COMPOSITIO MATHEMATICA

E. MICHAEL

Uniform AR's and ANR's

Compositio Mathematica, tome 39, nº 2 (1979), p. 129-139

<http://www.numdam.org/item?id=CM_1979__39_2_129_0>

COMPOSITIO MATHEMATICA, Vol. 39, Fasc. 2, 1979, pag. 129–139

UNIFORM AR's AND ANR's

E. Michael

1. Introduction

The purpose of this note is to introduce and study metric spaces which are uniform AR's or ANR's. Our definitions and results are quite natural, and appear to be potentially rather useful. Theorems 1.1 and 7.2 will be applied in [14].

It should be remarked that our uniform ANR's are not the same as J. Isbell's ANRU's [8] [9]. They are, however, equivalent to the uniform ANR's introduced by the author in an abstract (see [11]) in 1955. In that abstract, uniform ANR's were defined by means of condition (c) of our Theorem 7.1, and Theorem 1.1 and Proposition 1.4 of this paper were indicated there (for that definition) without proof.[1]

Recall that a metrizable space Y is an *AR* (resp. *ANR*) if and only if, whenever X is metrizable and $A \subset X$ is closed, then every continuous $f: A \to Y$ extends to a continuous $f': X \to Y$ (resp. $f': U \to Y$ for some open $U \supset A$ in X). Analogously, we call a metric[2] space (Y, d) a *uniform AR* (resp. *uniform ANR*) if, whenever (X, ρ) is a metric space and $A \subset X$ is closed, then every uniformly continuous $f: A \to Y$ extends to a continuous $f': X \to Y$ (resp. $f': U \to Y$ for some uniform neighborhood U of A in X) which is uniformly continuous at A. Here U is called a *uniform neighborhood* of A in X if $U \supset B_\epsilon(A)$ for[3] some $\epsilon > 0$, and f' is called *uniformly continuous at* A if to every $\epsilon > 0$ corresponds a $\delta(\epsilon) > 0$ such that $d(f'(x), f'(a)) < \epsilon$ whenever $\rho(x, a) < \delta(\epsilon)$ with $x \in X$ (resp. $x \in U$) and $a \in A$.[4]

[1] After this paper was completed, R. Engelking and H. Toruńczyk kindly informed the author that much of Lemma 2.1 and Theorem 1.1 was already obtained by Toruńczyk in [16, Section 2], and that some more precise estimates related to these results were obtained by him in [17, Lemma 3 and Theorem 4).

[2] We distinguish between a *metric* space (which carries a specific metric) and a *metrizable* space (which does not).

[3] $B_\epsilon(A)$ denotes $\{x \in X : \rho(x, A) < \epsilon\}$.

[4] Note that f' is *not* required to be uniformly continuous on X (resp. U).

0010–437X/79/05/0129-11 $00.20/0

It is clear that every uniform ANR is an ANR, and it is easy to check (or to conclude from Theorem 1.1 below) that the converse is true for compact spaces. In general, however, the converse is false. For example, the subspace $Y = \{1/n : n \in N\}$ of the real line R is an ANR (since it is discrete), but it is not a uniform ANR because $id_Y : Y \to Y$ cannot be extended to a continuous $f' : U \to Y$ for any neighborhood U of Y in $R - \{0\}$. An example of a *uniformly locally contractible* ANR which is not a uniform ANR will be given in Section 8.

THEOREM 1.1: *The following properties of a metric space (Y, d_0) are equivalent.*

(*a*) *Y is an AR (resp. ANR).*

(*b*) *There is a compatible metric $d \geq d_0$ on Y such that (Y, d) is a uniform AR (resp. uniform ANR).*

Let us now call a closed subspace Y of a metric space Z a *uniform retract* (resp. *uniform neighborhood retract*) of Z if there exists a retraction $r : Z \to Y$ (resp. $r : U \to Y$ for some uniform neighborhood U of Y in Z) which is uniformly continuous at Y.[5] Analogously to the situation in the non-uniform case [4, Theorem 7.1], we now have the following result, where we write $Z \sqsupset Y$ to denote that Z contains Y isometrically as a closed subset.

THEOREM 1.2: *The following properties of a metric space Y are equivalent.*

(*a*) *Y is a uniform AR (resp. uniform ANR).*

(*b*) *Y is a uniform retract (resp. uniform neighborhood retract) of every metric space $Z \sqsupset Y$.*

(*c*) *Y is a uniform retract (resp. uniform neighborhood retract) of some normed linear space $E \sqsupset Y$.*

REMARK: Observe that, by Theorem 1.2, every normed linear space is a uniform AR. More generally, that remains true for every convex subset of a normed linear space.

Some other characterizations of uniform ANR's will be given in Theorems 7.1 and 7.2.

PROPOSITION 1.3: *The following properties of a metric space Y are equivalent.*

[5] Such a retraction is called a *regular retraction* by H. Toruńczyk in [16, p. 53]. See Footnote 1.

(*a*) *Y is a uniform AR.*

(*b*) *Y is an AR and a uniform ANR.*[6]

PROPOSITION 1.4: *The following properties of a metric space Y are equivalent.*

(*a*) *Y is a uniform AR (resp. a uniform ANR).*

(*b*) *If Y^* is any metric space containing Y isometrically as a dense subset, then Y^* is a uniform AR (resp. uniform ANR).*

We conclude this introduction by considering a concept which is closely related to uniform ANR's. Call a metric space X a *weak uniform ANR* if to every $\epsilon > 0$ there corresponds a $\delta > 0$ such that, if X is metrizable and $A \subset X$ closed, then every continuous $f: A \to Y$ with diam $f(A) < \delta$ extends to a continuous $f': X \to Y$ with diam $f'(X) < \epsilon$.

The following characterization was essentially obtained by C. Pixley in [15, Theorem 3.1].

PROPOSITION 1.5 (*Pixley*): *The following properties of a metric space Y are equivalent.*

(*a*) *Y is a weak uniform ANR.*

(*b*) *Y is an ANR and uniformly locally contractible.*[7]

THEOREM 1.6: *Every uniform ANR is a weak uniform ANR, but the converse is false.*

Our paper is arranged as follows. After establishing two lemmas in Section 2, we prove Theorems 1.1 and 1.2 in Sections 3 and 4, and Propositions 1.3 and 1.4 in Sections 5 and 6. Section 7 is devoted to some further characterizations of uniform ANR's, and Section 8 proves Theorem 1.6.

2. Two lemmas

LEMMA 2.1: *Let (Y, d_0) be a metric ANR embedded as a closed subset of a normed linear space E, and let $r: G \to Y$ be a retraction with $G \supset Y$ open in E. Then there exists a compatible metric $d \geq d_0$ on*

[6] By a result of J. Dugundji [4, p. 366] and Theorem 1.1, the AR hypothesis can be weakened to assuming only that all the homotopy groups of Y are trivial.

[7] A metric space Y is *uniformly locally contractible* if to every $\epsilon > 0$ corresponds a $\delta > 0$ such that, for any $y \in Y$, $B_\delta(y)$ is contractible in $B_\epsilon(y)$.

Y with the following property: To every $\epsilon > 0$ corresponds a $\gamma(\epsilon) > 0$ such that, if $S \subset Y$ with diam $S < \gamma(\epsilon)$, *then* conv $S \subset G$ *and* diam $r(\text{conv } S) < \epsilon$.

PROOF: Define a relation $<$ on 2^Y by saying that $W < V$ if conv $W \subset G$ and $r(\text{conv } W) \subset V$. This $<$ is a *proper ordering* on 2^Y in the sense of [6, Definition 1]. Our conclusion will now follow from [6, Theorem 1], provided we can show that, whenever V is a neighborhood of y in Y, then $W < V$ for some neighborhood W of y in Y. But such a W is easily found by picking an open, convex $U \subset E$ such that $y \in U \subset r^{-1}(V)$, and letting $W = U \cap Y$. That completes the proof.

For use in future sections, we record here a result of R. Arens and J. Eells [1] (see also [13]).

LEMMA 2.2: *(Arens-Eells). Every metric space can be embedded as a closed subset in a normed linear space.*

3. Proof of Theorem 1.1

We shall only prove the ANR case; the AR case is similar.

$(a) \to (b)$. By Lemma 2.2, there is a normed linear space $E \sqsupset Y$. Let $r: G \to Y$ be a retraction, where $G \supset Y$ is open in E. Now let d be the metric on Y obtained in Lemma 2.1, and let us show that (Y, d) is a uniform ANR.

Suppose, therefore, that (X, ρ) is a metric space, $A \subset X$ closed, and $f: A \to Y$ uniformly continuous. We must extend f to a continuous $f': U \to Y$, with U a uniform neighborhood of A in X, such that f' is uniformly continuous at A.

For each $a \in A$, let

$$V_a = \{x \in X - A : \rho(x, a) < 2\rho(x, A)\}.$$

It is easy to check that $\{V_a : a \in A\}$ is an open cover of $X - A$, and it therefore has a locally finite (with respect to $X - A$) partition of unity $\{p_a : a \in A\}$ subordinated to it.

Since f is uniformly continuous, we can choose, for each $\epsilon > 0$, a $\beta(\epsilon) > 0$ such that $d(f(a), f(a') < \epsilon$ whenever $a, a' \in A$ and $\rho(a, a') < \beta(\epsilon)$. Let $\gamma(\epsilon)$ be as in Lemma 2.1, and let $\delta(\epsilon) = \frac{1}{4}\beta(\frac{1}{2}\gamma(\epsilon))$. Let $U = \{x \in X : \rho(x, A) < \delta(1)\}$, and define $f': U \to Y$ by

$$f'(x) = f(x) \qquad \text{if } x \in A,$$

$$f'(x) = r\Big(\sum_{a \in A} p_a(x) f(a)\Big) \quad \text{if} \quad x \in X - A.$$

It is easy to check that this f' is well defined, extends f, is continuous on $X-A$, and that $d(f'(x), f'(a))<\epsilon$ whenever $\rho(x, a)<\delta(\epsilon)$ with $x \in U$ and $a \in A$.

$(b)\to(a)$. By Lemma 2.2, there is a normed linear space $E \sqsupset Y$. By (b), Y is a neighborhood retract of E. Since E is an AR by Dugundji's extension theorem [4, Theorem 4.1], it follows that Y is an ANR.

That completes the proof.

4. Proof of Theorem 1.2

$(a)\to(b)$. Clear.

$(b)\to(c)$. This follows from Lemma 2.2.

$(c)\to(a)$. Suppose that Y is a neighborhood retract of a normed linear space $E \sqsupset Y$. (The proof for retracts is similar). Then the given metric d on Y satisfies all the conditions of Lemma 2.1, so the proof of Theorem 1.1 $(a)\to(b)$ goes through unchanged to show that Y must be a uniform ANR.

Remark: The above proof shows that, in 1.2(b), the assignment $\epsilon\to\delta(\epsilon)$ which makes the retraction $r:Z\to Y$ (resp. $r:U\to Y$) uniformly continuous at Y can be chosen to depend only on Y and not on Z. Moreover, one can choose the $\delta(\epsilon)$ and U so that $U\supset B_{\delta(\epsilon)}(Y)$ for all $\epsilon>0$.

5. Proof of Proposition 1.3

$(a)\to(b)$. By Theorem 1.1 $(b)\to(a)$.

$(b)\to(a)$. Suppose that X is a metric space, $A\subset X$ closed, and $f:A\to Y$ uniformly continuous. By (b), f extends to a continuous $g:U\to Y$, with U a uniform neighborhood of A in X and g uniformly continuous at A. Pick a uniform open neighborhood V of A in X such that $\bar{V}\subset U$. Since Y is an AR, the map $g|(\bar{V}-V)$ extends to a continuous $h:X-V\to Y$. Now define $f':X\to Y$ by letting $f'|\bar{V}=g$ and $f'|X-V=h$. This f' is the required extension of f.

6. Proof of Proposition 1.4

Since $(b)\to(a)$ is trivial, it sufficies to prove $(a)\to(b)$, and we do this only for uniform ANR's. We will use the characterization in Theorem 1.2(b).

So suppose that Y is a uniform ANR, and that $Y^*\supset Y$ isometrically with Y dense in Y^*. Suppose that $Z \sqsupset Y^*$. Let $Z'=$

$Z-(Y^*-Y)$. Then Y is closed in Z', so there exists a retraction $r':U'\to Y$, with U' a uniform neighborhood of Y in Z' and r' uniformly continuous at Y. Let $U=U'\cup(Y^*-Y)$, and define $r:U\to Y^*$ by letting $r(z)=r'(z)$ if $z\in U'$ and $r(z)=z$ if $z=Y^*-Y$. It is easy to check that U is a uniform neighborhood of Y^* in Z (since U' is a uniform neighborhood of Y in Z'), and that r is a retraction which is uniformly continuous at Y^*.

7. Further characterizations of uniform ANR's

Our first result in this section shows that being a uniform ANR is equivalent to two properties which are formally quite a bit weaker. Condition 7.1(c) is due to S. Lefschetz, who showed in [10, p. 83] that it is equivalent to 7.1(a) for *compact* metric spaces. A non-uniform version of the equivalence of 7.1(a) and 7.1(c) was obtained by J. Dugundji in [5, Theorem 13.4]; our proof that 7.1(c) implies 7.1(a) is similar in spirit to Dugundji's proof of the analogous non-uniform implication in [5].

THEOREM 7.1. *The following properties of a metric space Y are equivalent.*

(a) *Y is a uniform ANR.*

(b) *To every $\epsilon>0$ corresponds a normed linear space $E\sqsupset Y$ and a retraction $r:U\to Y$ of a uniform neighborhood U of Y in E onto Y such that $d(r(z),z)<\epsilon$ for all $z\in U$.*

(c) *To every $\epsilon>0$ corresponds a $\delta>0$ such that, if K is a simplicial complex and $L\subset K$ a subcomplex containing all the vertices of K, then every continuous[8] $f:L\to Y$ such that* diam $f(\sigma\cap L)<\delta$ *for every simplex σ of K extends to a continuous $f':K\to Y$ such that* diam $f'(\sigma)<\epsilon$ *for every simplex σ of K.*

PROOF (a)→(b): This follows from the easy part of Theorem 1.2, since 7.1(b) is formally weaker than 1.2(c).

(b)→(c). Let $\epsilon>0$ be given. Choose $E\sqsupset Y$ and $r:U\to Y$ as in (b), but with ϵ replaced by $\frac{1}{2}\epsilon$. Pick $\delta>0$ so that $\delta<\frac{1}{2}\epsilon$ and conv $S\subset U$ whenever $S\subset Y$ and diam $S<\delta$. Now let K, $L\subset K$, and $f:L\to Y$ be as in (c). Since every convex subset of E is an AR [3, Theorem 4.1], f can be extended (by induction on the dimension of the simplices of K) to a continuous $g:K\to E$ such that $g(\sigma)\subset$ conv $f(\sigma\cap L)$ for every simplex σ of K. Then $g(K)\subset U$, and we define the required

[8] i.e., continuous on every (closed) simplex.

extension $f': K \to Y$ of f by $f' = r \circ f$. It is easy to check that this f' has all the desired properties.

$(c) \to (a)$. By Theorem 1.2, we need only show that Y satisfies 1.2(b). So suppose that $Z \sqsupset Y$, and let us show that Y is a uniform neighborhood retract of Z.

We begin by setting up the machinery used in the usual proof of Dugundji's extension theorem. For each $z \in Z - Y$, let $\rho(z) = \frac{1}{2}d(z, Y)$. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a locally finite (with respect to $Z - Y$) open refinement of $\{B_{\rho(z)}(z) : z \in Z - Y\}$ such that $U_\alpha \neq U_\beta$ when $\alpha \neq \beta$. For each α pick $z_\alpha \in Z$ such that $U_\alpha \subset B_{\rho(z_\alpha)}(z_\alpha)$, and pick $y_\alpha \in Y$ such that $d(y_\alpha, z_\alpha) < 3\rho(z_\alpha)$. The following assertion is now easily checked.

(*) If $u, v \in U_\alpha$ for some α, then $\rho(z_a) \leq d(z_\alpha, Y)$, $d(z_\alpha, y_\alpha) < 3d(u, Y)$, $d(v, y_\alpha) < 4d(u, Y)$, and $d(v, u) \leq d(u, Y)$.

For each $\epsilon > 0$, let $\delta(\epsilon)$ be as in (c). Choose $s_n > 0$ $(n = 1, 2, \ldots)$ so that $s_{n+1} < \frac{1}{2}s_n$ and $s_n < \frac{1}{8}\delta(\frac{1}{4}\delta(\frac{1}{n}))$ for all n. For all n, let

$$\begin{aligned} T_n &= \{z \in Z : s_{n+1} \leq d(z, Y) \leq s_n\}, \\ \mathcal{U}_n &= \{U \in \mathcal{U} : U \cap T_n \neq \emptyset\}, \\ \mathcal{V}_n &= \mathcal{U}_n \cap \mathcal{U}_{n-1} \quad (n \geq 2). \end{aligned}$$

Note that, by the first assertion of (*) and the assumption $s_{n+1} < \frac{1}{2}s_n$, $\mathcal{V}_n \cap \mathcal{V}_m = \emptyset$ if $m \neq n$.

For any $\mathcal{S} \subset \mathcal{U}$, let $N(\mathcal{S})$ denote the nerve of $\mathcal{S}$, and let $N_0(\mathcal{S})$ denote the 0-skeleton of $N(\mathcal{S})$ (i.e., $N_0(\mathcal{S}) = \mathcal{S}$).

Define $f : N_0(\mathcal{U}) \to Y$ by $f(U_\alpha) = y_\alpha$. By (*),

$$\operatorname{diam} f(\sigma \cap N_0(\mathcal{U}_n)) < \delta\left(\frac{1}{4}\,\delta\left(\frac{1}{n}\right)\right)$$

for each simplex σ of $N(\mathcal{U}_n)$. We now apply (c), with $K = N(\mathcal{V}_n)$ and $L = N_0(\mathcal{V}_n)$, to extend $f | N_0(\mathcal{V}_n)$ to a continuous $f_n : N(\mathcal{V}_n) \to Y$ such that $\operatorname{diam} f_n(\sigma) < \frac{1}{4}\delta(1/n)$ for each simplex σ of $N(\mathcal{V}_n)$.

For each n, define $L_n \subset N(\mathcal{U}_n)$ by

$$L_n = N_0(\mathcal{U}_n) \cup N(\mathcal{V}_n) \cup N(\mathcal{V}_{n+1}),$$

and define $g_n : L_n \to Y$ by letting $g_n | N_0(\mathcal{U}_n) = f | N_0(\mathcal{U}_n)$, $g_n | N(\mathcal{V}_n) = f_n$, and $g_n | N(\mathcal{V}_{n+1}) = f_{n+1}$. Then

$$\operatorname{diam} g_n(\sigma \cap K_n) < 2\left(4s_n + \frac{1}{4}\,\delta\left(\frac{1}{n}\right) \leq \delta\left(\frac{1}{n}\right)\right)$$

for every simplex σ of $N(U_n)$. Applying (c) with $K = N(\mathcal{U}_n)$ and $L = L_n$, we can therefore extend g to a continuous $h_n : N(\mathcal{U}_n) \to Y$ such that diam $h_n(\sigma) < 1/n$ for every simple σ of $N(\mathcal{U}_n)$.

Since $\mathcal{U}$ is a locally finite open cover of $Z - Y$, there is a canonical map $p : Z - Y \to N(\mathcal{U})$. Let $W = \{z \in Z ; d(z, Y) \leq s_1\}$, and define $r : W \to Y$ by letting $r(z) = z$ if $z \in Y$ and $r(z) = h_n(p(z))$ if $z \in T_n$. Clearly W is a uniform neighborhood of Y in Z, and it is easy to check that r is well defined and continuous, and thus a retraction from W onto Y.

It remains to show that r is uniformly continuous at Y. Suppose $z \in T_n$. Let σ be a simplex of $N(\mathcal{U}_n)$ containing $p(z)$ (so that every vertex of σ contains z), and let U_α be a vertex of σ. Then, again applying (*),

$$\begin{aligned} d(z, r(z)) &\leq d(z, y_\alpha) + d(y_\alpha, r(z)) \\ &\leq 4s_n + \text{diam } h_n(\sigma) < 4s_n + \frac{1}{n} < \frac{2}{n}. \end{aligned}$$

This implies that r is uniformly continuous at Y, and that completes the proof.

THEOREM 7.2: *The following properties of a metric space Y are equivalent.*

(a) Y is a uniform ANR.

(b) To every $\epsilon > 0$ corresponds a $\gamma = \gamma(\epsilon) > 0$ such that, if $Z \sqsupset Y$, if A is a closed subset of a metrizable space X, and if $g : X \to Z$ is continuous with $d(g(x), Y) < \gamma$ for all $x \in X$, then every continuous $f : A \to Y$ with $d(f(x), g(x)) < \gamma$ for all $x \in A$ extends to a continuous $f' : X \to Y$ with $d(f'(x), g(x)) < \epsilon$ for all $x \in X$.

Moreover, if Y is complete then (b) follows from (a) even if X is only assumed collectionwise normal.

PROOF: $(a) \to (b)$. Assume (a). Let $\epsilon > 0$ be given. By Theorem 1.2 and the remark following its proof in Section 4, we can find a $\beta > 0$ such that, if $E \sqsupset Y$, then there is a retraction $r : B_\beta(Y) \to Y$ with $d(z, r(z)) < \frac{1}{2}\epsilon$ for all $z \in B_\beta(Y)$. We may assume that $\beta < \epsilon$. Let $\gamma = \frac{1}{2}\beta$, and let us show that this γ has all the properties required by (b).

Suppose that $Z \supset Y$, $A \subset X$, $g : X \to Z$ and $f : A \to Y$ are as in (b). By Lemma 2.2, we can embed Z isometrically in a normed linear space E such that Y is closed in E. Let $U = \{z \in E : d(z, Y) < \beta\}$, where d is the metric on E. By choice of β, there is a retraction $r : U \to Y$ such that $d(r(z), z) < \frac{1}{2}\epsilon$ for all $z \in U$. It will now suffice to extend f to a continuous $h: X \to E$ such that $d(h(x), g(x)) < \gamma$ for all $x \in X$, for then $h(x) \in U$ for all $x \in X$ and we can let $f^* = r \circ h$. It is easy to check that this f^* will have the required properties.

To define the desired $h: X \to E$, let $f^*(x) = f(x) - g(x)$ for $x \in A$. Let $C = \{z \in E : \|z\| < \gamma\}$. Clearly f^* is continuous and $f^*(A) \subset C$. Since C is a convex subset of E and X is metrizable, we can extend f^* to a continuous $h^*: X \to C$ by Dugundji's extension theorem [4, Theorem 4.1] (If Y is complete, then we may assume that E is a Banach space, hence C is completely metrizable, so by [3, Theorem 2] the above extension h^* of f^* exists even if X is only collectionwise normal). We now define the required $h: X \to E$ by letting $h(x) = h^*(x) + g(x)$ for all $x \in X$.

$(b) \to (a)$. Assume (b). By Theorem 7.1, we need only show that Y satisfies 7.1(b). Let $\epsilon > 0$ be given. Pick any normed linear space $E \sqsupset Y$ (see Lemma 2.2). Let $\gamma(\epsilon)$ be as in (b), and let $U = B_{\gamma(\epsilon)}(Y)$. Let $g: U \to U$ be the identity map, and let $f = g|Y$. By (b), f can be extended to continuous $r: U \to Y$ such that $d(r(z), z) < \epsilon$ for all $z \in U$. This r satisfies the requirement of 7.1(b).

8. Proof of Theorem 1.6

That every uniform ANR is a weak uniform ANR follows from Theorem 7.2, since being a weak uniform ANR is clearly equivalent to the special case of 7.2(b) in which $Z = Y$ and g is a constant map (the proof of $7.2(a) \to (b)$ can be shortened in this case). That the converse is false follows (in view of Proposition 1.5) from the following example.

EXAMPLE 8.1: *There exists a locally compact, separable metric space Y which is an ANR and uniformly locally contractible, but which is not a uniform ANR.*

PROOF: We take Y to be a dense open subset of Borsuk's example [2, Theorem 11.1] of a compact metric space Y^* which is locally contractible but not an ANR.

Let $Q = I^\omega$, metrized by $d(x, y) = \Sigma_{i=1}^{\infty} 2^{-i}|x_i - y_i|$. For $n \geq 1$, let

$$Y_n = \left\{ y \in Q : \frac{1}{2^n} \leq y_1 \leq \frac{1}{2^{n-1}}, y_i = 0 \text{ for } i > n \right\}.$$

Clearly Y_n is homeomorphic to an n-cube; let $\dot{Y}_n$ be the boundary of the cube Y_n (i.e. $\dot{Y}_n$ consists of all $y \in Y_n$ such that $y_1 \in \{2^{-n}, 2^{-(n-1)}\}$ or

$y_i \in \{0, 1\}$ for some i with $1 < i \leq n$). Let $Y = \bigcup_{n=1}^{\infty} \dot{Y}_n$. We will show that this Y has the required properties.

First, Y is locally a finite dimensional polytope, hence locally an ANR, and thus an ANR by [7, Theorem 19.3] (or [12, Proposition 4.1]). Next, Borsuk shows in [2, p. 125] that the closure Y^* of Y in Q is not an ANR, so Y cannot be a uniform ANR by Proposition 1.4 and Theorem 1.1$(b) \to (a)$. It therefore remains to show that Y is uniformly locally contractible.

Let $\epsilon > 0$ be given. Pick $n > 1$ so that $2^{-n} < \frac{1}{5}\epsilon$, and let

$$S = \left\{y \in Y : y_1 \geq \frac{1}{2^n}\right\}, \; T = \left\{y \in Y : y_1 \leq \frac{1}{2^{n-1}}\right\}.$$

Now S is a finite polytope, and thus uniformly locally contractible; pick $\gamma > 0$ so that, if $x \in S$, then $B_\gamma(x) \cap S$ is contractible over $B_\epsilon(x) \cap S$. Let $\delta = \min(\gamma, 2^{-(n+1)})$. We will show that, if $x \in Y$, then $B_\delta(x)$ is contractible over $B_\epsilon(x)$.

Let $V = B_\delta(x)$. Then diam $V \leq 2^{-n}$, so $V \subset S$ or $V \subset T$. If $V \subset S$, then our claim is true because $\delta \leq \gamma$. So we may suppose that $V \subset T$.

Let $A = \{y \in Q : y_n = 0\}$, $B = \{y \in Q : y_n = 1\}$. Then $d(a, b) \geq 2^{-n}$ whenever $a \in A$, $b \in B$; since $\delta \leq 2^{-(n+1)}$, it follows that $V \cap A = \emptyset$ or $V \cap B = \emptyset$. Suppose $V \cap B = \emptyset$; the case $V \cap A = \emptyset$ is similar. For each $y \in T$, define $y' \in T$ and $y^* \in Y_n$ by

$$y'_i = y_i \quad \text{if} \quad i \neq n, \qquad y'_n = 0;$$
$$y^*_i = y_i \quad \text{if} \quad i \leq n, \qquad y_i = 0 \quad \text{if} \quad y > n.$$

We now contract V to x by moving each $y \in V$ to x along five line-segments, namely from y to y' to y^* to x^* to x' to x. It is easy to check that each of these segments lies in Y, and that the distance between its end points – and thus its diameter – is $\leq 2^{-n}$. Hence the entire path lies in Y, and, since $2^{-n} < \frac{1}{5}\epsilon$, it lies in $B_\epsilon(x)$. That completes the proof.

REFERENCES

[1] R. ARENS and J. EELLS: On embedding uniform and topological spaces. *Pacific J. Math. 6* (1956) 397–403.

[2] K. BORSUK: *Theory of retracts*, Monografie Matematyczne 44, Polish Scientific Publishers, Warsaw, 1967.

[3] C.H. DOWKER: On a theorem of Hanner, *Ark. Mat. 2* (1952) 307–313.

[4] J. DUGUNDJI, An extension of Tietze's theorem, *Pacific J. Math. 1* (1951) 353–367.

[5] J. DUGUNDJI: Absolute neighborhood retracts and local connectedness in arbitrary metric spaces, *Compositio Math. 13* (1958) 229–246.

[6] J. DUGUNDJI and E. MICHAEL: On local and uniformly local topological properties, *Proc. Amer. Math. Soc.* 7 (1956) 304–307.

[7] O. HANNER: Retraction and extension of mappings of metric and non-metric spaces, *Ark. Mat. 2* (1952) 315–360.

[8] J. ISBELL: Uniform neighborhood retracts, *Pacific J. Math. 11* (1961) 609–648.

[9] J. ISBELL: *Uniform spaces*, Mathematical Surveys 12, American Mathematical Society, 1964.

[10] S. LEFSCHETZ, *Topics in topology*, Princeton University Press, Princeton, 1942.

[11] E. MICHAEL: On the completion of a uniformly LC^n or ANR metric space, *Bull. Amer. Math. Soc. 61* (1955), Abstract 847t, p. 571.

[12] E. MICHAEL: Local properties of topological spaces, *Duke Math. J. 21* (1954) 163–172.

[13] E. MICHAEL: A short proof of the Arens-Eells embedding theorem, *Proc. Amer. Math. Soc. 15* (1964) 415–416.

[14] E. MICHAEL: Continuous selections and finite-dimensional sets.

[15] C. PIXLEY: An example concerning continuous selections on infinite-dimensional spaces, *Proc. Amer. Math. Soc. 43* (1974), 237–244.

[16] H. TORUŃCZYK: Absolute retracts as factors of normed linear spaces, *Fund. Math. 86* (1974) 53–67.

[17] H. TORUŃCZYK: Factors of the Hilbert cube, *Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. 24* (1976), 757–765.

(Oblatum 19-VIII-1977 & 21-VIII-1978)

University of Washington
Seattle, Washington 98195
Department of Mathematics