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# $\pi$-ADIC EISENSTEIN SERIES FOR FUNCTION FIELDS

David Goss

## Contents



## Introduction

Let $k$ be a global field of characteristic $p > 0$, $\infty$ a fixed place and $d$ a positive integer. The purpose of this paper is to define, and begin to study, characteristic $p$ "holomorphic" modular forms for $(k, \infty, d)$. In particular, we define holomorphic Eisenstein series. As our functions have characteristic $p$ values, our theory is distinct from the ones developed by Weil, Harder and Langlands.

To accomplish this, we use the general framework of Drinfeld [2]. In this paper various rigid analytic spaces (in the sense of [13]–[17] and [19]) and moduli spaces associated to "elliptic modules" were introduced. These spaces act in a fashion similar to the classical Poincaré halfplane and moduli of elliptic curves. It is on these spaces that our theory is developed.

Section 1 introduces the basic concepts, i.e. elliptic modules, modular forms etc. As some of our proofs of results contained in [2] are sketched, the reader can refer there for more information. The

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ideas from that paper will be indicated by a "[2]". Our main result is the existence of various $q$-expansions in the case $d = 2$. We shall then see that, with interesting variations, much of the classical $SL(2, \mathbb{Z})$ theory goes through. For instance 1-forms correspond to forms of weight 2 with *double* zeros at each cusp.

Section 2 introduces the Eisenstein series. We show these come from algebraically defined modular forms over $k$. Further, we show these series give all necessary information for defining an elliptic module "analytically". In the case $d = 2$, we compute the $q$-expansions. In the case $d = 1$, we prove a rationality statement for certain "zeta-values". This result is similar to the one classically proven at the positive integers for the Riemann zeta function.

At this point, for simplicity of calculation, we let $k$ be the rational field $\mathbb{F}_q(T)$, and $\infty$ the place associated to $1/T$. One expects, however, that the results will be true in great generality.

Section 3, then, defines the Hecke operators. It is shown that various spaces of modular forms are stable under these operators. The effect on $q$-expansions is also shown. If $f$ is an eigenfunction for all the Hecke operators we introduce $L_f$, the corresponding $L$-series.

Section 4 gives a computation of the genera of the moduli curves. By previous theorems, one can then compute dimensions of spaces of forms of weight $\geq 2$ by the Riemann–Roch theorem. In particular we see trivially the existence of many forms of weight $\geq 2$. However, as classical, the dimension of weight one forms is not given by the Riemann–Roch theorem. Thus, the exact dimensions would be very interesting.

Section 4 also presents some evidence for a conjecture involving subgroups of the Jacobian generated by cusps.

We hope that the study of these forms will add not only to our knowledge of function fields but also to our understanding of modular forms in general.

As this paper is a version of the author's Harvard Thesis, he thanks his advisor B. Mazur. He also thanks the N.S.F.

## 1. Basic concepts

*Elliptic modules*

Let $q = p^n$, with $p$ a rational prime. We fix a smooth, projective, geometrically connected curve $C$ over $\mathbb{F}_q$ and a point $\infty$. We let $A$ be the affine ring of $C - \infty$ and $k$ the function field of $C$. Thus, $A$ is a Dedekind domain with finite class group. Its unit group is $\mathbb{F}_q^*$. If $v$ is any place of $k$, we denote by $|\ |_v$ the normalized valuation and by $k_v$ the associated local field. The letter $I$ will always denote an ideal of $A$ and we let $A[I^{-1}]$ be the affine ring of $\operatorname{Spec}(A) - V(I)$. The symbol $\square$ will denote the end of a proof.

Let $S$ be an A-scheme with a line bundle $L$.

DEFINITION 1.1: We denote the ring of $\mathbb{F}_q$-linear endomorphisms of $L$ by $\operatorname{End}_{\mathbb{F}_q}(L)$.

PROPOSITION 1.2: *Let $F$ be the $q^{th}$ power map; $F: z \mapsto z^q$. Then $\operatorname{End}_{\mathbb{F}_q}(L)$ is the ring of polynomials*

$$\sum a_i F^i, \quad \text{with } a_i \in \Gamma(S, L^{(1-q^i)}),$$

*under composition.* $\square$

REMARK 1.3: Since $(aF)(bF) = ab^q F^2$, $\operatorname{End}_{\mathbb{F}_q}(L)$ is not in general commutative.

Let $d \in \mathbb{N}^+$.

DEFINITION 1.4, [2]: An elliptic module $E$ of rank $d$ over $S$ is a pair $(L, \phi)$ consisting of an $S$-line bundle $L$ and an $\mathbb{F}_q$-homomorphism $\phi: A \to \operatorname{End}_{\mathbb{F}_q}(L)$ so that:

(1) if the cardinality of $A/(a)$ is $q^m$, then

$$\phi(a) = aF^0 + \sum_{i=1}^{dm} \phi_i(a) F^i.$$

(2) the section $\phi_{md}(a)$ is nowhere zero.

CONVENTION: If $L$ is the trivial bundle, we shall denote $E$ by $(\phi)$. Further, when the meaning is obvious, we shall sometimes call an elliptic module a "module".

EXAMPLE 1.5: Let $A = \mathbb{F}_q[T]$ and $S = \operatorname{spec}(K)$ where $K$ is a field. Then a rank $d$ elliptic module is determined by

$$\phi(T) = TF^0 + \sum_{i=1}^{d} c_i F^i,$$

where $c_i \in K$ and $c_d \neq 0$.

Let $E_i = (L_i, \phi_i)$ $i = 1, 2$ be two elliptic modules over $S$.

DEFINITION 1.6, [2]: A homomorphism from $E_1$ to $E_2$ is an element $P \in \mathrm{Hom}_{\mathbb{F}_q}(L_1, L_2)$ such that $P\phi_1 = \phi_2 P$.

Note that $P$ has a form similar to that given in 1.2.

REMARKS 1.7: (1) Taking the form of $P$ into account, non-zero homomorphisms can be seen only to occur between modules of the same rank. Further, the elements of $\mathbb{F}_q^*$ act as automorphisms. If $(\phi)$ is a module over a ring $R$ and $\alpha \in R^*$ then $\alpha F^0$ gives an isomorphism $(\phi) \simeq ((\alpha F^0)\phi(\alpha F^0)^{-1})$. We denote $((\alpha F^0)\phi(\alpha F^0)^{-1})$ by $(\alpha\phi\alpha^{-1})$.

(2) Under the above definitions elliptic modules over $S$ form a category.

Fix $E = (L, \phi)$ of rank $d$ over $S$. Via $\phi$, $A$ acts on $\Gamma(S, L)$.

DEFINITION 1.8, [2]: Let $E_I = \{\cap \ker \phi(i) \mid i \in I\}$.

PROPOSITION 1.9, [2]: *The set $E_I$ is a finite flat $A$-invariant group scheme. It is étale away from the fiber over $V(I)$.*

PROOF: The results are obvious for principal ideals. In general, we can find an ideal $J$, prime to $I$, so that $J \cdot I = (f)$. Thus, the result can be seen in general. $\square$

DEFINITION 1.10, [2]: A level $I$ structure is a homomorphism $\psi\colon (I^{-1}/A)^d \to \Gamma(S, L)$ so that for every $M$ in $V(I)$, $E_M$, as a divisor, coincides with the sum of divisors $\psi(\alpha)$, $\alpha \in (M^{-1}/A)^d$.

REMARK 1.11: Away from $V(I)$ a level $I$ structure is the same as an isomorphism

$$(I^{-1}/A)^d \times S \to E_I.$$

Now let $\#(V(I)) > 1$.

THEOREM 1.12, [2]: *The functor given by isomorphism classes of elliptic modules, $E = (L, \phi)$, of rank $d$ together with a level $I$ structure is representable by a scheme $M_I^d$.*

PROOF: Let $J_1, J_2$ be two distinct maximal ideals in $V(I)$. Over $A[J_i^{-1}]$, $i = 1, 2$, the level structure gives rise to a canonical trivialization of the underlying line bundle $L$. Let $\{x_1, \ldots, x_j\}$ by $\mathbb{F}_q$-algebra generators of $A$. Thus, over $A[J_i^{-1}]$, $i = 1, 2$, we can describe $(\phi)$ up to isomorphism by giving functions,

$$\{\phi_r(x_i)\} \quad \text{all } r, i,$$

subject to the elliptic module conditions and the level structure. Then patch. □

REMARKS 1.13: (1) Over $A[I^{-1}]$, the functor in the theorem is always representable by a scheme $M_I^d$. The proof is the same.

(2) The scheme $M_I^d$ is affine and of finite type over $\mathbb{F}_q$. Whenever no confusion will result, we shall denote by $M_I^d$ either the scheme or its affine algebra.

Fix now an A-algebra $R$. If $g: S' \to S$ is an $A$-morphism and $E = (L, \phi)$ is a module over $S$, we define the pullback $g^*(E)$ by pulling back the line bundle and its sections.

DEFINITION 1.14: Let $j \in \mathbb{Z}$. A modular form over $R$ of rank $d$, weight $j$ is a rule, $F$, which to each elliptic module $E = (L, \phi)$, of rank $d$ defined over an $R$-scheme $S$ assigns a section

$$F((E) \in \Gamma(L^{-\otimes j}),$$

subject to the following conditions:

(1) let $S' \xrightarrow{g} S$ be an $R$-map, then $F(g^*(E)) = g^*(F(E))$,

(2) suppose over $S'$ there is a nowhere-zero section $\beta$ of $g^*(L)$. Then the element

$$F(g^*(E)) \cdot \beta^{\otimes j} \in \Gamma(S', O_{S'})$$

depends only on the isomorphism class of $(E, \beta)$.

REMARKS 1.15: (1) A modular form $F$ of level $I$, weight $j$ is defined in exactly the same manner except it is a rule on pairs $(E, \psi)$, with $\psi$ a level I structure.

(2) By part 2 of Definition 1.14, the weight of a nontrivial form without level must be divisible by $q-1$.

EXAMPLES 1.16: (1) Let $a \in A$. Then $\phi_i(a)$ is a form of weight $q^i - 1$. If $A = \mathbb{F}_q[T]$, then $\phi_d(T)$ is nowhere zero. It is analogous to the classical $\Delta$.

(2) Let $\psi$ be a level $I$ structure and $0 \neq \alpha \in (I^{-1}/A)^d$. Then $\psi(\alpha)$ is a form of weight $-1$, level I. Over $A[I^{-1}]$ we can invert to get a form of weight 1, level I.

DEFINITION 1.17: Let $H^d$ be the graded ring of forms of rank $d$ over $A$. Let $H_I^d$ be the graded ring of forms of rank $d$, level I over $A$.

THEOREM 1.18: *The scheme* $\mathrm{Spec}(H^d)$ *represents isomorphism classes of pairs* $(E, \omega)$, *where* $E = (L, \phi)$ *is a module of rank d and* $\omega$ *is a nowhere-zero section of L. The scheme* $\mathrm{Spec}(H_I^d)$ *represents isomorphism classes of triples* $(E, \omega, \psi)$, *where* $\psi$ *is a level I structure.*

PROOF: As we now have trivializations, the proof of representability goes as before. One then notes that elliptic modules are defined by modular forms to conclude the proof. □

COROLLARY 1.19: *The rings* $H^d$, $H_I^d$ *are finitely generated* $\mathbb{F}_q$-*algebras.* □

REMARKS 1.20: (1) The natural map $\mathrm{Spec}(H_I^d) \to M_I^d$ is a principal $\mathbb{G}_m$-bundle.

(2) To give $\omega$ amounts to giving a nowhere zero relative differential on $L$.

The computations of Drinfeld give the following result for $M_I^d$. The general proof is immediate.

THEOREM 1.21, [2]: (1) *The rings* $H_I^d$, $H^d$, *and* $M_I^d$ *are regular* $\mathbb{F}_q$-*algebras of dimensions* $d+1$, $d+1$, *and d respectively.*

(2) *As A-algebras they are flat and, away from* $V(I)$, *smooth. Their relative dimensions are d, d and* $d-1$ *respectively.*

(3) *If* $I \mid J$ *the corresponding maps of moduli spaces are finite and flat.* □

*Analytic description of elliptic modules*

Let $[K : k_\infty] < \infty$, and let $K^s$ denote the separable closure of $K$.

DEFINITION 1.22, [2]: A $K$-lattice is a discrete, finitely generated $A$-submodule of $K^s$ which is $\mathrm{Gal}(K^s/K)$ stable. Its rank is that of the underlying projective module. Its type is the isomorphism class of the underlying projective module. We shall use the word lattice to mean $K$-lattice for some $K$.

DEFINITION 1.23, [2]: Let $N_1$, $N_2$ be two $K$-lattices of rank $d$. Then a morphism from $N_1$ to $N_2$ is an element $\alpha$ of $K$ so that $\alpha N_1 \subseteq N_2$.

THEOREM 1.24, [2]: *The category of elliptic modules of rank d over K is equivalent to the category of K-lattices of rank d.*

PROOF: (Sketch). Let $N$ be a rank $d$ $K$-lattice. We define

$$e_N(z) = z \prod_{\substack{\alpha \in N \\ \alpha \neq 0}} (1 - z/\alpha).$$

As $N$ is discrete, $e_N$ is entire.

As the characteristic is finite, we can write $N = \cup N_i$, with $N_i$ a finite group. Therefore, $e_N = \lim_i \{e_{N_i}\}$, with

$$e_{N_i}(z) = z \prod_{\substack{\alpha \in N_i \\ \alpha \neq 0}} (1 - z/\alpha).$$

As $e_{N_i}$ is a polynomial whose zeroes form an additive group, it is an additive polynomial, i.e. $e_{N_i}(z + y) = e_{N_i}(z) + e_{N_i}(y)$. Therefore, $e_N$ is additive and even $\mathbf{F}_q$-linear. Finally, the derivative $e'_N(z)$ is identically 1.

On points, $e_N$ gives an isomorphism $\bar{k}_\infty/N \simeq \bar{k}_\infty$. Thus, via $e_N$, $\bar{k}_\infty$ inherits a new $A$-module structure, $\phi$. By definition, for $a \in A$

$$e_N(az) = \phi(a)(e_N(z)).$$

Now,

$$e_N(az) \text{ and } ae_N(z) \prod_{\substack{\alpha \in a^{-1}N/N \\ \alpha \neq 0}} (1 - e_N(z)/e_N(\alpha))$$

have the same divisors and derivative. As the analysis is nonarchimedean, they are identical. Therefore,

$$\phi(a)(z) = az \prod_{\substack{\alpha \in a^{-1}N/N \\ \alpha \neq 0}} (1 - z/e_N(\alpha)).$$

The rank of $\phi$ is now seen to be $d$. As $N$ is a $K$-lattice, $(\phi)$ is an elliptic module defined over $K$, i.e., has coefficients in $K$.

Similarly, a morphism of lattices gives rise to a morphism of elliptic modules. To go the other way, let $\phi$ be an elliptic module defined over $K$. From the above one knows that the associated lattice function, $e_N$, (if it exists) is of the form

$$e_N(z) = z + \sum c_i z^{q^i}.$$

Let $a \in A$ be a non-unit. From $e_N(az) = \phi(a)(e_N(z))$, one finds the $c_i$ by induction. From this $e_N$ is seen to be entire. One then shows the difficult fact that

$$e_N(az) = \phi(a)(e_N(z))$$

for *all* $a \in A$. Let $N$ be the kernel of $e_N$. It is now not too difficult to see $N$ is a $K$-lattice of rank $d$.

A similar argument holds for morphisms of elliptic modules. □

COROLLARY 1.25: *If $N$ corresponds to $(\phi)$ then $\alpha N$, for $\alpha \in \bar{k}_\infty^*$, corresponds to $(\alpha\phi\alpha^{-1})$.* □

PROPOSITION 1.26: *There is a one to one correspondence between the following sets:*

(1) *isomorphism classes $(E, \omega)$, where $E = (\phi)$ is a $K$-elliptic module of rank $d$ and $\omega \in K^*$. ($\omega$ is a nowhere-zero section of $\mathbb{A}^1$ over $K$).*

(2) *the set of $K$-lattices of rank $d$.*

*This correspondence is given by $(\phi)$ goes to the lattice of $(\omega^{-1}\phi\omega)$.* □

The next result we prove for later use.

PROPOSITION 1.27: *The function $e_N$ gives an isomorphism $N\backslash\mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$ as rigid analytic spaces.*

PROOF: As $N$ acts by translations, only a finite group stabilizes any bounded disc around the origin. The quotient space is defined by dividing each disc by its stabilizer and patching. As $N$ is fixed point free the quotient space is smooth.

We know $e_N'$ is identically 1. Thus, $e_N$ is everywhere étale. By using Newton polygons, one can see that any bounded disc in the image of $e_N$ is covered by some other bounded disc. Thus the result follows from the main theorem of [13]. □

## *Group actions*

DEFINITION 1.28, [2]: We define $M^d = \varprojlim_{\substack{\text{all } I \text{ with} \\ \#(V(I))>1}} M_I^d$.

Let $A^f = \hat{A} \otimes k$ be the finite adeles.

PROPOSITION 1.29, [2]: (1) *There is a left action of $GL(d, A^f)/k^*$ on $M^d$.*

(2) *Let $U_I$ be the kernel of the map $GL(d, A) \to GL(d, \hat{A}/I)$. Then $U_I \backslash M^d \simeq M_I^d$.*

PROOF: (1) The scheme $M^d$ represents elliptic modules, $E = (L, \phi)$, or rank $d$ with a map $\psi: (k/A)^d \to \Gamma(L)$ such that $\psi$ is a level structure when restricted to $(I^{-1}/A)^d$ for all $I$ with $\#(V(I)) > 1$. The elements in $GL(d, \hat{A})$ just act on the level structure.

Now let $g \in GL(d, A^f)$ have coefficients in $\hat{A}$. As a map on

$$(k/A)^d = \bigoplus_v (k_v/A_v)^d,$$

its kernel is finite. As in the analytic case, but taking into account multiplicities, we can divide the universal module by the image of this subgroup under $\psi$. As the elements in $A$ act trivially we get the associated action of all of $GL(d, A^f)/d^*$.

(2) This is a consequence of the normality of the schemes $M_I^d$, 1.21.3 and Zariski's Main Theorem. □

DEFINITION 1.30: We let $M_d = GL(d, \hat{A}) \backslash M^d$.

PROPOSITION 1.31: *The scheme $M_d$ is flat and of finite type over $A$. It is normal and has normal generic fiber.* □

EXAMPLE 1.32: Let $A = \mathbf{F}_q[T]$ on $d = 2$. We define

$$j = \phi_1(T)^{q+1}/\phi_2(T).$$

Then $M_2 \simeq \operatorname{Spec}(A[j])$. Indeed the map $M_2 \to \operatorname{Spec}(A[j])$ can be seen to be proper by the valuative criterion. Further, up to isomorphism $j$ describes the module. Indeed, if $j \neq 0$ we can set $\phi_2(T) = 1$. The map is then seen to be birational. The result follows from Zariski's Main Theorem.

*Analytic description of the moduli spaces*

DEFINITION 1.33 ($\Omega^d$ is defined in [2]): We let

$$\Omega^d = \{x \in \mathbf{P}^{d-1}(\bar{k}_\infty) \mid x \text{ is not contained in any hyperplane defined over } k_\infty\}.$$

$$W^d = \{x \in \mathbb{A}^d(\bar{k}_\infty) \mid x \text{ is not contained in any hyperplane defined over } k_\infty\}.$$

Both spaces have a natural $GL(d, K_\infty)$ action on them.

Let $\{C_i\}$, $i = 1, \ldots, d$, be fractional ideals over $A$ and $Y = \bigoplus_i C_i$.

PROPOSITION 1.34: *If* $x = (x_1, \ldots, x_d) \in W^d$, *then* $N_x = \Sigma_i x_i C_i$ *is a lattice. Conversely, any lattice of type* $Y$ *arises in this fashion.* □

Drinfeld shows the following theorem for $\Omega^d$. The extension to $W^d$ is immediate.

THEOREM 1.35, [2]: (1) *The spaces* $\Omega^d$, $W^d$ *are admissible (in the sense of* [14], [15]) *open subsets of* $\mathbb{P}^{d-1}$, $\mathbb{A}^d$ *respectively. They are smooth.*

(2) *The right action of* $GL(d, k_\infty)$ *on these spaces is consistent with this rigid structure.*

(3) *If* $\Gamma \subseteq GL(d, k_\infty)$ *is discrete, then the quotients* $\Gamma\backslash\Omega^d$, $\Gamma\backslash W^d$ *exist as rigid analytic spaces.*

(4) *The rigid space* $\Omega^2$ *is geometrically connected.* □

Note that the map $W^d \to \Omega^d$ is the restriction of an algebraic map and so is rigid.

DEFINITION 1.36, [2]: Consider $GL(d, A^f)$ as a discrete set. Then we set

$$\tilde{\Omega}^d = GL(d, A^f) \times \Omega^d / GL(d, k)$$
$$\tilde{W}^d = GL(d, A^f) \times W^d / GL(d, k).$$

Let $\mathrm{Pic}^d(A)$ be the isomorphism classes of projective rank $d$ $A$-modules.

PROPOSITION 1.37: (1) *We have an isomorphism*

$$U_I\backslash\tilde{\Omega}^d \simeq \bigcup_{Y \in \mathrm{Pic}^d(A)} (\Omega^d \times GL(Y/IY))/GL(Y)$$

(2) *We have an isomorphism*

$$U_I\backslash\tilde{W}^d \simeq \bigcup_{Y \in \mathrm{Pic}^d(A)} (W^d \times GL(Y/IY))/GL(Y).$$

PROOF: Standard, (see [6]). □

DEFINITION 1.38: We call the images of $\Omega^d$, $W^d$, in the above decompositions, the components. Any function on a component is called a component function.

REMARKS 1.38: (1) Let $Y = \bigoplus_{i=1}^{d} C_i$ and $x \in W^d$ as in proposition 1.34. Via the construction $(Y, x) \mapsto N_x$, the right action of $GL(Y)$ is $(x, g) \mapsto x \cdot (g^t)^{-1}$. A similar statement is true for $\Omega^d$. Let $\tilde{Y}$ be the $A$-dual of $Y$. Notice that $g^t \in GL(\tilde{Y})$.

(2) One can check that $U_I$ has fixed-point free action. Thus the map

$$\tilde{\Omega}^d \to U_I \backslash \tilde{\Omega}^d$$

is étale. Consequently, $U_I \backslash \tilde{\Omega}^d$ is smooth.

Let $X$ by any $k_\infty$ scheme. If $X$ is quasi-projective, then $X$ has a natural rigid analytic structure. We denote this by $X_{an}$.

The above discussion and the analytic theory of elliptic modules makes it clear that there are maps

$$U_I \backslash \tilde{\Omega}^d \to (M_I^d \otimes k_\infty)_{an}$$

$$U_I \backslash \tilde{W}^d \to (\mathrm{Spec}(H_I^d) \otimes k_\infty)_{an}.$$

Drinfeld shows the following theorem for $M_I^d$. The general proof is similar.

THEOREM 1.39, [2]: *The above maps are isomorphisms of rigid analytic spaces.* □

COROLLARY 1.40: *Let $F$ be a modular form of rank $d$, level $I$, weight $j$ defined over $K$, with $[K:k_\infty] < \infty$. Then $F$ gives rise to a rigid analytic function $f$ on $U_I \backslash \tilde{W}^d$ defined over $K$. On each component we have $f(cx) = c^{-j} f(x)$ for all $x \in W^d$ and $c \in \bar{k}_\infty^*$.*

PROOF: This follows from the above and proposition 1.26. □

REMARK 1.41: Since the map $M_I^d \otimes k \to M_d \otimes k$ is finite Galois, Theorem 1.39, Corollary 1.40, and 1.43 descend to the case of the full modular group $GL(d, \hat{A})$.

DEFINITION 1.42: We let $\Gamma_Y = GL(Y)$ and

$$\Gamma_Y(I) = \ker(\Gamma_Y \to GL(Y/IY)).$$

From now on we view $\Omega^d$ as sitting in $\mathbb{A}^{d-1}$. In the usual fashion, the $f$ of Corollary 1.40 restricts to a function on $\Omega^d$. Indeed, we have the following description of $f$.

PROPOSITION 1.43: *Let $f$ be as in Corollary 1.40. We restrict $f$ to $\Omega^d$ by sending $x \in \Omega^d$ to $(x, 1) \in W^d$. Let $g \in \Gamma_Y(I)$ and let $g^t = \begin{pmatrix} G & b \\ c & d \end{pmatrix}$. We fix $G$ to be the upper left $(d-1)\times(d-1)$ minor and $d$ to be a scalar. Finally, using dot product notation, we have*

$$f(x \cdot g^t) = f\left(\frac{xG + b^t}{x \cdot c^t + d}\right)(x \cdot c^t + d)^{-j}. \quad \square$$

REMARK 1.44: Any rigid function satisfying the above transformation property is called an analytic modular form of level I, weight $j$, type $Y$. In order to see when such an $f$ comes from an algebraic $F$ we need cuspidal conditions. These we will give in the rank two cases in 1.79.

*$M_I^1$ and Tate uniformization*

We describe here the analog of the $p$-adic elliptic curve theory of Tate. Along the way, we describe the scheme $M^1$.

Let $R$ be a fixed complete d.v.r. over $A$, $(\pi)$ its maximal ideal and $K$ the fraction field. Let $E = (\phi)$ be an elliptic module over $k$ of rank $d$.

DEFINITION 1.45, [2]: We say $E$ has stable reduction if there exists $c \in K$ such that the following holds:

(1) the module $(c\phi c^{-1})$ has coefficients in $R$,

(2) the reduction modulo $(\pi)$ is an elliptic module of rank $\leq d$.

We say $E$ has good reduction if the rank remains constant.

REMARK 1.46: Let $\#(V(I)) > 1$. One can show that every module with level $I$ structure has stable reduction. If we work over $A[I^{-1}]$, then the result is true for any $I$. Therefore, in the rank one case, a level structure implies good reduction.

We can now sketch the proof of the following basic theorem.

THEOREM 1.47, ([2], [4]): *The ring $M^1$ is the ring of integers of the maximal abelian extension of $k$ split totally at $\infty$. The action of the ideles is that of class field theory.*

PROOF: We know from 1.46 that the map $M_I^1 \to \mathrm{Spec}(A)$ is proper.

It is therefore integral. The scheme $M_I^1$ is normal, by 1.21. At each finite place, by the explicit group action, one can identify the Frobenius. From this and the analytic theory we see $M^1$ is connected and splits totally at $\infty$. Finally, as the map is unramified outside of $V(I)$. Class Field Theory finishes the proof. □

REMARKS 1.48: (1) There is a subtlety in the construction that needs to be understood. The universal module over $M_I^1$ does not have the smallest field of definition possible for a rank one elliptic module. It has merely the smallest field of definition for a rank one module with level $I$ structure. If $(I, J) = 1$, the modules over $M_I^1$ and $M_J^1$ are different. This is in distinction to the classical case of cyclotomic fields.

(2) In a similar fashion, one can show the existence of a map $M_I^2 \rightarrow M_I^1$ with geometrically connected fibers.

DEFINITION 1.49, [2]: A $\phi$-lattice $N$ over $K$ is a finitely generated projective $A$ submodule of $K^s$, ($A$-action via $\phi$), so that the following hold:

(1) the group $N$ is $\mathrm{Gal}(K^s/K)$ stable,

(2) in any ball there are only finitely elements of $N$.

The rank of $N$ is its rank as an $A$-module.

PROPOSITION 1.50, [2]: *The isomorphism classes of $K$-elliptic modules of rank $d + d_1$ with the stable reduction are in one to one correspondence with the isomorphism classes $((\phi), N)$, where $(\phi)$ is a rank $d$ $K$-elliptic module with good reduction and $N$ a rank $d_1$ $\phi$-lattice.*

PROOF: Given $((\phi), N)$ we form the function

$$e_N(z) = z \prod_{\substack{\alpha \in N \\ \alpha \neq 0}} (1 - z/\alpha).$$

As before, we construct an elliptic module. By using division points, we see the rank is $d + d_1$.

The reverse direction is accomplished by formally showing the existence of $\phi$ and $e_N$. Then $e_N$ is shown to be entire and we let $N$ be its kernel. □

Let $M$ be a prime dividing $I$ and $J$ an arbitrary ideal. Over $M_I^1[M^{-1}]$, we have the universal module $(\phi)$ with trivial bundle. For any ring $R$, let $R((q))$ be the finite-tailed Laurent series. It will be always obvious whether $q$ is a number or function.

DEFINITION 1.51: Let $T(I, J, M)$ be the elliptic module over $(M_I^1 \otimes k)((q))$ associated to $((\phi), \phi(IJ)(1/q))$ by the proposition.

These modules are called the Tate objects. The reader will find it helpful to keep the corresponding construction for elliptic curves in mind.

PROPOSITION 1.52: *The module, $T(I, J, M)$, may be extended to a family over $M_I^1[M^{-1}]((q))$. It has nowhere-zero section* "1" *and a natural level I structure $\psi$.*

PROOF: The fact that $T(I, J, M)$ may be extended to such a family follows from the universality of proposition 1.50. The existence of $\psi$ follows from the fact $(\phi)$ has level structure and that our lattice is of the form $\phi(I)(\phi(J)(1/q))$. $\square$

REMARKS 1.53: (1) To give the equations for $T(I, J, M)$ amounts to computing various $q$-expansions of Eisenstein series. We shall describe this in 2.16.

(2) For the case of no level, one has similar constructions generically. The problem is to find elliptic modules with small fields of definition. However, in the case of $\mathbb{F}_q[T] = A$ we can be very specific. Let $\phi_\zeta$, for $\zeta \in \mathbb{F}_q^*$, be the module given by $\phi_\zeta(T) = TF^0 + \zeta F$. Then, $\phi_\zeta$ defines a family over $A$. If $J$ is any ideal, we can form the Tate object, $T(\zeta, J)$, associated to $(\phi_\zeta)$ and $J$. This object will have coefficients in $\mathbb{F}_q[T]((q))$ and nowhere-zero section "1".

Let $F$ be an arbitrary modular form of level $I$, weight $j$ defined over a ring $H$.

DEFINITION 1.54: We call $F(T(I, J, M), \text{"1"}, \psi) \in H \otimes (M_I^1[M^{-1}]((q))$, the $q$-expansion of $F$ at the "cusp" $(I, J, M)$. We say $F$ is holomorphic at the cusp if the expansion contains no negative terms. We say $F$ is holomorphic, if it is holomorphic at each cusp.

Via the Tate objects, one can compactify the modular curves to get a scheme $\bar{M}_I^2$ proper over spec$(A)$ with the same regularity conditions as $M_I^2$. At each cusp we can choose a section of the universal bundle, $L_U$, with nonzero reduction. Indeed, by the Tate construction of the cusp we need only take a nonzero section of level $M$ for $(\phi)$. Thus, we can extend $L_U$ to $\overline{L_U}$ over $\bar{M}_I^2$. A holomorphic $F$ is the same as a section of $(\bar{L}_U)^{-\otimes j}$.

DEFINITION 1.55: We set $\omega = (\bar{L}_U)^{-1}$.

PROPOSITION 1.56: *The line bundle $\omega$ has positive degree.*

PROOF: The reciprical of any nontrivial section of level $I$ extends to a section of $\omega$. $\square$

COROLLARY 1.57: *These are no everywhere holomorphic forms of negative degree.* $\square$

COROLLARY 1.58: *The graded ring of holomorphic forms of level I, over A, is finitely generated.* $\square$

*Analytic description of the cusps*

By the Tate theory, the cusps are sections over $M_I^1[I^{-1}]$. Thus, they are rational over $k_\infty$. We shall show in 1.78 that analytically these correspond to equivalence classes of $\mathbb{P}^1(k)$ under various groups.

THEOREM 1.59: *Let D be a Dedekind domain, K its field of fractions and Y a rank $d \geq 2$ projective D-module. Then,*

$$K^*\backslash K \otimes Y/GL(Y)$$

*is in one to one correspondence with the ideal class group of D.*

PROOF: Let $Y = \bigoplus_{i=1}^d C_i$ and $\tilde{Y} = \bigoplus_{i=1}^d C_i^{-1}$ be the dual module. Any $x = (x_1, \ldots, x_d) \in K^d$ gives rise to a map $\tilde{x}$: $\tilde{Y} \to K$, by

$$(c_1, \ldots, c_d) \to \sum x_i c_i.$$

As before, the left action of $GL(\tilde{Y})$ on $\tilde{x}$ gives rise to a right action of $GL(Y)$ on $K^d$.

For $x \in \mathbb{P}^{d-1}(K)$, let $[x_1, \ldots, x_d]$ be homogeneous coordinates. We map $x$ to the ideal class of

$$\sum x_i C_i^{-1}.$$

By the above, this class is invariant of the $GL(Y)$ action. Since any ideal can be generated by at most two elements, it is easy to see this map is surjective.

Now, if $x_1, x_2 \in \mathbb{P}^{d-1}(K)$ map to the same class, we may assume they map to the same ideal $H$. As $H$ is projective, the map splits, i.e.

$$\tilde{Y} = Y_i \oplus H \quad i = 1, 2.$$

But, then $Y_1 \simeq Y_2$ by the theory of projective modules. Thus $x_1$ is equivalent to $x_2$ under $GL(Y)$. □

Now let $Y = C_1 \oplus C_2$ be of rank 2. We have $\mathrm{End}(Y)$ is matrices

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

with $a, d \in D$, $b \in C_2^{-1}C_1$ and $c \in C_1^{-1}C_2$. The group $GL(Y)$ consists of those elements of invertible determinant.

For any ideal $I$ of $D$, we let $U(I)$ be the image of the units of $D$ in $D/I$. Let "$-$" denote reduction mod $I$.

Proposition 1.60: *Let $x = [x_1, x_2]$, $w = [w_1, w_2] \in \mathbb{P}^1(K)$. Then $x$ is equivalent to $w$ under $\Gamma_Y(I)$ iff the following hold:*

(1) *the fractional ideals $x_1C_1^{-1} + x_2C_2^{-1}$ and $w_1C_1^{-1} + w_2C_2^{-1}$ are isomorphic,*

(2) *by using scalars, fix $(x_1, x_2)$, $(w_1, w_2)$ so that (1) is an equality. Let $H$ be this fractional ideal. Then there is a $u \in U(I)$ so that $(\bar{x}_1, \bar{x}_2)$ and $u(\bar{w}_1, \bar{w}_2)$ are equal as functions from $\tilde{Y}/I\tilde{Y} \to H/IH$.*

Proof: By the theory of projective modules, the image of the map $GL(Y) \to GL(Y/IY)$ consists of all those elements whose determinant is in $U(I)$. Let $S$ be this subgroup.

We can assume our equivalence class under $GL(Y)$ contains $\infty = [1, 0]$. Let $\Gamma_\infty$ be its stabilizer. Thus,

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \,\middle|\, a, c \text{ are units and } b \in C_2^{-1}C_1 \right\}.$$

Then, $\Gamma_\infty \backslash GL(Y)/\Gamma_Y(I) = \bar{\Gamma}_\infty \backslash S$. This translates to the theorem. □

Example 1.61: Let $D = A = \mathbb{F}[T]$. We pick $(x_1, x_2)$, $(w_1, w_2) \in A \oplus A$ so that $x_1$ is prime to $x_2$ and $w_1$ is prime to $w_2$. Then $x = [x_1, x_2]$ is equivalent to $w = [w_1, w_2]$ iff

$$(x_1, x_2) \equiv \zeta(w_1, w_2) (\mathrm{mod}\ I),$$

with $\zeta \in \mathbb{F}_q^*$.

*Some geometry*

For $x \in \mathbb{A}^{d-1}(\bar{k}_\infty)$, let $\|x\|$ be its max norm under the canonical extension of $|\ |_\infty$. Let $B^d$ be the boundary of $\Omega^d$.

DEFINITION 1.62: Let $x \in \Omega^d$ be a geometric point. We define

$$d(x) = \operatorname*{Min}_{b \in B^d}\{\|x - b\|\}.$$

LEMMA 1.63: *Suppose $\|x - y\| < d(x)$, then $d(x) = d(y)$.* □

PROPOSITION 1.64: *Let $x = (x_1, \ldots, x_{d-1}) \in \Omega^d$ and $c \in \mathbb{R}$. Then we have $d(x) \geq c$ iff for all $z = (z_1, \ldots, z_d)$ in $k_\infty^d$*

$$\left|\sum x_i z_i\right|_\infty \geq c \operatorname*{Max}_{1 \leq i \leq d-1}\{|z_i|_\infty\}.$$

PROOF: Suppose $d(x) \geq c$. Pick $z$ with $z_j \neq 0$ for some $j < d$ and let for $i < d$, $|z_i|_\infty$ be the maximum. We put

$$b_j = z_j/z_i \quad \text{for all } j.$$

Define $r = (r_1, \ldots, r_{d-1})$ by $r_n = x_n$, $n \neq i$, and

$$r_i = -(b_1 x_1 + \cdots + \underbrace{b_i x_i}_{\text{skip}} + \cdots + b_d).$$

Now, $r$ satisfies the equation with $(b_1, \ldots, 1, \ldots, b_d)$ and so belongs to $B^d$. Further $\|x - r\| = |x_i - r_i|_\infty$. As $d(x) \geq c$, $|x_i - r_i|_\infty \geq c$. To finish multiply by $|z_i|_\infty$.

For the converse, let $r = (r_1, \ldots, r_{d-1}) \in B^d$. Then $r$ satisfies an equation of the form

$$0 = b_1 r_1 + \cdots + b_d, \quad b_i \in k_\infty.$$

Thus $|b_1(x_1 - r) + \cdots + b_{d-1}(x_{d-1} - r_{d-1})|_\infty = |b_1 x_1 + \cdots + b_d|_\infty$. By hypothesis, the last number is $\geq c\,\mathrm{Max}_{1<i<d-1}\{|b_i|_\infty\}$. But the first is $\leq \mathrm{Max}_{1 \leq i \leq d-1}\{|b_i|_\infty |x_i - r_i|_\infty\}$. Thus, for some $i$, $|x_i - r_i|_\infty \geq c$. □

Put $q_1 = \#(\mathcal{O}_\infty/(\pi_\infty))$, where $\mathcal{O}_\infty$ is the maximal compact at infinity and $(\pi_\infty)$ its maximal ideal.

PROPOSITION 1.65: *Let $\{c_i\}$ be a collection of rational numbers,*

*$c_{i+1} > c_i$ and $c_i \to \infty$ as $i \to \infty$. Let*

$$U_i = \{x \mid d(x) \geq q_1^{-c_i}\}.$$

*Then the cover of $\Omega^d$ by $\{U_i\}$ is admissible.*

PROOF: To see that the subsets are admissible, we may assume we are in the unit polydisc. Let $1 \geq q_i^{-v_i} \geq q_1^{-n}$. Further, let $R$ be a finite $q_1^{-n}$ net for the compact subset $H$ of $(k_\infty)^d$ given by

$$\{z \mid |z_i|_\infty \leq 1 \text{ all } i, \text{ with equality for some } j < d\}.$$

For each $r \in H$, define a function

$$f_r(x) = \sum_{i=1}^{d-1} r_i x_i + r_d.$$

By proposition 1.64 and the fact $\|x\| \leq 1$, we see that $U_i \cap \{\text{unit polydisc}\}$ is the inverse image of the annulus of smaller radius $q_1^{-c_i}$ under $\Pi_{r \in R} f_r$.

To see the cover is admissible, let $B$ be any Tate algebra and $\varphi\colon \operatorname{Max}(B) \to \Omega^d$ a morphism. We need to see that the image of $\varphi$ is contained in $U_i$ for some $i$. Again, we may suppose that the image is contained in the unit polydisc. For $r \in H$, let

$$g(r) = \operatorname*{Min}_{b \in \operatorname{Max}(B)} \{|f_r(\varphi(b))|_\infty\}.$$

Thus, $g$ is continuous, never zero on a compact set and so is bounded below. □

In a similar fashion, one can show the following propositions.

PROPOSITION 1.66: *For each $x$ in $\Omega^d$ there is a $b \in B^d$ with $\|x - b\| = d(x)$.* □

PROPOSITION 1.67: *The set $\{x \mid d(x) = q_1^c\}$, $c \in Q$, is an admissible open of $\Omega^d$.* □

The group $k_\infty^*$ acts on the above subsets by $d(ax) = |a|_\infty d(x)$. Whether any two distinct equivalence classes are isomorphic is unknown.

*Stabilizers of cusps*

Let $Y = C_1 \oplus C_2$ be a rank two projective $A$-module.

LEMMA 1.68: *Let $\alpha \in \Gamma_Y(I)$, then* $\det(\alpha) = 1$.

PROOF: One knows $\det(\alpha)$ is a unit in $A$. As the units inject into $A/I$, for any $I \neq A$, we see $\det(\alpha) = 1$. □

Let $b \in \mathbb{P}^1(k)$. Pick $\rho_b \in SL(2, k)$ so that $\rho_b(b) = \infty$.

PROPOSITION 1.69: *Let $\Gamma_b \subseteq \Gamma_Y(I)$ be the stabilizer of $b$. Then, $\rho_0 \Gamma_b \rho_b^{-1}$ is the group of translations by some fractional ideal $C_b$.*

PROOF: Let $g \in \rho_b \Gamma_b \rho_b^{-1}$. One has $g(\infty) = \infty$ and so $g$ has eigenvalues in $A$. As these eigenvalues are units they must be 1, and so our group acts as translations by $C_b$. One then checks that $C_b$ is a fractional ideal. □

REMARK 1.70: By the techniques of the proof of theorem 1.59 it is not difficult to find the isomorphism class of $C_b$. Further, in a natural way, $C_b = IJ_b$, where $J_b$ depends only on $b$ and $\rho_b$.

PROPOSITION 1.71: *Let $\alpha = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2, k_\infty)$ and $z \in \Omega^2$. Then, for any $y \in B^2$ with $cy + d \neq 0$,*

$$|\det(\alpha)|_\infty |z - y|_\infty = |\alpha(z) - \alpha(y)|_\infty |cz + d|_\infty |cy + d|_\infty.$$

PROOF: We have,

$$\begin{pmatrix} z & 1 \\ y & 1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} cz + d & 0 \\ 0 & cy + d \end{pmatrix} \begin{pmatrix} \alpha(z) & 1 \\ \alpha(y) & 1 \end{pmatrix}.$$

Take determinants and norms. □

PROPOSITION 1.72: *Let $Y = C_1 \oplus C_2$ be a rank 2 projective module and let*

$$\alpha = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(Y)$$

*with $c \neq 0$. Then there is a number $r \in \mathbb{R}$ independent of $\alpha$ so that $d(z) > r$ implies $d(\alpha(z)) < r$.*

PROOF: By standard arguments, $k_\infty/C$ is compact for any fractional ideal $C$. Further, there is a number $m \in \mathbb{R}$ so that for all $z \in k_\infty$ there exists $a \in C$ with $|z-a|_\infty < m$.

Now choose $m$ for the case $C = A$ and let $d(z) > \operatorname{Max}\{m, 2\}$. Pick $y' \in A$ so that $|z-y'|_\infty = d(z)$ and $cy'+d \neq 0$. By proposition 1.71,

$$d(z) \geq |\alpha(z)-\alpha(y')|_\infty |cy'+d|_\infty |cz+d|_\infty.$$

But,

$$|cz+d|_\infty = |c|_\infty |z+d/c|_\infty \geq |c|_\infty d(z).$$

Therefore,

$$|c|_\infty^{-1} \geq |\alpha(z)-\alpha(y')|_\infty |cy'+d|_\infty.$$

Finally $|cy'+d|_\infty$ is bounded below. The proposition now follows easily. □

COROLLARY 1.73: *There is an $i$ so that the stabilizer of $U_i$ (as defined in proposition 1.65) under $\rho_b \Gamma_b \rho_b^{-1}$ is the group of translations by $C_b$.* □

DEFINITION 1.74: We define

$$e_b(z) = z \prod_{\substack{\alpha \in C_b \\ \alpha \neq 0}} (1 - z/\alpha)$$

and $q_b = 1/e_b$.

LEMMA 1.75: (1) *By logarithmic differentiation, $q_b(z) = \Sigma_{\alpha \in C_b} (z+\alpha)^{-1}$.*
(2) *As $d(z) \to \infty$, $q_b \to 0$ uniformly.* □

We can now prove the following basic result.

THEOREM 1.76: *At the* cusp $b$, *$q_b$ is an analytic uniformizer.*

PROOF: Since $k_\infty/C_b$ is compact, $e_b$ is bounded above on $k_\infty$. Consequently, $q_b$ has a punctured disc around the origin in its image.

Now choose $m$ for $C_b$ as in the proof of 1.72. We want to see that for all $N \gg 0$, the image of $d(z) > N$ contains a punctured disc around the origin. To see this let $N > m$ and let $d(z) < N$. Then there is a $y \in C_b$ so that $|z - y|_\infty < N$. As $e_b$ is invariant under translations by $C_B$ and as it is entire, the image of $d(z) < N$ is bounded. The statement for $q_b$ follows.

By 1.73 we have the existence of an $N_1$ so that only translations stabilize $d(z) > N_1$. As we know from 1.27 that $C_b \backslash \mathbb{A}^1 \simeq \mathbb{A}^1$ via $e_b$, the result follows. $\square$

Let $f$ be an analytic modular form of level $I$, weight $j$, type $Y$.
If

$$\rho_b^{-1} = \frac{az + b}{cz + d},$$

then

$$f(\rho_b^{-1}(z))/(cz + d)^j$$

determines an analytic germ $\Sigma_{n=-\infty}^{\infty} a_n q_\infty^n$ by the theorem.

DEFINITION 1.77: We call the series $\Sigma_{n=-\infty}^{\infty} a_n q_\infty^n$ the analytic $q$-expansion at $b$ of $f$.

As $\Omega^2$ is connected, a component modular form is determined by its expansion at one cusp.

*Connection with the algebraic definition*

Let $\beta \in \bar{k}_\infty^*$. We call the function, ${}_\beta q_b = q_{b/\beta}$, the $\beta$-normalized uniformizer. Similarly, we call $\Sigma\, a_n \beta^n {}_\beta q_b^n$, the $\beta$-normalized expansion of $f$.

Now choose $\beta$ so that $N_b = \beta C_b$ gives rise to the universal module $(\phi)$ of level $I$ defined over $M_I^1[M^{-1}]$, as in 1.51. Twisting by $\rho_b$ takes $C_1 z + C_2$, $z \in \Omega^2$, into lattices of the form $\bar{C}_1 z + \bar{C}_2$. Thus, $C_b = I\bar{C}_1^{-1}\bar{C}_2$. Finally, via $e_{N_b}$, the family of modules,

$$\mathbb{A}^1/(\beta \bar{C}_1 z + \bar{C}_2),$$

can be considered over $k_\infty(({}_\beta q_b))$.

PROPOSITION 1.78: *The above family has coefficients in $M_I^1[M^{-1}](({}_\beta q_b))$. Via the map $q \mapsto {}_\beta q_b$, this family is isomorphic to a Tate object $T(I, J, M)$ for some $J$. Further, any Tate object arises in this fashion.*

PROOF: The only difficult part is the last. This however, follows from 1.59. $\square$

By G.A.G.A. we now have the following basic theorem.

THEOREM 1.79: (1) *An analytically defined modular form $f$ of weight $j$ level $I$ comes from an algebraic $F$ iff at each cusp the expansion is finite-tailed. In this case at the cusp $b = (I, J, M)$, the $\beta$-normalized expansion of $f$, as above, and the $q$-expansion of $F$, via Tate objects, are the same.*

(2) *If at the cusps there are no negative terms, then $f$ comes from a section of $\omega^{\otimes j}$.*

(3) *The spaces of analytic modular forms, holomorphic at the cusps, are finite dimensional.* $\square$

Our purpose here is to characterize the forms of weight 2, level $I$.

THEOREM 1.80: *Let $\Omega^1$ be the bundle of 1-forms on $M_I^2 \otimes k$. Then,*

$$\omega^{\otimes 2} \simeq \Omega^1 \text{ (2 cusps).}$$

PROOF: Let $\alpha \in \Gamma_Y(I) \subseteq SL(Y)$. If $\alpha(z) = (az + b)/(cz + d)$, then an easy calculation shows that

$$dz = (cz + d)^2 d(\alpha(z)).$$

Thus, $dz$ is an equivariant nowhere zero section of $\omega^{-\otimes 2} \otimes \Omega^1$, considered as analytic sheaf.

Let $b$ be a cusp. One has $dq_b = d(e_b^{-1}) = -1 q_b^2 dz$. Thus, $dz$ extends to give the required isomorphism. $\square$

COROLLARY 1.81: *The degree of $\omega$ on $\overline{\Gamma_Y(I)\backslash\Omega^2}$ is $g + 1 - N$, where $g$ is the genus and $N$ the number of inequivalent cusps.* $\square$

REMARKS 1.82: (1) In a similar fashion one sees the derivative of a form of weight $pn$ is a form of weight $pn + 2$.

(2) Presumably there is a deformation theoretic proof of the theorem that would extend the result to $A[I^{-1}]$.

DEFINITION 1.83: Let $f$ be a holomorphic form of level $I$, weight $j$, type $Y$. We say $f$ is double-cuspidal iff at each cusp the zeroth and first term of its expansion vanish.

PROPOSITION 1.84: *The dimension of double-cusp forms of weight two, level I is the genus of* $\overline{\Gamma_Y(I)\backslash\Omega^2}$. $\square$

As one can show the existence of a map $\overline{M_I^2} \to M_I^1$ with geometrically connected fibers, all the components have the same genus.

*The full group*

For the full modular group $GL(Y)$ the theory has one important variant. At a cusp the stabilizer is of the form $\zeta z + C_b$ where $\zeta \in \mathbb{F}_q^*$.

DEFINITION 1.85: We let $e_b$ be $e_{C_b}$, $g_b = e_b^{-1}$ and $q_b = g_b^{(q-1)}$. One then sees that $q_b$ is an analytic uniformizer at $b$, etc.

## 2. Eisenstein series

Let $[K: k_\infty] < \infty$, $N$ a $K$-lattice of rank $d$ and $j \in \mathbb{N}^+$.

DEFINITION 2.1: (1) We define

$$E_d^j(N) = \sum_{\substack{\alpha \in N \\ 0 \neq \alpha}} \alpha^{-j} \in K.$$

(2) Let $\Psi$ be a level $I$ structure; $\Psi: (I^{-1}/A)^d \xrightarrow{\sim} I^{-1}N/N$. If $0 \neq x \in (I^{-1}/A)^d$ then we define

$$\underset{(x,I)}{E^j}(N) = \sum_{\substack{\alpha \in I^{-1}N \\ \alpha \equiv \Psi(x)(N)}} \alpha^{-j} \in K^s.$$

Note that if $(q-1) \nmid j$ then $E_d^j$ is identically 0. It is also obvious that $E_d^{jp^n} = (E_d^j)^{p^n}$, etc.

EXAMPLE 2.2: *Let* $d = 2$ *and* $Y = C_1 \oplus C_2$. Then for all $z \in \Omega^2$,

$$E_2^j(C_1 z + C_2) = \sum_{(0,0) \neq (c_1, c_2) \in Y} (c_1 z + c_2)^{-j}.$$

We now fix a $Y = \bigoplus_{i=1}^d C_i$. Given $x \in W^d$ we have the lattice $N_x$ of 1.34. Further, by using $I^{-1}Y/Y$, it is easy to equip the $N_x$ with continuously varying level structure. Thus, we can think of $E_d^j$, and $E^j_{(x,I)}$ as functions on $W^d$.

THEOREM 2.3: *The functions $E_d^j$, $E^j_{(x,I)}$ are rigid analytic on $W^d$.*

PROOF: We show the corresponding functions on $\Omega^d$ are rigid analytic. For this, we use the covering $\{U_i\}$ of 1.65. But, 1.64 makes it clear the series converge uniformly on $U_i$. $\square$

COROLLARY 2.4: *The functions $E_d^j$, $E^j_{(x,I)}$ are analytic modular forms for $\Gamma_Y$, $\Gamma_Y(I)$ respectively.* $\square$

*Algebraic definition of Eisenstein series*

In this section we show how the Eisenstein series arise from algebraic forms defined over $k$.

LEMMA 2.5: *Let $N$ be a lattice, and $e_N$ the corresponding function. Let $e_N(z) = z + \Sigma_{i=1}^{\infty} c_i(N) z^{q^i}$. Then $c_i$ is an algebraic form of weight $(q^i - 1)$ defined over $k$.*

PROOF: Given an elliptic module $(\phi)$ defined over a field containing $k$, we know how (Proof of 1.24) to construct the corresponding function $e_N$. The coefficients $c_i(N)$ are thus constructed via the $\phi_i(a)$ for $a \in A$. The weight follows trivially once we view $c_i$ as a lattice function. $\square$

PROPOSITION 2.6: *The functions $E_d^j$ arise from an algebraic modular form defined over $k$.*

PROOF: Let $q_N = e_N^{-1}$. By logarithmic differentiation, $q_N = \Sigma_{\alpha \in N} (z + \alpha)^{-1}$. We expand $(z + \alpha)^{-1}$ in its Laurent series about the origin by use of the geometric series. One sees the Laurent coefficients of the sum are Eisenstein series. The proposition follows by synthetic division on $q_N$ and from 2.5 upon comparing coefficients. $\square$

PROPOSITION 2.7: *Let $\Psi$ be the level structure of the universal module of rank $d$, level $I$. Let $0 \neq x \in (I^{-1}/A)^d$. Then the $k$-modular form $\Psi(x)^{-1}$ equals $E^1{}_{(x,I)}$ over $k_\infty$. The form $E^1{}_{(x,I)}$ is never-zero on $W^d$.*

PROOF: Let $N$ be a lattice and $0 \neq a \in I^{-1}N/N$. Note that $e_N(a)$ is a point of order $I$. We have

$$0 \neq \frac{1}{e_N(a)} = \sum_{\alpha \in N} (a + \alpha)^{-1}. \quad \square$$

PROPOSITION 2.8: *The forms* $E^j_{(x,I)}$ *arise from algebraic forms defined over* $k$.

PROOF: Let $a, N$ be as above. Then

$$\frac{1}{e_N(z)+e_N(a)} = \sum_{\alpha\in N} (z+(a+\alpha))^{-1}.$$

Further,

$$(e_N(z)+e_N(a))^{-1} = e_N(a)^{-1}\left(1+\frac{e_N(z)}{e_N(a)}\right)^{-1}$$
$$= e_N(a)^{-1} - \frac{e_N(z)}{e_N(a)^2} + \cdots.$$

Upon comparing Taylor coefficients at the origin, the result follows from 2.5 and 2.7. □

PROPOSITION 2.9: *We have* $E^j_{(x,I)} = (E^1_{(x,I)})^j$ *for* $j = p^r i$, $1 \leq i \leq q$.

PROOF: This follows from the expression for $(E^1_{(x,I)})^j$ given in the proof of 2.8. □

The next result should be viewed as a result of "zeta-values".

THEOREM 2.10: (1) *Let* $I \subseteq A$ *be a non-zero ideal. Then there is a* $\pi \in \bar{k}^*_\infty$ *so that for all i divisible by* $(q-1)$, $\pi^{-i} \sum_{0\neq\alpha\in I} \alpha^{-i}$ *is algebraic.*
(2) *Let* $x \in k$ *but not* $I$. *Then there is a* $\pi \in \bar{k}^*_\infty$ *so that for all* $i$, $\pi^{-i} \sum_{\alpha\in I} (\alpha + x)^{-i}$ *is algebraic.*

PROOF: It follows from 1.47 that there exists a $\pi$ such that $I \cdot \pi$ gives rise to an elliptic module with algebraic coefficients. The results follow from 2.6 and 2.9. □

REMARK 2.11: For $\mathbf{F}_q[T]$, 2.10.1 (and much more) was proven by Carlitz in 1935 (see [3]).

VARIANTS 2.12: Let $J \subseteq A$ be a non-zero ideal and $N$ a lattice. We define

$$E^j(J, N) = \sum_{0\neq\alpha\in JN} \alpha^{-j}.$$

As $J$ runs through $\mathrm{Pic}^1(A)$, these functions are conjugate under Galois action. Similar constructions exist for level $I$.

*Expression of the forms $\phi_i(a)$ in terms of the $E_d^j$*

Our purpose here is to show how to compute the forms $\phi_i(a)$ in terms of Eisenstein series. Thus, analytically, these series provide all the data needed to define elliptic modules.

Let $N$ be a rank $d$ lattice and $e_N(z) = \Sigma\, \alpha_i z^{q^i}$. As, $e'_N(z)$ is identically 1, $e_N$ has a composition inverse $\log_N(z) = z + \Sigma_{i\geq 1}\, \beta_i z^{q^i}$. Set $\beta_0 = 1$. Let $z/e_N(z) = \Sigma\, \gamma_j z^j$.

The following lemma is due to Carlitz, (see [3], 2.3.4).

LEMMA 2.13: *If $j = q^h - q^i$, then $\gamma_j = \beta_{h-i}^{q^i}$.* □

Now let $(\phi)$ be the module associated to $N$ and $a \in A$. Further, let $\phi(a) = aF^0 + \Sigma_{i=1}^m\, \phi_i(a)F^i$.

LEMMA 2.14: *For $a \in A$, $a\ \log_N(z) = \log_N(\phi(a)(z))$.*

PROOF: We know $e_N(az) = \phi(a)(e_N(z))$. Thus $az = \log_N(\phi(a)e_N(z))$, and so, a $\log_N(z) = \log_N(\phi(a)(z))$. □

PROPOSITION 2.15: *For $a \in A$,*

$$(a - a^{q^n})\beta_n = \sum_{\substack{i+j=n \\ i\geq 1}} \beta_j \phi_i(a)^{q^j}.$$

PROOF: We use 2.14 to find the $\beta_i$ in terms of the $\phi_i(a)$. □

THEOREM 2.16: *The forms $\phi_i(a)$ are polynomials, with coefficients in $k$, in the $E_d^j$.*

PROOF: This follows by induction from 2.13, 2.15 and the fact the $\gamma_j$ are Eisenstein series.

## *$q$-expansions* $(d = 2)$

We present here the $q$-expansions for the rank 2 Eisenstein series. Our purpose is to give the calculation and then the analysis of the coefficients. The "additive harmonic analysis" to be used here is based on forms of Newton's rules for expanding power sums of roots of polynomials and the next proposition.

PROPOSITION 2.17: *Let $N$ be a $K$-lattice and $h \in \mathbb{N}^+$. Further let $\zeta_h(z) = \Sigma_{\alpha \in N} (z+\alpha)^{-b}$. Then there is a monic polynomial $P_h \in K[x]$, of degree $h$, so that $P_h(q_N(z)) = \zeta_h(z)$. Further, $P_h(0) = 0$ and its coefficients are $F_q$-polynomials in the Taylor coefficients of $e_N$.*

PROOF: Let $w$ be another indeterminate. As a function of $w$, $e_N(w) - e_N(z) = e_N(w-z) = (w-z)\Pi(1-(w-z)/\alpha)$. Taking logarithmic derivatives w.r.t. $w$, we find

$$\sum (w-(z+\alpha))^{-1} = (e_N(w) - e_N(z))^{-1} = -q_N(1-e_N(w)q_N)^{-1}.$$

Now, expand both sides about $w = 0$ and compare coefficients. □

COROLLARY OF PROOF 2.18: *If $N$ gives rise to a module defined over a field $F$, then $P_h \in F[x]$.* □

Let $Y = C_1 \oplus C_2$ and $y = (y_0, y_1) \in Y$. Further, we let $I$ be a non-zero ideal of $A$ which may be $A$ itself. For now, put

$$E(z) = \sum_{\substack{(c_1,c_2)\in Y-\{0\} \\ (c_1,c_2)\equiv y(IY)}} (c_1 z + c_2)^{-j}.$$

Finally, let $q_\infty(z) = \Sigma_{\alpha \in IC_2C_1^{-1}} (z+\alpha)^{-1}$. In the next theorem, we give the $q_\infty$-computation for $E(z)$. As $Y$ and $I$ may vary, we see this computation is completely general.

THEOREM 2.19: *The Eisenstein series are holomorphic at the cusps.*

PROOF: It is enough to compute for general $Y$ at $\infty$. Let

$$D = \sum_{0 \neq c_2 \equiv a_2(IC_2)} c_2^{-j}.$$

Then $D$ is the constant term which appears iff $a_1 \equiv 0$. Now let $c_1 \neq 0$. Then

$$\sum_{c_2 \equiv a_2(IC_2)} (c_1 z + c_2)^{-j} = c_1^{-j} \sum_{c_2 \equiv a_2} (z + c_2/c_1)^{-j}.$$

We let $\{x_i^{c_1}\}$ be a set of representatives of $C_2/(c_1IC_2C_1^{-1})$, and $P_j = \Sigma_{i=1}^{j} r_i x^i$ be as in 2.17 for $I = N$. Thus,

$$c_1^{-j}\sum_{c_2\equiv a_2(IC_2)}(z+c_2/c_1)^{-j}=c_1^{-j}\sum_{x_i^{c_1}}P_j(q_\infty(z+a_1/c_1+x_i^{c_1}/c_1))$$
$$=c_1^{-j}\sum_{x_i^{c_1}}\sum_{i=1}^{j}r_iq_\infty^i(z+a_1/c_1+x_i^{c_1}/c_1).$$

Now, $q_\infty(z+a_1/c_1+x_i^{c_1}/c_1)=(e_\infty(z)+e_\infty(a_1/c_1+x_i^{c_1}/c_1))^{-1}=q_\infty(z)(1+e_\infty(a_1/c_1+x_i^{c_1}/c_1)(q_\infty(z))^{-1}$. Therefore we can compute the expansion of $q_\infty^i(z+a_1/c_1+x_i^{c_1}/c_1)$ as a function of $q_\infty$. Summing over $c_1$ finishes the proof. □

Thus, we have the expansion $E(z)=\Sigma\, a_nq_\infty^n$.

REMARK 2.20: In the case $I=A$, $a=0$, the above computation is in terms of the $g_\infty$ of 1.85. However, as only powers of $g_\infty$ divisible by $(q-1)$ can occur, the answer is in terms of the $q_\infty$ of 1.85.

PROPOSITION 2.21: *For each $a_n$, there are only finitely many $c_1$ that contribute a nonzero term. This term can be calculated via the rank one module $(\phi)$ associated to $e_\infty$.*

PROOF: Fix an $i\le j$. We see that the expansion $\Sigma\, b_nq_\infty^n$ of $q_\infty^i(z+a_1/c_1+x_i^{c_1}/c_1)$ involves power sums of the roots of $\phi(c_1)(x)-e_\infty(a_1)=0$. For each $n$ the power is independent of $c_1$. Thus, we can compute via Newton. The key point is that as $|c_1|_\infty\to\infty$, the gap between the two highest nonzero coefficients of $\phi(c_1)$ tends to infinity also. Thus, by Newton, for $|c_1|_\infty\gg 0$, $c_1$ contributes 0. □

LEMMA 2.22: *Let $\beta$ be a normalizing factor and $(\Psi)$ the module associated to $\beta IC_2C_1^{-1}$. Then the $\beta$-normalized expansion may be calculated via $(\Psi)$ and its division points.*

PROOF: One checks that via normalization the $P_j$ corresponding to $IC_1^{-1}C_2$ goes to the $P_j$ corresponding to $M=\beta IC_1^{-1}C_2$. Further, one checks that the power sum of roots of $\phi(c_1)-e_\infty(a_1)$ is taken to the corresponding sum of roots of $\Psi(c_1)-e_M(\beta a_1)$. □

This completes the calculation. By choosing $\beta$ properly, we see the expansions reflect the fact that the Eisenstein series are defined over $k$. As a very non-obvious result we have the following:

THEOREM 2.23: *Let $A=\mathbf{F}_q[T]$, and $E$ an Eisenstein series for*

*$GL(2, A)$. We normalize the expansion so as to compute via the modules $(\phi_\zeta)$ of 1.53.2. Then the coefficients belong to k and have bounded denominators.*

PROOF: The corresponding Tate object is defined over $A((q))\otimes k$. Thus all algebraic $k$-modular forms have bounded denominators in their expansions at $T(\zeta, J)$. □

Finally, by 2.16, one is now able (in theory) to compute the equations for the Tate objects.

## 3. Hecke operators

For simplicity of calculation, from now on we let $A = \mathbb{F}_q[T]$. Thus we can drop the subscripts denoting the type of the lattice.

We define here the Hecke operators for a component form $f$ on $\Omega^2$, with or without level. For the full group $GL(2, A)$, we shall see that the Eisenstein series are eigenfunctions for the Hecke operators. Further, we shall see the space of cusp forms is stable. In the case of level, the spaces of Eisenstein series, cusp forms, and double cusp forms will be seen to be stable.

*The full group*

The situation here is similar to that of ([11], pp. 98–104), "tensored with A".

Let $N$ be a rank two lattice and $I$ an ideal. Let $N^1 \supseteq N$ be another rank two lattice and $H \subseteq A$ a proper ideal.

DEFINITION 3.1: We say $N$ is of index $H$ is $N^1$ if $N^1/N \simeq J_1^{-1}/A \oplus J_2^{-1}/A$ with $J_1, J_2$ ideals and $H = J_1J_2$.

DEFINITION 3.2: As a formal sum, we define $T_H(N) = \Sigma N_i$ over all $N_i$ containing $N$ of index $H$.

PROPOSITION 3.3: (1) *If $H, J$ are relatively prime then $T_HT_J = T_{HJ}$.*
(2) *If $P$ is a prime, then $T_{P^i} = T_{P^{i-1}}T_P = (T_P)^i$.*

PROOF: (1) is obvious. To see (2) we perform the standard calculation and we see the multiplicities are $\equiv 0(q)$, and so in $A$. □

CONVENTION: If $I$ is an ideal, $i$ will be its unique monic generator.

In general, lower case letters will denote monic polynomials if they denote elements of $A$. We denote the degree of $i$ by $D(i)$.

Let $F$ on $W^2$ be a form of weight $(q-1)r=j$ for $GL(2,A)$. Let $f$ be the corresponding function on $\Omega^2$, which we assume holomorphic at the cusp $\infty$. Thus, $f(z)=\Sigma_{n\geq 0}a_nq_\infty^n$, with $q_\infty = g_\infty^{(q-1)}$.

DEFINITION 3.4: For $H\subseteq A$, $N\subseteq W^2$, we define $T_HF(N)=\Sigma\, F(N_i)$.

LEMMA 3.5: *Let $N=Aw_1+Aw_2$. Any lattice containing $N$, with index $H$, has a unique basis of the form,*

$$e_1=d^{-1}w_1+B/hw_2,\quad e_2=a^{-1}w_2,$$

*with $ad=h$ and $D(B)<D(d)$.* □

REMARK 3.6: In terms of $f$, definition 3.4 becomes

$$T_Hf(z)=\sum_{\substack{a,ad=h\\ D(B)<D(d)}} a^jf\left(\frac{d^{-1}z+B/h}{a^{-1}}\right)=\sum_{\substack{a,ad=h\\ D(B)<D(d)}} a^jf\left(\frac{az+B}{d}\right).$$

THEOREM 3.7: *We have $T_Hf(z)$ is a form of weight $j$ for $GL(2,A)$ and is holomorphic at infinity. If $f$ is a cusp form then so is $T_Hf$.*

PROOF: Everything follows once we know the expansion for $T_Hf$. We compute the expansion in terms of $g_\infty$ and, as usual, only powers divisible by $q-1$ can occur. The computation is the next two lemmas.

LEMMA 3.8: *Let $0\neq a\in A$ and let $\alpha_1,\ldots,\alpha_n$ be representatives of $A/(a)$. Then $g_\infty(az)=1/a\ \Sigma_{\alpha_i}(g_\infty(z)-e_\infty(\alpha_i/a)g_\infty^2+\cdots)$.*

PROOF: By now, this is standard. □

LEMMA 3.9: *The sum $\Sigma_{D(B)<D(d)}\,q_\infty(az+B/d)^n$ may be computed in terms of $q_\infty(az)$ by Newton's formula.*

PROOF: Let $(\phi)$ be the module associated to $e_\infty$. Then, as $d$ varies, $q_\infty(az+B/d)$ runs over the reciprocal roots of $\phi(d)(x)=e_\infty(az)$. Let $\phi(d)=\Sigma_{n=0}^m a_nx^{q^n}$. Then by easy algebra the reciprocal roots satisfy the equation,

$$X^{q^m} - \sum a_n g_\infty(az) X^{(q^m - q^n)} = 0.$$

We can now use Newton. □

It is easy to see that the computations above commute with normalization.

*The case of level I*

Let $g = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in SL(2, A)$ and $f$ a modular form of level $I$ weight $j$ on $\Omega^2$. Here we only work with inhomogeneous $f$.

DEFINITION 3.10: We set $f \mid g = (Cz + D)^{-2} f\left(\dfrac{Az + B}{Cz + D}\right)$.

For $d \in A$ prime to $I$, we let $R_d$ be any element in $SL(2, A)$ congruent to $\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}^{-1}(I)$. These exist for general reasons, as in 1.60.

PROPOSITION 3.11: *The map $f \mapsto f \mid R_d$ is a representation of $(A/I)^*$ on the space of forms of weight j, level I and holomorphic at the cusps.* □

REMARK 3.12: If $I$ is a product of distinct primes, then, as $(\#(A_I)^*_1 q) = 1$, this representation decomposes according to characters. In general, it is not known whether this is true.

Let $H$ be an ideal prime to $I$.

DEFINITION 3.13: We set $T_H f = \sum\limits_{\substack{a, ad = h \\ D(B) < D(d)}} a^r f \mid R_d \left(\dfrac{d^{-1}z + B/hi}{a^{-1}}\right)$.

Again, one sees that if $H, J$ are prime to $I$ then $T_H T_J = T_{HJ}$.

THEOREM 3.14: *If f is holomorphic, cuspidal or double cuspidal, then the same is true for $T_H f$.*

PROOF: In general, the map $f \mapsto f \mid R_d$ takes the expansion of $f$ at one cusp to that of another. Then, via a computation totally similar to that of 3.7 and a close inspection of Newton's rules, the theorem follows. □

*Behavior of Eisenstein series under the Hecke operators*

Since the proof of the theorem here is completely similar to that of the classical case, (see [10], pp. IV-32 to IV-36), we shall merely quote it. Note that we know directly the existence of Eisenstein series of weights 1 and 2, whereas classically this is a delicate problem.

Let $I \subseteq A$ be an ideal possibly equal to $A$.

DEFINITION 3.15: For proper $I$, we let $\mathscr{E}(j, I)$ be the space generated over $k_\infty$ by Eisenstein series of level $I$, weight $j$. For $I = A$ we let $\mathscr{E}(j, A) = \{k_\infty \cdot E_2^j\}$.

THEOREM 3.16: (1) *Let $\sigma(I)$ be the number of inequivalent cusps. Then, in the natural fashion, $\mathscr{E}(j, I) \simeq k_\infty^{\sigma(I)}$, for proper $I$.*

(2) *If $J \mid I$, then $\mathscr{E}(j, J) \subseteq \mathscr{E}(j, I)$.*

(3) *The space $\mathscr{E}(j, I)$ is stable under the Hecke operators.* $\square$

EXAMPLE 3.17: Let $E_2^j$ be a nonzero Eisenstein series for the full group. One has easily (see [11], p. 104) that $T_I E_2^j = i^j$ for all $I$.

*L-Series*

Let $f$ be a form for the full group and suppose $f$ is an eigenfunction for all Hecke operators, i.e. $T_H f = c(H) f$ all $H$. Let $c(h) = c(H)$.

DEFINITION 3.18: We define, as a formal sum,

$$L_f = \sum_{n \in A} c(n) n^{-s} = \prod_{r \text{ prime}} (1 - c(r) r^{-s}).$$

VARIANT 3.19: If $f$ is a form of level $I$ and an eigenfunction for all $T_J$, $J$ prime to $I$, we let

$$L_f = \sum_{\substack{n \text{ monic} \\ \text{prime to } I}} c(n) n^{-s} = \prod_{r \text{ prime} \nmid I} (1 - c(r) r^{-s})^{-1}.$$

EXAMPLE 3.20: We have $L_{E_2^j} = \Sigma_{n \in A} n^{j-s}$. One sees easily that in the $k_\infty$ topology $L_{E_2^j}$ converges for $s \gg 0$, $s \in \mathbb{Z}$.

CONJECTURE 3.21: For all eigenforms $f$, $L_f$ converges for $s \in \mathbb{Z}$, $s \gg 0$.[1]

[1] In a forthcoming paper, to appear in *Inventiones*, we show how to define these typefunctions on a continuous space, where they can have analytic continuation and interpolation at finite primes.

## 4. Computations and examples

Our first goal is to give a formula for the genus of $\overline{\Gamma(I)\backslash\Omega^2}$, for all proper $I \subseteq \mathbf{F}_q[T]$. Since we know from 1.32 that $\bar{M}_2$ is a curve of genus zero, we can use Hurwitz's formula.

LEMMA 4.1: (1) *The order of* $PGL(2, \mathbf{F}_q[T])/\Gamma(I)$ *is*

$$q^{3D(i)} \prod_{\substack{r \mid i \\ r \text{ prime}}} (1 - q^{-2D(r)}).$$

(2) *The number of inequivalent cusps is*

$$(q-1)^{-1} q^{2D(i)} \prod_{\substack{r \mid i \\ r \text{ prime}}} (1 - q^{-2D(r)}).$$

PROOF: These are standard coset computations. $\square$

LEMMA 4.2: *There is only one elliptic point for* $GL(2, A)$. *At each point of* $\Gamma(I)\backslash\Omega^2$ *above it, the isotropy group has order* $q+1$. *The ramification is tame.*

PROOF: Up to isomorphism, the rank two module given by $\phi(T) = TF^0 - F^2$ is the only one with extra automorphisms. Its automorphism group has order $q^2 - 1$. The usual number is $q - 1$. $\square$

Let $q_\infty = g_\infty^{(q-1)}$ be the standard uniformizer at the cusp at $\infty$. Let $b$ be a cusp for $\Gamma(I)$ and $q_b$ the uniformizer. Although the ramification is wild we still have:

LEMMA 4.3: *The order of zero of* $\mathrm{d}q_\infty/\mathrm{d}q_b$ *is* $q^\alpha - 2$, *where* $\alpha = D(i) + 1$.

PROOF: Express $g_\infty$ in terms of $q_b$. Then use Newton to find the order of zero of $g_\infty$ and the order of zero of $\mathrm{d}g_\infty/\mathrm{d}q_b$. $\square$

THEOREM 4.4: *The genus g of* $\overline{\Gamma(I)\backslash\Omega^2}$ *is*

$$1 + (q^2 - 1)^{-1} q^{2D(i)} (q^{D(i)} - q - 1) \cdot \prod_{\substack{r \mid i \\ r \text{ prime}}} (1 - q^{-2D(r)}).$$

PROOF: This follows from Hurwitz's formula. □

COROLLARY 4.5: *We have*

$$\deg(\omega) = (q^2-1)^{-1} q^{3D(i)} \prod_{\substack{r \mid i \\ r \text{ prime}}} (1 - q^{-2D(r)}).$$

PROOF: This follows from 1.81, 4.1.2, and 4.4. □

COROLLARY 4.6: (1) *For* $j > 1$, $\omega^{\otimes j}$ *has degree* $> 2g+1$.
(2) *The graded ring of* $k^1$*-holomorphic forms is generated by the forms of weight* 1, 2 *and* 3.

PROOF: (1) is obvious. (2) is a consequence of (1) and the results of [25]. □

Let $DC(j, I)$ (resp. $C(j, I)$) be the double-cusp forms (resp. cusp forms) of weight $j$, level $I$.

COROLLARY 4.7: (1) *We have* $\dim DC(2, I) = g$. *If* $j > 2$ *then*

$$\dim DC(j, I) = [(j-1)q^{3D(i)} - q^{2D(i)+1} - q^{2D(i)}](q^2-1)^{-1} \times \prod_{\substack{r \mid i \\ r \text{ prime}}} (1 - q^{-2D(r)}).$$

(2) *If* $j \geq 2$, *then*

$$\dim C(j, I) = (j-1)q^{3D(i)}(q^2-1)^{-1} \cdot \prod_{\substack{r \mid i \\ r \text{ prime}}} (1 - q^{-2D(r)}).$$

PROOF: This is now a consequence of the Riemann–Roch Theorem. □

For the forms of weight one, we can only show the following theorem.

THEOREM 4.8: *We have* $\Gamma(\bar{M}_I^1, \omega) = DC(1, I) \oplus \xi(1, I)$.

PROOF: We know $\xi(1, I)$ has dimension equal to the number of inequivalent cusps. Thus, the theorem follows from Riemann–Roch and duality. □

The construction of a nonzero element in $DC(1, I)$ would be very interesting.

Finally, we know $\underset{(x,I)}{E^1}$ has its divisor supported on the cusps. Let $b_1, b_2$ be two cusps for $\Gamma(I)$.

CONJECTURE 4.9: There is a modular function $f$, formed out of weight one Eisenstein series, so that

$$(f) = n((b_1) - (b_2))$$

for some $n \in \mathbb{N}$.

EXAMPLE 4.10: Let $q = 2$, $I = (T^2)$. Then, $\overline{M_I^2}$ is a curve of genus 5. Here we may take $n = 16 = 2^4$. As a (special?) corollary, the Jacobian has a nonzero point of order 2.

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