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THE UNRAMIFIED PRINCIPAL SERIES OF $\mathfrak{p}$-ADIC GROUPS I. THE SPHERICAL FUNCTION

W. Casselman

It will be shown in this paper how results from the general theory of admissible representations of $\mathfrak{p}$-adic reductive groups (see mainly [7]) may be applied to give a new proof of Macdonald's explicit formula for zonal spherical functions ([9] and [10]). Along the way I include many results which will be useful in subsequent work.

Throughout, let k be a non-archimedean locally compact field, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ its prime ideal, and q the order of the residue field.

If $\mathbf{H}$ is any algebraic group defined over k, H will be the group of its k-rational points.

For any k-analytic group H, let $C_c^\infty(H)$ be the space of locally constant functions of compact support: $H \to \mathbf{C}$. For any subset X of H, let ch_X or $ch(X)$ be its characteristic function (which lies in $C_c^\infty(H)$ if X is compact and open).

Fix a connected reductive group $\mathbf{G}$ defined over k. Let $\tilde{\mathbf{G}}$ be the simply connected covering of its derived group $\mathbf{G}^{\mathrm{der}}$, $\mathbf{G}^{\mathrm{adj}}$ the quotient of $\mathbf{G}$ by its centre, and $\psi : \tilde{\mathbf{G}} \to \mathbf{G}$ the canonical homomorphism. If $\mathbf{H}$ is any subgroup of $\mathbf{G}$, let $\tilde{\mathbf{H}}$ be its inverse image in $\tilde{\mathbf{G}}$.

Fix also a minimal parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Let $\mathbf{A}$ be a maximal split torus contained in $\mathbf{P}$, $\mathbf{M}$ the centralizer of $\mathbf{A}$, $\mathbf{N}$ the unipotent radical of $\mathbf{P}$, and $\mathbf{N}^-$ the unipotent radical of the parabolic opposite to $\mathbf{P}$. Let Σ be the roots of $\mathbf{G}$ with respect to $\mathbf{A}$, ${}^{nd}\Sigma$ the subset of nondivisible roots, Σ^+ the positive roots determined by $\mathbf{P}$, Δ the simple roots in Σ^+, W the Weyl group. For any $\alpha \in \Sigma$, let $\mathbf{N}_\alpha$ be the subgroup of $\mathbf{G}$ constructed in §3 of [2] (its Lie algebra is $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$).

Let δ be the modulus character of $P : mn \to |\det Ad_{\mathfrak{n}}(m)|$. Let w_ℓ be the longest element of W.

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If H is a compact group, $\mathcal{P}_H$ is the projection operator onto H-invariants.

In §1 I shall give an outline of the results from Bruhat-Tits that I shall need. Complete proofs have not yet appeared, but the necessary facts are not difficult to prove when **G** is split (see [8]) or even unramified – i.e. split over an unramified extension of k. There is no serious loss if one restricts oneself to unramified **G**, since any reductive group over a global field is unramified at almost all primes, and important applications will be global. As far as understanding the main ideas is concerned, one may assume **G** split. This will simplify both arguments and formulae considerably.

Since the first version of this paper was written, Matsumoto's book [12] has appeared with another proof of Macdonald's formula, in a more general form valid not just for the spherical functions on $\mathfrak{p}$-adic groups but for those related to more general Hecke algebras.

1. The structure of G

Let $\mathcal{B}$ be the Bruhat-Tits building of $\tilde{G}$. (Refer to [6], Chapter II of [10], and [13].)

There exists in $\mathcal{B}$ a unique apartment $\mathcal{A}$ stabilized by $\tilde{A}$. The stabilizer $\tilde{\mathcal{N}}$ of $\mathcal{A}$ in $\tilde{G}$ is equal to the normalizer $N_{\tilde{G}}(\tilde{A})$; let $\nu : \tilde{\mathcal{N}} \to \mathrm{Aut}(\mathcal{A})$ be the corresponding homomorphism. The dimension of $\mathcal{A}$ over $\mathbf{R}$ is equal to that of $\tilde{A}$ over k, say r, and the image of $\tilde{A}$ with respect to ν is a free group of rank r. Therefore the translations are precisely those elements of $\mathrm{Aut}(\mathcal{A})$ commuting with $\nu(\tilde{A})$, so that the inverse image of the translations is $\tilde{M}$. The kernel of ν is the maximal compact open subgroup $\tilde{M}_0$ of $\tilde{M}$. Let $\tilde{A}_0$ be $\tilde{A} \cap \tilde{M}_0$, which is maximal compact and open in $\tilde{A}$.

There exists on $\mathcal{A}$ a canonical affine root system Σ_{aff}. Let W_{aff} be the associated affine Weyl group. Choose once and for all in this paper a special point $x_0 \in \mathcal{A}$, let Σ_0 be the roots of Σ_{aff} vanishing at x_0, and let W_0 be the isotropy subgroup of W_{aff} at x_0. Then Σ_0 is a finite reduced root system and W_0 its Weyl group. The homomorphism ν is a surjection from $\tilde{\mathcal{N}}$ to W_{aff}, and therefore induces isomorphisms of $\tilde{\mathcal{N}}/\tilde{M}_0$ with W_{aff} and of $\tilde{\mathcal{N}}/\tilde{M}$ with W_0. It also induces an injection of $\tilde{A}/\tilde{A}_0$ into $\mathcal{A} : a \to \nu(a)x_0$, and one may therefore identify Σ_0 with a root system in the vector space $\mathrm{Hom}(\tilde{A}/\tilde{A}_0, \mathbf{R})$. The map taking the rational character α to the function $a \mapsto -\mathrm{ord}_{\mathfrak{p}}(\alpha(a))$ allows one also to identify Σ with a root system in $\mathrm{Hom}(\tilde{A}/\tilde{A}_0, \mathbf{R})$. The two root systems one thus obtains are not necessarily the same or even homothetic, but what is true is that each $\alpha \in \Sigma$ is a positive multiple of a unique root $\lambda(\alpha)$ in Σ_0. The map λ is a bijection between ${}^{\mathrm{nd}}\Sigma$ and Σ_0. Let Σ_0^+, Δ_0 correspond to Σ^+, Δ. Let $\mathscr{C}$ be the vectorial chamber $\{\alpha(x) > 0$ for all $\alpha \in \Sigma_0^+\}$, and let C be the affine chamber of $\mathscr{A}$ contained in $\mathscr{C}$ which has x_0 as vertex.

Let $\tilde{B}$ be the Iwahori subgroup fixing the chamber C. It also fixes every element of C.

For each $\alpha \in \Sigma_{\mathrm{aff}}$, let $\tilde{N}(\alpha)$ be the group $\{n \in \tilde{N} \mid nx = x$ for all $x \in \mathscr{A}$ with $\alpha(x) \geq 0\}$. Then:

(1) $$\tilde{N}(\alpha+1) \subsetneq \tilde{N}(\alpha);$$

(2) For any $g \in \tilde{\mathcal{N}}$, $g\tilde{N}(\alpha)g^{-1} = \tilde{N}(\nu(g)\alpha)$;

(3) For any $\alpha \in {}^{\mathrm{nd}}\Sigma$, the group $\tilde{N}_\alpha$ is the union of the $\tilde{N}(\lambda(\alpha)+i)$ $(i \in Z)$;

(4) $$\tilde{N}(-\alpha) - \tilde{N}(-\alpha+1) \subseteq \tilde{N}_\alpha \nu^{-1}(w_\alpha)\tilde{N}_\alpha;$$

(5) If $\tilde{N}_0 = \Pi\tilde{N}(\alpha)(\alpha \in \Sigma_0^+)$ and $\tilde{N}_1^- = \Pi\tilde{N}(-\alpha+1)(\alpha \in \Sigma_0^+)$ then one has the *Iwahori factorization* $\tilde{B} = \tilde{N}_1^- \tilde{M}_0 \tilde{N}_0$.

As a consequence of (2):

(6) For $m \in \tilde{M}$ and $\alpha \in \Sigma_0$, $m\tilde{N}(\alpha+i)m^{-1} = \tilde{N}(\alpha+i-\alpha(\nu(m)x_0))$.

Let $\tilde{\alpha}$ be the dominant root in Σ_0, and let S_{aff} be $\{w_\alpha \mid \alpha \in \Delta_0$ or $\alpha = \tilde{\alpha} - 1\}$. Then $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter group, and in fact $(\tilde{G}, \tilde{B}, \tilde{\mathcal{N}}, S_{\mathrm{aff}})$ is an affine Tits system.

Recall that the Hecke algebra $\mathscr{H}(\tilde{G}, \tilde{B})$ is the space of all compactly supported functions $f: \tilde{G} \to \mathrm{C}$ which are right- and left-$\tilde{B}$-invariant, endowed with the product given by convolution. (Here $\tilde{B}$ is assumed to have measure 1, so that $ch(\tilde{B})$ is the identity of this algebra.) As a linear space it has the basis $\{ch(\tilde{B}w\tilde{B})$ $(w \in W_{\mathrm{aff}})\}$.

(7) If $w \in W_{\mathrm{aff}}$ has the reduced expression $w = w_1 \cdots w_p$ $(w_i \in S_{\mathrm{aff}})$ then $ch(\tilde{B}w\tilde{B}) = \Pi ch(\tilde{B}w_i\tilde{B})$.

For any $w \in W_{\mathrm{aff}}$, define $q(w)$ to be $[\tilde{B}w\tilde{B} : \tilde{B}]$. Then

(8) $$ch(\tilde{B}w_\alpha\tilde{B})^2 = (q(w_\alpha)-1)ch(\tilde{B}w_\alpha\tilde{B}) + q(w_\alpha)ch(\tilde{B}) \quad (\alpha \in S_{\mathrm{aff}})$$

For any $\alpha \in \Sigma_0$, define

(9) $$a_\alpha = w_\alpha \circ w_{\alpha-1}.$$

It is a translation of $\mathscr{A}$ whose inverse image in $\tilde{M}$ is a coset of $\tilde{M}_0$, and I shall often treat it as if it were an element of this coset. Because of (6),

$$a_\alpha \tilde{N}(\alpha + i) a_\alpha^{-1} = \tilde{N}(\alpha + i + 2) \tag{10}$$

or, in other words, $a_\alpha(\alpha) = \alpha - 2$.

1.1. Remark: There is another way to consider a_α which may be more enlightening. If $\tilde{G}$ is of rank one, then $\tilde{M}/\tilde{M}_0$ is a free group of rank one over Z, and a_α is the coset of $\tilde{M}_0$ which generates this group and takes $-\mathscr{C}$ into itself. If $\tilde{G}$ is not necessarily of rank one and $\alpha \in \Delta_0$, then the standard parabolic subgroup associated to $\Delta - \{\lambda^{-1}(\alpha)\}$ has the property that its derived group is of rank one and again simply connected ([3] 4.3) and a_α for $\tilde{G}$ is the coset of $\tilde{M}_0$ containing the a_α for this group. If α is not necessarily in Δ_0, there will exist $w \in W_0$ such that $\beta = w^{-1}\alpha \in \Delta_0$; let $a_\alpha = w a_\beta w^{-1}$. If G is split, the construction is even simpler; let a_α be the image of a generator of $\mathfrak{p}$ with respect to the *co-root* $\alpha_*: \mathbf{G}_m \to \tilde{\mathbf{G}}$.

It is always true that:

$$\text{For any } w \in W_0,\ w a_\alpha w^{-1} = a_{w\alpha}. \tag{11}$$

For each $\alpha \in \Sigma_{\text{aff}}$, let

$$q_\alpha = [\tilde{N}(\alpha - 1) : \tilde{N}(\alpha)]. \tag{12}$$

Because of (10), $q_{\alpha+2}$ is always the same as q_α, but it is not necessarily the same as $q_{\alpha+1}$. Macdonald ([10] III) defines the subset Σ_1 with $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_0 \cup \frac{1}{2}\Sigma_0$; $\alpha/2$ (for $\alpha \in \Sigma_0$) lies in Σ_1 if and only if $q_{\alpha+1} \neq q_\alpha$. He proves that Σ_1 is a root system, and for each $\alpha \in \Sigma_0$ defines $q_{\alpha/2}$ to be $q_{\alpha+1}/q_\alpha$. Then:

$$\text{For } \alpha \in \Sigma_0, [\tilde{N}(\alpha + 1) : \tilde{N}(\alpha + m + 1)] = q_{\alpha/2}^{[m/2]} q_\alpha^m; \tag{13}$$

$$\text{For } \alpha \in \Delta_0,\ q(w_\alpha) = q_{\alpha/2} q_\alpha; \tag{14}$$

When $\tilde{G}$ has rank one and $\alpha > 0$,

$$\delta(a_\alpha) = 1/[\tilde{N}(\alpha) : a_\alpha \tilde{N}(\alpha) a_\alpha^{-1}] = q_{\alpha/2}^{-1} q_\alpha^{-2}. \tag{15}$$

It may happen that $q_{\alpha/2} < 1$. For example, if $\tilde{G}$ has rank one then there are two possible inequivalent choices of the special point, and if q_α is not always equal to $q_{\alpha+1}$ then for one of these choices $q_{\alpha/2}$ will be <1, for the other >1. The second choice is better in some sense; the corresponding maximal compact subgroup is what Tits [13] calls hyperspecial. In general, a simple argument on root hyperplanes will show that there is always some choice of x_0 which assures $q_{\alpha/2} \geq 1$ for all $\alpha > 0$.

This completes my summary of the simply connected case.

The algebraic group of automorphisms of $\mathbf{G}$ contains $\mathbf{G}^{\mathrm{adj}}$, and therefore there is a canonical homomorphism from $\mathbf{G}$ to $\mathrm{Aut}(\tilde{\mathbf{G}})$. Thus G acts on $\tilde{G}: x \mapsto {}^g x$. If X is a compact subset of $\tilde{G}$, so is ${}^g X$, so that this action of G preserves what [6] calls the *bornology* of G. By [6], 3.5.1. the morphism $\psi: \tilde{G} \to G$ is *$\tilde{B}$-adapted.* This means ([6] 1.2.13) that for each $g \in G$ the subgroup ${}^g\tilde{B}$ is conjugate in $\tilde{G}$ to $\tilde{B}$, or that there exists $h \in \tilde{G}$ such that $hBh^{-1} = \psi^{-1}(g\psi(\tilde{B})g^{-1}) = {}^g\tilde{B}$. The action of G on $\tilde{G}$ therefore induces one of G on $\mathcal{B}$.

The stabilizer of $\mathcal{A}$ in G is $\mathcal{N} = N_G(A)$. Let here, too, ν be the canonical homomorphism: $\mathcal{N} \to \mathrm{Aut}(\mathcal{A})$. The inverse image of the translations is M.

Theorem 3.19 of [4] and its proof assert that the inclusion of M into G induces an isomorphism of $M/\psi(\tilde{M})Z_G$ with $G/\psi(\tilde{G})Z_G$, hence that every $g \in G$ may be expressed as $m\psi(\tilde{g})$ with $m \in M$, $\tilde{g} \in \tilde{G}$. Since $m\mathcal{A} = \mathcal{A}$, this implies that one may choose the h above so that simultaneously $h\tilde{B}h^{-1} = {}^g\tilde{B}$ and $h\tilde{\mathcal{N}}h^{-1} = {}^g\tilde{\mathcal{N}}$. Therefore ψ is $\tilde{B}-\tilde{\mathcal{N}}$-adapted ([6] 1.2.13).

Since $\tilde{\mathcal{N}}/\tilde{M} \cong \mathcal{N}/M \cong W$, ψ is of *connected type* ([6] 4.1.3). Let $G_1 = \{g \in G \mid |\chi(g)| = 1$ for all rational characters $\chi: G \to G_m\}$. If G^{der} is the derived group of G, then $\psi(\tilde{G}) \subseteq G^{\mathrm{der}} \subseteq G_1$; [4] 3.19 implies that $\psi(\tilde{G})$ is closed in G and $G^{\mathrm{der}}/\psi(\tilde{G})$ compact, while it is clear that G_1/G^{der} is compact. Therefore $G_1/\psi(\tilde{G})$ is compact.

Let

$$B = \{g \in G_1 \mid gx = x \text{ for all } x \in C\}$$

$$K = \{g \in G_1 \mid gx_0 = x_0\}.$$

Since $\tilde{B}$ is compact, so is $\psi(\tilde{B})$ and furthermore $B \cap \psi(\tilde{G}) = \psi(\tilde{B})$. Therefore since $G_1/\psi(\tilde{G})$ is compact, so is B. Since $B \subseteq K$ and K/B is finite, K is also compact. The subgroup K is what [6] calls a *special, good, maximal bounded* subgroup of G.

Let $\mathcal{N}_K = \mathcal{N} \cap K$ and $M_0 = M \cap K = M \to B$. The injection of $\mathcal{N}_K/M_0$ into W is an isomorphism ([6] 4.4.2). *From now on I assume every representative of an element of W to lie in K.* Such a representative is determined up to multiplication by an element of M_0.

The triple $(K, B, \mathcal{N}_K)$ form a Tits system with Weyl group W, and therefore

(16) K is the disjoint union of the BwB $(w \in W)$;

(17) $$[K:B] = \Sigma\, [BwB:B] = \Sigma\, q(w)\ (w \in W).$$

The group G has the *Iwasawa decomposition* ([6] 4.4.3)

(18) $$G = PK$$

and a refinement:

(19) G is the disjoint union of the PwB $(w \in W)$.

Let

$$M^- = \{m \in M \mid m^{-1}\mathscr{C} \subseteq \mathscr{C}\};$$
$$A^- = A \cap M^-.$$

The group A^- is also $\{a \in A \mid |\alpha(a)| \leq 1\}$ for $\alpha \in \Delta\}$, so that this terminology agrees with that of [7].

The group G has the *Cartan decomposition* ([6] 4.4.3):

(20) $$G = KM^-K.$$

Let ξ be the canonical homomorphism ([6] 1.2.16) from G to the group of automorphisms of $\mathscr{A}$ taking C to itself, and let $G_0 = G_1 \cap \ker(\xi)$. The triple $(G_0, B, \mathcal{N} \cap G_0)$ form a Tits system with affine Weyl group isomorphic to W_{aff}, and ψ induces an isomorphism between the Hecke algebras $\mathscr{H}(\tilde{G}, \tilde{B})$ and $\mathscr{H}(G_0, B)$ ([6] 1.2.17). Define Ω to be the subgroup of $\mathcal{N}/M_0$ of elements taking C to itself. Then elements of Ω normalize B, and hence for any $\omega \in \Omega$, $w \in W_{\text{aff}}$

(21) $$ch(B\omega B)ch(BwB) = ch(B\omega wB)$$

in $\mathscr{H}(G, B)$. Furthermore the group $\mathcal{N}/M_0$ is a semi-direct product of Ω and W_{aff}, and

(22) G is the disjoint union of the BxB $(x \in \mathcal{N}/M_0)$.

(In fact, $(G, B, \mathcal{N})$ form a *generalized* Tits system – see [8].) As a corollary of (7), (8), (21), and the isomorphism between $\mathscr{H}(\tilde{G}, \tilde{B})$ and $\mathscr{H}(G_0, B)$:

1.2. PROPOSITION: *In any finite-dimensional module over $\mathcal{H}(G, B)$ each $ch(BxB)$ $(x \in \mathcal{N})$ is invertible.*

For $\alpha \in \Sigma_{\mathrm{aff}}$, define $N(\alpha)$ to be $\psi(\tilde{N}(\alpha))$. Since $\psi \mid \tilde{N}$ is an isomorphism with N, all the properties stated earlier for the $\tilde{N}(\alpha)$ hold also for the $N(\alpha)$. In particular, for example:

(23) B has the Iwahori factorization $B = N_1^- M_0 N_0$.

From now on let $P_0 = M_0 N_0$.

There is a nice relationship between the Bruhat decompositions of G and K:

1.3. PROPOSITION: *For any $w \in W$*
(a) $BwB \subseteq \bigcup PxP$ $(x > w)$;
(b) $BwB \cap PwP = P_0 w N_0$.

PROOF: I first claim that $BwB = BwN_0$. To see this, observe that the Iwahori factorization of B gives

$$BwB = BwN_1^- M_0 N_0 = BwN_1^- N_0$$

but then $wN_1^- = wN_1^- w^{-1} \cdot w \subseteq Bw$.

Next,

$$BwN_0 = P_0 N_1^- wN_0$$

and

$$\begin{aligned} N_1^- w &= w_\ell N_1 w_\ell^{-1} \cdot w \\ &\subseteq Pw_\ell P \cdot Pw_\ell wP \\ &\subseteq \bigcup Pw_\ell yP (y < w_\ell w) \end{aligned}$$

by Lemma 1, p. 23, of [5]. But according to the Appendix, $y < w_\ell w$ if and only if $w_\ell y > w$, and this proves 1.3(a).

For (b), it suffices to show that for $n^- \in N_1^-$, if $n^- w \in PwP$ then $n^- \in wPw^{-1}$. But if $n^- w \in PwP = PwN$, one has $n^- w = pwn$ with $p \in P$, $n \in N$ and then $n^- = p \cdot wnw^{-1}$. As is well known, elements of the group wNw^{-1} factor uniquely according to $wNw^{-1} = (wNw^{-1} \cap N)(wNw^{-1} \cap N^-)$. Hence $n^- \in wNw^{-1} \cap N^-$.

In the rest of this paper, the notation will be slightly different. The main point is that it is clumsy to have to refer to both the Bruhat-Tits system Σ_0 and the system Σ arising from the structure of G as a reductive algebraic group. Therefore I shall often confound $\alpha \in {}^{\mathrm{nd}}\Sigma$

with $\lambda(\alpha) \in \Sigma_0$ – referring for example to q_α instead of $q_{\lambda(\alpha)}$, etc. Also I shall write $N_{\alpha,i}$ (for $\alpha \in {}^{\mathrm{nd}}\Sigma$) instead of $N(\alpha + i)$, and refer to a_α as an element of G or a coset of M_0, when what I really mean is $\psi(a_\alpha)$.

2. Elementary properties of the principal series

If σ is a complex character of M – i.e. any continuous homomorphism from M to $\mathbf{C}^\times$ – it is said to be *unramified* if it is trivial on M_0. Because the group M/M_0 is a free group of rank r, the group $X_{nr}(M)$ of all unramified characters of M is isomorphic to $(\mathbf{C}^\times)^r$. This isomorphism is non-canonical, but the induced structure of a complex analytic group is canonical.

I assume all characters of M to be unramified from now on.

The character χ of M determines as well one of P, since $M \cong P/N$. The *principal series* representation of G induced by this (which is itself said to be unramified) is the right regular representation R of G on the space $I(\chi) = \mathrm{Ind}(\chi \mid P, G)$ of all locally constant functions $\phi : G \to C$ such that $\phi(pg) = \chi\delta^{1/2}(p)\phi(g)$ for all $p \in P$, $g \in G$. This representation is admissible ([7] §3).

Define the G-projection $\mathcal{P}_\chi$ from C_c^∞ onto $I(\chi)$:

$$\mathcal{P}_\chi(f)(g) = \int_P \chi^{-1}\delta^{1/2}(p) f(pg)\, \mathrm{d}p$$

Here and elsewhere I assume P to have the left Haar measure according to which meas $P_0 = 1$.

For each $w \in W$, let $\phi_{w,\chi} = \mathcal{P}_\chi(ch_{BwB})$, and let $\phi_{K,\chi} = \mathcal{P}_\chi(ch_K)$. (I shall often omit the reference to χ). Thus ϕ_w is identically 0 off PwB and $\phi_w(pwb) = \chi\delta^{1/2}(p)$ for $p \in P$, $b \in B$.

2.1. Proposition: *The functions $\phi_{w,\chi}(w \in W)$ form a basis of $I(\chi)^B$.*

This is because G is the disjoint union of the open subsets PwB (1.9)).

2.2. Corollary: *The function $\phi_{K,\chi}$ is a basis of $I(\chi)^K$.*

Of course this also follows directly from the Iwasawa decomposition.

Recall from [7] §3 that if (π, V) is any admissible representation of

G then $V(N)$ is the subspace of V spanned by $\{\pi(n)v - v \mid n \in N, v \in V\}$, and that the *Jacquet module* V_N is the quotient $V/V(N)$. If V is finitely generated as a G-module then V_N is finite-dimensional ([7] Theorem 3.3.1). Since $V(N)$ is stable under M, there is a natural smooth representation π_N of M on V_N.

According to [7] Theorem 6.3.5, if $V = I(\chi)$ then V_N has dimension equal to the order of W. This suggests:

2.3. PROPOSITION: *The canonical projection from $I(\chi)^B$ to $I(\chi)_N$ is a linear isomorphism.*

I shall give two proofs of this. The first describes the relationship between $I(\chi)^B$ and $I(\chi)_N$ in more detail, but the second shows this proposition to be a corollary of a much more general result.

The first: it is shown in §6.3 of [7] that one has a filtration of $I(\chi)$ by P-stable subspaces I_w $(w \in W)$, decreasing with respect to the partial order on W mentioned in the Appendix. The space I_w consists of the functions in $I(\chi)$ with support in $\cup\, PxP$, $(x > w)$ and clearly $I_x \subseteq I_y$ when $y < x$. According to Proposition 1.3(a), ϕ_w lies in I_w. Each space $(I_w)_N/\Sigma\,(I_x)_N$ $(x > w,\ x \neq w)$ is one-dimensional ([7] 6.3.5), and the map on I_w which takes ϕ to

$$\int_{w^{-1}Nw\cap N\setminus N} \phi(wn)\,\mathrm{d}n$$

induces a linear isomorphism of this space with $\mathbf{C}$. It is easy to see, then, from Proposition 1.3(b) that the image of ϕ_w with respect to this map is non-trivial, and this proves 2.3.

For the second proof:

2.4. PROPOSITION: *If (π, V) is any admissible representation of G, then the canonical projection from V^B to $V_N^{M_0}$ is a linear isomorphism.*

PROOF: Because B has an Iwahori factorization with respect to P, Theorem 3.3.3 of [7] implies surjectivity.

For injectivity, suppose $v \in V^B \cap V(N)$. Then Lemma 4.1.3 of [7] implies the existence of $\epsilon > 0$ such that $\pi(ch_{BaB})v = 0$ for $a \in A^-(\epsilon)$ (where $A^-(\epsilon) = \{a \in A \mid |\alpha(a)| < \epsilon \text{ for all } \alpha \in \Delta\}$). Apply Proposition 1.2.

This proof of injectivity is Borel's (see Lemma 4.7 of [1]).

Proposition 2.4 may be strengthened to give as well a relationship between the structure of V^B as a module over the Hecke algebra $\mathcal{H}(G, B)$ and that of V_N as a smooth representation of M:

2.5. PROPOSITION: *Let (π, V) be an admissible representation of G, $v \in V$ with image $u \in V_N$. Then for any $m \in M^-$ the image of $\pi(ch_{BmB})v$ in V_N is equal to* $\operatorname{meas}(BmB)\pi_N(m)v$.

PROOF: If $v \in V^B$, then because $m^{-1}N_1^- m \subseteq N_1^-$ (1.6), $\pi(m)v \in V^{M_0 N_1^-}$. Jacquet's First Lemma ([7] 3.3.4) implies that $v_0 = \operatorname{meas}(BmB)^{-1}\pi(ch_{BmB})v = \mathcal{P}_B(\pi(m)v)$ and $\pi(m)v$ have the same image in V_N.

There are two more results one can derive from Proposition 2.4.

2.6. PROPOSITION: *If (π, V) is any irreducible admissible representation of G with $V^B \neq 0$, then there exists a G-embedding of V into some unramified principal series. Conversely, if V is any non-trivial G-stable subspace of an unramified principal series, then $V^B \neq 0$.*

PROOF: Recall the version of Frobenius reciprocity given as 3.2.4 in [7]:

$$\operatorname{Hom}_G(V, \operatorname{Ind}(\chi \mid P, G)) \cong \operatorname{Hom}_M(V_N, \chi\delta^{1/2}).$$

If V is a subspace of $I(\chi)$ then the left-hand side is non-trivial, hence the right-hand side. This means that $V_N^{M_0} \neq 0$, and by 2.4 neither is V^B trivial. If $V^B \neq 0$ on the other hand, then 2.4 implies that $V_N^{M_0} \neq 0$. Since it is finite-dimensional, there exists some one-dimensional M-quotient, hence by Frobenius reciprocity a G-morphism into an unramified principal series.

2.7. PROPOSITION: *The G-module $I(\chi)$ is generated by $I(\chi)^B$.*

PROOF: If U is the quotient of $I(\chi)$ by the G-space generated by $I(\chi)^B$, then $U^B = 0$. The linear dual of U^B is canonically isomorphic to $\tilde{U}^B$, where $\tilde{U}$ is the space of the admissible representation contragredient to U (see §2 of [7]), and hence $\tilde{U}^B = 0$ as well. But since U is a quotient of $I(\chi)$, $\tilde{U}$ is a subspace of $I(\chi^{-1})$, which is the contragredient of $I(\chi)$ ([7] 3.1.2). Proposition 2.6 implies that $\tilde{U}$ is trivial and therefore also U.

3. Intertwining operators

Assume in this section that all characters χ of M are *regular* – i.e. that whenever $w \in W$ is such that $w\chi = \chi$ then $w = 1$.

With this condition satisfied, it is shown in §6.4 of [7] that for each $x \in K$ representing $w \in W$ there exists a unique G-morphism $T_x : I(\chi) \to I(w\chi)$ such that for all $\phi \in I(\chi)$ with support in $\cup PyP$ $(y \not< w^{-1}) \cup Pw^{-1}P$

$$(1) \qquad T_x\phi(1) = \int_{wNw^{-1}\cap N\setminus N} \phi(x^{-1}n)\,dn.$$

Here $wNw^{-1} \cap N\setminus N$ is assumed to have the Haar measure such that the orbit of $\{1\}$ under N_0 has measure 1. Since χ is unramified, one sees easily that T_x is independent of the choice of $x \in K$ representing w, and one may call it T_w. Furthermore, it is shown in §6.4 of [7] that T_w varies holomorphically with χ in the sense that for a fixed $f \in C_c^\infty(G)$ and $g \in G$, $T_w(\mathcal{P}_\chi f)(g)$ is a holomorphic function of χ. Finally, every G-morphism from $I(\chi)$ to $I(w\chi)$ is a scalar multiple of T_w.

The operator T_w is in particular a B-morphism and a K-morphism, so it takes $I(\chi)^B$ to $I(w\chi)^B$ and $I(\chi)^K$ to $I(w\chi)^K$. Therefore it takes $\phi_{K,\chi}$ to a scalar multiple of $\phi_{K,w\chi}$.

For each $\alpha \in \Sigma$, define

$$c_\alpha(\chi) = \frac{(1 - q_{\alpha/2}^{-1/2} q_\alpha^{-1}\chi(a_\alpha))(1 + q_{\alpha/2}^{-1/2}\chi(a_\alpha))}{1 - \chi(a_\alpha)^2}.$$

3.1. THEOREM: *One has*

$$T_w(\phi_{K,\chi}) = c_w(\chi)\phi_{K,w\chi}$$

where

$$c_w(\chi) = \Pi c_\alpha(\chi) \quad (\alpha > 0,\ w\alpha < 0).$$

PROOF: Step (1). Assume G to be of semi-simple rank one, α the single non-multipliable positive root, and $w = w_\alpha$ the single non-trivial element of W. Since $\phi_K(1) = 1$, and one knows $T_w(\phi_K)$ to be a multiple of ϕ_K, it suffices to calculate $T_w(\phi_K)(1)$. Since $K = B \cup BwB$, $\phi_K = \phi_1 + \phi_w$, and one only need evaluate $T_w(\phi_1)(1)$ and $T_w(\phi_w)(1)$ separately.

Evaluating the second is simple, since ϕ_w has support in PwP, and in fact $\phi_w(wn) = 1$ if $n \in N_0$ and 0 if $n \in N - N_0$:

$$T_w(\phi_w)(1) = \int_N \phi(wn)\,\mathrm{d}n$$
$$= \int_{N_0} \mathrm{d}n = 1.$$

As for the first, since T_w varies holomorphically with χ it suffices to calculate $T_w(\phi_1)(1)$ for all χ in some open set of $X_{nr}(M)$. Define $\Phi = \Phi_\chi$ on PwP:

$$\Phi(n_1 m w n_2) = \chi^{-1}\delta^{1/2}(m).$$

For $f \in C_c^\infty(PwP) \subseteq C_c^\infty(G)$,

$$T_w(\mathcal{P}_\chi f(1) = \int_{PwP} \Phi(x) f(x)\,\mathrm{d}x.$$

Here the measure adopted on PwP is the restriction of a Haar measure on G with the normalization condition that meas $P_0 w N_0 = 1$ (note that PwP is open in G). This formula actually makes sense for all $f \in C_c^\infty(G)$ under certain conditions on χ:

3.2. LEMMA: *If $|\chi(a)| < 1$ for all regular elements of A^-, then for every $f \in C_c^\infty(G)$ the integral*

$$\int_{PwP} \Phi_\chi(x) f(x)\,\mathrm{d}x$$

converges absolutely and is equal to $T_w(\mathcal{P}_\chi(f))(1)$. If $f = ch_B$, then it is equal to $c_\alpha(\chi) - 1$.

PROOF: It suffices to let f be the characteristic function of a set of the form $N_n^- X$, where X is an open subgroup of P_0 and N_n^- ($n \geq 1$) is the subgroup of §1. This is because every function in $C_c^\infty(G)$ is a linear combination of (1) a function in $C_c^\infty(PwP)$ and (2) right P-translates of such characteristic functions. For $f = ch(N_n^- x)$, the above integral is equal to

$$\int_{N_n^- X} \Phi_\chi(x)\,\mathrm{d}x = [P_0 : X]^{-1} \int_{N_n^-} \phi_\chi(x)\,\mathrm{d}x$$

where the measure on N_n^- is such that meas $N_1^- = [BwB : B]^{-1} = (q_\alpha q_{\alpha/2})^{-1}$. This may be not quite obvious – it is because the Haar measure adopted on G is $(q_\alpha q_{\alpha/2})^{-1}$ times the one in which meas $B = 1$, $B = N_1^- P_0$, and $\Phi(xp) = \Phi(x)$ for $p \in P_0$.

Recall from 1.(1), 1.(3), and 1.(4) that

$$N_n^- = (N_n^- - N_{n+1}^-) \cup (N_{n+1}^- - N_{n+2}^-) \cup \cdots$$

and

$$N_m^- - N_{m+1}^- \subseteq N a_\alpha^{-m} w_\alpha N.$$

Therefore the integral above is equal to

$$\sum_n^\infty [\text{meas}(N_m^- - N_{m+1}^-)] \chi(a_\alpha)^m \delta^{1/2}(a_\alpha)^{-m}.$$

From (1.(13) one sees that

$$\text{meas } N_m^- = q_{\alpha/2}^{-[m+1/2]} q_\alpha^{-m} \quad (m \geq 1)$$

and from (1.(15) that

$$\delta(a_\alpha) = q_{\alpha/2}^{-1} q_\alpha^{-2}.$$

When $|\chi(a_\alpha)| < 1$, therefore, it is easy to deduce that the above sum is dominated by an absolutely convergent geometric series.

When $f = ch_B$, $m = 1$. The sum may be calculated explicitly by breaking it up into even and odd terms, thus concluding the proof.

For χ such that $|\chi(a_\alpha)| < 1$, the functional Λ induces a functional λ on $I(\chi)$ such that

$$\lambda(R(p)f) = \chi^{-1}\delta^{1/2}(p)\lambda(f).$$

By Frobenius reciprocity, it corresponds to a G-morphism from $I(\chi)$ to $I(w\chi)$. This must be a scalar multiple of T_w, and since for $f \in C_c^\infty(PwP)$

$$\Lambda(f) = T_w(f)(1)$$

it corresponds exactly to T_w. Therefore when $|\chi(a_\alpha)| < 1$, and by analytic continuation for all regular χ,

$$T_w(\phi_1)(1) = c_\alpha(\chi) - 1 \quad \text{and} \quad T_w(\phi_K)(1) = c_\alpha(\chi).$$

Step (2). Let G be arbitrary, but $w = w_\alpha$, $\alpha \in \Delta$, again. In this case, each ϕ_w with $w \neq 1$, w_α lies in the complement of $P \cup Pw_\alpha P$

$$T_{w_\alpha}(\phi_w)(1) = \int_{w_\alpha N w_\alpha^{-1} \cap N = N} \phi_w(w_\alpha n)\, dn = 0$$

and $T_{w_\alpha}(\phi_1)(1)$ and $T_{w_\alpha}(\phi_{w_\alpha})(1)$ may be calculated exactly as in Step (1). Since $\phi_K = \Sigma\, \phi_w$, the theorem is proven in this case.

Step (3). Proceed by induction on the length of w. Let $\Psi_w = \{\alpha > 0 \mid w\alpha < 0\}$. Then if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ (a) $\Psi_{w_1 w_2} = w_2^{-1}\Psi_{w_1} \cup \Psi_{w_2}$ and (b) $T_{w_1 w_2} = T_{w_1} T_{w_2}$, and applying these will conclude the proof.

3.3. Remark: When G is split, each $q_\alpha = q$ and each $q_{\alpha/2} = 1$. In this case,

$$c_\alpha(\chi) = \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}.$$

I won't use it in this paper, but it will be useful elsewhere to have this partial generalization:

3.4. Theorem: *If $\alpha \in \Delta$ and $\ell(w_\alpha w) > \ell(w)$, then*

$$T_{w_\alpha}(\phi_w) = (c_\alpha(\chi) - 1)\phi_w + q_\alpha^{-1} q_{\alpha/2}^{-1} \phi_{w_\alpha w}$$
$$T_{w_\alpha}(\phi_{w_\alpha w}) = \phi_w + (c_\alpha(\chi) - q_\alpha^{-1} q_{\alpha/2}^{-1})\phi_{w_\alpha w}.$$

Proof: One has

$$\begin{aligned} T_{w_\alpha}(\phi_1)(w_\alpha) &= [Bw_\alpha B : B]^{-1} R(ch(Bw_\alpha B)) T_{w_\alpha}(\phi_1))(1) \\ &= q_\alpha^{-1} q_{\alpha/2}^{-1} T_{w_\alpha}(\phi_{w_\alpha})(1) \\ &= q_\alpha^{-1} q_{\alpha/2}^{-1}. \end{aligned}$$

Since in the rank one case $T_{w_\alpha}(\phi_K) = c_\alpha(\chi)\phi_K$, one also has

$$T_{w_\alpha}(\phi_{w_\alpha})(w_\alpha) = c_\alpha(\chi) - q_\alpha^{-1} q_{\alpha/2}^{-1}$$
$$T_{w_\alpha}(\phi_{w_\alpha})(1) = 0 \quad \text{for } w \neq 1, w_\alpha.$$

Therefore, since T_{w_α} takes any ϕ_w into a linear combination of ϕ_w's:

$$T_{w_\alpha}(\phi_1) = (c_\alpha(\chi) - 1)\phi_1 + q_\alpha^{-1} q_{\alpha/2}^{-1} \phi_{w_\alpha}$$
$$T_{w_\alpha}(\phi_{w_\alpha}) = \phi_1 + (c_\alpha(\chi) - q_\alpha^{-1} q_{\alpha/2}^{-1})\phi_{w_\alpha}.$$

The theorem follows from this because $R(ch(BwB))\phi_1 = \phi_w$ and $R(ch(BwB))\phi_{w_\alpha} = \phi_{w_\alpha w}$.

This result tells the effect of T_{w_α} on $I(\chi)^B$, but to find a reasonable way to describe the effect of every T_w on $I(\chi)^B$ seems rather difficult.

As a consequence of Theorem 3.1 one has:

3.5. PROPOSITION: (a) *The operator T_w is an isomorphism if and only if $c_{w^{-1}}(w\chi)c_w(\chi) \neq 0$.*

(b) $\operatorname{Ind}(\chi)$ *is irreducible if and only if $c_{w_\ell}(w_\ell\chi)c_{w_\ell}(\chi) \neq 0$.*

PROOF: The operator $T_{w^{-1}} \circ T_w$ is a scalar multiple of the identity on $I(\chi)$. This scalar must be $c_{w^{-1}}(w\chi)c_w(\chi)$ by Theorem 3.1. If it is not 0, then T_w has an inverse. If it is 0, then either $T_w(\phi_K)$ or $T_{w^{-1}}(\phi_K) = 0$. If the first, T_w clearly has no inverse. If the second, then the image of T_w cannot be all of $\operatorname{Ind}(w\chi)$, and again has no inverse.

For (b), apply (a) and [7] 6.4.2.

3.6. PROPOSITION: *Assume that $q_{\alpha/2} \geq 1$ for all $\alpha > 0$. If $|\chi(a_\alpha)| < 1$ for all $\alpha > 0$, then ϕ_K generates $I(\chi)$.*

As I have mentioned earlier, the assumption $q_{\alpha/2} \geq 1$ amounts to restricting the initial choice of the special point x_0 – or, in other words, the subgroup K. When G is simply connected and of rank one, for example, and $q_{\alpha/2} \neq 1$ then the Proposition is true for one choice of K but not the other.

PROOF: Let U be the quotient of $I(\chi)$ by the G-space generated by ϕ_K. If $U \neq 0$, it will have an irreducible G-quotient (since it is finitely generated by Proposition 2.7). According to [7] 6.3.9 there will exist a G-embedding of this irreducible quotient into some $I(w\chi)$, and the composite map from $I(\chi)$ to $I(w\chi)$ must be a non-zero multiple of T_w. Since $U^K = 0$, $T_w(\phi_K) = 0$. Therefore $c_w(\chi) = 0$, and for some $\alpha > 0$ either $\chi(a_\alpha) = q_\alpha q_{1/2}$ or $\chi(a_\alpha) = -q_{1/2}$, contradicting the assumption.

This is the $\mathfrak{p}$-adic analogue of a well known result of Helgason on real groups.

I want now to introduce a new basis of $I(\chi)^B$ (still under the assumption that χ is regular). Recall from Proposition 2.3 that $I(\chi)^B \cong I(\chi)_N$, and again from §6.4 of [7] that $I(\chi)$ is isomorphic to the direct sum $\bigoplus (w\chi)\delta^{1/2}$. Explicitly, the maps

$$\Lambda_w : \phi \to T_w(\phi)(1)$$

form a basis of eigenfunctions of the dual of $I(\chi)_N$ with respect to the action of U. Let $\{f_w\} = \{f_{w,\chi}\}$ be the basis of $I(\chi)^B$ dual to this – thus

$$\Lambda_w(f_x) = \begin{cases} 0 & (x \neq w) \\ 1 & (x = w) \end{cases}$$

It is an unsolved problem and, as far as I can see, a difficult one to express the bases $\{\phi_w\}$ and $\{f_w\}$ in terms of one another. This is directly related to the problem I mentioned at the end of the proof of Theorem 3.4. The only fact which is simple is:

3.7. PROPOSITION: *One has* $f_{w_\ell} = \phi_{w_\ell}$.

PROOF: For $w \neq w_\ell$,

$$T_w(\phi_{w_\ell})(1) = \int_{wNw^{-1}\cap N\setminus N} \phi_{w_\ell}(w^{-1}n)\,\mathrm{d}n = 0$$

because $\mathrm{supp}(\phi_{w_\ell}) \subseteq Pw_\ell P$, while

$$\begin{aligned} T_{w_\ell}(\phi_{w_\ell})(1) &= \int_N \phi_{w_\ell}(w_\ell n)\,\mathrm{d}n \\ &= \int_{N_0} \mathrm{d}n = 1. \end{aligned}$$

Also, by the definition of the $\{f_w\}$ and Theorem 3.1:

3.8. LEMMA: *One has*

$$\phi_K = \Sigma\, c_w(\chi) f_w.$$

It follows immediately from the definition of the $\{f_w\}$ and Proposition 2.5 that:

3.9. LEMMA: *One has* $\pi(ch_{BmB})f_w = \mathrm{meas}(BmB)(w\chi)\delta^{1/2}(m)f_w$ *for all* $m \in M^-$.

4. The spherical function

As I have mentioned earlier, the contragredient of $I(\chi)$ is $I(\chi^{-1})$. Consider the matrix coefficient

$$\Gamma_\chi(g) = \langle R(g)\phi_{K,\chi}, \phi_{K,\chi^{-1}}\rangle.$$

According to [7] 3.1.3 this is also equal to

$$\int_K \phi_{K,\chi}(gk)\phi_{K,\chi^{-1}}(k)\,\mathrm{d}k = \int_K \phi_{K,\chi}(gk)\,\mathrm{d}k$$

where meas $K = 1$. The function Γ_χ is called the zonal spherical function corresponding to χ. It satisfies

(1) $$\Gamma_\chi(1) = 1;$$

(2) $$\Gamma_\chi(k_1gk_2) = \Gamma_\chi(g) \quad \text{for all } k_1, k_2 \in K \quad \text{and} \quad g \in G.$$

4.1. PROPOSITION: *For any $w \in W$, $\Gamma_{w\chi} = \Gamma_\chi$.*

PROOF: The matrix coefficient Γ_χ is the only matrix coefficient of $I(\chi)$ satisfying (1) and (2). As such, it is determined by the isomorphism class of $I(\chi)$. But since by Proposition 3.5 the representations $I(\chi)$ and $I(w\chi)$ are generically isomorphic, $\Gamma_\chi = \Gamma_{w\chi}$ generically as well; since Γ_χ clearly depends holomorphically on χ, $\Gamma_\chi = \Gamma_{w\chi}$ for all χ.

Define

$$\begin{aligned}\gamma(\chi) &= c_{w_\ell}(w_\ell\chi)\\ &= \prod_{\alpha>0} \frac{(1 - q_{\alpha/2}^{-1/2}q_\alpha^{-1}\chi(a_\alpha)^{-1})(1 + q_{\alpha/2}^{-1/2}\chi(a_\alpha)^{-1})}{1 - \chi(a_\alpha)^{-2}}\end{aligned}$$

Note that because of the Cartan decomposition, Γ_χ is determined by its restriction to M^-.

4.2. THEOREM (Macdonald): *If χ is regular then for all $m \in M^-$*

$$\Gamma_\chi(m) = Q^{-1}\Sigma\gamma(w\chi)((w\chi)\delta^{1/2})(m) \quad (w \in W)$$

where

$$Q = \Sigma q(w)^{-1} \quad (w \in W).$$

PROOF: One has

$$\phi_K = \Sigma c_w(\chi) f_w,$$

therefore

$$\begin{aligned}\Gamma_\chi(m) &= \mathscr{P}_K(R(m)\phi_K)(1)\\ &= \Sigma c_w(\chi)\mathscr{P}_K(R(m)f_w)(1)\\ &= \Sigma c_w(\chi)\mathscr{P}_K(\mathscr{P}_B R(m)f_w)(1)\end{aligned}$$

(since $B \subseteq K$)

$$= \Sigma c_w(\chi)(w\chi)\delta^{1/2}(m)\mathscr{P}_K f_w(1)$$

(by Proposition 3.9).

By Proposition 3.7,

$$\begin{aligned}\mathscr{P}_K f_{w_\ell} = \mathscr{P}_K \phi_{w_\ell} &= \text{meas}(Bw_\ell B)\phi_K\\ &= Q^{-1}\phi_K\end{aligned}$$

(by (1.9) and the remarks preceding it). Therefore the term in the sum above corresponding to w_ℓ is $Q^{-1}c_{w_\ell}(w_\ell\chi)$. By the W-invariance of Γ_χ (Proposition 4.1) and the linear independence of the χ's ([10] 4.5.7) this implies the theorem.

4.3. Remark: The general theory of the asymptotic behavior of matrix coefficients (§4 in [7]) asserts the existence of $\epsilon > 0$ such that ϕ_K is a linear combination of the characters $(w\chi)\delta^{1/2}$ on $A^-(\epsilon)$. Macdonald's formula makes this explicit.

Appendix

Let Σ be a root system, Σ^+ a choice of positive roots, and (W, S) the corresponding Coxeter group. For $x, y \in W$, define $x < y$ to mean that y has a reduced decomposition $y = s_1 \cdots s_n$, where s_i is the elementary reflection associated to the simple root α_i, and $x = s_{i_1} \cdots s_{i_m}$ with $1 \le i_1 < \cdots < i_m \le n$. According to Lemma 3.7 of [3] (an easy application of the exchange condition of [5] Chapter IV, §1.5) one may take m to be the length of x in W. If $x < y$, then $\ell(x) \le \ell(y)$, and $\ell(x) = \ell(y)$ if and only if $x = y$.

Let w_ℓ be the longest element in W. The following is, I believe, essentially due to Steinberg ([11] Exercise (a) on p. 128).

A.1. Proposition: *Let $x, y \in W$ be given. The following are equivalent*:

(a) $x < y$;

(b) $x^{-1} < y^{-1}$;

(c) *One has* $y = xw_1 \cdots w_r$, *where* w_i *is the re fection associated to the root* $\theta_i > 0$, *and* $xw_1 \cdots w_{i-1}(\theta_i) > 0$;

(d) $w_\ell x > w_\ell y$.

PROOF: (a)$\Leftrightarrow$(b) is immediate.

For (c)$\Rightarrow$(a): Suppose that y has the reduced decomposition $y = s_1 \cdots s_n$, and assume at first that $y = xw$, where w is the reflection corresponding to the root $\theta > 0$, and $x(\theta) > 0$. Then $y(\theta) = x(-\theta) < 0$, so that according to [5] Cor. 2, p. 158, there exists i such that $\theta = s_n \cdots s_{i+1}(\alpha_i)$. Then $w = (s_n \cdots s_{i+1})s_i(s_n \cdots s_{i+1})^{-1}$ and $x = s_1 \cdots s_{i-1}s_{i+1} \cdots s_n$, so that indeed $x < y$.

In the general case, let $y = xw_1 \cdots w_v$ as in (c), and let $y_i = xw_1 \cdots w_{i-1}$ for each i. By what I have just shown, $y = y_r > y_{r-1} > \cdots > x$, and since $<$ is clearly transitive, $x < y$.

(a)$\Rightarrow$(c): Proceed by induction on the length of x. If $\ell(x) = 0$, then $x = 1$ and $y = s_1 \cdots s_n$, where by [5] Cor. 2, p. 158, one has $s_1 \cdots s_{i-1}(\alpha_i) > 0$.

In general, say $x = s_{i_1} \cdots s_{i_m}$ is a reduced decomposition of x. Let $x' = s_{i_2} \cdots s_{i_m}$, $y' = s_{i_1+1} \cdots s_n$. Then $\ell(x') < \ell(x)$ and $x' < y'$, so that by the induction hypothesis $y' = x'w_1' \cdots w_r'$ as in (c), say w_i' corresponding to θ_i'. One now has

$$\begin{aligned} y &= s_1 \cdots s_{i_1}y' \\ &= s_1 \cdots s_{i_1}x'w_1' \cdots w_r' \\ &= s_1 \cdots s_{i_1-1}xw_1' \cdots w_r' \end{aligned}$$

Let $k = i_1 - 1$ for convenience. Then

$$\begin{aligned} y &= s_1 \cdots s_k x \\ &= x \cdot (x^{-1}s_kx)((s_kx)^{-1}s_{k-1}(s_kx)) \cdots ((s_2 \cdots s_kx)^{-1}s_1(s_2 \cdots s_kx)). \end{aligned}$$

Let θ_j be the root $(s_{j+1} \cdots s_kx)^{-1}(\alpha_j)$, w_j correspond to θ_j. One has

$$y = xw_kw_{k-1} \cdots w_1$$

and further (1) $\theta_j = (x^{-1}s_k \cdots s_{j+1})(\alpha_j) > 0$ according to [5] Cor. 2, p. 158, since by assumption on the original y one has $\ell(s_j \cdots s_kx) > \ell(s_{j+1} \cdots s_kx)$; (2) $xw_k \cdots w_{j+1}(\theta_j) = s_{j+1} \cdots s_kx(\theta_j) = \alpha_j > 0$.

(c)$\Leftrightarrow$(d): One has $y = xw_1 \cdots w_r$ as in (c) $\Leftrightarrow x < y \Leftrightarrow x^{-1} < y^{-1} \Leftrightarrow y^{-1} = x^{-1}w'_1 \cdots w'_s$ as in (c) $\Leftrightarrow y = w'_s \cdots w'_1 x \Leftrightarrow w_\ell y = w_\ell w'_s w_\ell^{-1} \cdots w_\ell x \Leftrightarrow (w_\ell y)^{-1} = (w_\ell x)^{-1}(w_\ell w'_1 w_\ell^{-1}) \cdots (w_\ell w'_s w_\ell^{-1})$. Note that $w_\ell w'_i w_\ell^{-1}$ is the reflection associated to $\bar{\theta}'_i = w_\ell(-\theta'_i)$.

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