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p*-LOCAL UNCONDITIONAL STRUCTURE OF BANACH SPACES

Y. Gordon

Abstract

There are Banach spaces which fail to have *p*-local unconditional structure (*p*-l.u.st.) for any $p, \infty > p > 0$. In particular, there exist *n*-dimensional Banach spaces $E_n, n = 1, 2, \dots$, whose *p*-l.u.st. constants are “almost” the largest possible theoretical value $\min\{n^{1/2}, n^p\}$. The *p*-l.u.st. constant is smaller and not equivalent to the usual l.u.st. constant.

1. Introduction

Given any $\infty > p \geq 0$, let η_p be the ideal norm defined in the following manner: If $T \in L(E, F)$ is a bounded operator from a Banach space E to a Banach space F which can be written the form $Tx = \sum_{i \geq 1} A_i x$ ($x \in E$), where A_i ($i = 1, 2, \dots$) are in the class $\mathcal{F}(E, F)$ of the finite-rank operators from E to F , then

$$\eta_p(T) = \inf \sup \left\| \sum_{i=1}^N \pm (r(A_i))^p A_i \right\|$$

where $r(A)$ denotes the rank of an operator A , the supremum ranges over all choices of \pm signs and integers N , and the infimum is taken over all the possible representations of the operator T .

$\eta_p(T)$ is a non-decreasing function of p , and η_p is a Banach ideal norm, that is has the following properties:

(1) η_p is a norm and $\eta_p(E, F) = \{T \in L(E, F); \eta_p(T) < \infty\}$ is a Banach space under the norm η_p .

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(2) $\eta_p(T) = \|T\|$ whenever $r(T) = 1$.

(3) If $u \in L(G, E)$, $T \in \eta_p(E, F)$, $v \in L(F, H)$, then $vTu \in \eta_p(G, H)$ and $\eta_p(vTu) \leq \|v\| \|u\| \eta_p(T)$.

We recall some well known facts about a general Banach ideal norm α which may be found in [9]. If $T \in L(E, F)$, the adjoint ideal norm $\alpha^*(T)$ is defined as the least C such that the inequality

$$\text{trace}(SvTu) \leq C \|v\| \|u\| \alpha(S)$$

holds for any finite-dimensional normed spaces X and Y , $u \in L(X, E)$, $v \in L(F, Y)$ and $S \in L(Y, X)$. If X and Y are finite-dimensional normed spaces, the dual space $(\alpha(X, Y))'$ can be naturally identified with $\alpha^*(Y, X) = (L(Y, X), \alpha^*)$ via the identity $\langle T, S \rangle = \text{trace}(ST)$ for $T \in \alpha(X, Y)$, $S \in \alpha^*(Y, X)$. Hence, if $T \in L(E, F)$, $\alpha^{**}(T) = \sup \alpha(vTu)$, where the supremum ranges over all finite-dimensional normed spaces X and Y , $u \in L(X, E)$ and $v \in L(F, Y)$ with $\|u\| = \|v\| = 1$. From this we get immediately that $\eta_p^{**}(T) = \eta_p^{**}(T')$ for every operator $T \in L(E, F)$.

If in the definition of $\eta_p(T)$, T is further restricted only to representations for which $r(A_i) = 1$ for all i , then the corresponding resulting norm which is independent of p was called in [8] the weakly nuclear norm of T and denoted by $\eta(T)$. It follows that $\|\cdot\| \leq \eta_p \leq \eta_q \leq \eta$ for $0 \leq p < q < \infty$, and since $\eta_0 = \|\cdot\|$ on finite dimensional spaces, taking double adjoints we obtain $\eta_0^{**} = \|\cdot\|^{**} = \|\cdot\| \leq \eta_p^{**} \leq \eta_q^{**} \leq \eta^{**}$.

Using ultraproducts it can be shown (see for example [16]) that $T \in \eta^{**}(E, F)$ if and only if $j_F T$ factors through some Banach lattice, more precisely, $\eta^{**}(T) = \inf \|v\| \|u\|$, where the infimum ranges over all Banach lattices L and $u \in L(E, L)$, $v \in L(L, F'')$, satisfying $j_F T = vu$, where $j_F: F \rightarrow F''$ is the canonical inclusion. Thus, if T is a map on, or, into, a norm one complemented subspace of a Banach lattice, then $\|T\| = \eta_p^{**}(T) = \eta^{**}(T)$. If $T = I_E$ the identity operator on a Banach space E , $\eta^{**}(I_E)$ is generally better known as the local unconditional structure (l.u.st.) constant of E which is usually denoted by $x_u(E)$ [7]. If α is an ideal norm, $\alpha(E)$ denotes $\alpha(I_E)$. For $p > 0$, $\eta_p^{**}(E)$ will be called the p -l.u.st. constant of E . $x(E)$ will denote the unconditional basis constant of E .

If $\dim(E) = n$ and $0 \leq p < q < \infty$, then trivially we get from the definitions

$$1 \leq \eta_p(E) \leq \eta_q(E) \leq \eta(E) = x_u(E) \leq x(E) \leq d(E, \ell_2^n) \leq \sqrt{n},$$

and also, since $r(A_i) \leq n$, $\eta_q(E) \leq n^{q-p} \eta_p(E)$. Moreover, the represen-

tation $I_E = I_E$ shows that $\eta_p(E) \leq n^p$, thus $\eta_p(E) \leq \min\{n^{1/2}, n^p\}$ for all $p \geq 0$.

The main result here shows that the last inequality is asymptotically “almost” the best possible. There exists a sequence $E_n, n = 1, 2, \dots$, of n -dimensional spaces for which $\eta_p(E_n) \geq \min\{an^{1/2}, an^p\} \exp(-\sqrt{\log n})$ where a is an absolute positive constant. Since the exponential factor tends to zero more slowly than any negative power of n , this implies that if $p \neq q$ and $0 < p < \frac{1}{2}$, then η_p and η_q are not equivalent ideal norms, and in particular η_p and η are not equivalent ideal norms. Since $\eta_p(E_n) \rightarrow \infty$ as $n \rightarrow \infty$, this also implies there exists a reflexive separable Banach space which fails to have p -l.u.st. for all $p > 0$.

Regarding x_u it was proved in [3] that there is an absolute constant $c > 0$ and a sequence of spaces $F_n, \dim(F_n) = n$, such that $x_u(F_n) \geq c\sqrt{n}$. Our result therefore is of interest for the smaller p -l.u.st. constants η_p . We do not know if $\eta_p(E)$ can be asymptotically equivalent to $\min\{n^{1/2}, n^p\}$ for a sequence of spaces $E_n, \dim(E_n) = n$. It is also an open question whether for $q > p \geq \frac{1}{2}$ $\eta_p(E)$ and $\eta_q(E)$ are always equivalent when $\dim(E) < \infty$; the same question is also open for the constants $x_u(E)$ and $x(E)$. It was proved recently by Johnson, Lindenstrauss and Schechtman, that there exists a Banach spaces E with $x_u(E) = \infty$, that is E does not have local unconditional structure, yet E has an unconditional Schauder decomposition into 2-dimensional spaces. This fact implies that $x_u(E)$ and $\eta_p(E)$ are not equivalent since $\eta_p(E)$ is finite for such spaces. Also unknown is whether many of the spaces which fail l.u.st. also fail p -l.u.st. for some $p > 0$. Does $L_q(\infty > q \geq 1)$ have a subspace without p -l.u.st.? G. Pisier proved that if $p > 2, L_q$ has a subspace without l.u.st. (See [15] for $q > 4$; for $2 < q$ we know of an unpublished proof).

To obtain the lower estimates for $\eta_p(E_n)$ we use the characterization of the adjoint norm η_p^* proved in the next section and an inequality due to S. Chevet which was communicated to us by G. Pisier who has used the inequality to prove that l.u.st. constant x_u of

$$\underbrace{\ell_1^n \hat{\otimes}_\epsilon \ell_1^n \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon \ell_1^n}_{2^k}$$

chosen in some appropriate relation to N , and where α is any scalar $< \frac{1}{2}$, and $c_\alpha > 0$ is a constant depending only on α .

2. p -local unconditional structure

A characterization of η_p^* is given by the following proposition.

PROPOSITION 1: *If p, C are non-negative constants and $T \in L(E, F)$, then the following statements are equivalent:*

- (1) $\eta_p^*(T) \leq C$.
- (2) $\sum_{i=1}^n \text{trace}(TA_i) \leq C \max_{\pm} \|\sum_{i=1}^n \pm (r(A_i))^p A_i\|$ for any choice of $\{A_i\}_{i=1}^n \subset \mathcal{F}(F, E)$.
- (3) If $K_{E'}$ denotes the w^* -closure of the extreme points of the unit ball of E' equipped with the w^* topology, there exists a probability measure on the compact topological product space $K = K_{E'} \times K_{F'}$ such that for every $A \in \mathcal{F}(F, E)$ holds the inequality

$$\text{trace}(TA) \leq C(r(A))^p \int_K |\langle A'(x'), y'' \rangle| d\mu(x', y'').$$

PROOF: Let C_i ($i = 1, 2, 3$) denote a constant C which appears in the inequality of statement (i). Let X, Y be finite-dimensional spaces and $\epsilon > 0$, and let $S = \sum_{i=1}^n A_i$ where $A_i \in L(Y, X)$ are chosen to satisfy $(1 + \epsilon)\eta_p(S) \geq \max_{\pm} \|\sum_{i=1}^n \pm (r(A_i))^p A_i\|$. Then, for any $u \in L(X, E)$, $v \in L(F, Y)$, we get

$$\begin{aligned} \text{trace}(SvTu) &= \sum_{i=1}^n \text{trace}(A_i v T u) = \sum_{i=1}^n \text{trace}(u A_i v T) \\ &\leq C_2 \max_{\pm} \left\| \sum_{i=1}^n \pm (r(u A_i v))^p u A_i v \right\| \\ &\leq C_2 \|u\| \|v\| \max_{\pm} \left\| \sum_{i=1}^n \pm (r(A_i))^p A_i \right\| \\ &\leq C_2 \|u\| \|v\| (1 + \epsilon) \eta_p(S), \end{aligned}$$

this implies $\inf C_1 \leq C_2(1 + \epsilon)$, therefore $\inf C_1 \leq \inf C_2$.

Given arbitrary $B_i \in \mathcal{F}(F, E)$, $i = 1, 2, \dots, n$,

$$\begin{aligned} \sum_{i=1}^n \text{trace}(TB_i) &\leq C_3 \int_K \sum_{i=1}^n (r(B_i))^p |\langle B_i'(x'), y'' \rangle| d\mu \\ &\leq C_3 \sup \left\{ \sum_{i=1}^n (r(B_i))^p |\langle B_i'(x'), y'' \rangle|; \|x'\| = \|y''\| = 1 \right\} \\ &= C_3 \max_{\pm} \left\| \sum_{i=1}^n \pm (r(B_i))^p B_i \right\|, \end{aligned}$$

hence $\inf C_2 \leq C_3$.

If $A \in \mathcal{F}(F, E)$, let $\tilde{A} \in C(K)$ be the function defined by: $\tilde{A}(x', y'') = \langle A'(x'), y'' \rangle (r(A))^p$, and denote by M the convex hull of the set $\{C_2 \tilde{A}; A \in \mathcal{F}(F, E), \text{trace}(TA) = 1\}$. Statement (2) implies that M is disjoint from the set $N = \{f \in C(K), f < 1\}$ which is also convex and contains the open unit ball of $C(K)$, therefore there exists a probability measure $\mu \in M(K) = (C(K))'$ such that $\mu(g) \geq 1$ for all $g \in M$, this shows that $\inf C_3 \leq C_2$.

Let now $\{A_i\}_{i=1}^n \subset \mathcal{F}(F, E)$, and consider the space $X = \text{span}\{A_i(y); y \in F, i = 1, 2, \dots, n\}$. Let $u: X \rightarrow E$ be the inclusion map, and $S = \sum_{i=1}^n A_i$ be the map of F into X , S' maps X' to F' and $(S')_a$ will denote the map S' of X' onto $S'(X')$. Let j be the inclusion of $S'(X')$ in F' , then $v = j'j_F$ maps F to $Y = (S'(X'))'$. Both X and Y are now finite-dimensional spaces, and

$$\begin{aligned} \sum_{i=1}^n \text{trace}(TA_i) &= \text{trace}(vTu(S')'_a) \\ &\leq C_1 \|u\| \|v\| \eta_p((S')'_a) \\ &\leq C_1 \max_{\pm} \left\| \sum_{i=1}^n \pm A_i (r(A_i))^p \right\| \end{aligned}$$

the last inequality is because $\|u\| = \|v\| = 1$ and the fact that if we denote by \tilde{A}_i the operator A_i' considered as a map of Y to X , then

$(S')'_a = \sum_{i=1}^n \tilde{A}_i$ and so

$$\eta_p((S')'_a) \leq \max_{\pm} \left\| \sum_{i=1}^n \pm (r(\tilde{A}_i))^p \tilde{A}_i \right\| = \max_{\pm} \left\| \sum_{i=1}^n \pm (r(A_i))^p A_i \right\|.$$

Therefore, $\inf C_2 \leq C_1$, and the proof is complete. □

We need a preliminary lemma which was used in [6].

LEMMA 2: *If $x_i, y_i, i = 1, 2, \dots, m$, are arbitrary vectors in ℓ_2^n , then $n \sum_{i=1}^m \sum_{j=1}^m \langle x_i, y_j \rangle^2 \geq (\sum_{i=1}^m \langle x_i, y_i \rangle)^2$.*

PROOF: Without loss of generality we can assume $\{y_i\}_{i=1}^m$ are fixed such that the operator $T = \sum_{i=1}^m y_i \otimes y_i$ has rank n . We shall maximize the function $f(x_1, x_2, \dots, x_m) = \sum_{i=1}^m \langle x_i, y_i \rangle$ subject to the constraint $\sum_{i=1}^m \sum_{j=1}^m \langle x_i, y_j \rangle^2 = 1$. At the maximum point, the function

$$\varphi \equiv \sum_{i=1}^m \langle x_i, y_i \rangle - \lambda \left(\sum_{i=1}^m \sum_{j=1}^m \langle x_i, y_j \rangle^2 - 1 \right)$$

satisfies $\partial\varphi/\partial x_{ik} = 0$, where $x_i = (x_{ik})_{k=1}^n$. This yields $y_i = 2\lambda \sum_{j=1}^m \langle x_i, y_j \rangle y_j = 2\lambda T(x_i)$, hence $f = \sum_{i=1}^m \langle x_i, y_i \rangle = 2\lambda \sum_{i=1}^m \sum_{j=1}^m \langle x_i, y_j \rangle^2 = 2\lambda \cdot T = \sum_{i=1}^m y_i \otimes y_i = 2\lambda \sum_{i=1}^m y_i \otimes T x_i$, therefore $I_{\ell_2^n} = 2\lambda \sum_{i=1}^m x_i \otimes y_i$ and taking trace, $n = 2\lambda \sum_{i=1}^m \langle x_i, y_i \rangle = 4\lambda^2$, so that $2\lambda = \sqrt{n}$. \square

Given Banach spaces E and F let $E \hat{\otimes}_\epsilon F$ denote the completion of the tensor product space $E \otimes F$ under the ϵ -norm, that is the ordinary norm induced on it as a subspace of $L(E', F)$. $E \hat{\otimes}_\pi F$ denotes the completion of $E \otimes F$ under the π -norm, that is, on $E \otimes F$ the norm $|\cdot|_\pi$ is defined as

$$\left| \sum_{i=1}^n x_i \otimes y_i \right|_\pi = \sup \left\{ \sum \langle y_i, Bx_i \rangle; \|B\| \leq 1, B \in L(E, F') \right\}.$$

If k is a positive integer, E_ϵ^k will denote the space $\underbrace{E \hat{\otimes}_\epsilon E \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon E}_k$,

and for $\{x_i\}_{i=1}^k \subset E$, $\vec{x} = \otimes_{i=1}^k x_i = x_1 \otimes x_2 \otimes \cdots \otimes x_k$ will be a k tensor in E_ϵ^k . If $u_i \in L(E, E)$ are isometries on E ($i = 1, 2, \dots, k$), let $\vec{u} = \otimes_{i=1}^k u_i = u_1 \otimes u_2 \otimes \cdots \otimes u_k$ be the isometry of E_ϵ^k defined by: $\vec{u}(\vec{x}) = \otimes_{i=1}^k u_i(x_i)$. This definition makes \vec{u} also an isometry on $E_\pi^k = \underbrace{E \hat{\otimes}_\pi E \cdots \hat{\otimes}_\pi E}_k$. We shall denote by π_p ($1 \leq p < \infty$) the p -absolutely

summing ideal norm [14].

LEMMA 3: Let E have a normalized symmetric basis $\{e_i\}_{i=1}^n$, and let $T: E \rightarrow \ell_2^n$ be the basis to basis map, $T(e_i) = (\underbrace{0, \dots, 0}_i, 1, 0, \dots, 0)$, $i = 1, 2, \dots, n$. Let $A = \sum_{i_1=1}^n \dots \sum_{i_k=1}^n a_{i_1, i_2, \dots, i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ be any norm-one element in E_ϵ^k , then

$$\eta_{1/2}^*(E_\epsilon^k) \sum |a_{i_1, i_2, \dots, i_k}| \leq (n\sqrt{2}\pi_1(T))^k.$$

PROOF: By Pietsch [14] there exists a probability measure μ on $K_{E'}$ such that for every $x = \sum_{i=1}^n \xi_i e_i \in E$

$$\|Tx\|_2 = \left(\sum_{i=1}^n \xi_i^2 \right)^{1/2} \leq (\pi_1(T) \int_{K_{E'}} |\langle x, x' \rangle| d\mu(x')).$$

Let $\vec{d}\mu = \underbrace{d\mu \times d\mu \times \cdots \times d\mu}_k$ be the product measure on the set of

extreme points of the unit ball of $(E_\epsilon^k)' = (E')^k$,

$$K_{(E')^k_\pi} = \left\{ \vec{x}' = \bigotimes_{i=1}^k x'_i; x'_i \in K_{E'} \right\}.$$

Let $u = \sum_{i=1}^m A_i \otimes B_i$ be any rank- m operator in $L(E_\epsilon^k, E_\epsilon^k)$, where $A_i \in (E_\epsilon^k)'$ and $B_i \in E_\epsilon^k$. Suppose A_i and B_i have the representations

$$A_i = \sum_{i_1=1}^n \dots \sum_{i_k=1}^n a_{i_1, \dots, i_k}^{(i)} e'_{i_1} \otimes \dots \otimes e'_{i_k}, \quad \text{and}$$

$$B_i = \sum_{i_1=1}^n \dots \sum_{i_k=1}^n b_{i_1, \dots, i_k}^{(i)} e_{i_1} \otimes \dots \otimes e_{i_k}.$$

Let $\epsilon^{(i)} = (\epsilon_j^{(i)})_{j=1}^n$, $\epsilon_j^{(i)} = \pm 1$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, n$, and $g_{\epsilon^{(i)}}$ be the isometry of E defined by $g_{\epsilon^{(i)}}(e_j) = \epsilon_j^{(i)} e_j$. Let $\pi^{(i)}$ be any permutation of the integers $\{1, 2, \dots, n\}$, and $g_{\pi^{(i)}}$ be the isometry of E defined by $g_{\pi^{(i)}}(e_j) = e_{\pi^{(i)}(j)}$. Set $g_i = g_{\epsilon^{(i)}} g_{\pi^{(i)}}$, and let $\vec{g}_{\epsilon, \pi} = \bigotimes_{i=1}^k g_i$ be the isometry of E_ϵ^k .

Denote by Av_ϵ and Av_π the averages with respect to signs and permutations, that is, if $f(\epsilon^{(1)}, \dots, \epsilon^{(k)})$ is a real function then

$$Av_\epsilon(f) = 2^{-nk} \sum_{\epsilon} f(\epsilon^{(1)}, \dots, \epsilon^{(k)})$$

where the sum is taken over all possible distinct elements $(\epsilon^{(1)}, \epsilon^{(2)}, \dots, \epsilon^{(k)})$; and similarly for a function $h(\pi^{(1)}, \dots, \pi^{(k)})$

$$Av_\pi(h) = (n!)^{-k} \sum_{\pi} h(\pi^{(1)}, \dots, \pi^{(k)})$$

where the sum ranges over all $(n!)^k$ possible choices of $(\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(k)})$.

We shall find a lower bound for the integral

$$I = Av_\epsilon Av_\pi \int_{K_{(E')^k_\pi}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon, \pi}(A) \rangle| d\vec{\mu}(\vec{x}')$$

which will give the claim of the lemma. First observe that

$$\begin{aligned} & |\langle u'(\vec{x}'), \vec{g}_{\epsilon, \pi}(A) \rangle| = \\ & = \left| \sum_{\ell_1, \dots, \ell_k=1}^n \langle e_{\ell_1}, x'_1 \rangle \dots \langle e_{\ell_k}, x'_k \rangle \sum_{i_1, \dots, i_k=1}^n \sum_{j_1, \dots, j_k=1}^n \langle g_1(e_{i_1}), e'_{j_1} \rangle \right. \\ & \quad \left. \dots \langle g_k(e_{i_k}), e'_{j_k} \rangle a_{i_1, \dots, i_k} \sum_{i_1=1}^m a_{j_1}^{(i)} \dots \sum_{i_k=1}^m b_{\ell_1}^{(i)} \dots \ell_k \right|. \end{aligned}$$

Integrating with respect to $d\mu(x'_1)$ first we get

$$\int_{K_{E'}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon, \pi}(A) \rangle| d\mu(x'_1) \geq (\pi_1(T)^{-1}) \left[\sum_{\ell_1=1}^n \left(\sum_1 \right)^2 \right]^{1/2}$$

where

$$\begin{aligned} \sum_1 = & \sum_{\ell_2, \dots, \ell_k=1}^n \langle e_{\ell_2}, x'_2 \rangle \dots \langle e_{\ell_k}, x'_k \rangle \sum_{i_1, \dots, i_k=1}^n \sum_{j_1, \dots, j_k=1}^n \langle g_1(e_{i_1}), e'_{j_1} \rangle \\ & \dots \langle g_k(e_{i_k}), e'_{j_k} \rangle a_{i_1, \dots, i_k} \sum_{i=1}^m a_{j_1, \dots, j_k}^{(i)} b_{\ell_1, \dots, \ell_k}^{(i)}. \end{aligned}$$

Next, integrating with respect to $d\mu(x'_2)$ and using the fact, which we shall also use throughout, that

$$\int \left[\sum_{\ell_1=1}^n \left(\sum_1 \right)^2 \right]^{1/2} d\mu(x'_2) \geq \left[\sum_{\ell_1=1}^n \left(\int \left| \sum_1 \right| d\mu(x'_2) \right)^2 \right]^{1/2},$$

we obtain

$$\int_{K_E} \int_{K_{E'}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon, \pi}(A) \rangle| d\mu(x'_1) d\mu(x'_2) \geq (\pi_1(T))^{-2} \left[\sum_{\ell_1, \ell_2=1}^n \left(\sum_2 \right)^2 \right]^{1/2}$$

where $\sum_2 = \sum_{\ell_3, \dots, \ell_k=1}^n \langle e_{\ell_3}, x'_3 \rangle \dots$ (the ... represent the same terms which appear in \sum_1).

If we continue to integrate with respect to $d\mu(x'_3)$ and so on, finishing with $d\mu(x'_k)$ we obtain

$$\int_{K_{(E)_{\pi}^k}} |\langle u'(\vec{x}'), \vec{g}_{\epsilon, \pi}(A) \rangle| d\vec{\mu}(\vec{x}') \geq (\pi_1(T))^{-k} \left[\sum_{\ell_1, \dots, \ell_k=1}^n \left(\sum_k \right)^2 \right]^{1/2}$$

where

$$\begin{aligned} \sum_k = & \sum_{i_1, \dots, i_k=1}^n \sum_{j_1, \dots, j_k=1}^n \langle g_1(e_{i_1}), e'_{j_1} \rangle \dots \langle g_k(e_{i_k}), e'_{j_k} \rangle a_{i_1, \dots, i_k} \\ & \cdot \sum_{i=1}^m a_{j_1, \dots, j_k}^{(i)} b_{\ell_1, \dots, \ell_k}^{(i)}. \end{aligned}$$

Let $\sigma^{(i)} = \pi^{(i)-1}$, i.e. $\sigma^{(i)}(r) = j$ iff $\pi^{(i)}(j) = r$, then

$$\langle g_i(e_{i_1}), e'_{j_1} \rangle = \langle \epsilon_{j_1}^{(1)} e_{\pi^{(1)}(i_1)}, e'_{j_1} \rangle = \begin{cases} 0; & i \neq \sigma^{(1)}(j_1) \\ \epsilon_{j_1}^{(1)}; & i_1 = \sigma^{(1)}(j_1) \end{cases}.$$

Using Khintchine's inequality [17]

$$Av_{\epsilon^{(1)}} \left| \sum_{i,j_1} c_{i,j_1} \epsilon_{j_1}^{(1)} \right| \geq 2^{-1/2} \left(\sum_{j_1=1}^n \left(\sum_{i=1}^n c_{i,j_1} \right)^2 \right)^{1/2}$$

and averaging over all $\epsilon^{(1)}, \epsilon^{(2)}, \dots, \epsilon^{(k)}$, we get

$$2^{k/2} Av_{\epsilon} \left| \sum_k \right| \geq \left[\sum_{j_1, \dots, j_k=1}^n (a_{\sigma^{(1)}(j_1), \dots, \sigma^{(k)}(j_k)})^2 \left(\sum_{i=1}^m a_{j_1, \dots, j_k}^{(i)} b_{\ell_1, \dots, \ell_k}^{(i)} \right)^2 \right]^{1/2}.$$

Hence,

$$\begin{aligned} & Av_{\epsilon} \int_{K_{(E)_{\pi}^k}} |\langle u'(\bar{x}'), \bar{g}_{\epsilon, \pi}(A) \rangle| d\bar{\mu}(\bar{x}') \geq \\ & \geq (\sqrt{2}\pi_1(T))^{-k} \left[\sum_{\ell_1, \dots, \ell_k=1}^n \sum_{j_1, \dots, j_k=1}^n (a_{\sigma^{(1)}(j_1), \dots, \sigma^{(k)}(j_k)})^2 \right. \\ & \quad \left. \times \left(\sum_{i=1}^m a_{j_1, \dots, j_k}^{(i)} b_{\ell_1, \dots, \ell_k}^{(i)} \right)^2 \right]^{1/2}. \end{aligned}$$

Now we shall average over all permutations, and use the fact that

$$Av_{\pi} |a_{\sigma^{(1)}(j_1), \dots, \sigma^{(k)}(j_k)}| = n^{-k} \sum_{i_1, \dots, i_k} |a_{i_1, \dots, i_k}|,$$

this gives the following estimate for I

$$\begin{aligned} & (n\sqrt{2}\pi_1(T))^k I \geq \\ & \geq \sum_{i_1, \dots, i_k=1}^n |a_{i_1, \dots, i_k}| \left[\sum_{\ell_1, \dots, \ell_k=1}^n \sum_{j_1, \dots, j_k=1}^n \left(\sum_{i=1}^m a_{j_1, \dots, j_k}^{(i)} \right. \right. \\ & \quad \left. \left. \times b_{\ell_1, \dots, \ell_k}^{(i)} \right)^2 \right]^{1/2} \geq \sum |a_{i_1, \dots, i_k}| m^{-1/2} |\text{trace}(u)| \end{aligned}$$

the last inequality follows from Lemma 2. The proof is completed by applying Proposition 1 for $p = \frac{1}{2}$ while noting that $\bar{g}_{\epsilon, \pi}(A)$ are norm-one elements of E_{ϵ}^k and the \bar{x}' which appear in I are norm-one elements of the dual space $(E')_{\pi}^k$. □

LEMMA 4: *With the notation of Lemma 3,*

$$(\sqrt{2}\pi_1(T))^k \eta_{1/2}(E_{\epsilon}^k) \geq \sum |a_{i_1, i_2, \dots, i_k}|.$$

PROOF: This follows immediately from Lemma 3 and the obvious inequality $\alpha(F)\alpha^*(F) \geq \dim(F)$ for any ideal norm α and finite-dimensional space F . \square

We shall next use the following inequality due to S. Chevet [1], for the sake of completeness we include the proof. If $\{x_i\}_{i=1}^n \subset E$, we shall denote by $\epsilon_2(\{x_i\}) = \sup\{(\sum_{i=1}^n |\langle x_i, x' \rangle|^2)^{1/2}; x' \in E', \|x'\| = 1\}$.

LEMMA 5: If $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ are elements in Banach spaces E and F respectively, and g_{ij} ($i, j = 1, 2, \dots, n$) is a sequence of equidistributed, independent, orthonormal random Gaussian variables, then

$$(*) \quad \frac{\Lambda}{2} \leq E \left(\left\| \sum_{ij} g_{ij} x_i \otimes y_j \right\|_{E \otimes F} \right) \leq \sqrt{2} \Lambda$$

where $\Lambda = \epsilon_2(\{y_j\})E(\|\sum_i g_{i,1} x_i\|) + \epsilon_2(\{x_i\})E(\|\sum_j g_{i,1} y_j\|)$.

PROOF: Let $T = \{(\xi, \eta); \xi \in E', \eta \in F', \|\xi\| = \|\eta\| = 1\}$. For each $t = (\xi, \eta) \in T$, define the random variables

$$X_t = \sum_{ij} g_{ij} \langle x_i, \xi \rangle \langle y_j, \eta \rangle \quad \text{and}$$

$$Y_t = \alpha \sum_j g_{j,1} \langle y_j, \eta \rangle + \beta \sum_i g_{i,2} \langle x_i, \xi \rangle$$

where $\alpha = \epsilon_2(\{x_i\})$, $\beta = \epsilon_2(\{y_j\})$. It is easy to see that if $s = (\xi_1, \eta_1) \in T$, then

$$\begin{aligned} E(|X_t - X_s|^2) &= \sum_{ij} (\langle x_i, \xi \rangle \langle y_j, \eta \rangle - \langle x_i, \xi_1 \rangle \langle y_j, \eta_1 \rangle)^2 \\ &= \sum_{ij} (\langle x_i, \xi - \xi_1 \rangle \langle y_j, \eta \rangle + \langle x_i, \xi_1 \rangle \langle y_j, \eta - \eta_1 \rangle)^2 \\ &\leq 2 \sum_{ij} (\langle x_i, \xi - \xi_1 \rangle \langle y_j, \eta \rangle)^2 + (\langle x_i, \xi_1 \rangle \langle y_j, \eta - \eta_1 \rangle)^2 \\ &= 2E(|Y_t - Y_s|^2) \end{aligned}$$

hence $E(|X_t - X_s|^2) \leq 2E(|Y_t - Y_s|^2)$. By a result due to Sudakov ([2], Corollaire 2.1.3) this implies $E(\vee_T X_t) \leq \sqrt{2}E(\vee_T Y_t)$, from which the right hand side of (*) follows.

For the other side, pick $\xi_0 \in E'$, $\|\xi_0\| = 1$, such that $\alpha = \epsilon_2(\{x_i\})$, and define the random variables $Z_\eta = \sum_{ij} g_{ij} \langle x_i, \xi_0 \rangle \langle y_j, \eta \rangle$ and $W_\eta = \alpha \sum_{j=1}^n g_{j,1} \langle y_j, \eta \rangle$. Then

$$E(|Z_\eta - Z_{\eta_1}|^2) = \alpha^2 \sum_j \langle y_j, \eta - \eta_1 \rangle^2 = E(|W_\eta - W_{\eta_1}|^2),$$

so again $E(\sup_{\|\eta\|=1} Z_\eta) = E(\sup_{\|\eta\|=1} W_\eta)$, but

$$\begin{aligned} E\left(\left\|\sum_{ij} g_{i,j}x_i \otimes y_j\right\|_{E\hat{\otimes}_\epsilon F}\right) &\geq E\left(\left\|\sum_{ij} g_{i,j}\langle x_i, \xi_0\rangle y_j\right\|\right) \\ &= E\left(\sup_{\|\eta\|=1} Z_\eta\right) = E\left(\sup_{\|\eta\|=1} W_\eta\right) \\ &= \alpha E\left(\left\|\sum_i g_{i,1}y_i\right\|\right), \text{ and similarly, } \geq \beta E\left(\left\|\sum_i g_{i,1}x_i\right\|\right), \end{aligned}$$

hence the left side of (*). □

THEOREM 6: *Let $\{e_i\}_{i=1}^n$ be a symmetric basis for a Banach space E , and let $T : E \rightarrow \ell_2^n$ be the natural basis to basis map. Then, if $E_\epsilon^{2k} =$*

$$\underbrace{E \hat{\otimes}_\epsilon E \hat{\otimes}_\epsilon \dots \hat{\otimes}_\epsilon E}_{2^k}$$

$$\eta_{1/2}(E_\epsilon^{2k})(\|T^{-1}\| \pi_1(T))^{2k} \geq \sqrt{\frac{2}{\pi n}} n^{2k} 2^{-3k/2-2k^{-1}}.$$

PROOF: For each integer $k = 1, 2, \dots$, let I_k denote a set consisting of n^{2^k} elements, and let $\{b_\nu\}_{\nu \in I_k}$ denote the natural basis of $E_\epsilon^{2^k}$ (each b_ν has the form $e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}$, where $1 \leq i_j \leq n$). Consider the random vectors of $E_\epsilon^{2^k} = E_\epsilon^{2^{k-1}} \hat{\otimes}_\epsilon E_\epsilon^{2^{k-1}}$ of the form $A^k = \sum_{\alpha, \beta \in I_{k-1}} g_{\alpha, \beta} b_\alpha \otimes b_\beta$. By Lemma 5

$$E(\|A^k\|_{E_\epsilon^{2^k}}) \leq 2\sqrt{2}\epsilon_2(\{b_\alpha\}_{\alpha \in I_{k-1}})E(\|A^{k-1}\|_{E_\epsilon^{2^{k-1}}}).$$

Since $\{\otimes_{i=1}^{2^{k-1}} x'_i; x'_i \in K_{E'}\}$ is the set of extreme points of the unit ball of $(E_\epsilon^{2^{k-1}})' = (E')_\pi^{2^{k-1}}$, we have

$$\begin{aligned} \epsilon_2(\{b_\alpha\}_{\alpha \in I_{k-1}}) &= \sup\left\{\left(\sum_{i_1, \dots, i_{2^{k-1}}=1}^n \langle e_{i_1}, x'_1 \rangle \dots \langle e_{i_{2^{k-1}}}, x'_{2^{k-1}} \rangle\right)^2; \right. \\ &\left. x'_i \in K_{E'}\right\} = \left[\sup\left\{\left(\sum_{i=1}^n \langle e_i, x' \rangle^2\right)^{1/2}; x' \in K_{E'}\right\}\right]^{2^{k-1}} = \|T^{-1}\|^{2^{k-1}}, \end{aligned}$$

hence we get the reduction formula

$$E(\|A^k\|_{E_\epsilon^{2^k}}) \leq 2\sqrt{2}\|T^{-1}\|^{2^{k-1}}E(\|A^{k-1}\|_{E_\epsilon^{2^{k-1}}}),$$

and so

$$E(\|A^k\|_{E_\epsilon^{2^k}}) \leq (2\sqrt{2})^k \|T^{-1}\|^{2^{k-1}+2^{k-2}+\dots+2^0} E\left(\left\|\sum_{i=1}^n g_i e_i\right\|_{E'}\right)$$

$$\begin{aligned}
 &= (2\sqrt{2})^k \|T^{-1}\|^{2k-1} \mathbf{E} \left(\left\| \sum_{i=1}^n g_i e_i \right\| \right) \\
 &\leq (2\sqrt{2})^k \|T^{-1}\|^{2k} \mathbf{E} \left(\left(\sum g_i^2 \right)^{1/2} \right) \\
 &\leq (2\sqrt{2})^k \|T^{-1}\|^{2k} \sqrt{n}.
 \end{aligned}$$

On the other hand, $\mathbf{E}(\sum_{\alpha, \beta \in I_{k-1}} |g_{\alpha, \beta}|) = \sqrt{2/\pi} n^{2k}$, hence

$$\begin{aligned}
 &\sup \left(\sum |a_{i_1, \dots, i_{2k}}| / \left\| \sum a_{i_1, \dots, i_{2k}} e_{i_1} \otimes \dots \otimes e_{i_{2k}} \right\|_{E_\epsilon^{2k}} \right) \\
 &\geq \mathbf{E} \left(\sum_{\alpha, \beta \in I_{k-1}} |g_{\alpha, \beta}| \right) / \mathbf{E}(\|A^k\|) \\
 &\geq \sqrt{\frac{2}{\pi}} n^{2k} / (2/\sqrt{2})^k \sqrt{n} \|T^{-1}\|^{2k},
 \end{aligned}$$

and the inequality of Lemma 4 establishes the Theorem. □

REMARK: More generally, the same proof can show that if E_i ($i = 1, 2, \dots, 2^k$) are n_i -dimensional spaces with symmetric bases, and if $T_i: E_i \rightarrow \ell_2^{n_i}$ are the natural basis to basis maps, then for $F = E_1 \hat{\otimes}_\epsilon E_2 \hat{\otimes}_\epsilon \dots \hat{\otimes}_\epsilon E_{2^k}$,

$$\eta_{1/2}(F) \prod_{j=1}^{2^k} \|T_j^{-1}\| \pi_1(T_j) \geq \sqrt{2/\pi} \prod_{j=1}^{2^k} n_j / 2^{k/2+2^{k-1}} \left(\sum_{j=1}^{2^k} \sqrt{n_j} \right).$$

It is not essential that 2^k spaces appear in F , however, if the number is not 2^k then the bottom line on the right hand side of the inequality will be different.

EXAMPLE: If $E = \ell_2^n$, then $\|T^{-1}\| = 1$, $\pi_1(T) \leq \sqrt{\pi n}/2$ [5], hence if $E_N = \underbrace{\ell_2^n \hat{\otimes}_\epsilon \dots \hat{\otimes}_\epsilon \ell_2^n}_{2^k}$ where $N = n^{2^k}$,

$$\eta_{1/2}(E_N) \geq \sqrt{2/\pi} \sqrt{n^{2^k} / (\sqrt{\pi})^{2^k} 2^{3k/2} \sqrt{n}} \geq a \sqrt{N} e^{-\sqrt{\log N}}$$

where $a > 0$ is constant, and k, n are chosen in an appropriate relation to N . If $0 < p < \frac{1}{2}$, then $\eta_p(E_N) \geq N^{p-1/2} \eta_{1/2}(E_N) \geq a N^p e^{-\sqrt{\log N}} \rightarrow \infty$ as $N \rightarrow \infty$.

COROLLARY 7: If E is an n -dimensional normed space such that $d(E, \ell_2^n) < (n/\pi)^{1/2}$, then $\eta_{1/2}(E_\epsilon^k) \rightarrow \infty$ as $k \rightarrow \infty$.

PROOF: First observe that if $k > \ell$ then E_ϵ^ℓ is isometric to a norm one-complemented subspace of E_ϵ^k , therefore the ideal property of the norm $\eta_{1/2}$ implies that $\eta_{1/2}(E_\epsilon^k)$, $k = 1, 2, \dots$, is a nondecreasing sequence. Clearly $d(E_\epsilon^k, (\ell_2^n)_\epsilon^k) \leq (d(E, \ell_2^n))^k$, hence using the estimates for $\eta_{1/2}((\ell_2^n)_\epsilon^k)$ we obtain

$$\eta_{1/2}(E_\epsilon^{2k}) \geq \eta_{1/2}((\ell_2^n)_\epsilon^{2k}) / (d(E, \ell_2^n))^{2k} \geq \sqrt{2/\pi} \left(\frac{\sqrt{n/\pi}}{d(E, \ell_2^n)} \right)^{2k} / 2^{3k/2} \sqrt{n}$$

which tends to ∞ with k .

REMARKS: (1) It may be true that $\eta_{1/2}(E_\epsilon^k) \rightarrow \infty$ as $k \rightarrow \infty$ whenever $d(E, \ell_2^n) < \sqrt{n}$. It is obviously false if $d(E, \ell_2^n) = \sqrt{n}$, as in the case $E = \ell_\infty^n$.

(2) If $1 < q < \infty$, and $c > |\frac{1}{2} - 1/q|$, there exists N such that if $n \geq N$ and E is any m -dimensional subspace of $L_q(\mu)$, then $\eta_c(E_\epsilon^k) \rightarrow \infty$ as $k \rightarrow \infty$. The reason for this is that $d(E, \ell_2^n) \leq n^{||1/2 - 1/q|}$ by [13], so if $c \leq \frac{1}{2}$

$$\eta_c(E_\epsilon^{2k}) \geq (c - 1/2)^{2k} \eta_{1/2}(E_\epsilon^{2k})$$

which tends to ∞ by applying the inequality in the proof of Corollary 7.

THEOREM 8: *There exists a reflexive separable Banach space E with both $\eta_p(E)$ and $\eta_p^{**}(E)$ infinite for all values of $p > 0$.*

PROOF: Let E_N be the space in the Example, and let $E = (\sum_{i=1}^\infty \oplus E_{N_i})_{\ell_2}$ where $N_i \rightarrow \infty$, $N_i = (n_i)^{2k_i}$, which the proper relation maintained between n_i and k_i with N_i . Since E_{N_i} is norm one complemented in E , it follows that

$$\eta_p(E) \geq \eta_p^{**}(E) \geq \eta_p^{**}(E_{N_i}) = \eta_p(E_{N_i}) \geq a N_i^p \exp(-\sqrt{\log N_i}) \xrightarrow{i \rightarrow \infty} \infty.$$

□

In order to estimate $\pi_1(T)$ for general spaces it may be useful to apply the following proposition.

PROPOSITION 9: *For any $0 < p, q, r < \infty$ there exist constants $a_{r,q}, b_{p,q} > 0$ such that for any Banach space E and any operator $T : E \rightarrow \ell_2^n$*

(1) $b_{p,q}^{-1} \pi_p(T) \leq \sqrt{n} (\int_{S_n} \|T'x\|^q dm(x))^{1/q} \leq a_{r,q}^{-1} \pi_r(T')$, where $S_n = \{x \in \ell_2^n; \|x\|_2 = 1\}$ and $dm(x)$ is the rotation invariant normalized measure on S_n .

(2) If E' is a subspace of an L_s -space, $1 \leq s < \infty$, and if $0 < p \leq s \leq r < \infty$ and $0 < q < \infty$, the inequalities of (1) becomes equivalence relations and the constants of equivalence are independent of n, T and E .

(3) If $\dim(E) = n$ and E has a symmetric basis and $T : E \rightarrow \ell_2^n$ is the basis to basis map, then all the values $I_q = (\int_{S_n} \|T'x\|^q dm(x))^{1/q}$ ($0 < q < \infty$) are equivalent and the constants of equivalence are independent of n and E .

PROOF: (1) By [14] there exists a probability measure μ on S_n such that for all $x \in \ell_2^n$

$$\|T'x\| \leq \pi_q(T') \left(\int_{S_n} |\langle x, x' \rangle|^q d\mu(x') \right)^{1/q}$$

hence by integrating with respect to dm

$$I_q \leq \pi_q(T') \left(\int_{S_n} \int_{S_n} |\langle x, x' \rangle|^q d\mu(x') dm(x) \right)^{1/q} = \pi_q(T') / \pi_q(\ell_2^n)$$

since $(\int_{S_n} |\langle x, x' \rangle|^q dm(x))^{1/q} = \|x'\|_2 (\pi_q(\ell_2^n))^{-1}$ [5]. Since $\pi_q(\ell_2^n) \sim \sqrt{n}$, and $\pi_q(T')$ is a non-increasing function of q , and I_q is a non-decreasing function of q , the right hand side of (1) readily follows.

Without loss of generality we may assume that T' is a 1-1 map, and define the probability measure ν on the unit ball $B_{E'}$ of E' by

$$\int_{B_{E'}} f d\nu = \int_{S_n} f(T'x / \|T'x\|) \|T'x\|^q dm(x) / \int_{S_n} \|T'x\|^q dm(x)$$

for $f \in C(B_{E'})$. Taking $f = |\langle \xi, \cdot \rangle|^q$ where $\xi \in E$, we obtain

$$\begin{aligned} \int_{B_{E'}} |\langle \xi, x' \rangle|^q d\nu(x') &= \int_{S_n} |\langle T\xi, x \rangle|^q dm(x) / \int_{S_n} \|T'x\|^q dm(x) \\ &= \|T\xi\|^q / (\pi_q(\ell_2^n))^q \int_{S_n} \|T'x\|^q dm(x), \end{aligned}$$

therefore, $\pi_q(T) \leq \pi_q(\ell_2^n) (\int_{S_n} \|T'x\|^q dm(x))^{1/q}$, and as above this proves the left hand side of (1).

(2) Let $j : E' \rightarrow L_s$ be an isometric embedding, then by [12] $\pi_s(jT') \leq \pi_s((jT'))$, that is

$$\pi_s(T') = \pi_s(jT') \leq \pi_s(T''j') \leq \pi_s(T'') = \pi_s(T),$$

and (2) follows from the inequalities

$$\begin{aligned} \pi_s(T') &\leq \pi_s(T) \leq \pi_p(T) \leq b_{p,q} \sqrt{n} I_q \leq b_{p,q} a_{r,q}^{-1} \pi_r(T') \\ &\leq b_{p,q} a_{r,q}^{-1} \pi_s(T'). \end{aligned}$$

(3) Set $|x|_E = \|T'x\|$ for vectors $x \in \ell_2^n = (R^n, \|\cdot\|_2)$. Since E is symmetric, by [10] $\|T\| \|T^{-1}\| = d(E, \ell_2^n)$, and $d(E, \ell_2^n) \leq \sqrt{n}$, so $a\|x\|_2 \leq |x|_E \leq b\|x\|_2$ for all $x \in \ell_2^n$, where $b/a \leq \sqrt{n}$. From the remark following Lemma 2.7 in [4], the values $I_q = (\int_{S_n} |x|_E^q dm(x))^{1/q}$ ($0 < q < \infty$) are all equivalent to the Levy mean M^* , which is by definition the unique number such that $m(\{x \in S_n; |x|_E \leq M^*\}) \leq \frac{1}{2}$ and $m(\{x \in S_n; |x|_E \geq M^*\}) \leq \frac{1}{2}$, that is, there exist absolute positive constants a_q, b_q such that $a_q M^* \leq I_q \leq b_q M^*$. □

COROLLARY 10: *If $\dim(E) = n$, there are absolute constants $a, b > 0$ such that for any $T : E \rightarrow \ell_2^n$*

$$a \pi_1(T) \leq \sqrt{n} \int_{S_n} \|T'x\| dm(x) \leq b \pi_1(T) \sqrt{\ln n} x_u(E).$$

PROOF: Let γ_p denote the best factorization through an L_p -space norm [9]. Interpolation technique as in Theorem 7 [11] shows that $\pi_p(T') \leq n^{1/p} \gamma_\infty(T')$. Since $\gamma_\infty(T') = \gamma_1(T) \leq x_u(E) \pi_1(T)$ [7], and $\pi_p(\ell_2^n) \geq c \sqrt{n/p}$ [5], we obtain

$$\begin{aligned} \int_{S_n} \|T'x\| dm(x) &\leq \left(\int_{S_n} \|T'x\|^p dm(x) \right)^{1/p} \leq \pi_p(T') / \pi_p(\ell_2^n) \\ &\leq \sqrt{p/n} c^{-1} n^{1/p} \pi_1(T) x_u(E) \end{aligned}$$

and the estimate follows by taking $p = \ln n$, and from Proposition 9. □

COROLLARY 11: *Let $E_N = \underbrace{\ell_p^n \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon \ell_p^n}_{2^k}$, $N = n^{2^k}$, where $1 \leq p \leq 2$. There exists $b > 0$, such that $\eta_{1/2}(E_N) \geq b \sqrt{N} e^{-\sqrt{\log N}}$ for the proper relation between n, k with N .*

PROOF: Factor $T : \ell_p^n \xrightarrow{A} \ell_1^n \xrightarrow{B} \ell_2^n$, where A, B are the inclusions. Then $\|T^{-1}\| = n^{1/p-1/2}$, and $\pi_1(T) \leq \|A\| \pi_1(B) = \sqrt{2} n^{1-1/p}$, since $\pi_1(B) = \sqrt{2}$ is the Khintchine constant. Therefore, $\|T^{-1}\| \pi_1(T) \leq \sqrt{2n}$. The proof is concluded as in the Example following Theorem 6. □

REMARK: Since the distance between E_ϵ^k and $(\ell_p^n)_\epsilon^k$ is $\leq (d(E, \ell_p^n))^k$, it follows from the estimates of Corollary 10, Theorem 6 and the inequality

$$\eta_{1/2}(E_\epsilon^k) \geq \eta_{1/2}((\ell_p^n)_\epsilon^k)(d(E, \ell_p^n)^{-k},$$

that if E is any n -dimensional Banach space such that $\inf_{1 \leq p \leq 2} d(E, \ell_p^n) < \sqrt{n}/2$, then $\eta_{1/2}(E_\epsilon^{2k}) \rightarrow \infty$ as $k \rightarrow \infty$. □

Denote by $r_i(t)$, the i -th Rademacher function on $[0, 1]$.

COROLLARY 12: Let $1 < p \leq 2 \leq q < \infty$, $1/p < 1/q + \frac{1}{2}$. Assume F is a Banach space of type p and cotype q . Then, for any c satisfying $c > 1/p - 1/q$, there exists an integer N such that if $n > N$ and if E is any n -dimensional symmetric subspace of F , then $\eta_c(E_\epsilon^k) \rightarrow \infty$ as $k \rightarrow \infty$.

PROOF: Let α be the type- p constant and β be the cotype- q constant of F respectively. Let $\{e_i\}_{i=1}^n$ denote the symmetric basis of a subspace $E \subset F$, and $\{e'_j\}_1^n$ be the biorthogonal functionals. The inequality

$$\left(\sum_i^n |\xi_i|^q \right)^{1/q} \leq \beta \int_0^1 \left\| \sum_{i=1}^n \xi_i r_i(t) e_i \right\| dt = \beta \left\| \sum_{i=1}^n \xi_i e_i \right\|$$

implies that $\|\sum \xi_i e_i\| \leq \beta \|\xi\|_q$. By Proposition 9

$$\begin{aligned} b_{1,q}^{-1} n^{-1/2} \pi_1(T) &\leq \left(\int_{S_n} \|T'x\|^{q'} dm(x) \right)^{1/q'} \leq \beta \left(\int_{S_n} \|\xi\|_q^{q'} dm(\xi) \right)^{1/q'} \\ &= \beta n^{1/q'} (\pi_q(\ell_2^n))^{-1}, \end{aligned}$$

hence $\pi_1(T) \leq \beta c_q n^{1/q'}$, where $c_q > 0$ is constant. Select scalars $\{x_i\}_{i=1}^n$ so that $\sum x_i^2 = 1$ and $\|T^{-1}\| = \|\sum x_i e_i\|$. Then,

$$\|T^{-1}\| = \int_0^1 \left\| \sum x_i r_i(t) e_i \right\| dt \leq \alpha \|x\|_p \leq \alpha n^{1/p-1/2},$$

therefore $\|T^{-1}\| \pi_1(T) \leq \alpha \beta c_q n^{1/p-1/q+1/2}$. Combining this with the inequality $\eta_c(E_\epsilon^{2k}) \geq \eta_{1/2}(E_\epsilon^{2k})(n^{2k})^{c-1/2}$, and Theorem 6, establishes that for $c > 1/p - 1/q$ and n sufficiently large, $\eta_{1/2}(E_\epsilon^{2k})(n^{2k})^{c-1/2} \rightarrow \infty$ as $k \rightarrow \infty$. □

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