit formula for this restriction.

5.1. Lemma: *For $a \in A$, $a \notin A^-$, $W_\chi(a) = 0$.*

Proof: For $n \in N_0$, $a \in A$,

$$\begin{aligned} W_\chi(an) &= W_\chi(a) \\ &= W_\chi(ana^{-1} \cdot a) \\ &= \psi(ana^{-1}) W_\chi(a). \end{aligned}$$

For $a \notin A^-$, $|\alpha(a)| > 1$ for some $\alpha \in \Delta$. Then for some $n \in N_{\alpha,0}$ one has $\psi_\alpha(ana^{-1}) \neq 1$, so that $W_\chi(a) = 0$.

The idea we use from this point on is extremely close to that used in [3] to find an explicit formula for the spherical function. We use notation and results from [3] freely. (The common point of both problems is simply that in each case one has for each χ a certain unique functional on $I(\chi)$ invariant under N_0; we shall explain this more precisely later.) Because W_χ is holomorphic in χ, we may assume χ to be a regular character.

Let $\{f_w\}$ be the basis of $I(\chi)^B$ which is as a linear space canonically isomorphic to $I(\chi)_N$, dual to the maps $f \to T_w(f)(1)$. By definition, then,

$$\varphi_K = \sum c_w(\chi) f_w.$$

For $a \in A^-$, [3] 2.5 implies

$$\mathcal{P}_{N_0}(R_a \varphi_K) = \sum c_w(\chi)(w\chi)\delta^{1/2}(a) f_w$$

Hence

$$\begin{aligned} W_\chi(a) &= \Omega_\chi(R_a \varphi_K) \\ &= \Omega_\chi(\mathcal{P}_{N_0}(R_a \varphi_K)) \\ &= \sum c_w(\chi) \Omega_\chi(f_w)(w\chi)\delta^{1/2}(a). \end{aligned}$$

Because $\varphi_{w_l} = f_{w_l}$ ([3] 3.7),

$$\Omega_\chi(f_{w_l}) = 1,$$

but the other values of $\Omega_\chi(f_w)$ must be obtained by using a functional equation for W_χ.

5.2. Lemma: *For any $w \in W$, χ such that $\zeta_w(\chi) \neq 0$,*

$$W_{w\chi} = (\zeta_w(\chi^{-1})/\zeta_w(\chi))\, W_\chi.$$

Proof: Let $F_\chi : \mathrm{Ind}(\chi) \to \mathrm{Ind}(\mathbf{C}_\psi \mid N, G)$ be the G-morphism corresponding to Ω_χ. Thus

$$F_\chi(\varphi)(g) = \Omega_\chi(R_g\varphi)$$

for all $g \in G$, $\varphi \in I(\chi)$. According to 4.3,

$$F_{w\chi}(T_w\varphi) = (\zeta_w(\chi^{-1})/\eta_w(\chi))F_\chi(\varphi).$$

Now $W_\chi = F_\chi(\varphi_K)$; therefore

$$F_{w\chi}(T_w\varphi_K) = c_w(\chi)\, W_{w\chi} = (\zeta_w(\chi)/\eta_w(\chi))\, W_{w\chi}$$

on the one hand and

$$= (\zeta_w(\chi^{-1})/\eta_w(\chi))\, W_\chi$$

on the other. For generic χ, therefore,

$$W_{w\chi} = (\zeta_w(\chi^{-1})/\zeta_w(\chi))\, W_\chi.$$

This is Jacquet's functional equation for the $\mathcal{P}$-adic Whittaker function in [5].

Let $\zeta(\chi)$ be $\zeta_{w_l}(\chi)$.

5.3. Corollary: *The function $W_\chi/\zeta(\chi)$ is W-invariant as a function of χ.*

Proof: It must be shown, by 5.2, that

$$\frac{\zeta(w\chi)}{\zeta(\chi)} = \frac{\zeta_w(\chi^{-1})}{\zeta_w(\chi)}.$$

The left hand side is

$$\frac{\prod_{\alpha>0}\zeta_\alpha(w\chi)}{\prod_{\alpha>0}\zeta_\alpha(\chi)}=\frac{\prod_{\alpha>0}\zeta_{w^{-1}\alpha}(\chi)}{\prod_{\alpha>0}\zeta_\alpha(\chi)}$$

$$=\frac{\prod_{w\alpha>0}\zeta_\alpha(\chi)}{\prod_{\alpha>0}\zeta_\alpha(\chi)}$$

$$=\frac{\prod_{\substack{\alpha<0\\ w\alpha>0}}\zeta_\alpha(\chi)}{\prod_{\substack{\alpha>0\\ w\alpha<0}}\zeta_\alpha(\chi)}=\frac{\zeta_w(\chi^{-1})}{\zeta_w(\chi)}.$$

We now have

$$\frac{W_\chi}{\zeta(\chi)}=\sum_{w\in W}(c_w(\chi)/\zeta(\chi))\Omega_\chi(f_w)(w\chi)\delta^{1/2}.$$

The coefficient for $w=w_l$ is

$$\frac{c_{w_l}(\chi)}{\zeta(\chi)}=\frac{1}{\eta_{w_l}(\chi)}=\prod_{\alpha>0}\frac{1}{1-\chi(a_\alpha)^{d_\alpha}}$$

where

$$d_\alpha=\begin{cases}1 & (q_{\alpha/2}=1)\\ 2 & (q_{\alpha/2}\neq 1)\end{cases}.$$

This coefficient may also be expressed as

$$\prod_{\alpha>0}\frac{1}{1-(w_lw_l\chi)(a_\alpha)^{d_\alpha}}.$$

Because of the linear independence of characters on A^- and the W-invariance of $W_\chi/\zeta(\chi)$, one must therefore have

$$\frac{W_\chi}{\zeta(\chi)}=\sum_{w\in W}\prod_{\alpha>0}\frac{1}{1-(w_\chi)(a_\alpha)^{-d_\alpha}}(w\chi)\delta^{1/2}.$$

This expression can be reduced further. Let

$$\epsilon(\chi)=\prod_{\alpha>0}\frac{1}{1-\chi(a_\alpha)^{-d_\alpha}}.$$

Then for $\beta \in \Delta$

$$\begin{aligned}\epsilon(w_\beta\chi) &= \prod_{\alpha>0} \frac{1}{1-\chi(a_{w_\beta\alpha})^{-d_\alpha}} \\ &= \prod_{\substack{\alpha>0\\ \alpha\neq\beta}} \frac{1}{1-\chi(a_\alpha)^{-d_\alpha}} \cdot \frac{1}{1-\chi(a_\beta)^{d_\beta}} \\ &= -\chi(a_\beta)^{d_\beta}\epsilon(\chi).\end{aligned}$$

Hence in general

$$\epsilon(w\chi) = (\operatorname{sgn} w) \cdot \prod_{\substack{\alpha>0\\ w\alpha<0}} \chi(a_\alpha)^{-d}\alpha \cdot \epsilon(\chi)$$

and

$$W_\chi = \zeta(\chi) \cdot \epsilon(\chi) \cdot \sum_{w\in W} (\operatorname{sgn} w) \cdot \prod_{\substack{\alpha>0\\ w\alpha<0}} \chi(a_\alpha)^{-d_\alpha} \cdot (w\chi)\delta^{1/2}.$$

When $a = 1$:

$$\begin{aligned}W_\chi(a) &= \zeta(\chi) \cdot \prod_{\alpha>0} \frac{1}{1-\chi(a_\alpha)^{-d_\alpha}} \cdot \sum (\operatorname{sgn} w) \prod_{\substack{\alpha>0\\ w\alpha<0}} \chi(a_\alpha)^{-d_\alpha} \\ &= \zeta(\chi).\end{aligned}$$

All in all we have

5.4. Theorem: *For $a \in A^-$,*

$$W_\chi(a) = \zeta(\chi) \cdot \prod_{\alpha>0} \frac{1}{1-\chi(a_\alpha)^{-d_\alpha}} \cdot \sum_{w\in W} (\operatorname{sgn} w) \prod_{\substack{\alpha>0\\ w\alpha<0}} \chi(a_\alpha)^{-d_\alpha}(w\chi)\delta^{1/2}(a).$$

In particular, $W_\chi(1) = \zeta(\chi)$.

To repeat,

$$\zeta(\chi) = \prod_{\alpha>0} \begin{cases} 1-q^{-1}\chi(a_\alpha) & (q_{\alpha/2}=1) \\ (1+q_{\alpha/2}^{-1/2}\chi(a_\alpha))(1-q_{\alpha/2}^{-1/2}q_\alpha^{-1}\chi(a_\alpha)) & (q_{\alpha/2}\neq 1). \end{cases}$$

5.5. REMARK: It may be helpful if we summarize our argument, and especially to bring out the points which it has in common with the argument in [3].

(1) In each case, one has a functional $F_\chi : I(\chi) \to \mathbf{C}$ which (a) depends holomorphically on χ, (b) is unique, in some sense, up to scalar multiplication, (c) is N_0-invariant. Here the functional is Ω_χ, in [3] it is the map $f \to \mathcal{P}_K(f)(1)$.

(2) Because of the uniqueness property, whenever χ is regular one has

$$T_w^*(F_{w_\chi}) = \gamma_w(\chi) F_\chi$$

for some scalar $\gamma_w(\chi)$. This scalar is multiplicative in w since T_w is, and can be calculated explicitly for an elementary reflection.

(3) One is looking for a formula for the function $\Phi_\chi(\mathrm{a}) = F_\chi(R_a \varphi_K)$, $a \in A^-$. Because of holomorphicity, it suffices to find the formula when χ is regular; in this case, the result in (2) gives some relation between Φ_χ and $\Phi_{w\chi}$.

(4) The value of $F_\chi(\varphi_{w_l}) = F_\chi(f_{w_l})$ can be calculated explicitly.

(5) One expresses φ_K in terms of the basis $\{f_w\}$:

$$\varphi_K = \sum_w c_w(\chi) f_w$$

then continues

$$\mathcal{P}_{N_0}(R_a \varphi_K) = \sum c_w(\chi)(w\chi)\delta(a) f_w$$

$$\Phi_\chi(a) = \sum c_w(\chi)(w\chi)\delta(a) F_\chi(f_w)$$

The term $F_\chi(f_{w_l})$ is known, and one uses the functional equation from (3) to find the other coefficients.

6. Whittaker functions and Jacquet modules

The explicit formula of §5 suggests properties of the functions in any Whittaker model of an admissible representation. The properties of W_χ that we wish to generalize here are (1) it vanishes off A^-; (2) on A^- it is the restriction of an A-finite function.

We again allow G to be an arbitrary reductive group. Let $\psi = \Pi \psi_\alpha$ be any character of N (not necessarily principal). Let (π, V) be an admissible representation of G.

6.1. PROPOSITION: *Let ψ be a principal character of N, $\Omega : V \to \mathbf{C}_\psi$ an N-morphism. For any $v \in V$ there exists $\epsilon > 0$ such that whenever $|\alpha(a)| > \epsilon^{-1}$ for some $\alpha \in \Delta$, $\Omega(\pi(a)v) = 0$.*

PROOF: Choose a compact open subgroup N_0 of N fixing V. For any $n \in N_0$, $a \in A$:

$$\begin{aligned}\Omega(\pi(a)\pi(n)v) &= \Omega(\pi(a)v)\\ &= \Omega(\pi(ana^{-1})\pi(a)v)\\ &= \psi(ana^{-1})\Omega(\pi(a)v).\end{aligned}$$

Since ψ is principal, there exists $\epsilon > 0$ such that for whenever $|\alpha(a)| > \epsilon^{-1}$ for some $\alpha \in \Delta$, $aN_0a^{-1} \not\subseteq \ker(\psi)$. For such an a, $\Omega(\pi(a)v) = 0$.

Another way of saying this that the support on A of any Whittaker function lies in a translate of A^-. Our next result says that there is some other translate of A^- on which the function is A-finite.

Let u be a vector of the Jacquet module V_N. Choose a compact open subgroup K_0 of G with the Iwahori factorization $K_0 = N_0^- M_0 N_0$ such that u is fixed by M_0 and ψ is trivial on N_0. Let v be the canonical lifting of u in V^{K_0} (refer to §4 of [2]).

6.2. LEMMA: *If $K_0' \subseteq K_0$ is any smaller group with Iwahori factorization and $v' \in V^{K_0'}$ the corresponding canonical lifting of u, then v' and v have the same image in $V_{\psi,N}$.*

Here it is not necessary to assume ψ principal.

PROOF. By Proposition 4.1.8 of [2],

$$v = \mathcal{P}_{N_0}(v').$$

But then v and v' clearly have the same image in $V_{\psi,N}$ since $\psi \equiv 1$ on N_0.

The correspondence $u \mapsto v \mapsto v \bmod V_\psi(N)$ thus defines a *canonical* linear map Φ from V_N to $V_{\psi,N}$.

6.3. PROPOSITION: *Let $\Omega : V \to \mathbf{C}_\psi$ be an N-morphism, v an element of V and u its image in V_N. There exists $\epsilon > 0$ such that whenever $|\alpha(a)| < \epsilon$ for all $\alpha \in \Delta$,*

$$\Omega(\pi(a)v) = (\Omega \circ \Phi)(\pi_N(a)u).$$

PROOF: Choose K_0 for v and u as above. By definition of the canonical lifting in [2] there exists $\epsilon > 0$ such that $\mathcal{P}_{K_0}(\pi(a)v) = \mathcal{P}_{N_0}(\pi(a)v)$ is the canonical lifting of $\pi_N(a)u$ whenever $|\alpha(a)| < \epsilon$ for all $\alpha \in \Delta$. But then one has only to apply the definition of Φ.

It happens that the two translates of A^- used in 6.1 and 6.3 coincide in the case of the Whittaker function W_χ, and indeed for the Whittaker function attached to any element of $I(\chi)^B$. One can see this directly from the proofs of 6.1 and 6.3.

What we have proven so far about the support of a Whittaker function W can be summarized in a picture, representing A rather figuratively as a plane, A^- as a cone. In fact one can also say something about the behaviour all over A; we have left it to last because it is not so simple to formulate as the above.

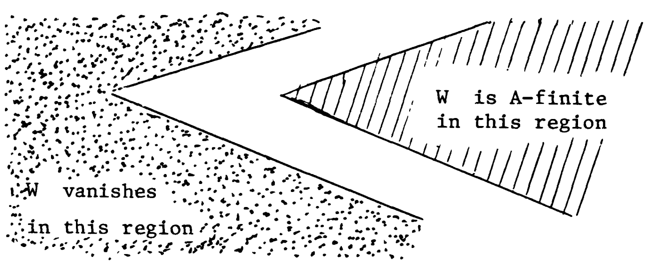

Fig 1.

Let θ be a subset of Δ, ψ_θ the character of $M_\theta \cap N$ defined as $\Pi\psi_\alpha(\alpha \in \theta)$. Just as above, one can define a canonical linear map Φ_θ from $(V_{N_\theta})_{\psi_\theta, M\cap N_\theta}$ to $V_{\psi,N}$, and almost exactly the same proof yields:

6.4. Proposition: *Let $\Omega : V \to \mathbf{C}_\psi$ be an N-morphism, $v \in V$ with image $u \in V_{N_\theta}$. There exists $\epsilon > 0$ such that whenever $|\alpha(a)| < \epsilon$ for all $\alpha \in \Delta - \theta$,*

$$\Omega(\pi(a)v) = (\Omega \circ \Phi_\theta)(\pi_{N_\theta}(a)u).$$

In other words, in the direction of A_θ^- any function for V eventually becomes a Whittaker function for V_{N_θ}.

One well known consequence:

6.5. Corollary: *If (π, V) is absolutely cuspidal then any Whittaker function associated to V has compact support on G modulo N.*

One might note that in general the canonical maps Φ_θ are neither injective nor surjective.

One might also note that the above argument will show that whenever $\Omega : V \to \mathbf{C}$ is a functional invariant under some compact open subgroup $N_0 \subseteq N$ then one can prove $\Omega(\pi(a)v)$ is A-finite on some translate of A^{-}.

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