COMPOSITIO MATHEMATICA

L. S. HUSCH
I. IVANŠIĆ

Shape domination and embedding up to shape

Compositio Mathematica, tome 40, nº 2 (1980), p. 153-166
<http://www.numdam.org/item?id=CM_1980__40_2_153_0>

COMPOSITIO MATHEMATICA, Vol. 40, Fasc. 2, 1980, pag 153–166

SHAPE DOMINATION AND EMBEDDING UP TO SHAPE

L.S. Husch* and I. Ivanšić

In [2; p. 354], K. Borsuk poses the following question: If X is a metric compactum such that X is shape dominated by a compactum $Y \subseteq \mathbf{R}^q$, q-dimensional Euclidean space, then does there exist a compactum $Z \subseteq \mathbf{R}^q$ which has the same shape as X? In this note we show [Theorem 12] that the answer is yes for a certain class of compacta. This class of compacta was considered by I. Ivanšić [9] who defined it in non-shape theoretic terms and proved an "embedding up to shape" theorem. We give a shape theoretic description of this class and study the structure of neighborhoods of nice embeddings of members of this class. This work is strongly motivated by Siebenmann's thesis [15] and we shall often appeal to it; [16] and [17] contain many of the concepts which we use from [15].

Since the submission of this paper, A. Kadlof [22] has constructed a continuum Y in $\mathbf{R}^3$ which shape dominates a continuum X which does not embed up to shape in $\mathbf{R}^3$. The authors express their gratitude to the referee for his comments which have led to a shortening of some of the proofs and generalization of the results.

We shall use the Mardešić–Segal approach to shape theory [12]; we use the language of pro-category theory. We refer the reader to [10] which contains most of the definitions we need. Let CW_0 [Ho-CW_0] denote the category whose objects are finite connected pointed CW-complexes and whose morphisms are pointed continuous maps [pointed homotopy classes of continuous maps]. The objects of the category pro-CW_0 are inverse systems $\mathbf{X} = (X_i, p_{ij}, \Lambda)$ where $X_i \in$ object CW_0 and the bonding maps p_{ij} are morphisms in CW_0; in this paper we shall only consider index sets Λ which are subsets of the positive integers. A morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y} = (Y_i, q_{ij}, \Gamma)$ consists of an order-preserving function $f: \Gamma \to \Lambda$ and a family $\{f_i : i \in \Gamma)$ of morphisms in

* Research was conducted while the first author was on leave at the University of Zagreb supported under an exchange program between the National Academy of Sciences (U.S.A.) and Yugoslav Council of Academies.

0010-437X/80/02/0153-14$00.20/0

CW_0, $f_i: X_{f(i)} \to Y_i$ such that if $i \le j$, then $q_{ij} f_j = f_i p_{f(i)f(j)}$. Two morphisms $\mathbf{f}, \mathbf{g}: \mathbf{X} \to \mathbf{Y}$ are homotopic, $\mathbf{f} \simeq \mathbf{g}$, if, for each $i \in \Gamma$, there exists $j \ge f(i)$, $g(i)$ such that $f_i p_{f(i)j} = g_i p_{g(i)j}$. The identity morphism $\mathbf{1}: \mathbf{X} \to \mathbf{X}$ and the composition of two morphisms are defined naturally (see [10], [12]). The category pro-Ho-CW_0 is defined similarly; if $\mathbf{X}$ is an object in pro-CW_0, then we will abuse notation by letting $\mathbf{X}$ also designate the corresponding element of pro-Ho-CW_0. Let X and Y be separable metric pointed continua. Then there exists objects $\mathbf{X}$ and $\mathbf{Y}$ in pro-CW_0 such that the inverse limits $\varprojlim X_i$ and $\varprojlim Y_i$ are homeomorphic to X and Y respectively. X and Y have the same *pointed shape*, designated shape $X =$ shape Y if there exist morphisms $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ and $\mathbf{g}: \mathbf{Y} \to \mathbf{X}$ in pro-Ho-CW_0 such that $\mathbf{fg} \simeq \mathbf{1}$ and $\mathbf{gf} \simeq \mathbf{1}$. If we do not necessarily have the relationship $\mathbf{fg} \simeq \mathbf{1}$, we say that X is *pointed shape dominated* by Y, designated shape $X \le$ shape Y.

The *fundamental dimension* of a separable metric continuum X, $Fd(X)$, is defined to be the minimum of dimensions, dim Y, of all separable metric continua Y such that shape $X =$ shape Y.

Let X be an n-dimensional continuum in $\mathbf{R}^q$; X is 1-ULC *embedded* if for each $\epsilon > 0$ there exists $\delta > 0$ such that any map of the 1-sphere into $\mathbf{R}^q - X$ whose image has diameter $< \delta$ extends to a map of the 2-cell into $\mathbf{R}^q - X$ whose image has diameter $< \epsilon$.

We shall use tools from PL (= piecewise linear) topology. [7] and [14] are good references. int, bdry and cl will denote interior, boundary and closure, respectively. Since we almost always work with pointed spaces and pointed maps, we will often "ignore" the basepoint in our notation.

Let X be a pointed metric continuum. X has *stable* pro-π_1 if there exists an object $\mathbf{X}$ in pro-CW_0 such that shape $(\varprojlim X_i) =$ shape (X) and the associated object $\{\pi_1(X_i)\}$, the induced system of fundamental groups, is isomorphic in pro-groups, shortly pro-isomorphic to a group G; i.e. there exist morphisms $\mathbf{f}: \{\pi_1(X_i)\} \to \{G\}$ and $\mathbf{g}: \{G\} \to \{\pi_1(X_i)\}$ in the category pro-groups ($\{G\}$ denotes the inverse system with one set) such that $\mathbf{fg} \simeq \mathbf{1}$ and $\mathbf{gf} \simeq \mathbf{1}$.

PROPOSITION 1: *Let X be a* 1-ULC *embedded n-dimensional continuum in q-dimensional Euclidean space $\mathbf{R}^q$, $q - n \ge 3$, $q \ge 5$. Then there exists a sequence of closed connected* PL *neighborhoods $\{U_i\}$ of X in $\mathbf{R}^q$ such that*

1.1. $U_i \subset \text{int}\, U_{i-1}$ *for all i*;

1.2. $\cap\, U_i = X$;

1.3. $\pi_j(U_i, \text{bdry}\, U_i) = 0$ *for all $j \le q - n - 1$ and all i.*

In addition, if X has stable pro-π_1 *which is pro-isomorphic to a finitely presented group, then* $\{U_i\}$ *can be chosen so that*

1.4. *the inclusion* $U_i \subset U_{i-1}$ *induces an isomorphism* $\pi_1(U_i) \to \pi_1(U_{i-j})$ *for all i.*

PROOF: We need the following result due to Štanko [19]. If X satisfies the hypotheses of Proposition 1 and if K is a closed polyhedron in $\mathbf{R}^q$ with $\dim K \leq q-n-1$, then there exists an arbitrarily small ambient isotopy of $\mathbf{R}^q$ with support arbitrarily close to $X \cap K$ which moves K off of X.

Let W be a compact connected PL-neighborhood of X in $\mathbf{R}^q$. Suppose that bdry W contains components W' and W''. Let α be a PL arc in W such that W' and W'' meet α in an endpoint of α. By the above mentioned result of Štanko, we may assume that $\alpha \cap X = \phi$. By removing a small regular neighborhood of α, we obtain a closed connected neighborhood of X with one less boundary component. Hence, by induction, we can find a sequence of closed connected PL-neighborhoods $\{W_i\}$ of X in $\mathbf{R}^q$ such that bdry W_i is connected and both 1.1 and 1.2 are satisfied.

By using Štanko as above, it can be shown that the inclusion $W_i - X \subset W_i$ induces isomorphisms on fundamental groups. If X has stable pro-π_1 which is pro-isomorphic to a finitely presented group, the sequence $\{\pi_1(W_i)\}$ is isomorphic in the category of pro-groups to a finitely presented group. By putting these two results together, $\{\pi_1(W_i - X)\}$ is also isomorphic in the category of pro-groups to a finitely presented group. It is straightforward to check that the latter condition implies Siebenmann's condition that $\{\pi_1(W_i - X)\}$ is essentially constant [16, p. 204] or stable [15, p. 14]; hence, by [15; Theorem 3.10] or [16; Proposition 1.9], we can modify $\{W_i\}$ so that 1.4 is also satisfied and, in addition, so that bdry $W_i \subseteq W_i$ induces isomorphisms of fundamental groups.

Let W_1' be the dual $(q-n-1)$-skeleton of some triangulation of W_1. By Štan'ko, we may assume that $W_1' \cap X = \phi$, and, hence, X lies in the interior of a regular neighborhood U_1 of the n-skeleton of some triangulation of W_1. Note that $\pi_j(U_1, \text{bdry } U_1) = 0$ for $j \leq q-n-1$. Choose W_{i1} so that $W_{i1} \subset \text{int } U_1$ and repeat the construction to obtain $U_2 \subset \text{int } U_1$. The Proposition follows by induction.

THEOREM 2: *Let X be a continuum which has fundamental dimension n; then there exists a tower* $\{X_i, \phi_i\}$ *in* pro-CW_0 *such that*

2.1. *dimension* $X_i \leq n$;

2.2. shape $(\underleftarrow{\lim}\, X_i) =$ shape X.

In addition, if X has stable pro-π_1 *pro-isomorphic to a finitely* *presented group, the tower can be chosen so that*

2.3. *ϕ_i induces an isomorphism $\pi_1 X_{i+1} \to \pi_1 X_i$ for each i.*

Mardešić [11] has obtained 2.1 and 2.2.

PROOF: Let Y be a continuum such that $\dim Y = n$ and shape $Y = $ shape X. By [8] and [19], there exists a $1-\text{ULC}$ embedding $\phi: Y \to \mathbf{R}^{2n+1}$. Let $\{U_i\}$ be a sequence of closed PL-neighborhoods of $\phi(Y)$ in $\mathbf{R}^{2n+1}$ as given in Proposition 1.

Since, by 1.3, the pairs $(U_i, \text{bdry } U_i)$ are n-connected for all i, by [21] there exist n-dimensional complexes $X_i \subset U_i$ such that U_i collapses to X_i. Let $r_i: U_i \to X_i$ be a retraction which is homotopic to the identity map on U_i. Let $\phi_i = r_i \mid X_{i+1}: X_{i+1} \to X_i$. It is easily checked that $\{X_i, \phi_i\}$ is a tower in CW_0 which is isomorphic in pro-Ho-CW_0 to $\{U_i\}$ and, hence, shape $(\varprojlim X_i) = $ shape X. Since, by 1.4, the inclusion map $U_{i+1} \subset U_i$ induces an isomorphism of fundamental groups and since the inclusion $X_{i+1} \subset U_{i+1}$ and the retraction $r_i: U_i \to X_i$ are homotopy equivalences, ϕ_i also induces an isomorphism of fundamental groups for each i, in the special case when X has stable pro-π_1 pro-isomorphic to a finitely presented group.

COROLLARY 3: *If a continuum X has stable* pro-π_1 *pro-isomorphic to a finitely presented group and $FdX = n \geq 3$, then X can be embedded up to shape in $\mathbf{R}^{2n}$.*

PROOF: Corollary follows immediately from Theorem 2 of [9] since every mapping $X_1 \to \mathbf{R}^{2n}$ induces an epimorphism of fundamental groups.

Let $\{X_i, p_i\}$ and $\{Y_i, q_i\}$ be towers in pro-CW_0. Suppose that $\{f_i\}: \{X_i\} \to \{Y_i\}$ is a morphism in pro-Ho-CW_0; by [18; p. 404] we may assume that each f_i is a level-preserving cellular map. Let $M(f_i)$ be the reduced mapping cylinder of f_i [-i.e., if $a_i \in Y_i$ and $x_i \in X_i$ are the base points, then $M(f_i)$ is obtained from the disjoint union $(X_i \times [0, 1]) \cup Y_i$ by identifying $(x, 1)$ and $f_i(x)$ for all $x \in X_i$ and by shrinking $(x_i \times [0, 1]) \cup \{a_i\}$ to a point m_i.] Define $\alpha_i: X_i \to M(f_i)$ and $\beta_i: Y_i \to M(f_i)$ by $\alpha_i(x) = (x, 0)$ and $\beta_i(y) = y$. Note that $M(f_i) \in$ object CW_0. The bonding maps $p_i: X_{i+1} \to X_i$, $q_i: Y_{i+1} \to Y_i$ and a homotopy between $f_i p_i$ and $q_i f_{i+1}$ induce bonding maps $\lambda_i: M(f_{i+1}) \to M(f_i)$ such that $\lambda_i \alpha_{i+1} = \alpha_i p_i$ and $\lambda_i \beta_{i+1} = \beta_i q_i$ (for example, the construction in the proof of Theorem 7 in [10, Part I] can be slightly modified to produce such maps λ_i). Let $\pi_j(f_i) = \pi_j(M(f_i), \alpha_i(X_i), m_i)$; by construction λ_i

induces a homomorphism (a function, if $j = 1$) $\pi_j(f_{i+1}) \to \pi_j(f_i)$ for each i. $\{f_i\}$ is said to be ***shape*** *r-connected* if, for each $1 < j \leq r$, the tower of groups $\{\pi_j(f_i)\}$ is isomorphic in the category of pro-groups to the trivial group and for $j = 1$, $\{\pi_1(f_i)\}$ is isomorphic in the category of pro-pointed sets to the trivial pointed set. The shape r-connectedness of $\{f_i\}$ can be equivalently described in the following way: $\{f_i\}$ is shape r-connected if it induces an isomorphism of homotopy pro-groups of $\{X_i\}$ and $\{Y_i\}$, denoted by $\pi_j(\mathbf{X})$ and $\pi_j(\mathbf{Y})$, for each $1 \leq j < r$ and an epimorphism for $j = r$ in the category of pro-groups. Namely, the above construction gives us the morphisms $\{\alpha_i\}:\{X_i\} \to \{M(f_i)\}$ and $\{\beta_i\}:\{Y_i\} \to \{M(f_i)\}$ in pro-Ho-CW$_0$, where α_i and β_i are inclusions, $\{\beta_i\}$ admits a shape inverse $\{g_i\}:\{M(f_i)\} \to \{Y_i\}$ and $\{f_i\} = \{g_i\}\{\alpha_i\}$ holds. Therefore, in the exact sequence of homotopy pro-groups (e.g. [10, Part I] p. 56) of the pair $\{M(f_i), X_i\}$

$$\cdots \longrightarrow \pi_k(\mathbf{X}) \xrightarrow{\alpha_*} \pi_k(\mathbf{M(f)}) \xrightarrow{j_*} \pi_k(\mathbf{M(f)}, \mathbf{X}) \xrightarrow{\partial} \pi_{k-1}(\mathbf{X}) \longrightarrow \cdots$$

where we identify X_i with $\alpha_i(X_i)$ and $j_i: M(f_i) \to (M(f_i), X_i)$ is the inclusion, one can replace $\pi_k(\mathbf{M(f)})$ by $\pi_k(\mathbf{Y})$ for each k. This way we obtain the following exact sequence of homotopy pro-groups

$$\cdots \longrightarrow \pi_k(\mathbf{X}) \xrightarrow{f_*} \pi_k(\mathbf{Y}) \xrightarrow{j_*\beta_*} \pi_k(\mathbf{M(f)}, \mathbf{X}) \longrightarrow \pi_{k-1}(\mathbf{X}) \longrightarrow \cdots$$

induced by a morphism $\{f_i\}:\{X_i\} \to \{Y_i\}$. Now, by [10, Part II] if we have in the category of pro-groups a short exact sequence

$$1 \longrightarrow \cdot \xrightarrow{\kappa} \cdot \longrightarrow 1$$

then κ is an isomorphism, and also, in an exact sequence of the type

$$\cdots \longrightarrow \cdot \xrightarrow{\kappa} \cdot \longrightarrow 1$$

κ is an epimorphism. The stated equivalence on r-connectedness of $\{f_i\}$ is now obvious.

Let X be a continuum. We say that X has *shape finite r-skeleton* ($r \geq 1$) if there exists a finite connected pointed CW-complex K ($K \in$ object CW$_0$) and an object $\{X_i\}$ in pro-CW$_0$ such that $X = \varprojlim X_i$, and a morphism $\{f_i\}:\{K\} \to \{X_i\}$ such that $\{f_i\}$ is shape r-connected. Note that if the latter property is valid for one system $\{X_i\}$, then for each system $\{X'_i\}$ such that $X = \varprojlim X'_i$, there exists a shape r-connected morphism $\{f'_i\}:\{K\} \to \{X'_i\}$.

We leave to the reader to check that if X and Y are continua with the same shape and if X has a shape finite r-skeleton, then Y also has a shape finite r-skeleton. Note that if K is a CW complex such that the r-skeleton of K is finite, then K has a shape finite r-skeleton.

REMARK 4: If a continuum X has shape finite r-skeleton and $r \geq 2$, then X has stable pro-π_1 pro-isomorphic to $\pi_1(K)$ which is a finitely presented group; if $r = 1$, then X is pointed 1-movable.

THEOREM 5: *Let X be a continuum of fundamental dimension $n \geq 3$. The following are equivalent for $2 \leq r < n$.*

5.1. *X has a shape finite r-skeleton.*

5.2. *There exists an object $\{X_i\}$ in* pro-CW$_0$ *such that* $\dim X_i \leq n$ *for all i,* shape $(\varprojlim X_i) =$ shape X, *and the bonding maps $X_{i+1} \to X_i$ are r-connected for all i.*

PROOF: First let us assume 5.1. Let $\{Z_i\}$ be a tower in pro-CW$_0$ which satisfies 2.1, 2.2 and 2.3. Let K be an object in CW$_0$ such that there exists a shape r-connected morphism $\{f_i\}:\{K\} \to \{Z_i\}$. Since the induced system of groups $\{\pi_j(f_i)\}$ is isomorphic in pro-groups to the trivial group for all $j \leq r$, by choosing a subsequence, if necessary, we may assume that the induced homomorphisms $\pi_j(f_{i+1}) \to \pi_j(f_1)$ are the zero homomorphisms for all i and all $1 < j \leq r$. In the case $j = 1$, we want $\pi_1(f_{i+1}) \to \pi_1(f_i)$ to be the constant map. By chasing around the following commutative diagram

$$\begin{array}{ccccccccc}
\pi_2(f_{i-1}) & \longrightarrow & \pi_1 K & \xrightarrow{(f_{i-1})_*} & \pi_1 Z_{i-1} & \xrightarrow{(\beta_{i-1})_*} & \pi_1(f_{i-1}) & \longrightarrow & 1 \\
\uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle 0} & & \\
\pi_2(f_i) & \longrightarrow & \pi_1 K & \xrightarrow{(f_i)_*} & \pi_1 Z_i & \xrightarrow{(\beta_i)_*} & \pi_1(f_i) & \longrightarrow & 1
\end{array}$$

where the rows are exact and the vertical maps are induced by the bonding maps, one can show that each f_i induces an isomorphism $\pi_1 K \to \pi_1 Z_i$.

By Theorem A of [20], $\pi_2(f_1)$ is a finitely generated $\mathbf{Z}\pi_1 K$-module. Since $\pi_2 Z_1$ is mapped onto $\pi_2(f_1)$, let $\gamma_i : S^2 \to Z_i$, $i = 1, 2, \ldots, m$, be cellular maps of the 2-sphere into Z_1 whose classes in $\pi_2 Z_1$ are mapped onto a set of generators of $\pi_2(f_1)$. Let Z_1' be the CW-complex obtained from Z_1 by attaching m 3-cells by means of the mappings γ_i. Denote by $f_1' : K \to Z_1'$ the mapping induced by f_1. Then this construction implies that the inclusion $(M(f_1), K) \to (M(f_1'), K)$ induces the trivial homomorphism $\pi_2(f_1) \to \pi_2(f_1')$.

Let D_i be a closed 3-cell which lies in the interior of the 3-cell which was attached by the map γ_i. Let $Z' = cl(Z_1' - \bigcup_{i=1}^m D_i)$; note that Z' deformation retracts to Z_1. By m applications of van Kampen's Theorem [3], it follows that the inclusion $Z_1 \subseteq Z_1'$ induces isomorphisms of fundamental groups. Let $p: \tilde{Z}_1' \to Z_1'$ be the universal covering of Z_1' and let $\tilde{Z}' = p^{-1}(Z')$, $\tilde{\Sigma} = p^{-1}(\bigcup_{i=1}^m \text{bdry } D_i)$ and $\tilde{D} = p^{-1}(\bigcup_{i=1}^m D_i)$. From the Mayer–Vietoris sequence, we obtain the exact sequence

$$H_2\left(\tilde{\Sigma}\right) \to H_2(\tilde{Z}') \oplus H_2(\tilde{D}) \to H_2(\tilde{Z}_1') \to H_1\left(\tilde{\Sigma}\right)$$

from which it follows that the inclusion $\tilde{Z}' \subseteq \tilde{Z}_1'$ induces an epimorphism $H_2(\tilde{Z}') \to H_2(\tilde{Z}_1')$. By Hurewicz's Theorem [18; p. 397] and covering space theory [18; p. 377] it follows that the inclusion $Z' \subseteq Z_1'$ induces an epimorphism $\pi_2(Z') \to \pi_2(Z_1')$ and, thus, $Z_1 \subseteq Z_1'$ induces an epimorphism $\pi_2(Z_1) \to \pi_2(Z_1')$. The following diagram

$$\begin{array}{ccccccc} \pi_2 K & \longrightarrow & \pi_2 Z_1 & \longrightarrow & \pi_2(f_1) & \longrightarrow & 1 \\ \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle 0} & & \\ \pi_2 K & \longrightarrow & \pi_2 Z_1' & \longrightarrow & \pi_2(f_1') & \longrightarrow & 1 \end{array}$$

shows at once that $\pi_2(f_1') = 1$.

By an induction argument, one can find a finite CW-complex $X_1 \supseteq Z_1$ such that the induced map $g_1: K \to X_1$ is r-connected. [We can use a Mayer–Vietoris sequence argument as above to show that when we add on higher dimensional cells we do not "undo" the connectivity of the map which we have already achieved.] Notice that the map $Z_2 \to X_1$ induced by the bonding map $Z_2 \to Z_1$ still induces an isomorphism on fundamental groups.

We now try to do a similar construction for $f_2: K \to Z_2$ but now we have to exercise more care. By Theorem A of [20] again, $\pi_2(f_2)$ is a finitely generated $\mathbf{Z}_{\pi_1}K$-module. Consider the following commutative diagram

$$\begin{array}{ccccccc} \pi_2 K & \longrightarrow & \pi_2 Z_1 & \longrightarrow & \pi_2(f_1) & \longrightarrow & 1 \\ \uparrow{\scriptstyle id} & & \uparrow & & \uparrow{\scriptstyle 0} & & \\ \pi_2 K & \longrightarrow & \pi_2 Z_2 & \longrightarrow & \pi_2(f_2) & \longrightarrow & 1. \end{array}$$

Let $\Gamma_1, \ldots, \Gamma_s$ be classes in $\pi_2 Z_2$ which map onto a set of generators of $\pi_2(f_2)$. Since $\pi_2(f_2) \to \pi_2(f_1)$ is the trivial homomorphism, the images of Γ_i in $\pi_2 Z_1$ must lie in the image of $\pi_2 K \to \pi_2 Z_1$. Suppose that Γ'_i is a class in $\pi_2 K$ which maps onto the image of Γ_i in $\pi_2 Z_1$. Let Γ''_i be the image of Γ'_i in $\pi_2 Z_2$. Note that $\{\Gamma_i - \Gamma''_i\}$ still maps onto a set of generators of $\pi_2(f_2)$. We choose our attaching maps $\gamma_i : S^2 \to Z_2$ so that $\gamma_i \in \Gamma_i - \Gamma''_i$.

Now when we attach 3-cells to Z_2, using γ_i, to obtain Z'_2, we note that the image of $\Gamma_i - \Gamma''_i$ in $\pi_2 Z_1$ is trivial and, hence, we can extend the bonding map $Z_2 \to Z_1$ to a map $Z'_2 \to Z_1$. We continue with this type of alterations to Z_2 to obtain an object X_2 in CW_0 which contains Z_2 such that the inclusion $Z_2 \subseteq X_2$ induces isomorphisms on fundamental groups, the map $g_2 : K \to X_2$ induced by f_2 is r-connected and the bonding map $Z_2 \to Z_1$ extends to a mapping $X_2 \to Z_1$.

By induction and the maximality principle, we obtain a sequence $\{X_i\}$ of objects in CW_0 and mappings $g_i : K \to X_i$ such that (A) $Z_i \subseteq X_i$ and the inclusion induces isomorphisms of fundamental groups, (B) g_i is r-connected and (C) the bonding maps $Z_{i+1} \to Z_i$ extend to a mapping $\epsilon_i : X_{i+1} \to Z_i$. By composing ϵ_i with the inclusion $Z_i \subseteq X_i$, we obtain an object $\{X_i\}$ in pro-CW_0. It is straightforward to check that shape $(\varprojlim X_i) =$ shape $(\varprojlim Z_i)$. $\{g_i\} : \{K\} \to \{X_i\}$ is a morphism in pro-Ho-CW_0 and by using the fact that each g_i is r-connected, one can easily check that the bonding maps $X_{i+1} \to X_i$ are also r-connected. Since $r < n$, dimension of $X_i \leq n$ for each i and, hence, we have 5.2.

Remark 6: Note that the restriction $r < n$ is used only to obtain the fact that dim $X_i \leq n$. Hence if $r \geq n$, then we get 5.2 with the modification that dim $X_i \leq r + 1$.

Now let us assume 5.2 and let us choose an object $\{X_i\}$ in pro-CW_0 as in 5.2. Obviously X has pro-π_1 pro-isomorphic to $\pi_1(X_1)$ which is finitely presented. Let K be the r-skeleton of X_1 and let $f_1 : K \to X_1$ be the inclusion map. By the cellular approximation theorem [18; p. 404], f_1 is r-connected. If $e_i : X_{i+1} \to X_i$ is the bonding map, then we want to define $f_2 : K \to X_2$ such that $e_1 f_2$ is pointed-homotopic to f_1. Express $K = \bigcup_{j=0}^{m} k_j$ as the union of cells such that k_0 is the base-point and $\dim k_j \leq \dim k_{j+1}$ for all j. We define f_2 inductively on j; $f_2(k_0)$ is the base-point of X_2. Suppose that f_2 is defined on $\bigcup_{j=0}^{t-1} k_j$ so that $e_1 f_2$ is pointed-homotopic to $f_1 \mid \bigcup_{j=0}^{t-1} k_j$. Let $s = \dim k_t$; by definition of CW-complex, there exists a continuous map $\phi : [0, 1]^s \to cl(k_t)$ such that $\phi \mid (0, 1)^s : (0, 1)^s \to k_t$ is a homeomorphism. Let $\phi_0 = \phi \mid \mathrm{bdry}[0, 1]^s$; then $f_2 \phi_0$ represents an element of $\pi_{s-1}(X_2)$ [we may assume that $\phi([0, 1]^{s-1} \times \{0\}) \cup (\mathrm{bdry}[0, 1]^{s-1} \times [0, 1/2]))$ is the base-point of K]. Since $e_1 f_2$ is homotopic to f_1 and e_1 is r-connected, $f_2 \phi_0$

represents the trivial element of $\pi_{s-1}(X_2)$. Hence there exists an extension of f_2, $F_2: \bigcup_{j=0}^{t} k_j \to X_2$. Unfortunately e_1F_2 need not be homotopic to $f_1 \mid \bigcup_{j=0}^{t} k_j$. Let us assume that F_2 is chosen so that $F_2\phi([0,1]^{s-1} \times [0, 1/2])$ is the base point of X_2. Let $\xi: \bigcup_{j=0}^{t-1} k_j \times [0,1] \to X_1$ be a pointed homotopy such that $\xi_0 = e_1f_2$ and $\xi_1 = f_1$; by the homotopy extension property for CW-complexes [18; p. 29, 402], ξ can be extended to a homotopy (which we will denote also by ξ) of $\bigcup_{j=0}^{t} k_j \times [0,1]$ to X_1 such that $\xi_0 = e_1F_2$. Consider the mappings $\xi_1\phi$ and $f_1\phi$ which map $[0,1]^s$ into X_1; $\xi_1\phi \mid \text{bdry}[0,1]^s = f_1\phi \mid \text{bdry}[0,1]^s$. Define

$$\mu: \text{bdry}[0,1]^{s+1} \to X_1$$

by

$$\mu(x,t) = \begin{cases} f_1\phi(x) & (x,t) \in [0,1]^s \times \{0\} \\ \xi_1\phi(x) & (x,t) \in ([0,1]^s \times \{1\}) \cup (\text{bdry}[0,1]^s \times [0,1]). \end{cases}$$

Note that μ represents an element of $\pi_s X_1$; if μ represented the trivial element, then we would have a homotopy between e_1F_2 and f_1. Suppose that μ does not represent the trivial element. Since e_1 is r-connected, we can find $\mu': [0,1]^s \to X_2$ representing an element of $\pi_s X_2$ whose image under the homomorphism induced by e_1 is the negative of the class containing μ. We now redefine F_2 on $\phi([0,1]^{s-1} \times [0, 1/2])$ so that $F_2\phi \mid [0,1]^{s-1} \times [0,1/2]$ represents the class of μ'. Now it is straightforward to check that, with this new F_2, e_1F_2 and f_1 are homotopic. Hence, by induction, we get a map $f_2: K \to X_2$ such that e_1f_2 and f_1 are homotopic. It is easy to check that f_2 is r-connected. By another induction argument and the maximality principle, we get a tower of maps $\{f_i\}: \{K\} \to \{X_i\}$ such that each f_i is r-connected. Hence $\{f_i\}$ is shape r-connected.

REMARK 7: If X is a continuum which has the pointed-shape of a finite complex, then it is easy to verify that all the homotopy pro-groups of X are stable and X has a shape finite r-skeleton for all r. Conversely, if X has a shape finite r-skeleton for all r, and X has finite fundamental dimension ≥ 3, then it follows as a Corollary of Theorem 5 (see Remark 5) and [4; Thm. 1.1] that X is a pointed-fundamental ANR and there is an obstruction in $\tilde{K}_0(\check{\pi}_1(X))$ whose vanishing is a necessary and sufficient condition in order that X has the pointed-shape of a finite complex.

We now rephrase an embedding theorem of Ivanšić [9] in shape theoretic terms.

THEOREM 8: *Let M be a* PL-*manifold of dimension q and let X be a continuum which has fundamental dimension n, $q-n\geq 3$, has stable* pro-π_1 *which is* pro-*isomorphic to a finitely presented group and has a shape finite $(2n-q+1)$-skeleton. If there exists a shape map $\{f_i\}: X\to M$ which is shape $(2n-q+1)$-connected, then there exists a compactum $Z\subseteq M$ such that* shape X = shape Z.

PROOF: Let us first consider the case when $2n-q+1\geq 2$. By Theorem 5, there exists an object $\{X_i\}$ in pro-CW_0 such that dim $X_i\leq n$ for all i, shape $(\varprojlim X_i)$ = shape X and the bonding maps $X_{i+1}\to X_i$ are $(2n-q+1)$-connected for all i. By hypotheses, there exists a morphism $\{f_i\}:\{X_i\}\to\{M\}$ which is shape $(2n-q+1)$-connected. It is easily checked that $f_i: X_i\to M$ is $(2n-q+1)$-connected. The result now follows from [9].

If $2n-q+1=1$, we use Theorem 2 instead of Theorem 5 in the above argument. If $2n-q+1\leq 0$, then Y actually embeds in M by [8] where Y is an n-dimensional continuum with shape X = shape Y.

THEOREM 9: *If Y is a continuum which has stable* pro-π_n *and X is a continuum such that* shape $X\leq$ shape Y, *then X has stable* pro-π_n.

PROOF: The proof is motivated by the work of Edwards and Geoghegan [5] who use the work of Atiyah and Segal [1]. Let Q be the functor from the category of pro-groups to the category of topological groups which sends the system of groups $\{G_\alpha\}$ to its inverse limit which is topologized as a subgroup of $\Pi_\alpha G_\alpha$ where each G_α is given the discrete topology. Let P be the functor from the category of topological groups to the category of pro-groups which sends the group G to the system $\{G/I_\alpha \mid I_\alpha$ is an open subgroup of $G\}$.

Since Y has stable pro-π_n and shape $X\leq$ shape Y, there exists a group G and morphisms $\{f_i\}:\{\pi_n X_i\}\to\{G\}$ and $\{g_i\}:\{G\}\to\{\pi_n X_i\}$ such that $\{X_i\}$ is an object in pro-CW_0 with shape $(\varprojlim X_i)$ = shape X and $\{g_i\}\cdot\{f_i\}$ = {identity}. We will now show that $\{\pi_n X_i\}$ satisfies the Mittag–Leffler condition: for each i, there exists $j\geq i$ such that, for all $k\geq j$, image(e_{ik}) = image(e_{ij}) where $e_{ij}: \pi_n X_j\to\pi_n X_i$ is the bonding map. Given i, there exists $j\geq i$ such that $g_i f_m e_{mj} = e_{ij}$. Suppose $k\geq j$; clearly image$(e_{ik})\subseteq$ image(e_{ij}) by definition of an inverse system. Since $e_{ij} = g_i f_m e_{mj} = e_{ik} g_k f_m e_{mj}$, image $e_{ik}\supseteq$ image e_{ij}.

$Q(\{G\})$ is a discrete group and, since Q is a functor, $Q(\{g_i\})\circ Q(\{f_i\}) = Q(\{g_i\}\circ\{f_i\})$ = Q({identity}) = identity. Hence $Q(\{\pi_n X_i\})$ is a retract of a discrete group and, thus, is discrete.

Therefore $PQ(\{\pi_n X_i\})$ is equivalent in the category of pro-groups to a group. But, by Proposition 2 of [5] (or [1]), $\{\pi_n X_i\}$ is equivalent in the category of pro-groups to $PQ(\{\pi_n X_i\})$.

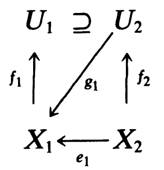
The following is well-known [5].

COROLLARY 10: *If Y is a fundamental* ANR, *then Y has stable* pro-π_n.

REMARK 11: If, in Theorem 9, we assume that Y has stable pro-π_1 which is pro-isomorphic to a finitely-presented group G, then X has stable pro-π_1 which is pro-isomorphic to a retract H of G and, by Lemma 1.3 of [20], H is finitely presented.

THEOREM 12: *Let $Y \subseteq \mathbf{R}^q$ be an n-dimensional continuum and let X be a continuum such that fundamental dimension of $X \leq n$, X has a shape finite $(2n-q+1)$-skeleton and shape $X \leq$ shape Y. If $n < 2/3(q-1)$ and $n \geq 3$, then there exists a compactum $Z \subseteq \mathbf{R}^q$ such that* shape $Z =$ shape X.

PROOF: If $q \geq 2n+1$, then we can find $Z \subseteq \mathbf{R}^q$ such that shape $Z =$ shape X by [8] since $Fd(X) = n$. Hence, we assume that $q < 2n+1$ and, therefore, $2(q-n-1)+1 < q$. By [19], we may assume that $Y \subseteq \mathbf{R}^q$ is $1-$ULC embedded. By Proposition 1, there exists a sequence $\{U_i\}$ of closed connected PL-neighborhoods of Y in $\mathbf{R}^q$ such that 1.1 through 1.3 are satisfied. By Remark 4, X has stable pro-π_1 pro-isomorphic to a finitely presented group Suppose that $(2n-q+1) \geq 2$. By Theorem 5, there exists an object $\{X_i\}$ in pro-CW$_0$ such that $\dim X_i \leq n$ for all i, shape $(\varprojlim X_i) =$ shape X and the bonding maps $X_{i+1} \to X_i$ are $(2n-q+1)$-connected for all i. Since shape $X \leq$ shape Y, there exist morphisms $\{f_i\}:\{X_i\} \to \{U_i\}$ and $\{g_i\}:\{U_i\} \to \{X_i\}$ such that $\{g_i\} \circ \{f_i\} \simeq \{\text{identity}\}$. By choosing subsequences, if necessary, we may assume that we have the following homotopy commutative diagram



where e_1 is the bonding map. Since the induced map e_{1*} on fundamental groups is an isomorphism f_{2*} is a monomorphism and g_{1*} is an epimorphism. Note that $\rho = f_{2*}e_{1*} - 1g_{1*}$ is a retraction of $\pi_1(U_2)$ onto the image of f_{2*}. From the proof of Lemma 1.3 of [20], there exists a finite number of non-trivial elements of $\pi_1(U_2)$, $\alpha_1, \ldots, \alpha_m$, such that the normal closure of $\{\alpha_i\}$ in $\pi_1(U_2)$ is the kernel of ρ.

Let $\Phi_1 : S^1 \to \text{bdry } U_2$ be a continuous map such that Φ_1 represents $p(\alpha_1)$. Since $\mathbf{R}^q$ is contractible, Φ_1 can be extended to the 2-cell D^2, $\Phi_1 : D^2 \to \mathbf{R}^q$. Since $\pi_j(U_2, \text{bdry } U_2) = 0$ for $j \leq q - n - 1$ and $q - n \geq 3$, we can homotope Φ_1 so that we may assume $\Phi_1(D^2) \subseteq cl(\mathbf{R}^q - U_2)$. By the simplicial approximation theorem and general position, we may assume that Φ_1 is a PL-embedding such that $\Phi_1(D^2) \cap \text{bdry } U_2 = \Phi_1(S^1)$. Let N be a regular neighborhood of $\Phi_1(D_2)$ in $cl(\mathbf{R}^q - U_2)$ such that $N \cap \text{bdry } U_2$ is a regular neighborhood of $\Phi_1(S^1)$ in bdry U_2.

Let $V_1' = U_2 \cup N$. By van Kampen's Theorem [3], the inclusion $U_2 \subseteq V_1'$ induces an epimorphism $\pi_1(U_2) \to \pi_1(V_1')$ whose kernel is the normal closure of $p(\alpha_1)$. Since α_1 lies in the kernel of ρ, the composition $X_2 \xrightarrow{f_2} U_2 \subseteq V_1'$ induces a monomorphism $\pi_1 X_2 \to \pi_1 V_1'$. By standard surgery calculations [15; p. 27], $\pi_j(V_1', \text{bdry } V_1') = 0$ for $j \leq q - n - 1$. By doing similar modifications for each $p(\alpha_i)$, we eventually obtain a PL-manifold $V_1 \supseteq U_2$ such that if β_1 denotes the composition $X_2 \xrightarrow{f_2} U_2 \subset V_1$, then β_1 induces an isomorphism of fundamental groups and $\pi_j(V_1, \text{bdry } V_1) = 0$ for $j \leq q - n - 1$.

By Theorem A of [20], $\pi_2(\beta_1)$ is a finitely generated $\mathbf{Z}\pi_1 X_2$-module. We essentially repeat now the argument above: let $\Phi : S^2 \to \text{bdry } V_1$ represents an element of $\pi_2(V_1)$ which maps onto a generator of $\pi_2(\beta_1)$ and extend to $\Phi : D^3 \to cl(\mathbf{R}^q - V_1)$ since $\pi_j(V_1, \text{bdry } V_1) = 0$ for $j \leq q - n - 1$ and $n < \frac{2}{3}(q - 1)$ implies that $2 \leq 2n - q + 1 < q - n - 1$. Since $2(q - n - 1) + 1 < q$, by general position, we may assume that Φ is a PL-embedding such that $\Phi(D^3) \cap \text{bdry } V_1 = \Phi(S^2)$. Let N be a regular neighborhood of $\Phi(D^3)$ in $cl(\mathbf{R}^q - V_1)$ such that $N \cap \text{bdry } V_1$ is a regular neighborhood of $\Phi(S^2)$ in bdry V_1. Let $V_2' = V_1 \cup N$.

Instead of van Kampen's theorem, we use the Mayer-Vietoris sequence of the universal covering space of V_2' to show that $\pi_2(V_1) \to \pi_2(V_2')$ is an epimorphism whose kernel is the submodule generated by the class of Φ (see [15; p. 27]). Again by standard surgery calculations, (A) the mapping $\beta_1' : X_2 \to V_2'$ induced by β_1 induces isomorphisms of fundamental groups; (B) $\pi_2(\beta_1')$ has fewer generators (as a $\mathbf{Z}\pi_1 X_2$-module) than $\pi_2(\beta_1)$ and (C) $\pi_j(V_2', \text{bdry } V_2') = 0$ for $j \leq q - n - 1$. By induction, we can add a finite number of such handles to V_1 to obtain V_2 so that the induced map $\beta_2 : X_2 \to V_2$ is 2-connected and $\pi_j(V_2, \text{bdry } V_2) = 0$ for $j \leq q - n - 1$.

Now we proceed to construct, by induction, V_{2n-q+1} such that the induced map $\beta_{2n-q+1} : X_2 \to V_{2n-q+1}$ is $(2n - q + 1)$-connected. To make the above argument work we need the inequality $(2n-q+1) < (q-n-1)$. By [9] or Theorem 8, there exists a compactum $Z \subseteq V_{2n-q+1}$ such that shape $Z =$ shape X.

The remaining case $2n-q+1=1$ follows immediately from Remark 4 and [9].

Recall McMillan's *cellularity criterion* (CC) [13]. $Y \subseteq \mathbf{R}^q$ satisfies CC if each neighborhood U of Y contains a neighborhood V of Y such that every map S^1 into $V-Y$ is homotopically trivial in $U-Y$.

COROLLARY 13: *Let $Y \subseteq \mathbf{R}^q$, $q \geq 5$, be an n-dimensional continuum which satisfies* (CC). *If X satisfies the same conditions as in Theorem 11, then X can be embedded up to shape into $\mathbf{R}^q$.*

PROOF: It suffices to show that Y has stable pro-π_1. Since $Y \subseteq \mathbf{R}^q$ satisfies CC there is a sequence of neighborhoods V_i such that hold:

13.1. $V_{i+1} \subseteq \operatorname{int} V_i$;

13.2. $\bigcap_i V_i = Y$;

13.3. $\pi_1(V_i - Y) = 1$ for all i.

But 13.3 says that $\{\pi_1(V_i - Y)\}$ is essentially constant, and hence by Theorem 3.10 of [15] we can modify V_i's getting U_i's such that $Y = \bigcap_i U_i$ and the inclusions $U_{i+1} \subseteq U_i$ induce isomorphisms $\pi_1(U_{i+1}) \to \pi_1(U_i)$ for all i. This just says that Y has stable pro-π_1 pro-isomorphic to $\pi_1(U_1)$.

COROLLARY 14: *Let $Y \subseteq \mathbf{R}^q$ be an n-dimensional continuum and let X be a continuum which has the shape of a finite complex of dimension $\leq n$ and* shape $X \leq$ shape Y. *If $n < 2/3(q-1)$ and $n \geq 3$, then there exists a compactum $Z \subseteq \mathbf{R}^q$ such that* shape $Z =$ shape X.

REFERENCES

[1] M.F. ATIYAH and B.G. SEGAL: Equivalent K-theory and completion. *J. Differential Geometry 3* (1969) 1–18.

[2] K. BORSUK: *Theory of shape.* Mathematical Monographs Vol. 59 Polish Scientific Publishers, Warsaw 1975.

[3] R.H. CROWELL and R.H. FOX: *Introduction to knot theory.* Ginn and Co., New York 1963.

[4] D.A. EDWARDS and R. GEOGHEGAN: The stability problem in shape and a Whitehead theorem in pro-homotopy. *Trans. Amer. Math. Soc. 214* (1975) 261–278.

[5] D.A. EDWARDS and R. GEOGHEGAN: Compacta weak shape equivalent to ANR's. *Fund. Math. 90* (1976) 115–124.

[6] R. GEOGHEGAN and R.R. SUMMERHILL: Concerning the shapes of finite dimensional compacta. *Trans. Amer. Math. Soc. 179* (1973) 281–292.

[7] J.F.P. HUDSON: *Piecewise linear topology.* W.A. Benjamin, Inc., New York 1969.
[8] W. HUREWICZ and H. WALLMAN: *Dimension theory.* Princeton University Press, Princeton 1948.
[9] I. IVANŠIĆ: Embedding compacta up to shape. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* (to appear)
[10] S. MARDEŠIĆ: On the Whitehead theorem in shape theory I, II. *Fund. Math. 91* (1976) 51–64; 93–103.
[11] S. MARDEŠIĆ: On covering dimension and inverse limits of compact spaces. *Illinois J. Math. 4* (1960) 278–291.
[12] S. MARDEŠIĆ: and J. SEGAL: Shapes of compacta and ANR systems. *Fund. Math. 72* (1971) 61–68.
[13] D.R. MCMILLAN JR: A criterion for cellularity in a manifold. *Ann. of Math. (2)79* (1964) 327–337.
[14] C.P. ROURKE and B.J. SANDERSON: *Introduction to piecewise-linear topology.* Springer-Verlag, Inc. New York 1972.
[15] L.C. SIEBENMANN: The obstruction to finding a boundary for an open manifold of dimension greater than five. *Thesis. Princeton Univ.*, Princeton 1965; order number 66-5012, University microfilms Ltd., 300 N. Zeeb Rd., Ann Arbor, Mich. 48106, USA.
[16] L.C. SIEBENMANN: On detecting open collars. *Trans. Amer. Math. Soc. 142* (1969) 201–227.
[17] L.C. SIEBENMANN, L. GUILLOU and H. HÄHL: Les voisinages ouvertes réguliers: critères homotopiques d'existence. *Ann. Sci. École Norm. Sup. (3)7* (1974) 431–461.
[18] E.H. SPANIER: *Algebraic topology.* McGraw-Hill Book Co., New York 1966.
[19] M.A. ŠTAN'KO: Approximation of compacta in E^n in codimension greater than two. *Mat. Sb. (N.S) 90(132)* (1973) 625–636: [*Math. USSR-Sb. 19* (1973) 615–626].
[20] C.T.C. WALL: Finiteness conditions for CW-complexes. *Ann. of Math. 81* (1965) 56–69.
[21] C.T.C. WALL: Geometrical connectivity I. *J. London Math. Soc. (2)3* (1971) 597–604.
[22] A. KADLOF (to appear).

(Oblatum 5-IV-1977 & 19-I-1979)

Department of Mathematics
University of Tennessee
Knoxville, Tenn. 37916

Department of Mathematics
University of Zagreb
41001 Zagreb, P.P. 187