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R. J. PLYMEN

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THE LAPLACIAN AND THE DIRAC OPERATOR IN INFINITELY MANY VARIABLES

R.J. Plymen

Introduction

Let E be an even-dimensional real Hilbert space, and let $O(E)$ be the full orthogonal group of E . Let σ be the spin representation of $O(E)$ on the space S of spinors. Then σ is a projective irreducible unitary representation of $O(E)$ on S . The subgroup $SO(E)$ also acts projectively and splits S into two irreducible subspaces S^+ and S^- , the spaces of positive and negative spinors. Let f be a spinor field, i.e. a map from U , an open set in E , to S . Then the Dirac operator P sends f to another spinor field, provided that f is differentiable. If f is twice differentiable then

$$P^2f(x) = \Delta f(x)$$

where Δ is the spinor Laplacian: the spinor Laplacian simply acts as the Laplacian on each scalar component of $f(x)$. Further, P exchanges positive and negative spinor fields.

In this article, we recast the whole of the above paragraph when E is infinite-dimensional. Subject to restrictions and qualifications which we state explicitly, the assertions in the above paragraph remain true in the infinite-dimensional case.

At the infinite-dimensional level, the spin representation was discovered by Shale and Stinespring [18]. These authors worked with a holomorphic spinor representation of the C^* -Clifford algebra. This representation is unitarily equivalent to a Fock representation of the C^* -Clifford algebra; this point of view is taken by the present author in [14], where the subject of spinors in Hilbert space is taken a little further.

Again at the infinite-dimensional level, the theory of the projective tensor product and the associated nuclear operators provides the perfect framework for the Dirac operator. Grothendieck has written an elaborate account of this theory [8]. We shall need only a small fragment of this theory, including the universal property of the projective tensor product, which replaces continuous multilinear maps by continuous linear maps. The projective tensor product of the Hilbert space E with itself must be sharply distinguished from the ordinary Hilbert space tensor product: for the former is isomorphic to the space of trace-class operators in the trace-norm, whereas the latter is isomorphic to the space of Hilbert-Schmidt operators in the Hilbert-Schmidt norm, by a classic result of Schatten [16].

Concerning differential calculus in Banach space, we shall need only the chain rule for the first derivative of a composite function, and the fact that the n th derivative of a vector-valued function is a symmetric multilinear map. Our references here are Cartan [3] or Dieudonné [4].

In 1967, Gross published an account of potential theory on Hilbert space which incorporates the Laplacian Δ_∞ , defined as the trace of the second Frechet derivative. The correction term which makes Δ_∞ formally self-adjoint and negative-definite is discussed, in the context of abstract Wiener space, by Elworthy [5, p.163]. This more sophisticated version of the Laplace operator has been studied, in a slightly different context, by Umemura [19]. In particular, its eigenvalues and eigenspaces can be found: the eigenvectors are Fourier-Hermite polynomials. We have not been able to find a “Dirac operator” associated with this version of the Laplacian.

The theory of the projective tensor product provides the right framework for a basis-free account of nuclear operators, the Laplacian Δ_∞ , the spinor Laplacian, and our Dirac operator P . We are extremely grateful to the referee whose comments led to much improved basis-free proofs in the present article.

In section 1 we give a brief review of the Dirac operator, based on section 1.1 of Hitchin’s article [10], in order to elucidate our opening paragraph. We use his notation P for the Dirac operator. In section 2, we describe the projective tensor product. In section 3, we define the Clifford multiplication and the Dirac operator at the infinite-dimensional level. In section 4, we find a formula for the n th power of the Dirac operator (Theorem 1), give a basis-free definition of the spinor Laplacian, and prove that the square of the Dirac operator is equal to the spinor Laplacian (Theorem 2). In section 5, we discuss very briefly the half-spinors and their relation to the Dirac operator.

We would like to thank the staff at the Mathematics Institute, University of Warwick, for their hospitality when the author was a visitor in the spring of 1977, and where this work was begun. In particular, we would like to thank David Elworthy for drawing our attention to Grothendieck's memoir [8].

1. Review of classical Dirac operator

Let E be an even-dimensional real Hilbert space. Then factoring out the ideal generated by elements of the form $x \otimes x - \|x\|^2.1$ in the tensor algebra $\otimes E$, we get a finite-dimensional algebra $\mathcal{A}(E)$, the *Clifford algebra* of E . We have $E \subset \mathcal{A}(E)$ such that $x^2 = \|x\|^2.1$. Suppose $\dim E = 2k$, then the complexification $\mathcal{A}(E) \otimes_{\mathbb{R}} \mathbb{C}$ is a full matrix algebra $\text{End } S$, where S is a 2^k -dimensional complex vector space. We thus have a linear map

$$m : E \otimes_{\mathbb{R}} S \longrightarrow S$$

the *Clifford multiplication*. Let f be a differentiable map from U , an open set in E , to S . Then $f'(x)$ lies in $\text{Hom}_{\mathbb{R}}(E, S)$ for each x in U . But $\text{Hom}_{\mathbb{R}}(E, S) \cong E \otimes_{\mathbb{R}} S$ so we have

$$f' : U \longrightarrow E \otimes_{\mathbb{R}} S$$

Then Pf is the compound map

$$U \xrightarrow{f'} E \otimes_{\mathbb{R}} S \xrightarrow{m} S$$

That is, $Pf = m \circ f'$. The linear operator P is the *Dirac operator*.

Let $\{e_1, \dots, e_{2k}\}$ be an orthonormal base in E . Then

$$f'(x) = \sum e_i \otimes f_i(x)$$

where $f_i(x) = f'(x) \cdot e_i = \partial f(x) / \partial x_i \cdot e_i$ so that

$$Pf(x) = \sum c_j f_j(x)$$

where

$$c_i c_j + c_j c_i = \delta_{ij}$$

This is a standard expression of the Dirac operator in terms of the partial derivatives $f_i(x)$ and the c_i -matrices. Note that the c_i -matrices act on the space S of spinors.

Let $\{\phi_1(x), \phi_2(x), \dots\}$ be the 2^{k+1} components of $f(x)$ with respect to an orthonormal basis in S viewed as a real space. If f is a twice differentiable map from U to S the space of spinors, then we have

$$P^2f(x) = \{\Delta\phi_1(x), \Delta\phi_2(x), \dots\}$$

where Δ is the Laplacian. The operator P^2 is therefore the *spinor Laplacian*. Since P is self-adjoint, P and P^2 have the same null space. P is an elliptic differential operator and its kernel is the finite-dimensional space of *harmonic spinors*.

The spin group $\text{Spin}(2k)$ lies in $\mathcal{A}(E)$. In fact, $\text{Spin}(2k)$ comprises all "words" $x_1 \dots x_{2n}$ of even length in $\mathcal{A}(E)$ where x_1, \dots, x_{2n} are unit vectors in E . We have an exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}(2k) \xrightarrow{\rho} \text{SO}(2k) \longrightarrow 1$$

with $\rho(x_1 \dots x_{2n}) = \tau(x_1) \dots \tau(x_{2n})$ where $\tau(x)$ is reflection in the hyperplane perpendicular to x . Since $\mathcal{A}(E)$ acts on S we have a unitary representation, the *spin representation*, of $\text{Spin}(2k)$ on S . This representation is not irreducible: it splits as a direct sum of two irreducible subrepresentations, the *half-spin representations*. The irreducible subspaces are denoted S^+ and S^- . Spinors in these subspaces are called *positive* and *negative* spinors. Let f be a *spinor field*, that is, a differentiable map from U to S . If $f(x)$ lies in S^+ for all x in U , then $Pf(x)$ lies in S^- : if $f(x)$ lies in S^- , then $Pf(x)$ lies in S^+ . In this sense we say that the Dirac operator exchanges positive and negative spinor fields.

This article is a first step in the project of investigating the Dirac operator on infinite-dimensional spin-manifolds: this project was proposed by de la Harpe [9, p.260]. The reader will find information on spin-manifolds in [1, 2, 9, 10, 13].

2. The projective tensor product

Let E and F denote two fixed real separable Hilbert spaces and let G denote a real Banach space. The real Banach space of continuous linear maps from E to G is denoted $\mathcal{L}(E; G)$. The algebraic tensor

product of E and G is denoted $E \odot G$. There exists a natural map $u \longrightarrow \tilde{u}$ of $E \odot G$ into $\mathcal{L}(E; G)$ of which the image is the space of continuous linear maps of finite rank from E to G . If $u = x \otimes y$ then $\tilde{u} = \langle x, \cdot \rangle y$ where $\langle \cdot, \cdot \rangle$ is the inner product in E .

The projective tensor product $E \otimes G$ arises in the following way. We put the following norm on $E \odot G$:

$$\|u\|_1 = \inf \left\{ \sum \|x_i\| \cdot \|y_i\| : u = \sum x_i \otimes y_i \right\}$$

Then $\|\cdot\|_1$ is a cross-norm in the sense that

$$\|x \otimes y\|_1 = \|x\| \cdot \|y\|$$

for all vectors x in E and y in G . The completion of $E \odot G$ is denoted $E \otimes G$. The map $u \longrightarrow \tilde{u}$ extends by continuity to a continuous linear map of norm ≤ 1 from $E \otimes G$ to $\mathcal{L}(E; G)$, still denoted $u \longrightarrow \tilde{u}$:

$$(2.1) \quad \|\tilde{u}\| \leq \|u\|_1$$

LEMMA 1: *The natural map $E \otimes G \longrightarrow \mathcal{L}(E; G)$ is injective.*

PROOF: This result is well-known: see Grothendieck [8, chapter I, Proposition 35]. It depends on the fact that the Hilbert space E enjoys the approximation property: the identity map can be approximated, uniformly on every precompact set in E , by continuous linear maps of finite rank.

The image of the natural map $E \otimes G \longrightarrow \mathcal{L}(E; G)$ is the vector space of *nuclear* operators. When $G = E$, the image of the map

$$E \otimes E \longrightarrow \mathcal{L}(E; E)$$

is the vector space of nuclear (i.e. trace-class) operators on E , and the $\|\cdot\|_1$ norm on $E \otimes E$ corresponds to the trace norm on the vector space of trace-class operators. This is a classic result of Schatten [16, p.119].

Let U be an open set in E and let $f: U \longrightarrow F$ be differentiable. Thus we have

$$f': U \longrightarrow \mathcal{L}(E; F)$$

where f' is the Frechet derivative. Now $f'(x)$ is nuclear for all x in U

if and only if there is a map $\phi_1: U \rightarrow E \otimes F$ such that $f' = \alpha_1 \circ \phi_1$, where α_1 is the natural map $E \otimes F \rightarrow \mathcal{L}(E; F)$. If $f'(x)$ is nuclear for all x in U then such a map ϕ_1 exists and is unique, by Lemma 1.

Let m be a continuous bilinear map from $E \times F$ to F . By the universal property of the projective tensor product, m determines uniquely a continuous linear map $r_1: E \otimes F \rightarrow F$ such that $r_1(x \otimes y) = m(x, y)$ for all x in E and y in F .

Suppose again that $f'(x)$ is nuclear for all x in U . We have the following commutative triangles:

$$\begin{array}{ccc}
 & E \otimes F & \\
 \phi_1 \nearrow & & \downarrow \alpha_1 \\
 U & \xrightarrow{f'} & \mathcal{L}(E; F)
 \end{array}
 \qquad
 \begin{array}{ccc}
 & E \otimes F & \\
 \nearrow & & \downarrow r_1 \\
 E \times F & \xrightarrow{m} & F
 \end{array}$$

Let P be defined by

$$Pf = r_1 \circ \phi_1$$

The Dirac operator arises when we make a concrete choice for the multiplication map m .

3. The Dirac operator

We now specify the multiplication map m . Let \mathcal{A} be the CAR algebra over E and let J be a complex structure in E . So J is an orthogonal operator such that $J^2 = -1$. Let H_J, π_J, Ω_J be the Fock space, representation, vacuum vector determined by J . It is known that two Fock representations π_{J_1} and π_{J_2} are unitarily equivalent if and only if $|J_1 - J_2|$ is Hilbert-Schmidt [12]. In what follows we choose and fix a Fock representation π_J .

We take F to be H_J as a real Hilbert space, i.e. we restrict scalars from \mathbb{C} to \mathbb{R} and take the inner product in F to be the real part of the inner product in H_J . Norms of vectors are thereby unaffected. We shall need the following properties of the Fock representation.

Since $E \subset \mathcal{A}$, π_J determines by restriction a continuous linear map

$$\pi_J: E \rightarrow \mathcal{L}(F; F)$$

Each unit vector in E is represented by a symmetry (self-adjoint unitary) and

$$(3.1) \quad \pi_J(x)\pi_J(y) + \pi_J(y)\pi_J(x) = 2 \langle x, y \rangle$$

DEFINITION 1: $m(x, y) = \pi_J(x)y$ with x in E and y in F .

Now

$$\begin{aligned} \|m(x, y)\| &= \|\pi_J(x)y\| \\ &= \|x\| \cdot \|\pi_J(x/\|x\|)y\| \end{aligned}$$

Now $x/\|x\|$ is a unit vector provided that $x \neq 0$ so we have

$$\|m(x, y)\| = \|x\| \cdot \|y\|$$

for all x in E and y in F . So m is certainly a continuous bilinear map from $E \times F$ to F , and determines uniquely a continuous linear map r_1 from $E \otimes F$ to F .

As in section 2, we denote by α_1 the natural map from $E \otimes F$ to $\mathcal{L}(E; F)$.

DEFINITION 2: Let U be an open set in E , and let f be a differentiable map from U to F such that $f'(x)$ is nuclear for all x in U . Then the Dirac operator P is given by

$$Pf = r_1 \circ \phi_1$$

where ϕ_1 is the unique map from U to $E \otimes F$ such that $f' = \alpha_1 \circ \phi_1$.

The CAR algebra \mathcal{A} is sometimes called the *C*-Clifford algebra*, the relations (3.1) are the *Clifford relations* and we shall call m the *Clifford multiplication*. This terminology is compatible with that in Hitchin's article [10].

In this section, E and F are real separable infinite-dimensional Hilbert spaces, hence there is a unitary map V from F onto E . Let $T \in E \otimes F$, and let \tilde{T} be the corresponding nuclear operator from E to F . Then $A = V\tilde{T}$ is a nuclear operator on E . Let $B = (A^*A)^{1/2}$. Then there exists an orthonormal basis $\{e_n\}$ such that

$$Be_n = \lambda_n e_n \quad n = 1, 2, 3, \dots$$

and $\sum \lambda_n < \infty$. Now

$$\|\tilde{T}e_n\| = \|Ae_n\| = \langle A^*Ae_n, e_n \rangle^{1/2} = \langle B^2e_n, e_n \rangle^{1/2} = \|Be_n\| = \lambda_n$$

hence the series $\sum e_n \otimes \tilde{T}e_n$ is absolutely summable in $E \otimes F$. Let

$S = \sum e_n \otimes \tilde{T}e_n$. Then

$$\tilde{S}(e_k) = \sum \langle e_n, e_k \rangle \tilde{T}e_n = \tilde{T}e_k \quad k = 1, 2, 3, \dots$$

hence $S = T$ by Lemma 1. Thus

$$(3.2) \quad T = \sum e_n \otimes \tilde{T}e_n$$

Let $f: U \rightarrow F$ be a differentiable map such that $f'(x)$ is nuclear for all x in U , and let $f' = \alpha_1 \circ \phi_1$. Let $T = \phi_1(x)$ so that $\tilde{T} = f'(x)$. Let

$$f_n(x) = f'(x) \cdot e_n$$

Then we have

$$\phi_1(x) = \sum e_n \otimes f_n(x)$$

Thus

$$Pf(x) = r_1 \phi_1(x) = \sum \pi_j(e_n) f_n(x)$$

Let $c_n = \pi_j(e_n)$. Then we have

$$(3.3) \quad c_i c_j + c_j c_i = 2\delta_{ij}$$

$$c_i^2 = 1$$

by the Clifford relations (3.1). Furthermore, c_i is an analogue in infinitely many dimensions of the original γ -matrices of Dirac. In fact the matrix of c_i with respect to an orthonormal basis in F is an infinite matrix which is symmetric and orthogonal. The vector $f_n(x)$ is the n th partial derivative of f at x evaluated at e_n [3, p.34]. So we have an expression for the Dirac operator in terms of the partial derivatives and the c -matrices:

$$Pf(x) = \sum_{n=1}^{\infty} c_n f_n(x).$$

As in the finite-dimensional case, the c -matrices act on the space of spinors [14].

4. The Spinor Laplacian

Let $\otimes^n E = E \otimes \dots \otimes E$ (n factors) and $\otimes^0 E = \mathbf{R}$. Let α_1 be the natural map from $E \otimes F$ to $\mathcal{L}(E; F)$ and in general let α_{n+1} be the natural map from $E \otimes G$ to $\mathcal{L}(E; G)$ where $G = (\otimes^n E) \otimes F$, $n = 0, 1, 2, \dots$

Let U be an open set in E and let D_0 be the set of all maps from U to F .

Let D_1 be the set of all f such that f is differentiable and $f'(x)$ is nuclear for all x in U . By Lemma 1, there exists uniquely $\phi_1: U \rightarrow E \otimes F$ such that $f' = \alpha_1 \circ \phi_1$.

Let D_2 be the set of all f in D_1 such that ϕ_1 is differentiable and $\phi_1'(x)$ is nuclear for all x in U . By Lemma 1, there exists uniquely $\phi_2: U \rightarrow E \otimes E \otimes F$ such that $\phi_1' = \alpha_2 \circ \phi_2$.

In general let D_{n+1} be the set of all f in D_n such that ϕ_n is differentiable and $\phi_n'(x)$ is nuclear for all x in U . By Lemma 1, there exists uniquely $\phi_{n+1}: U \rightarrow E \otimes (\otimes^n E) \otimes F$ such that $\phi_n' = \alpha_{n+1} \circ \phi_{n+1}$. Thus D_n is a real vector space and $D_{n+1} \subset D_n$ for each $n \geq 0$.

We recall that r_1 is a continuous linear map from $E \otimes F$ to F . The map r_n is defined by

$$r_{n+1} = 1 \otimes r_1: (\otimes^n E) \otimes (E \otimes F) \rightarrow (\otimes^n E) \otimes F$$

for all $n \geq 0$. Thus r_n reduces the rank of tensors from $n + 1$ to n .

Let

$$\rho_n: \mathcal{L}(E; (\otimes^n E) \otimes F) \rightarrow \mathcal{L}(E; (\otimes^{n-1} E) \otimes F)$$

be defined by $\rho_n(\lambda) = r_n \circ \lambda$. It is clear that

$$(4.1) \quad \alpha_n \circ r_{n+1} = \rho_n \circ \alpha_{n+1}$$

since each composite map is linear and continuous and sends the rank $n + 2$ tensor $x_1 \otimes \dots \otimes x_{n+1} \otimes y$ to the continuous linear map $\langle x_1, \dots, x_2 \otimes \dots \otimes \pi_J(x_{n+1})y \rangle$.

THEOREM 1: *If $f \in D_n$ then $Pf \in D_{n-1}$ and*

$$P^n f = r_1 \circ r_2 \circ \dots \circ r_n \circ \phi_n \quad (n \geq 1)$$

PROOF: If $f \in D_n$ then there exist ϕ_1, \dots, ϕ_n such that

$$f' = \alpha_1 \circ \phi_1, \dots, \phi_{n-1}' = \alpha_n \circ \phi_n$$

Let $g = Pf$. Then $g = r_1 \circ \phi_1$ by definition of P . Using the chain rule and (4.1) we have

$$g'(x) = r_1 \circ \phi_1'(x) = \rho_1 \alpha_2 \phi_2(x) = \alpha_1 r_2 \phi_2(x)$$

so that

$$g' = \alpha_1 \circ \theta_1 \text{ where } \theta_1 = r_2 \circ \phi_2$$

Then

$$\begin{aligned} \theta_1'(x) &= r_2 \circ \phi_2'(x) = \rho_2 \alpha_3 \phi_3(x) = \alpha_2 r_3 \phi_3(x) \\ \theta_1' &= \alpha_2 \circ \theta_2 \text{ where } \theta_2 = r_3 \circ \phi_3 \end{aligned}$$

and

$$\begin{aligned} \theta_{n-2}'(x) &= r_{n-1} \circ \phi_{n-1}'(x) = \rho_{n-1} \alpha_n \phi_n(x) = \alpha_{n-1} r_n \phi_n(x) \\ \theta_{n-2}' &= \alpha_{n-1} \circ \theta_{n-1} \text{ where } \theta_{n-1} = r_n \circ \phi_n \end{aligned}$$

Hence $g \in D_{n-1}$ by definition of D_{n-1} . Furthermore we have

$$\begin{aligned} Pf &= r_1 \circ \phi_1 \\ P^2f &= Pg = r_1 \circ \theta_1 = r_1 \circ r_2 \circ \phi_2 \end{aligned}$$

At the n th step of this iterative process we have, by the chain rule, the definition of D_n , the $(n-1)$ -fold application of (4.1), and the definition of P ,

$$\begin{aligned} (r_1 \circ \dots \circ r_{n-1} \circ \phi_{n-1})'(x) &= r_1 \circ \dots \circ r_{n-1} \circ \phi_{n-1}'(x) \\ &= \rho_1 \dots \rho_{n-1} \alpha_n \phi_n(x) \\ (r_1 \circ \dots \circ r_{n-1} \circ \phi_{n-1})' &= \alpha_1 \circ r_2 \circ \dots \circ r_n \circ \phi_n \\ P^n f &= r_1 \circ r_2 \circ \dots \circ r_n \circ \phi_n \end{aligned}$$

This ends the proof.

The real vector space $D_\infty = \bigcap D_n$ is an invariant domain for P . The formula

$$P^n f = r_1 \circ r_2 \circ \dots \circ r_n \circ \phi_n$$

gives the iterated Dirac operator in terms of the iterated Clifford multiplication and ϕ_n , which is essentially the n th Frechet derivative $f^{(n)}$ of f . For

$$f^{(n)}(x) \in \mathcal{L}_n(E; F)$$

where $\mathcal{L}_n(E; F)$ is the Banach space of continuous multilinear maps from $E \times \dots \times E$ to F . By the universal property of the projective tensor product we have

$$\mathcal{L}_n(E; F) \cong \mathcal{L}(\otimes^n E; F)$$

Then $\phi_n(x)$ is the unique tensor of rank $n + 1$ in $(\otimes^n E) \otimes F$ corresponding to $f^{(n)}(x)$.

We now discuss the Laplacian Δ_∞ [6]. This operator fits well into the framework of the present article. Let u be a map from U to \mathbf{R} . Then we have, identifying E with its dual E^* ,

$$u : U \longrightarrow \mathbf{R}, u' : U \longrightarrow E, u'' : U \longrightarrow \mathcal{L}(E; E)$$

Now the continuous bilinear functional from $E \times E$ to \mathbf{R} which sends x_1, x_2 to $\langle x_1, x_2 \rangle$ determines uniquely a continuous linear functional τ on $E \otimes E$ such that $\tau(x_1 \otimes x_2) = \langle x_1, x_2 \rangle$. Let α be the natural map from $E \otimes E$ to $\mathcal{L}(E; E)$. Then the *trace* of $\alpha(w)$ is by definition $\tau(w)$ for all w in $E \otimes E$.

DEFINITION 3: Let U be an open set in E and let u be a twice differentiable map from U to \mathbf{R} such that $u''(x)$ is nuclear for all x in U . Then the Laplacian $\Delta_\infty u$ is given by

$$\Delta_\infty u = \tau \circ v$$

where v is the unique map from U to $E \otimes E$ such that $u'' = \alpha \circ v$.

DEFINITION 4: Let $f \in D_2$. The *spinor Laplacian* is defined by the equation

$$\Delta_\infty f = (\tau \otimes 1) \circ \phi_2$$

Let $f(x) = \sum_n \langle f_n(x), y_n \rangle y_n$ be the Fourier expansion of $f(x)$ in F with respect to the orthonormal basis $\{y_n\}$. Let $u_n(x) = \langle f_n(x), y_n \rangle$. We claim that

$$\langle \Delta_\infty f(x), y_n \rangle = \Delta_\infty u_n(x)$$

i.e. that the spinor Laplacian acts as the Laplacian on each Fourier component of $f(x)$. This claim will follow immediately from the following

THEOREM 2: *Let $f \in D_2$. Then*

$$y^* \circ \Delta_\infty \circ f = \Delta_\infty \circ y^* \circ f \quad (y^* \in F^*)$$

and

$$P^2 f = \Delta_\infty f$$

PROOF: Here, F^* denotes the continuous dual of F . We claim that the following equations are true:

$$(4.2) \quad y^* \circ \alpha_1(w) = (1 \otimes y^*)(w) \quad w \in E \otimes F$$

$$(4.3) \quad (1 \otimes y^*) \circ \alpha_2(t) = \alpha(1 \otimes 1 \otimes y^*)(t) \quad t \in E \otimes E \otimes F$$

$$(4.4) \quad y^* \circ \tau \otimes 1 = \tau \circ (1 \otimes 1 \otimes y^*) \quad \text{on } E \otimes E \otimes F$$

$$(4.5) \quad \tau \otimes 1 = r_1 \circ r_2 \quad \text{on } (E \otimes E)_s \otimes F$$

We prove (4.5). Here $(E \otimes E)_s$ denotes the subspace of $E \otimes E$ consisting of symmetric rank 2 tensors. Now

$$(\tau \otimes 1)(x_1 \otimes x_2 \otimes y + x_2 \otimes x_1 \otimes y) = 2 \langle x_1, x_2 \rangle y$$

$$r_1 r_2(x_1 \otimes x_2 \otimes y + x_2 \otimes x_1 \otimes y) = \pi_J(x_1) \pi_J(x_2) y + \pi_J(x_2) \pi_J(x_1) y$$

Hence, by the Clifford relations (3.1), $\tau \otimes 1$ and $r_1 \circ r_2$ agree on tensors of the form $x_1 \otimes x_2 \otimes y + x_2 \otimes x_1 \otimes y$. By linearity and continuity, they agree on $(E \otimes E)_s \otimes F$. The proofs of (4.2), (4.3), (4.4) are similar.

Let $u = y^* \circ f$. By the chain rule, the definition of D_1 , and (4.2) we have

$$\begin{aligned} u'(x) &= y^* \circ f'(x) \\ &= y^* \circ \alpha_1 \phi_1(x) \end{aligned}$$

$$\begin{aligned}
 &= (1 \otimes y^*)\phi_1(x) \\
 u' &= (1 \otimes y^*) \circ \phi_1
 \end{aligned}$$

By the chain rule, the definition of D_2 , and (4.3) we have

$$\begin{aligned}
 u''(x) &= (1 \otimes y^*) \circ \phi_1'(x) \\
 &= (1 \otimes y^*) \circ \alpha_2 \phi_2(x) \\
 &= \alpha(1 \otimes 1 \otimes y^*)\phi_2(x) \\
 u'' &= \alpha \circ (1 \otimes 1 \otimes y^*) \circ \phi_2
 \end{aligned}$$

where α is the natural map $E \otimes E \rightarrow \mathcal{L}(E; E)$. By definition of the Laplacian, and (4.4) we have

$$\begin{aligned}
 \Delta_\infty u &= \tau \circ (1 \otimes 1 \otimes y^*) \circ \phi_2 \\
 &= y^* \circ (\tau \otimes 1) \circ \phi_2
 \end{aligned}$$

Hence $\Delta_\infty u = y^* \circ \Delta_\infty f$ by definition of the spinor Laplacian. By Theorem 1 we have

$$P^2 f = r_1 \circ r_2 \circ \phi_2$$

But $\phi_2 \in (E \otimes E)_s \otimes F$ since $f''(x)$ is a *symmetric* bilinear map from $E \times E$ to F [3, p.59]. Applying (4.5) we get

$$\begin{aligned}
 P^2 f &= (\tau \otimes 1) \circ \phi_2 \\
 &= \Delta_\infty f.
 \end{aligned}$$

EXAMPLE 1: Let $T \in (E \otimes E)_s$ and let \tilde{T} be the corresponding symmetric nuclear operator on E . Let y be a fixed vector in F and let $f(x) = \langle \tilde{T}x, x \rangle y$. The following equations are easy to verify successively:

$$\begin{aligned}
 \phi_1(x) &= 2\tilde{T}x \otimes y \\
 \phi_2(x) &= 2T \otimes y \\
 \Delta_\infty f(x) &= 2Tr(\tilde{T})y \\
 Pf(x) &= r_1 \phi_1(x) = 2r_1(\tilde{T}x \otimes y) \\
 (Pf)'(x) &= \alpha_1 \theta_1(x) \text{ where } \theta_1(x) = 2r_2(T \otimes y)
 \end{aligned}$$

$$\begin{aligned}
P^2f(x) &= r_1\theta_1(x) \\
&= 2r_1r_2(T \otimes y) \\
&= 2(\tau \otimes 1)(T \otimes y) \\
&= 2Tr(T)y \\
P^2f(x) &= \Delta_\infty f(x) = 2Tr(\tilde{T})y = \text{constant} \\
P^n f(x) &= 0 \quad n \geq 3 \\
f &\in D_\infty
\end{aligned}$$

With y_1, \dots, y_n vectors in F and $\tilde{T}_1, \dots, \tilde{T}_n$ symmetric nuclear operators on E and

$$f(x) = \sum_j \langle \tilde{T}_j x, x \rangle y_j$$

we have similarly

$$P^2f(x) = 2 \sum_j Tr(\tilde{T}_j)y_j = \Delta_\infty f(x).$$

5. The half-spin representations

Let \mathcal{A} be the CAR algebra over E and let J be a complex structure in E . Let π_J, H_J, Ω_J be the Fock representation, space, vacuum vector determined by J . Let F denote H_J as a real Hilbert space.

Let $SO(E)_1$ be the special nuclear orthogonal group. Then $SO(E)_1$ comprises all orthogonal operators T on E such that $I - T$ is nuclear and $\det(T) = 1$. Also $SO(E)_1$ is a Banach-Lie group and its universal cover $\text{Spin}(E)_\infty$ is a subset of \mathcal{A}^+ , the even CAR algebra. The spin representation Δ_J is by definition the restriction of π_J to $\text{Spin}(E)_\infty$. Now F splits into two invariant subspaces F^+ and F^- ; the corresponding subrepresentations are denoted Δ_J^+ and Δ_J^- . So we have

$$\Delta_J = \Delta_J^+ \oplus \Delta_J^-$$

The vectors in F , regarded as a $\text{Spin}(E)_\infty$ -module, are called *spinors*; those in F^+ or F^- are called *1/2-spinors*. For more background material, the reader may consult [9, 11, 12, 14, 15].

Now \mathcal{A}^+ is a simple C^* -algebra [17] and leaves F^+ invariant. Therefore Δ_j^+ is a *faithful* representation of the spin group $\text{Spin}(E)_\infty$. Since the elements of $\text{Spin}(E)_\infty$ are unitary elements of \mathcal{A}^+ , it is clear that Δ_j^+ is a faithful norm continuous unitary representation of $\text{Spin}(E)_\infty$; the same is true of Δ_j^- . These two representations are also irreducible [14].

Let Q^+, Q^- be the projections on F^+, F^- . Let $f \in D_1$, $f^+(x) = Q^+f(x), f^-(x) = Q^-f(x)$. Then we have

$$Pf^+(x) \in F^- \text{ and } Pf^-(x) \in F^+.$$

For

$$\begin{aligned} (f^+)'(x) &= Q^+ \circ f'(x) \\ &= Q^+ \circ \alpha_1 \phi_1(x) \end{aligned}$$

It is clear that

$$(5.1) \quad \alpha_1(1 \otimes Q^+)(u) = Q^+ \circ \alpha_1(u) \quad u \in E \otimes F$$

By (5.1) we have

$$\begin{aligned} (f^+)'(x) &= \alpha_1(1 \otimes Q^+) \phi_1(x) \\ (f^+)' &= \alpha_1 \circ (1 \otimes Q^+) \circ \phi_1 \\ (Pf^+)'(x) &= r_1(1 \otimes Q^+) \phi_1(x) \in F^- \end{aligned}$$

since we have $r_1: E \otimes F^+ \longrightarrow F^-$. Similarly, $(Pf^-)'(x) \in F^+$.

EXAMPLE 2: Let $f(x) = \langle \tilde{T}x, x \rangle \Omega_j$ with \tilde{T} a symmetric nuclear operator on E . Then, by Example 1, we have

$$\begin{aligned} \phi_1(x) &= 2\tilde{T}x \otimes \Omega_j \\ Pf(x) &= 2\pi_j(\tilde{T}x)\Omega_j \\ P^2f(x) &= 2Tr(\tilde{T})\Omega_j. \end{aligned}$$

Thus $f(x) \in F^+, Pf(x) \in F^-$ and $P^2f(x) \in F^+$.

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Department of Mathematics
The University
Manchester M13 9PL
England