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ON THE CHOW GROUP OF CERTAIN TYPES OF FANO THREEFOLDS

S. Bloch and J.P. Murre

Let X be either a quartic threefold in 4-dimensional projective space $\mathbf{P}_4$, or the intersection of a quadric and a cubic hypersurface in $\mathbf{P}_5$, or the intersection of three quadrics in $\mathbf{P}_6$. In this paper we study the Chow group $A^2(X)$ of such a variety X. The group $A^2(X)$ is the group of those *rational equivalence classes* of cycles of codimension two which are *algebraically equivalent* to zero. We work over an algebraically closed field k of characteristic different from two or three.

Our paper is, to a considerable degree, inspired by a paper of Tjurin [24]. In that paper Tjurin studies the intermediate Jacobian of Fano threefolds of index one and defined over the field of complex numbers $\mathbf{C}$. A Fano threefold X is a smooth projective variety of dimension 3 with ample anticanonical class $-K_X$; it is of index one if $-K_X = r \cdot D$ in $\mathrm{Pic}(X)$, with D a divisorclass containing a positive divisor, implies $r = 1$.* Tjurin defines a generalized Prym variety, a principally polarized abelian variety associated with a couple (J, σ) consisting of a Jacobian variety J of a curve and an endomorphism σ of J satisfying a certain quadratic equation (see §7); in case σ is an involution one gets the usual Prym variety studied by Mumford [14]. In the case of a Fano threefold X the generalized Prym is obtained from the 1-dimensional family of lines on X and the incidence relation between them. Tjurin asserts that the intermediate Jacobian of a Fano threefold is isomorphic to its generalized Prym, but his argument is seriously flawed and yields only an isogeny (see footnote 6). Broadly speaking, we want to apply Tjurin's ideas to $A^2(X)$ as well as to the intermediate Jacobians, and to plug the gap in his arguments.

The three types of threefolds mentioned in the beginning are the most natural examples of Fano threefolds of index one. Our results about them are similar to the results in [16] and [17], namely we show that the Chow group $A^2(X)$ is isomorphic to the group of points of the

* See also the recent paper [25].

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generalized Prym of Tjurin and that the polarization of the Prym is closely related to the cupproduct on $H^3_{ét}(X, \mathbf{Q}_l)$. In fact, more precisely, we show that the Chow group $A^2(X)$ has a *regular, universal isomorphism* into that Prym (see 8.18 for the definitions and 8.24 for the main result).

Our method is as follows. Firstly we show that $A^2(X)$ is isomorphic to the group of points of some abelian variety. We call this fact *weak-representability*; this notion has been introduced in [4] and there it has also been shown that this property is true for the quartics. For the two other types mentioned, weak-representability follows easily from the unirationality, cf [16]. Furthermore let $A^2(X)(l)$ be the inductive limit of the groups of torsion elements of $A^2(X)$ of order l^n (l prime number $\neq \mathrm{Char}(k)$), then there is an isomorphism between $A^2(X)(l)$ and $H^3_{ét}(X, Q_l/\mathbf{Z}_l(2))$. In fact there is a natural map (the cycle map for torsion cycles) between these two groups for any smooth projective variety X; this is shown by the first author in [5]. The existence of this map depends on the proof by Deligne of the Weil conjectures, but in our case it could be constructed by ad hoc methods similar as the ones used in [17]. Our method consists now in combining weak-representability with this isomorphism for torsion cycles. In this way we reduce questions for the Chow group to questions of étale cohomology. Next, using local constantness of étale cohomology we reduce the question to the generic Fano defined over $\mathbf{C}$ and thus use Tjurin's result. For a typical example see 8.11.

In principle our method should work also for the other Fano threefolds of index one provided some preliminary questions have been settled (unirationality, family of lines, etc.). On the other hand we have restricted (so far) our attention to "sufficiently general" Fano's (of the type considered), for instance we need smoothness of the family of lines. (Note that there exists interesting examples of Fano threefolds where this family is *not* smooth, see [23]).

For the case of the intersection of three quadrics in $\mathbf{P}_6$ the $A^2(X)$ has also been studied by Beauville [3], however from a somewhat different point of view.

A short description of the different sections: §1 and 2 contain preliminary results, mostly well-known, but for which there is usually not a clear-cut reference. In §1 we study the family of lines on X. We have followed a useful paper of Barth and Van de Ven [2]; we like to thank them for use of their – as yet – unpublished manuscript. §2 deals with unirationality questions for two of the types under consideration, well-known classically and due to Enriques, see [8] and [3], and a result developed by one of the authors for the quartics [4]. §3 deals systematically with the notion of weak-representability (called representability in [4]), §4 with the cycle map for torsion cycles and §5 with the relation with the (classical) intermediate Jacobian (in case $k = \mathbf{C}$). In §6 and 7 we reproduce the part of Tjurin's results relevant for our purpose; in §7 it was necessary to fill a gap in Tjurin's paper (see footnote 6). Finally: §8 contains the main results, namely the results on the Chow group and the polarization as mentioned above.

Notations:

X_3^4 denotes a 3-dimensional variety of degree 4 in $\mathbf{P}_4$.

$X_3^6 = Q \cdot C$ denotes a 3-dimensional variety in $\mathbf{P}_5$ which is an intersection of a quadric hypersurface Q and a cubic hypersurface C.

$X_3^8 = Q \cdot Q' \cdot Q''$ denotes a 3-dimensional variety in $\mathbf{P}_6$ which is an intersection of three quadric hypersurfaces.

Q_n: a quadric hypersurface in $\mathbf{P}_{n+1}$.

C_n: a cubic hypersurface in $\mathbf{P}_{n+1}$.

In a notation like V_n^d, n denotes the dimension and d the degree of V.

$\mathrm{Gr}(n, m)$ is the Grassmann variety of $\mathbf{P}_m$'s in $\mathbf{P}_n$ $(m < n)$.

§1. Lines on some special Fano varieties

Let V be a variety in projective space $\mathbf{P}_n$. Consider

(1) $$F(V) = \{l;\, l \text{ a line on } V\} \subset \mathrm{Gr}(n, 1),$$

the *variety of lines on V*.

PROPOSITION 1.1: *Let V be either X_3^4, or $X_3^6 = Q \cdot C$, or $X_3^8 = Q \cdot Q' \cdot Q''$. Then we have*:

i) *$F(V) \neq \emptyset$.*

ii) *$F(V)$ is connected.*

iii) *Let U be the variety parametrizing the varieties V of the above type (for each of the three cases respectively). Then there exists an open, non-empty set $U_0 \subset U$ such that for $V \in U_0$ the $F(V)$ is a smooth curve.*

REMARK 1.2: For $V = X_3^4$ this was proved by Barth and Van de Ven [2]. In fact Barth and Van de Ven proved the following more general result (with $k = \mathbf{C}$): let $V = X_{n-1}^d$ be a hypersurface of degree d in $\mathbf{P}_n$. Then:

i) $F(V) \neq \emptyset$ if $d + 3 \leq 2n$.

ii) $F(V)$ is connected if $d + 5 \leq 2n$ or if $n = 4$, $d = 4$.

iii) There exists (with similar notations as above) a non-empty, open set $U_0 \subset U$ such that for $V \in U_0$ we have $F(V)$ smooth and of dimension $2n - d - 3$.

Since we don't need this result for arbitrary hypersurfaces we present here only a proof for proposition 1.1; this proof is a straightforward generalization of the proof of Barth and Van de Ven for the case $n = 4$, $d = 4$. We need the following auxiliary result (see [2] for the case $m = 4$):

LEMMA 1.3: *Let $A = (a_{ij})$ be a matrix with $(m-1)$-columns and m-rows. Consider the matrix*

$$\tilde{A} = \underbrace{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right)}_{2m-2} \begin{array}{l} \updownarrow\ m-3 \\ \updownarrow\ 3 \\ \updownarrow\ m-3 \end{array}$$

(i.e. add $m-3$ rows of zeros, etc., as indicated). Then the matrices A for which rank $(\tilde{A}) < 2m-3$ form a cone in the space $k^{m(m-1)}$ of codimension at least 2.

PROOF: Put $W = k^m$ and consider the map

$$\varphi : W^{m-1} = \underbrace{W \times \cdots \times W}_{(m-1)\text{-times}} \longrightarrow \Lambda^{m-1}(W) \simeq W,$$

defined by $\varphi(w_1, \ldots, w_{m-1}) = w_1 \wedge \cdots \wedge w_{m-1}$. In particular, consider $A = (A_1, \ldots, A_{m-1}) \in W^{m-1}$, where $A_i \in W$ with $A_i = \Sigma_{j=1}^m a_{ji} e_j$, then we have

$$A \mapsto \varphi(A) = (\alpha_1, \ldots, \alpha_m) \in \Lambda^{m-1}(W),$$

with

$$\varphi(A) = \alpha_1 e_2 \wedge \cdots \wedge e_m + \alpha_2 e_1 \wedge e_3 \wedge \cdots \wedge e_m + \cdots + \alpha_m e_1 \wedge \cdots \wedge e_{m-1},$$

with

$$\alpha_1 = \det \begin{pmatrix} a_{21} & \cdots & a_{2m-1} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mm-1} \end{pmatrix}, \text{ etc.}$$

Put $W^* = \{A; \varphi(A) \neq (0, \ldots, 0)\} \subset W^{m-1}$.

CLAIM 1: *$\varphi \mid W^*$ is locally a fibering; i.e. the map φ (or better, its differential $\delta\varphi$) has maximal rank ($= m$).*

PROOF: $\varphi(A) \neq (0)$ means that the rank of A equals $(m-1)$. After a change of coordinates we can assume

$$A = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \\ & & 1 \\ 00 & & 0 \end{pmatrix}.$$

The assertion follows now by an easy computation.

Next consider in k^{2m-3} the vector space V spanned by the column-vectors of the matrix $\tilde{A}$; let V_1 be the subspace spanned by the columns of the matrix

$$\begin{pmatrix} A \\ 0 \end{pmatrix} \updownarrow m-3,$$

and similarly V_2 the subspace spanned by the columns of

$$\begin{pmatrix} 0 \\ A \end{pmatrix} \updownarrow m-3.$$

Clearly $V = V_1 + V_2$. Finally put on k^{2m-3} the usual scalar form $\langle \cdot, \cdot \rangle$ obtained via the standard basis e_i $(i = 1, \ldots, 2m-3)$ and $\langle e_i, e_j \rangle = \delta_{ij}$. Then we have

$$V_1 = {}^t(\alpha_1, \ldots, \alpha_m, 0, \ldots, 0)^\perp \cap (\text{span } e_{m+1}, \ldots, e_{2m-3})^\perp$$

and similarly

$$V_2 = {}^t(0, \ldots, 0, \alpha_1, \ldots, \alpha_m)^\perp \cap (\text{span } e_1, \ldots, e_{m-3})^\perp$$

CLAIM 2: *In W^* we have*

$$\operatorname{rank}(\tilde{A}) < 2m-3 \Leftrightarrow \operatorname{rank}\begin{pmatrix} \alpha_{m-2} & \alpha_1 \\ \alpha_{m-1} & \alpha_2 \\ \alpha_m & \alpha_3 \end{pmatrix} < 2$$

PROOF: $V = V_1 + V_2$, and in W^* we have $\dim V_1 = m-1$, $\dim V_2 = m-1$. Hence $\operatorname{rank}(\tilde{A}) < 2m-3 \Leftrightarrow \dim V < 2m-3 \Leftrightarrow \dim(V_1 \cap V_2) > 1 \Leftrightarrow$ vectors$\{e_1, \ldots, e_{m-3}, e_{m+1}, \ldots, e_{2m-3}, {}^t(\alpha_1, \ldots, \alpha_m, 0, \ldots, 0), {}^t(0, \ldots, 0,$

$\alpha_1, \ldots, \alpha_m)\}$ lin. dependent $\Leftrightarrow$

$$\operatorname{rank}\begin{pmatrix} \alpha_{m-2} & \alpha_1 \\ \alpha_{m-1} & \alpha_2 \\ \alpha_m & \alpha_3 \end{pmatrix} < 2.$$

CLAIM $1 + 2 \Rightarrow$ LEMMA 1.3.

PROOF: Consider in $\Lambda^{m-1}(W) \simeq W$ the set

$$S = \left\{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix};\ \operatorname{rank}\begin{pmatrix} \alpha_{m-2} & \alpha_1 \\ \alpha_{m-1} & \alpha_2 \\ \alpha_m & \alpha_3 \end{pmatrix} < 2 \right\}.$$

Clearly $\operatorname{codim}_W(S) \geq 2$. Now consider $E = \varphi^{-1}(S) \subset W^{m-1}$; then since φ is a fibering in W^* by claim 1, we have also

$$\operatorname{codim}_{W^{m-1}}(E) \geq 2,$$

which proves the lemma by claim 2.

1.4: Turning to the proof of prop. 1.1, we still need several preparations. Let $V \subset \mathbf{P}_n$ be a complete intersection defined by the equations:

$$f_\alpha(X) = \sum_{i_0+i_1+\cdots+i_n=d_\alpha} a_{\alpha,i_0i_1\ldots i_n} X_0^{i_0} \ldots X_n^{i_n} = 0 \tag{3}$$
$$(\alpha = 1, \ldots, r)$$

Let U be the space parametrizing the varieties V; note that U is a projective space or a product of projective spaces. Consider the correspondence

$$\begin{tikzcd} G = \mathrm{Gr}(n, 1) & Z \arrow[l, "q"'] \arrow[d, "p"] \\ & U \end{tikzcd} \tag{4}$$

with

$$Z = \{(V, l);\ l \subset V\}.$$

Let $(V_0, l) \in Z$. Now $l_0 \subset V_0$ gives $\Sigma_\alpha\, (d_\alpha + 1)$ linear independent conditions on the coefficients of the equations (3) for V_0; i.e. $q^{-1}(l_0)$ has codimension $\Sigma(d_\alpha + 1)$ and is locally isomorphic to a linear space. Hence we have

$$\dim Z = 2(n-1) + \left\{ \dim U - \sum_\alpha (d_\alpha + 1) \right\}. \tag{5}$$

Note that we have the (trivial) necessary condition:

$$F(V) \neq \emptyset \, \forall V \Rightarrow 2(n-1) \geq \sum_{\alpha} (d_\alpha + 1).$$

Note also that, since $GL(n)$ operates transitively on G and since $q^{-1}(l_0)$ is non-singular, we have that Z is non-singular.

For $(V_0, l_0) \in Z$ consider the map of tangent spaces

$$(6) \qquad (\delta p): T_{(V_0, l_0)}(Z) \longrightarrow T_{V_0}(U).$$

We have $l_0 \subset \mathbf{P}_n$. *Assume now that V_0 is smooth along l_0.* Then we have an exact sequence of normal bundles:

$$(7) \qquad 0 \to N_{l_0/V_0} \longrightarrow N_{l_0/\mathbf{P}_n} \xrightarrow{\pi} N_{V_0/\mathbf{P}_n}/l_0 \longrightarrow 0,$$

and hence an exact sequence:

$$(8) \quad 0 \longrightarrow \Gamma(l_0, N_{l_0/V_0}) \longrightarrow \Gamma(l_0, N_{l_0/\mathbf{P}_n}) \xrightarrow{\pi} \Gamma(l_0, N_{V_0/\mathbf{P}_n}/l_0) \longrightarrow$$
$$\longrightarrow H^1(l_0, N_{l_0/V_0}) \longrightarrow H^1(l_0, N_{l_0/\mathbf{P}_n})$$
$$\| $$
$$H^1(l_0, \oplus \mathcal{O}_{l_0}(1)) = 0.$$

LEMMA 1.5: *The following conditions are equivalent*:

1. $F(V_0)$ *smooth in* l_0,
2. *The map* (δp) *of* (6) *surjective in* (V_0, l_0),
3. π *surjective in* (8),
4. $H^1(l_0, N_{l_0/V_0}) = 0$,
5. $\dim \Gamma(l_0, N_{l_0/P_n}) = \dim \Gamma(l_0, N_{l_0/V_0}) + \dim \Gamma(l_0, N_{V_0/P_n}/l_0)$,
6. $\operatorname{codim}\{\operatorname{Ker} \pi$ *in* $\Gamma(l_0, N_{l_0/\mathbf{P}_n})\} = \dim \Gamma(l_0, N_{V_0/\mathbf{P}_n}/l_0)$.

PROOF: 1⇔2 is the well-known criterium of multiplicity 1. 3⇔4⇔5⇔6: immediate from the exact sequence (8). It remains to be seen that 2⇔3.

After a change of coordinates we can assume that l_0 is spanned by the points $(1, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$. For $l \in \mathrm{Gr}(n, 1)$ in a neighborhood of l_0 we can assume that it is spanned by the points $(1, 0, x_2, \ldots, x_n)$ and $(0, 1, y_2, \ldots, y_n)$ and we may consider $(x, y) = (x_2, \ldots, y_n)$ as (local) coordinates on $G = \mathrm{Gr}(n, 1)$; let (X, Y) be the corresponding coordinates in the tangentspace $T_{l_0}(G)$. Furthermore we use the coefficients $a = (\ldots, a_{\alpha, i_0 i_1 \ldots i_n}, \ldots)$ of equations (3) as coordinates on the parameterspace U of V; let $A = (\ldots, A_{\alpha, i_0 \ldots i_n}, \ldots)$ be the corresponding coordinates in the tangentspace $T_{V_0}(U)$. Note that $l_0 \subset V_0$ implies for the coordinates of V_0:

$$a_{\alpha,i_0i_10\dots0} = 0 \qquad (i_0 + i_1 = d_\alpha\,;\ \alpha = 1,\dots,r).$$

Next we compute the equations of $T_{(V_0,l_0)}(Z)$ as subspace of $T_{V_0}(U) \times T_{l_0}(G)$. For a point on l we have

$$\begin{cases} \bar{x}_0 = \lambda \\ \bar{x}_1 = \mu \\ \bar{x}_i = \lambda x_i + \mu y_i \qquad (i = 2,\dots,n), \end{cases}$$

and $l \subset V$ gives – after substituting in (3) – the equations for Z:

$$\text{(9)} \qquad \sum_i a_{\alpha,i_0\dots i_n} \lambda^{i_0} \mu^{i_1} \dots (\lambda x_j + \mu y_j)^{i_j} \dots \equiv 0 \qquad \text{in } \lambda, \mu \qquad (\alpha = 1,\dots,r)$$

Differentiating of (9) gives the equations of $T_{V_0,l_0}(Z)$:

$$\text{(10)} \qquad \sum_{i_0+i_1=d_\alpha} A_{\alpha,i_0i_10\dots0} \lambda^{i_0}\mu^{i_1} + \sum_{j=2}^{n} \Big(\sum_{i_0+i_1=d_\alpha-1} a^{(j)}_{\alpha,i_0i_1} \lambda^{i_0} \mu^{i_1} (\lambda X_j + \mu Y_j) \Big) \equiv 0 \text{ in } \lambda,\mu \qquad (\alpha = 1,\dots,r),$$

where we have abbreviated

$$a_{i_0i_10\dots1\dots0} = a^{(j)}_{i_0i_1} \qquad \text{(1 on the } j\text{-th place)}.$$

Clearly if $(A, X, Y) \in T_{V_0,l_0}(Z)$, then $(\delta p)(A, X, Y) = A$. Now we have the following interpretation for (10): consider the map

$$\begin{array}{c} T_{V_0,l_0}(Z) \subset T_{V_0}(U) \times T_{l_0}(G) \\ \downarrow \\ \Gamma(V_0, N_{V_0/\mathbf{P}_n}) \times \Gamma(l_0, N_{l_0/\mathbf{P}_n}) \\ \downarrow {\scriptstyle \rho\times\pi} \\ \Gamma(l_0, N_{V_0/\mathbf{P}_n}/l_0) \end{array}$$

where π is from (8) and ρ is the restriction map (restriction of normal bundle to l_0). Also $N_{V_0/\mathbf{P}_n}/l_0 \simeq \Sigma_\alpha\, \mathcal{O}_{l_0}(d_\alpha)$. From this point of view the L.H.S. of (10) is a section in $\Gamma(l_0, N_{V_0/\mathbf{P}_n}/l_0)$; the first part is $\rho(A)$ and the second part is $\pi(X, Y)$. Moreover, then the equations (10) read:

$$\text{(10}'\text{)} \qquad \rho(A) + \pi(X, Y) = 0.$$

(2) $\Rightarrow$ (3): δp being surjective we can choose A arbitrary and solve (10); since ρ is surjective this means π is surjective.

(3) $\Rightarrow$ (2): Take A, i.e. $\rho(A)$, arbitrary; since π is surjective we can solve (10′), i.e. δp is surjective.

This completes the proof of Lemma 1.5.

1.6: The notations are as in (4). Let $(V_0, l_0) \in Z$. Consider in $q^{-1}(l_0)$ the following set:

$$E_{l_0} = \{V;\ l_0 \subset V \text{ and } \delta p \text{ is } \textit{not} \text{ surjective in } (V, l_0)\}.$$

LEMMA 1.6: *Suppose we are in the case $V = X_3^4$, resp. X_3^6, resp. X_3^8. Then E_{l_0} is closed. Moreover E_{l_0} is empty or has codimension at least* 2 *in $q^{-1}(l_0)$.*

PROOF: By lemma 1.5 δp is surjective iff

$$\operatorname{codim}(\ker \pi) = \sum_\alpha (d_\alpha + 1),$$

because since $N_{V_0/\mathbf{P}_n}/l_0 \simeq \bigoplus_\alpha \mathcal{O}_{l_0}(d_\alpha)$, we have $\dim \Gamma(l_0, N_{V_0/\mathbf{P}_n}/l_0) = \Sigma_\alpha (d_\alpha + 1)$. Hence δp is surjective iff the rank of the matrix of the equations for $\ker(\pi)$ is maximal; moreover according to the above these equations are given by (10′), or rather by (10), after putting $A_{\alpha;i_0i_1.0\ldots0} = 0$. Hence the equations are

$$(11) \qquad \begin{cases} \sum_{j=2}^{n} a^{(j)}_{\alpha;d_\alpha-1,0} X_j = 0 \\ \sum_{j=2}^{n} a^{(j)}_{\alpha;d_\alpha-2,1} X_j + \sum_{j=2}^{n} a^{(j)}_{\alpha;d_\alpha-1,0} Y_j = 0 \\ ------------------ \\ ------------------ \\ \sum_{j=2}^{n} a^{(j)}_{\alpha;0,d_\alpha-1} Y_j = 0 \end{cases}$$

for $\alpha = 1, \ldots, r$. Hence the matrix of (11) is of the type as discussed in lemma 1.3, i.e. of type

$$\tilde{A} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right)$$

with $(\Sigma_{\alpha=1}^{r} d_\alpha)$ – rows for A and $(n-1)$ – columns and $\Sigma_1^r (d_\alpha + 1)$ – rows for $\tilde{A}$. I.e., we have for m in lemma 3

$$\begin{cases} \sum_1^r d_\alpha = m \\ \sum_1^r (d_a + 1) = 2m - 3 \\ n = m \end{cases}$$

This is satisfied for $V = X_3^4$ with $n = 4$, $m = 4$, $d = 4$, $r = 1$, for $V = X_3^6$ with $n = 5$, $m = 5$, $r = 2$, $d_1 = 3$, $d_2 = 2$ and for $V = X_3^8$ with $n = 6$, $m = 6$, $r = 3$, $d_1 = d_2 = d_3 = 2$. Lemma 1.6 follows now from lemma 1.3.

1.7: Notation again as in (4), let $(V_0, l_0) \in Z$ and consider in $q^{-1}(l_0)$ the set

$$S_{l_0} = \{V;\, l_0 \subset V \text{ and } V \text{ singular in a point of } l_0\}.$$

Lemma 1.7: *For $V = X_3^4$, resp. $V = X_3^6$, resp. $V = X_3^8$, we have S_{l_0} closed and of codimension at least* 2 *in $q^{-1}(l_0)$.*

Proof: Closed and $\neq q^{-1}(l_0)$ easy. The assertion about the codimension is laborious and is left to the reader.

Corollary 1.8: *For $V = X_3^4$, resp. X_3^6, resp. X_3^8, we have that $\tilde{E}_{l_0} = E_{l_0} \cup S_{l_0}$ is closed and of codimension at least* 2 *in $q^{-1}(l_0)$.*

1.9: Proof of i) of prop. 1.1: $\tilde{E}_{l_0}$ from 1.8 is closed in $q^{-1}(l_0)$ and different from $q^{-1}(l_0)$. Hence δp is almost everywhere surjective, hence (with the notations of (4)) $p(Z) = U$.

1.10: Proof of iii) of prop. 1.1: Put

$$U_1 = \{V;\, V \text{ smooth}\} \subset U,$$

then U_1 is open. Consider now the diagram (4) with $Z \mid U_1$. The set where δp is *not* surjective is closed in $Z \mid U_1$ because it is locally defined by the rank of the matrix of (11). The projection of this set on

U_1 is *closed* because p is proper; let $U_0 \subset U_1$ be its complement. Then U_0 is open and it suffices to see that U_0 is non-empty. For that it suffices to see that $F(V)$ is smooth for V *generic* over k.

Suppose now that $l \in F(V)$ *singular* in l. We have $\text{tr.deg}_k(V) = \dim U$, but on the other hand if k' is the field obtained by adjoining the Plücker-coordinates of l to k then $\text{tr.deg}_{k'}(V) \leq \dim q^{-1}(l) - 2 = \{\dim U - \Sigma_\alpha (d_\alpha + 1)\} - 2$, because $V \in E_l$ and $\operatorname{codim}(E_l, q^{-1}(l)) \geq 2$, by 1.6. Hence $\dim U = \text{tr.deg}_k(V) \leq \text{tr.deg}_k k' + \text{tr.deg}_{k'}(V) \leq 2(n-1) + \dim U - \Sigma (d_\alpha + 1) - 2$. Hence

$$\sum (d_\alpha + 1) + 2 \leq 2(n-1)$$

and this is a contradiction for $V = X_3^4$, resp. $V = X_3^6$, resp. $V = X_3^8$. Hence $F(V)$ is smooth.
(Note: for the proof of iii) we have not used 1.7).

1.11: Proof of ii) of prop. 1.1. (again we follow [2]): By the connectedness principle it suffices to work in characteristic 0. Consider the Stein factorisation of (4):

$$\begin{tikzcd} Z \arrow[d, "p"'] \arrow[dr, "h"] & \\ U & W \arrow[l, "g"] \end{tikzcd}$$

W normal, g finite, $h^{-1}(w)$ connected for $w \in W$. Hence we have to prove $U = W$. In our case (i.e., $V = X_3^4$, etc.) we have U is a projective space or a product of projective spaces. Hence $\pi_1(U) = 0$. Hence $U = W$ or g is ramified over a divisor $D \subset U$. In the latter case there exists a divisor $D' \subset W$ such that g is ramified in the points of D'. Put $D'' = h^{-1}(D')$, then p is *not* smooth in the points of D'', (EGA IV 17.7.77). Take $(V_0, l_0) \in D''$ and consider $D''' = D'' \cap q^{-1}(l_0)$, then D''' is a divisor in $q^{-1}(l_0)$. This contradicts corollary 1.8 because now $D''' \subset \tilde{E}_{l_0}$. This completes the proof of 1.1.

1.12: Finally we need also the following result on fourfolds:

Lemma 1.12: *Consider $V = X_4^4$, resp. X_4^6, resp. X_4^8, and defined over the complex numbers* $\mathbf{C}$. *Then*:

i) $F(V) \neq \emptyset$,

ii) *For V generic $F(V)$ is smooth and of dimension* 3.

PROOF:

i) Follows from the easy fact that there is a line through every point $v \in V$ (cf. [24], Lecture 4).

ii) Look to the diagram (4) in the present case. Again Z is non-singular (see 1.4). Since we are now in char. 0, the assertion follows immediately from Sard's lemma ([15], p. 42) and a count of dimensions.

§2. Unirationality and related questions

PROPOSITION 2.1: *Let $X_3^6 = Q_4 \cdot C_4 \subset \mathbf{P}_5$ be smooth and with Q_4 smooth. Then X is unirational.*

PROOF: This was proved by Enriques [8]; however a "rigorous proof" requires a lot of details. We follow Enriques idea; see also [21].

LEMMA 2.2: *Let $Q_4 \subset \mathbf{P}_5$ be a smooth quadric and $P \in Q_4$. Then there exists two families of 2-dimensional linear spaces $\{L'_u\}_{u \in \Gamma'}$ and $\{L''_v\}_{v \in \Gamma''}$ such that:*

a) $P \in L'_u \subset Q_4$,

b) $\Gamma' \simeq \mathbf{P}_1$ *over the field* $k(P)$,

c) $L'_{u_1} \cap L'_{u_2} = \{P\}$ *if* $u_1 \neq u_2$.

Similarly for $\{L''_v\}$.

PROOF: We can assume that $Q_4 = \mathrm{Gr}(3, 1)$. Points P, T etc. on Q correspond with lines p, t, etc. in $\mathbf{P}_3$. Take $u \in p \subset \mathbf{P}_3$, then $L'_u = \{q; q$ line in $\mathbf{P}_3$ going through $u\}$. Similarly take a 2-plane v through p, then $L''_v = \{q; q \subset v \subset \mathbf{P}_3\}$. Then for Γ' we can take p, for Γ'' the pencil of planes through p. The properties a), b) and c) are now immediate.

LEMMA 2.3: *Let $P \in X$ as above. Then there exists two families $\{E'_u\}_{u \in \Gamma'}$ and $\{E''_v\}_{v \in \Gamma''}$ of plane curves of degree 3 such that*

a) $P \in E'_u \subset X$,

b) $\Gamma' \simeq \mathbf{P}_1$ *over the field* $k(P)$,

c) $E'_{u_1} \cap E'_{u_2} = \{P\}$ *if* $u_1 \neq u_2$,

d) *for u sufficiently general (over $k(P)$) we have that E'_u is smooth in P.*

Similarly for $\{E''_v\}$.

PROOF: X smooth $\Rightarrow X \not\supset$ a 2-plane ([3], 1.4.6). Hence for each u we have $L'_u \not\subset C_4$, hence $E'_u = L'_u \cap C_4$ is a plane curve of degree 3 ($\forall u$,

$\forall P \in X$). Hence we have a), b) and c) by 2.2. To prove d) let us denote by $T_P(—)$ the tangent space in P and consider first $T_P(Q_4) \cap Q_4$: this is a cone over a smooth quadric $Q_2 \subset \mathbf{P}_3$; take H general (over $k(P)$) then we have $T_P(Q_4) \cap Q_4 \cap H = Q_2^*$, with $Q_2^* \subset \mathbf{P}_3$ smooth. Now $\{L'_u\}$, resp. $\{L''_v\}$, resp. $\{L''_v\}$, can be obtained as follows: take the two families $\{l'_u\}$ and $\{l''_v\}$ of lines on Q_2^*, then L'_u is the join of P and l'_u and similar for L''_v. Now consider also $T_P(C_4)$, then $T_P(C_4) \not\supset Q_2^*$, for, since X is smooth we have $T_P(C_4) \cap T_P(Q_4) \cap H = T_P(X) \cap H$ is itself of dimension 2, hence $\not\supset Q_2^*$. Since $T_P(C_4) \not\supset Q_2^*$ we have $T_P(C_4) \not\supset l'_u$ for u sufficiently general. Hence for such u we have $T_P(C_4) \not\supset L'_u$, i.e. L'_u intersects C_4 transversally in P, which proves d).

2.4. Construction of the rational curve $V_1(P)$

Given $P \in X$, take the system $\{E_u\}$ ($= \{E'_u\}$ or $\{E''_v\}$; we take one of the two systems and drop the index now). Take u generic over $k(P)$ and consider E_u. Take the tangent to E_u in P; this tangent intersects E_u in a "third" point R_u. Take the locus $V_1(P)$ of R_u over $k(P)$. Clearly $V_1(P) \subset X$ and $V_1(P) \simeq \mathbf{P}_1$ over $k(P)$, because for the function field we have $k(P)(V_1(P)) = k(P)(R_u) \subset k(P)(u)$, with $u \in \Gamma \simeq \mathbf{P}_1$ over $k(P)$ by 2.3b.

LEMMA 2.5: $P \in V_1(P)$.

PROOF: By a specialization argument it will be sufficient to prove this for X generic and for P generic on X.

Consider as before the quadric surface

$$Q_2^* = T_P(Q_4) \cap Q_4 \cap H,$$

where H is a hyperplane, generic with respect to $k(P)$. Next consider the cubic C_4 of $X_3^6 = Q_4 \cdot C_4$ and consider

$$T_P(C_4) \supset T_P^*(C_4) \supset T_P^*(C_4) \supset T_P^{**}(C_4)$$

where T, T^* and T^{**} denote respectively the tangent space, the tangent cone and the cone of lines on C_4 through P. Now we claim that

$$(12) \qquad \begin{cases} T_P^*(C_4) \cap Q_2^* \text{ is a finite set (4 points)} \\ T_P^{**}(C_4) \cap Q_2^* = \emptyset \end{cases}$$

This is true since we are in a "generic situation": otherwise it would be false for any X_0 and any $P_0 \in X_0$, but fixing $P_0 \in Q_4^0$ we can easily take C_4^0 sufficiently general through P_0 such that (12) is true.

Take $S \in T_P^*(C_4) \cap Q_2^*$ and a line l_{u_0} on Q_2^* through S. Then (cf. proof of 2.3) l_{u_0} and P span $L_{u_0} \subset Q_4$ and consider the curve E_{u_0} of 2.3 given by $E_{u_0} = L_{u_0} \cap C_4$. Now $PS \not\subset T_P^{**}(C_4)$, hence $PS \not\subset E_{u_0}$, but on the other hand since $PS \subset T_P^*(C_4)$ we have that PS meets E_{u_0} in P with multiplicity 3 (as line in L_{u_0}). Hence the point R_{u_0} used in 2.4 in the construction of $V_1(P)$ is the point P itself and hence $P \in V_1(P)$.

LEMMA 2.6: *Let $P_1, P_2 \in X$ with $P_1 \neq P_2$. Consider the curves $E_u(P_1)$, resp. $E_t(P_2)$, of 2.3 constructed for the point P_1, resp. for P_2. Suppose $P_2 \in E_{u_0}(P_1)$ and $P_1 \in E_{t_0}(P_2)$. Then $E_{u_0}(P_1) = E_{t_0}(P_2)$ (as curves).*

PROOF: Following the interpretation of Q_4 as Gr(3, 1) as in 2.2 we have $P_i \leftrightarrow p_i$, a line in $\mathbf{P}_3$. $P_2 \in L_{u_0}(P_1)$ means that p_2 meets p_1 in S, say. But then $S \leftrightarrow$index u_0 (see description in proof of 2.2) and hence $L_{u_0}(P_1)$ is the collection of lines in $\mathbf{P}_3$ through S. Since $P_1 \in L_{t_0}(P_2)$, the same is true for $L_{t_0}(P_2)$; hence $L_{u_0}(P_1) = L_{t_0}(P_2)$, hence $E_{u_0}(P_1) = E_{t_0}(P_2)$.

LEMMA 2.7: *Let P_1 be generic on $V_1(P)$. Then $P \notin V_1(P_1)$.*

PROOF: $P_1 \in V_1(P) \Rightarrow P_1 \in E_U(P)$ for u generic over $k(P)$ (see construction 2.4). If $P \in V_1(P_1)$ then also $P \in E_{t_0}(P_1)$ for some t_0. Hence by 2.6 we have $E_u(P) = E_{t_0}(P_1)$. But now PP_1 is tangent to $E_u(P)$ in P by construction and, hence, is not tangent to $E_u(P) = E_{t_0}(P_1)$ in P_1 as it should be in case $P \in V_1(P_1)$.

LEMMA 2.8: *For P_1, P_2 independent generic on $V_1(P)$ we have $V_1(P_1) \neq V_1(P_2)$.*

PROOF: If $V_1(P_1) = V_1(P_2)$ then by specializing P_2 to P we get $V_1(P_1) = V_1(P)$. But $P \in V_1(P)$ by 2.5 and $P \notin V_1(P_1)$ by 2.7.

Construction 2.9: Let $V_2(P)$ be the locus of $V_1(P_1)$, where P_1 moves over $V_1(P)$.

It follows from 2.8 that $V_2(P)$ is a *surface* defined over $k(P)$. Also from 2.5 follows that $V_1(P) \subset V_2(P)$ and in particular $P \in V_2(P)$.

LEMMA 2.10: *The surface $V_2(P)$ is unirational over the field $k(P)$ (hence rational over k if P is a point defined over k).*

PROOF: By 2.4 we have a birational transformation $\lambda_1(P): \mathbf{P}_1 \overset{\sim}{\dashrightarrow} V_1(P)$ defined over $k(P)$. Consider now the rational map $\lambda_2(P): \mathbf{P}_2 \dashrightarrow V_2(P)$, defined over $k(P)$, as composition:

$$\mathbf{P}_2 \overset{\sim}{\dashrightarrow} \mathbf{P}_1 \times \mathbf{P}_1 \underset{\lambda_1(P)\times id}{\overset{\sim}{\dashrightarrow}} V_1(P) \times \mathbf{P}_1 \underset{\mu(P)}{\dashrightarrow} V_2(P),$$

where $\mu(P)$ is defined as follows: take a generic point P_1 of $V_1(P)$ over $k(P)$ and a generic point P_1' of $\mathbf{P}_1$ over $k(P, P_1)$, then $\mu(P_1, P_1') = \lambda(P_1)(P_1') = P^*$. Note that P^* is a generic point of $V_1(P_1)$ over $k(P, P_1)$ and hence a generic point of $V_2(P)$ over $k(P)$.

Construction 2.11: Let $V_3(P)$ be the locus of $V_1(P_2)$, where P_2 moves over $V_2(P)$. It follows from 2.5 that $V_3(P) \supset V_2(P)$.

LEMMA 2.12: $V_3(P) = X$.

PROOF (Cf [4]): If not then $V_3(P) = V_2(P)$. By Lefschetz theory $V_2(P) = X \cdot F$, with F *a hypersurface. Take* P_1 and P_2 independent generic on X over k. Now $V_2(P_1) \cap V_2(P_2) = X \cap F_1 \cap F_2 \neq \emptyset$ and in fact of dimension (at least) 1, i.e. $V_2(P_1) \cap V_2(P_2) = \Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots$, where Γ is a curve with irreducible components Γ_1, Γ_2, etc. Take R generic on Γ_1. Now $\Gamma_1 \subset V_2(P_1)$, hence $V_1(R) \subset V_3(P_1)$ by construction of $V_3(P_1)$ (see 2.11), i.e. $V_1(R) \subset V_3(P_1) = V_2(P_1)$. Similarly $V_1(R) \subset V_2(P_2)$. Hence $V_1(R) \subset V_2(P_1) \cap V_2(P_2) = \Gamma$ and since $R \in V_1(R)$ and $R \notin \Gamma_i$ $(i \geq 2)$ we have $V_1(R) = \Gamma_1$. Take R^* generic on Γ_1 and independent of R then we get $V_1(R) = \Gamma_1 = V_1(R^*)$, a contradiction by 2.7 and 2.5. Hence $V_3(P) = X$.

2.13. PROOF OF 2.1: Take a point P rational over k. Now we have a sequence of rational maps

$$\mathbf{P}_3 \overset{\sim}{\dashrightarrow} \mathbf{P}_2 \times \mathbf{P}_1 \underset{\lambda_2(P)\times id}{\dashrightarrow} V_2(P) \times \mathbf{P}_1 \underset{\Psi}{\dashrightarrow} V_3(P) = X,$$

where Ψ is defined as follows: take a generic point P_2 on $V_2(P)$ over k and P' independent generic on $\mathbf{P}_1$. Then $\Psi(P_2, P') = \lambda_2(P_2)(P') = P^*$, where $\lambda_2(P_2)$ is defined in the proof of 2.10. Note that P^* is generic on $V_1(P_2)$ over $k(P_2)$ and hence generic on $V_3(P)$ over k.

PROPOSITION 2.14: *Let* $X_3^8 = Q \cdot Q' \cdot Q'' \subset \mathbf{P}_6$. *Suppose X is smooth. Then X is unirational.*

PROOF: [3], 4.5.3.

2.15: Finally we turn to the case of a quartic $X_3^4 \subset \mathbf{P}_4$. Here unirationality is not known, however there is the following result:

PROPOSITION 2.15 ([4], *page* 14): *Let* $X_3^4 \subset \mathbf{P}_4$ *be a smooth quartic which is sufficiently general. Then there exists a couple* (X', f), *where* X' *is a smooth threefold and such that:*

a) $f: X' \dashrightarrow X$ *is a surjective rational map of finite degree d.*

b) X' *is a "conic-bundle" in the following sense:* $\exists$ *rational map* $g: X' \dashrightarrow S$, *with S a smooth surface and such that the geometric generic fibre of* $\underline{g}$ *is isomorphic to* $\mathbf{P}_1$ *(i.e. if* ξ *is generic on S then* $X'_{\bar{\xi}} = g^{-1}(\xi) \times \mathrm{Sp}\, \overline{k(\xi)} \xrightarrow{\sim} \mathbf{P}_1$).

c) *Moreover S is such that the Chow group* $A^2(S)$ *is weakly-representable (see* §3) *(resp. if* $k = \mathbf{C}$: *is isogenous to the classical intermediate Jacobian* $J^2_{cl}(S)$ *of S).*

§3. Weak-representability[1]

3.1. Some notations:

In this section V denotes a smooth, projective variety defined over an algebraically closed field k. $CH^i(V)$ is the *Chow group* of cycle-classes of *codimension i*, i.e., the cycles of codimension i modulo *rational equivalence.* $A^i(V) \subset CH^i(V)$ is the subgroup of classes *algebraically equivalent* to zero. Sometimes we work with *dimension i* instead of codimension i, then we write $CH_i(V)$ and $A_i(V)$ respectively. Finally $CH^i(V)_n$ and $A^i(V)_n$ denote the subgroups of classes of order n.

Let $K \supset k$ be an overfield. Write K as an increasing union of rings R_α smooth and of finite type over k, and let

$$CH^i(V_K) = \varinjlim_{\alpha} CH^i(V \times_k \mathrm{Sp}\, R_\alpha),$$

respectively

$$A^i(V_K) = \varinjlim_{\alpha} A^i(V \times_k \mathrm{Sp}\, R_\alpha).$$

When $K = \bar{K}$ these correspond to the groups obtained by viewing V_K as variety over K.

[1] See [4], there this notion is called representability.

In the following, objects are tacitly assumed to be defined over k unless specified. We denote rational equivalence by $\sim$; if $\mathfrak{A}$ is a cycle then $Cl(\mathfrak{A})$ means the class with respect to rational equivalence.

LEMMA 3.2: $CH^i(V) \hookrightarrow CH^i(V_K)$.

PROOF: It suffices to show $CH^i(V) \hookrightarrow CH^i(V \times \mathrm{Sp}\, R_\alpha)$. But R_α is of finite type over $k = \bar{k}$, so there exists a k-point $\mathrm{Sp}\, k \hookrightarrow \mathrm{Sp}\, R_\alpha$ and the restriction

$$CH^i(V \times \mathrm{Sp}\, R_\alpha) \longrightarrow CH^i(V \times \mathrm{Sp}\, k)$$

defines a splitting for the above arrow.

DEFINITION 3.3: *$A^i(V)$ is called weakly-representable if there exists a triple (Γ, T, B) such that*

a) *Γ is a smooth curve,*

b) *$T \subset CH^i(\Gamma \times V)$,*

c) *B is an algebraic subgroup of the jacobian variety $J(\Gamma)$ of Γ such that*

$$0 \longrightarrow B(K) \longrightarrow J(\Gamma)(K) \xrightarrow{T_*} A^i(V_K) \longrightarrow 0 \tag{13}$$

is exact for all $K = \bar{K} \supset k$.

Remarks:

1. By our convention Γ, B and T are defined over k.
2. T_* above is the homomorphism

$$T_*: A_0(\Gamma_K) \longrightarrow A^i(V_K),$$

induced by T.

3. The above means that, for any $K = \bar{K} \supset k$, we have the isomorphism

$$A^i(V_K) \xrightarrow{\sim} (J(\Gamma)/B)(K), \tag{14}$$

i.e., roughly: the Chow group is parametrized by the points of an abelian variety. Note also: if we write $J(\Gamma)/B$ then we mean always the quotient obtained by taking every component of B with multiplicity 1.

EXAMPLE: Using the theory of the Picard variety we see easily that $A^1(V)$ is weakly-representable (Note however that the notion of representability of Grothendieck is much stronger than the present notion of weak-representability).

LEMMA 3.4: *Let $A^i(V)$ be weakly-representable by a triple (Γ, T, B). Let W be a smooth variety and $Z \in CH^i(W \times V)$. Consider the corresponding map (after the choice of a base point $w_0 \in W$):*

$$Z_*: W \longrightarrow A^i(V),$$

defined by $w \mapsto Cl(Z(w) - Z(w_0))$. Then there exists a subvariety $\Omega \subset W \times J(\Gamma)/B$, defined over k, such that

$$\Omega = \{(w, Z_*(w));\ w \in W(K)\} \qquad \forall K = \bar{K} \supset k. \tag{15}$$

Remark:

Ω is "something" like the graph of a morphism. Since our notion of weak-representability is of set-theoretical nature this is the best we can hope for in arbitrary characteristic. We call $Z_*: W \to J(\Gamma)/B$ a *weak-morphism.*

COROLLARY 3.5: *In char. 0 the subvariety Ω is the graph of a morphism*

$$Z_*: W \longrightarrow J(\Gamma)/B.$$

PROOF OF 3.4: First take $K = k$. Denote by Ω' the *set of points* on $W \times J(\Gamma)/B$ defined by the relation (15). Furthermore take on W a generic point ξ over k. Let L be the algebraic closure of the field $k(\xi)$. By the definition of weak-representability applied to $A^i(V_L)$ there exists a point $\zeta \in J(\Gamma)(L)$ such that $Z_*(\xi) = T_*(\xi)$. Let η be the image of ζ in $(J(\Gamma)/B)(L)$ (in order to be concrete we can see η as the Chow point of the translate B_ζ of B in $J(\Gamma)$, cf. [6]). Let Ω be the *closure of* (ξ, η) in the k-topology of $W \times J(\Gamma)/B$; i.e. the locus of (ξ, η) over k in the sense of Weil.

Choose a cycle $\mathfrak{A}$ of degree 0 on Γ such that $\zeta = Cl(\mathfrak{A})$, i.e. $\eta = T_*(\zeta) = ClT(\mathfrak{A})$. Moreover, during this proof we denote by Z also a *representative cycle* of the class $Z \in CH^i(W \times V)$; then

$$\eta = T_*(\zeta) = Z_*(\xi)$$

means explicitly:

$$(16) \qquad T(\mathfrak{A}) \sim Z(\xi - w_0) \qquad \text{(rational equivalence!).}$$

CLAIM 1: $\Omega \subset \Omega'$.

PROOF: Let $(\xi', \eta') \in \Omega$; extend the specialization $(\xi, \eta) \to (\xi', \eta')$ over k (in the sense of Weil's Foundations) to $(\xi, \eta, \zeta, \mathfrak{A}) \to (\xi', \eta', \zeta', \mathfrak{A}')$ over k. Using the fact that *specialization preserves rational equivalence* [22] we get from (16):

$$T(\mathfrak{A}') \sim Z(\xi' - w_0),$$

i.e., $\eta' = T_*(\zeta') = Cl(\mathfrak{A}') = Cl(Z(\xi' - w_0)) = Z_*(\xi')$, i.e., $(\xi', \eta') \in \Omega'$.

CLAIM 2: $\Omega' \subset \Omega$.

PROOF: Let $(\xi', \bar{\eta}) \in \Omega'$. Extend the specialization $\xi \to \xi'$ over k to $(\xi, \eta, \zeta, \mathfrak{A}) \to (\xi', \eta', \zeta', \mathfrak{A}')$ over k; then the same argument gives $\eta' = \bar{\eta}$, i.e. $(\xi', \bar{\eta}) \in \Omega$.

Hence the variety Ω defined as the locus of (ξ, η) has the property (15) as far as the field k is concerned. Now take $K = \bar{K} \supset k$ and take ξ on W generic over K. Repeat the argument over K; let us write Ω'_K and Ω_K for the corresponding sets. We have $\Omega_K = \Omega'_K$ as we did see above. Clearly $\Omega_K \subset \Omega$ and moreover by 3.2 we have $\Omega' \subset \Omega'_K$; hence finally $\Omega_K \subset \Omega = \Omega' \subset \Omega'_K = \Omega_K$, i.e. $\Omega = \Omega_K$ has the required property.

COROLLARY 3.6: *Let $A^i(V)$ be weakly-representably by a triple (Γ, T, B). Let Γ_1 be a smooth curve and $Z \in CH^i(\Gamma_1 \times V)$. Consider the induced group homomorphism*

$$Z_* : J(\Gamma_1) \longrightarrow A^i(V).$$

Then there is an abelian subvariety $\Omega \subset J(\Gamma_1) \times J(\Gamma)/B$, defined over k, such that

$$(15') \qquad \Omega = \{(a, Z_*(a)); \quad a \in J(\Gamma_1)(K)\} \qquad \forall K = \bar{K} \supset k.$$

PROOF: The proof of the existence of an irreducible subvariety Ω with property (15') is entirely similar as the proof of 3.4. The fact that Ω is an abelian variety follows since Z_* is set-theoretically a group homomorphism.

COROLLARY 3.7: *The assumptions are as in* 3.6.

a) *There exists an algebraic subgroup* B_1 *of* $J(\Gamma_1)$ *such that* $\mathrm{Ker}(Z_*)(K) = B_1(K)$ $(\forall K = \bar{K} \supset k)$.

b) *There exists an abelian subvariety* $I_1 \subset J(\Gamma)/B$ *such that* $I_1(K) = \mathrm{Im}(Z_*)(K)$ $(\forall K = \bar{K} \supset k)$. *Both* B_1 *and* I_1 *are defined over* k.

PROOF: Take the projections (in the sense of algebraic varieties):

$$B_1 = \mathrm{pr}_1\{\Omega \cap (J(\Gamma_1) \times 0)\} \quad \text{and } I_1 = \mathrm{pr}_2\{\Omega\}.$$

COROLLARY 3.8: *Let* $A^i(V)$ *be weakly-representable by a triple* (Γ, T, B). *Let* $\Sigma \in CH^n(V \times V)$, *with* $n = \dim V$. *Then there exists an algebraic subgroup* $B_1 \subset J(\Gamma)$ (*resp. an abelian subvariety* $I \subset J(\Gamma)/B$), *defined over* k, *such that* $B_1(K) = \mathrm{Ker}(\Sigma_* \cdot \mathrm{T}_*)(\mathrm{K})$ (resp. $I(K) = \mathrm{Im}(\Sigma_* \cdot T_*)(K)$) *for all* $K = \bar{K} \supset k$.

PROOF: Apply cor. 3.7 to the couple $(\Gamma_1, Z) = (\Gamma, \Sigma \cdot T)$.

LEMMA 3.9: *Let* S *be a surface and assume that* $A^2(S)$ *is weakly-representable. Then the canonical homomorphism* $A^2(S) \xrightarrow{\varphi} \mathrm{Alb}(S)$ *is an isomorphism. Moreover it is a morphism of abelian varieties* $J(\Gamma)/B \to \mathrm{Alb}(S)$.

PROOF: Let (Γ, T, B) weakly-represents $A^2(S)$, then we have a *morphism* $T_*^{\mathrm{alb}} : J(\Gamma) \to \mathrm{Alb}(S)$ obtained via commutative diagram

$$\begin{tikzcd} J(\Gamma) \arrow[r, "T_*"] \arrow[dr, "T_*^{\mathrm{alb}}"'] & J(\Gamma)/B = A^2(S) \arrow[d, dashed, "\varphi"] \\ & \mathrm{Alb}(S) \end{tikzcd}$$

Now $\tilde{B} = \mathrm{Ker}\, T_*^{\mathrm{alb}}$ is an *algebraic* subgroup and contains B, hence (since the components of B are counted with multiplicity 1!) φ is a *morphism* $\varphi : J(\Gamma)/B \to \mathrm{Alb}(S)$.

On the other hand apply 3.4 to the couple $(W, Z) = (S, \Delta_S)$, where $\Delta_S \subset S \times S$ in the diagonal. By 3.4 there exists a weak-morphism $S \xrightarrow{\Delta_*} A^2(S) = J(\Gamma)/B$ (after choice of a base point $s_0 \in S$), hence, after applying a sufficient high power of the Frobenius automorphism F we get a *morphism* ρ:

$$\begin{tikzcd} S \arrow[r, "\Delta_*"] \arrow[rr, bend right, "\rho"'] & J(\Gamma)/B \arrow[r, "F^N"] & J(\Gamma)/B \end{tikzcd}$$

(of course in char. 0 the map Δ_* itself is already a morphism, cf. 3.5). By the universal mapping property of the Albanese variety there is a *morphism* ψ such that $\rho = \psi \cdot \Delta_*^{\text{alb}}$:

$$\begin{tikzcd} S \arrow[r, "\Delta_*"] \arrow[rr, bend left, "\rho"] \arrow[dr, "\Delta_*^{\text{alb}}"'] & A^2(S) = J(\Gamma)/B \arrow[r, "F^N"] \arrow[d, "\varphi"] & J(\Gamma)/B \\ & \text{Alb}(S) \arrow[ur, "\psi"'] & \end{tikzcd}$$

Now $\rho = \psi \cdot \Delta_*^{\text{alb}} = \psi \cdot \varphi \cdot \Delta_*$ and $\rho = F^N \cdot \Delta_*$. Since Δ_* generates $A^2(S)$ we have (set-theoretically) $\psi \cdot \varphi = F^N$ and since (settheoretically) $\text{Ker}(F^N) = 0$ we have $\text{Ker}(\varphi) = 0$ (setth.). Finally Δ_*^{alb} generates $\text{Alb}(S)$, hence φ is surjective. Hence φ is settheoretically an isomorphism.

3.10: Next we prove a number of "stability" results for the notion of weak-representability (cf. §1 of [4]).

PROPOSITION 3.10: *Let V and W be smooth projective varieties. Let $f: V \to W$ be a proper morphism, generically finite and of degree d. Then: $A^i(V)$ weakly-representable $\Rightarrow A^i(W)$ weakly-representable.*

PROOF: Let (Γ, T, B) be a triple for $A^i(V)$. Consider the homomorphisms:

$$J(\Gamma) \xrightarrow{T_*} A^i(V) \xrightarrow{f_*} A^i(W) \xrightarrow{f^*} A^i(V) \xrightarrow{f_*} A^i(W). \tag{17}$$

Using the fact that $f_* \cdot f^* = d$ (multiplication by d) and the fact that $A^i(-)$ is a divisible group (cf [4], proof of prop. 1.3) we have indeed that f_* is surjective. Using the weak-representability we have that the points of order d in $A^i(V)$ form a finite group. It follows that

$f_* \mid \mathrm{Im}(f^*)$ has finite kernel. Furthermore let Z be the graph of f; put $\Sigma = {}^tZ \cdot Z$ and $T' = Z \cdot \Sigma \cdot T$. By cor. 3.8 the kernel of $(\Sigma \cdot T)_*$ is an algebraic subgroup $B_1 \subset J(\Gamma)$. Since $f_* \mid \mathrm{Im}(f^*)$ has finite kernel it now follows that the kernel B_2 of T'_* is also an algebraic subgroup of $J(\Gamma)$ (and for the connected components we have $B_2^0 = B_1^0$). *Claim*: (Γ, T', B_2) weakly represents $A^i(W)$. For the field k we have the required exact sequence (13) immediately by the construction. Next take $K = \bar{K} \supset k$ and do the same construction, i.e. take the sequence (17) for K, etc. By 3.8 we have the same group B_1 for K and by 3.2 the same $\mathrm{Ker}(f_* \mid \mathrm{Im}(f^*))$, hence we find the same group B_2 for K.

PROPOSITION 3.11: *Let V be smooth projective and $Z \hookrightarrow V$ smooth. Consider $W = B_Z(V)$, i.e., W is obtained by blowing up V along Z. Then $A^2(V)$ weakly-representable $\Rightarrow A^2(W)$ weakly-representable.*

COROLLARY 3.12: *Assume resolution of singularities in dimension $\leq n$. Then weak-representability of $A^2(V)$ is a birational invariant property for varieties of dimension $\leq n$.*

COROLLARY 3.13: *Let V be a smooth, projective unirational threefold. Then $A^2(V)$ is weakly-representable. In particular $A^2(X_3^6)$ and $A^2(X_3^8)$ are weakly-representable (of course, with the type of varieties as described in the introduction).*

PROOF: Let $f: \mathbf{P}_3 \dashrightarrow V$ be a rational map of degree d. By [1] this can be completed to

$$\begin{tikzcd} & B_Z(\mathbf{P}_3) \arrow[dl, "h"'] \arrow[dr, "g"] & \\ P_3 \arrow[rr, dashed, "f"'] & & V \end{tikzcd}$$

with h a sequence of blowing-ups starting from $\mathbf{P}_3$ with smooth centers and g a morphism generically of finite degree. Apply 3.10 and 3.11.

COROLLARY 3.14: *Suppose we have a proper, surjective rational map $\pi: V \dashrightarrow C$, with* $\dim V = 3$ *and* $\dim C = 1$. *Assume that for the geometric generic fibre we have a birational map $V_{\bar{\eta}} \xrightarrow{\sim} \mathbf{P}_2$. Then $A^2(V)$ is weakly-representable.*

PROOF: Left to the reader; standard.

3.15. PROOF OF 3.11: For $W = B_Z(V)$, we have $A^2(W) = A^2(V) \oplus A^1(Z)$ (for instance [3] or [16]). Let (Γ_1, T_1, B_1), resp. (Γ_2, T_2, B_2), weakly-represent $A^2(V)$, resp. $A^1(Z)$. From the above decomposition we get from T_1, resp. from T_2, an induced cycleclass on $\Gamma_1 \times W$, resp. on $\Gamma_2 \times W$, which we denote – for simplicity – by the same letter T_1, resp. T_2. (In order to give a precise description it would be necessary to describe explicitely the relation between $A^2(W)$ and $A^2(V)$ and $A^1(Z)$; see for instance [17], p. 67 lemma 2). Consider on the surface $S = \Gamma_1 \times \Gamma_2$ the cycleclass

$$T' = (p_1 \times 1_W)^*(T_1) + (p_2 \times 1_W)^*(T_2) \in CH^2(S \times W).$$

Let $i: \Gamma_3 \hookrightarrow \Gamma_1 \times \Gamma_2$ be a general hyperplane section; put $T = i^*(T') \in CH^2(\Gamma_3 \times W)$. We have $J(\Gamma_3) \twoheadrightarrow J(\Gamma_1) \times J(\Gamma_2) = \mathrm{Alb}(S)$. From this it follows that $T_*: A^1(\Gamma_3) \to A^2(W)$ is surjective. Finally let q_1 and q_2 be the projections of $J(\Gamma_3)$ on $J(\Gamma_1)$ and $J(\Gamma_2)$ respectively. It follows now easily that $(\Gamma_3, T, q_1^*(B_1) \cap q_2^*(B_2))$ weakly-represents $A^2(W)$.

PROPOSITION 3.16: *Let $\pi: V \to S$, with V a smooth threefold, S a smooth surface (always everything projective) and π a proper, surjective rational map such that*:

a) *the geometric generic fibre $V_{\bar\eta} \xrightarrow{\sim} (\mathbf{P}_1)_{\bar\eta}$ (i.e. V is a "conic bundle", see* 2.15),

b) *$A^2(S)$ is weakly-representable.*
Then $A^2(V)$ is weakly-representable.

COROLLARY 3.17 ([4], thm. 3.1): *Let X_3^4 be a sufficiently general quartic hypersurface of degree* 4 *in* $\mathbf{P}_4$. *Then $A^2(X)$ is weakly-representable.*

PROOF: 2.15 + 3.16.

3.18. PROOF OF 3.16: For this we need some preparations:

Let G be a *quotientgroup* of $A^i(V)$ in the following sense: for every $K = \bar K \supset k$ there exists groups $G'(K)$ and $G(K)$ such that

$$(18)\qquad \begin{cases} \text{i) } 0 \to G'(K) \to A^i(V_K) \to G(K) \to 0 \text{ exact,} \\ \text{ii) for } K_2 = \bar K_2 \supset K_1 = \bar K_1 \supset k \text{ we have} \\ \qquad G'(K_1) = G'(K_2) \cap A^i(V_{K_1}) \end{cases}$$

Note that we now have $G(K_1) \hookrightarrow G(K_2)$ (compare with 3.2). Furthermore we say that G *is stable by specialization* if G' is stable by specialization, i.e. if for cycles $\mathfrak{A}$ and $\mathfrak{A}'$ we have:

$$\left.\begin{array}{l} Cl(\mathfrak{A}) \in G'(K) \\ \mathfrak{A} \to \mathfrak{A}' \text{ a specialization over } K \end{array}\right\} \Rightarrow Cl(\mathfrak{A}') \in G'(K). \tag{19}$$

Finally G is called *compatible with correspondences* if for any $\Sigma \in CH^n(V \times V)$, $n = \dim V$, we have a factorization

$$\begin{tikzcd} A^i(V) \arrow[r, "\Sigma_*"] \arrow[d] & A^i(V) \arrow[d] \\ G \arrow[r, dotted, "\Sigma_*"] & G \end{tikzcd} \tag{20}$$

DEFINITION 3.19: *Let G be a quotientgroup of $A^i(V)$ with the above properties* (18), (19) *and* (20). *Such a G is called weakly-representable if there exists a triple (Γ, T, B) as in* 3.3 *(and all defined over k) such that*

$$0 \longrightarrow B(K) \longrightarrow J(\Gamma)(K) \xrightarrow{T_*} G(K) \longrightarrow 0 \tag{21}$$

exact for all $K = \bar{K} \supset k$.

EXAMPLE: S surface, Alb(S) as quotient of $A^2(S)$ is weakly-representable. More generally if V has dimension n, Alb(V) as quotient of $A^n(V)$ is weakly-representable.

LEMMA 3.20: *Let G be a quotientgroup of $A^i(V)$ and weakly-representable (always G has tacitly the properties* (18), (19) *and* (20)). *Then lemma* 3.4 *and its corollaries remain true if $A^i(V)$ is replaced by G.*

PROOF: The same; the essential points are (18), (19) and (20).

3.21. SUPPLEMENT TO PROPOSITION 3.10: Let V and W be smooth projective varieties. Let $f: V \to W$ be a proper morphism, generically of finite degree d. Let G be a quotient-group of $A^i(V)$ which is weakly-representable. Assume we have a factorization

$$\begin{tikzcd} A^i(V) \arrow[rr, "f_*"] \arrow[dr] & & A^i(W) \\ & G \arrow[ur, "\bar{f}_*"'] & \end{tikzcd}$$

Then $A^i(W)$ is weakly-representable.

PROOF: Similar as for 3.10 using the diagram

$$\begin{tikzcd} J(\Gamma) \arrow[r, "T_*"] & A^i(V) \arrow[rr, "f_*"] \arrow[dr] & & A^i(W) \arrow[r, "f^*"] & A^i(V) \arrow[rr, "f_*"] \arrow[dr] & & A^i(W). \\ & & G \arrow[ur, "\bar{f}_*"'] & & & G \arrow[ur, "\bar{f}_*"'] & \end{tikzcd} \tag{17'}$$

3.22. PROOF OF 3.16: Firstly: blowing up some curves and points in V we may assume that π is a morphism (note that this does not change the generic fibre).

Now there is a commutative diagram:

$$\begin{tikzcd} & V'' = B_Z(V') \arrow[dl, "f"'] \arrow[d, "j"] \arrow[dd, bend left=60, "g"] \\ V \arrow[d, "\pi"'] & V' = S' \times \mathbf{P}_1 \arrow[l, dotted] \arrow[d, "p"] \\ S & S' \arrow[l, "h"] \end{tikzcd} \quad \text{(diagram 1).}$$

with Z a sequence of smooth curves with normal crossings (resp. points) and h and f generically of finite degree d. We have

$$A^2(V'') = A^2(S') \oplus \mathrm{Pic}^0(S') \oplus \mathrm{Pic}^0(Z). \tag{22}$$

Put

$$K(S') = \mathrm{Ker}\{A^2(S') \longrightarrow \mathrm{Alb}(S')\}, \tag{23}$$

and consider $g^*K(S') \subset A^2(V'')$. Finally put

$$G = A^2(V'')/g^*K(S') = \mathrm{Alb}(S') \oplus \mathrm{Pic}^0(S') \oplus \mathrm{Pic}^0(Z). \tag{24}$$

Then the quotientgroup G of $A^2(V'')$ has the properties (18), (19) and (20) and G is weakly-representable.

Next: from the commutative diagram

$$
\begin{tikzcd}
A^2(V'') \arrow[r, "f_*"] & A^2(V) \\
A^2(S') \arrow[u] \arrow[r, "h_*"] \arrow[d] & A^2(S) \arrow[u, "\pi^*"'] \arrow[d, equal] & \arrow[l] \text{by 3.9} \\
\mathrm{Alb}(S') \arrow[r, "h_*^{\mathrm{alb}}"] & \mathrm{Alb}(S)
\end{tikzcd}
\tag{25}
$$

we see that we have a factorization:

$$
\begin{tikzcd}
A^2(V'') \arrow[rr, "f_*"] \arrow[dr] & & A^2(V) \\
& G = A^2(V'')/g^*K(S') \arrow[ur, "\bar{f}_*"'] &
\end{tikzcd}
\tag{26}
$$

Now apply 3.21, this gives 3.16.

§4. The cycle map for torsion cycles

4.1 Notations

The assumptions and notations are the same as in 3.1. Fix a prime number $l \neq \mathrm{char}(k)$, and consider the l-adic étale cohomology groups

$$H^i(V, \mathbf{Q}_l/\mathbf{Z}_l(j)) = \varinjlim_{\nu} H^i(V, \boldsymbol{\mu}_{l^\nu}^{\otimes j}).$$

For any abelian group A, let $A_{l^\nu} = \mathrm{Ker}(A \xrightarrow{l^\nu} A)$ and $A(l) = \bigcup_{\nu=1}^{\infty} A_{l^\nu}$. In particular, we have

$$A^i(V)(l) \subset CH^i(V)(l).$$

4.2 The cycle map

From [5], we have the following result.

THEOREM 4.2: *With notations as above, there exist homomorphisms*

$$\lambda_l^q = \lambda_l(V): CH^q(V)(l) \longrightarrow H^{2q-1}(V, \mathbf{Q}_l/\mathbf{Z}_l(q)) \quad (1 \leq q \leq \dim V), \tag{27}$$

with the following properties:

a) λ_l^q *is compatible with specialization. Given a smooth projective family* $\mathcal{V} \to \mathrm{Sp}\, R$, *$R$ a valuation ring with geometric generic point s and special point s_0, both of characteristic prime to l, there exists specialization maps on cycles and cohomology, and a commutative diagram*

$$\begin{array}{ccc} CH^q(\mathcal{V}_s)(l) & \xrightarrow{\lambda^q} & H^{2q-1}(\mathcal{V}_s, \mathbf{Q}_l/\mathbf{Z}_l(q)) \\ \downarrow & & \wr\| \\ CH^q(\mathcal{V}_{s_0})(l) & \xrightarrow{\lambda^q} & H^{2q-1}(\mathcal{V}_{s_0}, \mathbf{Q}_l/\mathbf{Z}_l(q)). \end{array}$$

b) λ_l *is functorial with respect to correspondences. Given a cycle Γ on $V \times W$, we have a commutative diagram* ($\dim \Gamma = \dim W + q - r$)

$$\begin{array}{ccc} CH^q(V)(l) & \xrightarrow{\lambda^q} & H^{2q-1}(V, \mathbf{Q}_l/\mathbf{Z}_l(q)) \\ {\scriptstyle \Gamma_*}\downarrow & & \downarrow{\scriptstyle \Gamma_*} \\ CH^r(W)(l) & \xrightarrow{\lambda^r} & H^{2r-1}(W, \mathbf{Q}_l/\mathbf{Z}_l(r)). \end{array}$$

c) $\lambda_l^1 : CH^1(V)(l) \to H^1(V, Q_l/\mathbf{Z}_l(1))$ *is the natural map obtained by identifying* $CH^1(V) \simeq H^1(V, \mathbf{G}_m)$ *and using the Kummer sequence*

$$0 \longrightarrow \boldsymbol{\mu}_{l^\nu} \longrightarrow \mathbf{G}_m \xrightarrow{l^\nu} \mathbf{G}_m \longrightarrow 0.$$

d) *When* $k = \mathbf{C}$, *Griffiths* [9] *gives the torus* $J_{cl}^q(V) = H^{2q-1}(V, \mathbf{R})/H^{2q-1}(V, \mathbf{Z})$ *a complex structure and defines a map* (*the Abel-Jacoby map*)

$$\theta : A^q(V) \longrightarrow J_{cl}^q(V).$$

The maps $\theta \mid A^q(V)(l)$ *and* $\lambda^q \mid A^q(V)(l)$ *coincide.*

e) *Let* $\mathrm{Alb}(V)$ *denote the Albanese variety of V* (*dual to the Picard variety*). *We have* (*cf.* [10], 2*A*9, *no.* 6) $\mathrm{Alb}(V)(l) \simeq H^{2n-1}(V, \mathbf{Q}_l/\mathbf{Z}_l(n))$, $n = \dim V$. *Since* $CH^n(V)(l) \simeq A^n(V)(l)$ *consists of zero cycles of degree* 0, *there is a canonical map independent of choice of base point* $CH^n(V)(l) \to \mathrm{Alb}(V)(l)$. *The composition* $CH^n(V)(l) \to \mathrm{Alb}(V)(l) \to H^{2n-1}(V, \mathbf{Q}_l/\mathbf{Z}_l(n))$ *is the map* λ_l^n.

4.3. In the case of our special varieties X_3^4, X_3^6 and X_3^8 it will turn out that λ_l^2 is an isomorphism for almost all l. We will prove some stability results similar to 3.10, 3.11, and 3.16. In fact, it would be possible by these arguments to give a direct construction for λ in the cases $V = X_3^4$, X_3^6, X_3^8. However this is somewhat artificial and laborious. It will be convenient, henceforth, to take $A^q(V)(l)$ as the domain of λ_l^q. Thus λ_l^q an isomorphism means $\lambda_l^q\colon A^q(V)(\ell) \simeq H^{2q-1}(V, \mathbf{Q}_l/\mathbf{Z}_l(q))$.

PROPOSITION 4.4: *Let V, W be smooth projective varieties, $f: V \to W$ a proper morphism generically finite of degree d. Assume $\lambda_l^q(V)$ is an isomorphism for some fixed q and almost all l. Then $\lambda_l^q(W)$ is also.*

PROOF: We have $f_*f^* =$ multiplication by d, both on cycles and cohomology. For l such that $(l, d) = 1$ and $\lambda_l^q(V)$ is an isomorphism, we find

$$\begin{tikzcd} A^q(W)(l) \arrow[r, "f^*"] \arrow[d, "\lambda"] & A^q(V)(l) \arrow[r, two heads, "f_*"] \arrow[d, "\wr\|"] & A^q(W)(l) \arrow[d, "\lambda"] \\ H^{2q-1}(W, \mathbf{Q}_l/\mathbf{Z}_l(q)) \arrow[r, "f^*"] & H^{2q-1}(V, \mathbf{Q}_l/\mathbf{Z}_l(q)) \arrow[r, two heads, "f_*"] & H^{2q-1}(W, \mathbf{Q}_l/\mathbf{Z}_l(q)). \end{tikzcd}$$

An easy diagram chase shows $\lambda_l^q(W)$ an isomorphism. Q.E.D.

PROPOSITION 4.5: *Let V be smooth and projective, $Z \hookrightarrow V$ a smooth closed subvariety. Consider the blowup $W = B_Z(V)$. If $\lambda_l^2(V)$ is an isomorphism for almost all l, then $\lambda_l^2(W)$ is also.*

PROOF. By well-known decomposition theorems for cycles and cohomology on blowings up, we have

$$A^2(W) \simeq A^2(V) \oplus A^1(Z),$$

$$H^3(W, \mathbf{Q}_l/\mathbf{Z}_l(2) \simeq H^3(V, \mathbf{Q}_l/\mathbf{Z}_l(2)) \oplus H^1(Z, \mathbf{Q}_l/\mathbf{Z}_l(1)).$$

The map $\lambda_l^2(W)$ is compatible with this decomposition. Thus $\lambda_l^2(W)$ is an isomorphism whenever $\lambda_l^2(V)$ is an isomorphism and $A^1(Z)(l) = CH^1(Z)(l)$. Since the Neron-Severi group $\mathrm{CH}^1(Z)/A^1(Z)$ is finitely generated, this will be true for almost all l. Q.E.D.

COROLLARY 4.6: *Let V be a smooth, projective unirational threefold. Then $\lambda_l^2(V)$ is an isomorphism for almost all l.*

PROOF: Standard, using 4.4 and 4.5.

PROPOSITION 4.7: *Let $\pi: V \to S$ with V a smooth projective threefold, S a smooth projective surface, and π a proper surjective* *rational map. Assume: the geometric generic fibre $V_{\bar{\eta}} \simeq (\mathbf{P}^1)_{\bar{\eta}}$. Then $\lambda_l^2(V)$ is an isomorphism for almost all l.*

COROLLARY 4.8: *$\lambda_l^2(X_3^4)$ is an isomorphism for almost all l.*

PROOF OF 4.8: Combine 2.15, 4.4, and 4.7.

PROOF OF 4.7: We use the notations of diagram 1, 3.22. By 4.4 and 4.5 it suffices to show $\lambda_l^2(S' \times \mathbf{P}^1)$ is an isomorphism. We have

$$A^2(S' \times \mathbf{P}^1) \simeq A^2(S') \oplus A^1(S'),$$

$$H^3(S' \times \mathbf{P}^1, \mathbf{Q}_l/\mathbf{Z}_l(2)) \simeq H^3(S', \mathbf{Q}_l/\mathbf{Z}_l(2)) \oplus H^1(S', \mathbf{Q}_l/\mathbf{Z}_l(1)).$$

By Roitman's theorem [20], $\lambda_l^2(S')$ is an isomorphism for all $l \neq \mathrm{char}(k)$. Thus $\lambda_l^2(S' \times \mathbf{P}^1)$ will be an isomorphism whenever $l \neq \mathrm{char}(k)$ and $A^1(S')(l) = \mathrm{CH}^1(S')(l)$. Q.E.D.

REMARK: Note that, unlike the proof of 3.16, we have not had to assume $A^2(S)$ weakly representable.

§5. The Abel-Jacobi property. (cf also [4])

In this section we assume that we work over a field $k = \bar{k} \subset \mathbf{C}$ (as usual $\mathbf{C}$ denotes the complex numbers). Let as before V be a smooth, projective variety. Now we have the i-th (classical) intermediate Jacobian variety $J_{cl}^i(V_{\mathbf{C}})$ and a natural homomorphism (the so-called (classical) Abel-Jacobi map):

$$\theta^i : A^i(V_{\mathbf{C}}) \longrightarrow J_{cl}^i(V_{\mathbf{C}}). \tag{28}$$

DEFINITION 5.1: *We say that $A^i(V)$ has the Abel-Jacobi property if θ^i is an isogeny.*[2,3]

PROPOSITION 5.2: *Let $k = \mathbf{C}$. Suppose $A^i(V)$ has the Abel-Jacobi property. Then $A^i(V)$ is weakly-representable (see* [4], *prop.* 1.3)

[2] θ isogeny $\overset{\text{def}}{\Leftrightarrow}$ θ is surjective and has finite kernel.

[3] See [7] and [9] for the definition of J_{cl} and θ. Note that in case the Abel-Jacobi property is true we have automatically $H^{p,q} = 0$ if $|p-q| \neq 1$; hence there is in this case no difference between the definition of Weil and of Griffiths for the intermediate Jacobian.

PROOF: Consider the Abel-Jacobi map $\theta^i : A^i(V) \to J^i_{cl}(V)$. By assumption this is an isogeny, hence in particular it is onto. It is standard that we can find a curve Γ and $T \in CH^i(\Gamma \times V)$ such that the composition

$$J(\Gamma) \xrightarrow{T_*} A^i(V) \xrightarrow{\theta^i} J^i_{cl}(V)$$

is onto. Moreover this composition is a morphism, in particular $\mathrm{Ker}(\theta \cdot T_*)$ is an algebraic group.

Again: θ^i is an isogeny, hence it has finite kernel. Hence there exists an integer N and a homomorphism ψ such that we have a commutative diagram

$$\begin{tikzcd} A^i(V) \arrow[rr, "\cdot N"] \arrow[dr, "\theta"'] & & A^i(V) \\ & J^i_{cl}(V) \arrow[ur, "\psi"'] & \end{tikzcd}$$

Since $A^i(V)$ is divisible, ψ is onto. Since also $\theta \cdot \psi = N$ we have that $\ker(\psi)$ is finite. Hence also $B = \mathrm{Ker}(\psi \cdot \theta \cdot T_*)$ is an algebraic subgroup of $J(\Gamma)$. It follows now easily that the triple $(\Gamma, N \cdot T, B)$ weakly-represents $A^i(V)$, where $N \cdot T = T + \cdots + T$ (N times).

PROPOSITION 5.3: *Let V and W be smooth, projective varieties. Let $f: V \to W$ be a proper morphism, generically finite and of degree d. Then: $A^i(V)$ has the Abel-Jacobi property $\Rightarrow A^i(W)$ has the Abel-Jacobi property.*

PROOF: Look at the commutative diagram:

$$\begin{tikzcd} A^i(V) \arrow[r, two heads, "f_*"] \arrow[d, "\theta_V"'] & A^i(W) \arrow[r, "f^*"] \arrow[d, "\theta_W"'] & A^1(V) \arrow[d, "\theta_V"'] \\ J^i_{cl}(V) \arrow[r, two heads, "f_*"'] & J^i_{cl}(W) \arrow[r, "f^*"'] & J^i_{cl}(V) \end{tikzcd}$$

First note that θ_W is surjective. Next: let $a \in A^i(W)$ with $\theta_W(a) = 0$. Then $0 = f^* \cdot \theta_W(a) = \theta_V \cdot f^*(a)$; hence $f^*(a) \in \mathrm{Ker}(\theta_V)$. Now $\mathrm{Ker}(\theta_V)$ is finite by assumption. Furthermore $\mathrm{Ker}\{f^* : A^i(W) \to A^i(V)\}$ is killed by d. Hence there exists an integer N such that $\mathrm{Ker}(\theta_W)$ is killed by N. Hence we have a factorization

$$\begin{tikzcd}
A^i(W) \arrow[rr, two heads, ".N"] \arrow[dr, "\theta_W"'] & & A^i(W) \\
& J^i_{cl}(W) \arrow[ur, dashed, "\rho"'] &
\end{tikzcd}$$

We want to see $\mathrm{Ker}(\theta_W)$ is finite. Hence it suffices to see that $A^i(W)_N$ is finite. Let $b \in A^i(W_N)$, then there exists $b_1 \in A^i(W)_{N^2}$ such that $b = N \cdot b_1$; put $c = \theta_W(b_1)$, then $b = \rho(c)$ and $c \in J^i_{cl}(W)_{N^2}$. Since the number of points of order N^2 is finite on $J^i_{cl}(W)$ we have $A^i(W)_N$ finite.

DEFINITION 5.4: *Let G be a quotientgroup of $A^i(V)$ with the properties* (18), (19) *and* (20) *of* 3.18. *We say that G has the Abel-Jacobi property if there exists a factorization*

$$\begin{tikzcd}
A^i(V) \arrow[rr, "\theta"] \arrow[dr] & & J^i_{cl}(V) \\
& G \arrow[ur, "\bar{\theta}"'] &
\end{tikzcd}$$

with $\bar{\theta}$ an isogeny.

COROLLARY 5.5: *Let the assumptions be as in* 5.3. *Let G be a quotientgroup of $A^i(V)$ which has the Abel-Jacobi property. Suppose that f_* factorizes*

$$\begin{tikzcd}
A^i(V) \arrow[rr, "f_*"] \arrow[dr] & & A^i(W) \\
& G \arrow[ur, "\bar{f}_*"'] &
\end{tikzcd}$$

then $A^i(W)$ has the Abel-Jacobi property.

PROOF: Similar as above, consider the commutative diagram

$$\begin{tikzcd}
A^i(V) \arrow[rr, two heads, "f_*"] \arrow[dd, "\theta_V"'] \arrow[dr] & & A^i(W) \arrow[r, "f^*"] \arrow[dd, "\theta_W"] \arrow[drr, "\bar{f}^*"] & A^i(V) \arrow[dd] \arrow[dr] & \\
& G \arrow[ur, "\bar{f}^*"'] \arrow[dl] & & & G \arrow[dl, "\bar{\theta}_V"] \\
J^i_{cl}(V) \arrow[rr, two heads] & & J^i_{cl}(W) \arrow[r] & J^i_{cl}(V) &
\end{tikzcd}$$

where $\bar{f}^*$ is the composition. Note that again $\bar{f}_* \cdot \bar{f}^* = f_* \cdot f^* = d$. The proof is similar now.

PROPOSITION 5.6: *Let V be smooth, projective and $Z \hookrightarrow V$ smooth. Consider the blow up $W = B_Z(V)$. If $A^2(V)$ has the Abel-Jacobi property, then $A^2(W)$ has the Abel-Jacobi property.*

PROOF: Straightforward.

COROLLARY 5.7: *If V is a smooth, projective unirational threefold, then $A^2(V)$ has the Abel-Jacobi property.*

PROPOSITION 5.8: *Let $\pi : V \dashrightarrow S$, with V a smooth threefold, S a smooth surface and π a proper, surjective, rational map such that:*

i) *the geometric generic fibre $V_{\bar{\eta}} \xrightarrow{\sim} (\mathbf{P}_1)_{\bar{\eta}}$,*

ii) *$A^2(S)$ has Abel-Jacobi property.*
Then $A^2(V)$ has the Abel-Jacobi property.

PROOF: Note first that $A^2(S)$ is weakly-representable by 5.2. The proposition follows (see 3.22) from 5.5 applied to the quotientgroup G of $A^2(V'')$ where

$$G = \mathrm{Alb}(S') \oplus \mathrm{Pic}^0(S') \oplus \mathrm{Pic}^0(Z),$$

and note that G has Abel-Jacobi property.

COROLLARY 5.9 ([4], thm. 3.1): *If $V = X_3^4 \subset \mathbf{P}_4$ smooth, then $A^2(V)$ has the Abel-Jacobi property.*

PROOF: $2.15 + 5.8$.

§6. The classical intermediate Jacobian of a generic X_3^4, resp X_3^6, resp X_3^8

In this section we want to outline part of Tjurin's [24] ingenious constructions.

In this section the groundfield is $\mathbf{C}$ and we take $X = X_3^4 \subset \mathbf{P}_4$, resp

$X_3^6 = Q_4 \cdot C_4 \subset \mathbf{P}_5$, resp. $X_3^8 = Q_4 \cdot Q_4' \cdot Q_4'' \subset \mathbf{P}_6$, *generic* over the prime field.[4]

6.1: As before we consider

$$F(X) = \{l; l \text{ line on } X\}.$$

By prop. 1.1 the $F(X)$ is in our case a *smooth, connected* curve. Consider also the following diagram

$$\begin{array}{ccc} P(X) & \xrightarrow{\varphi} & X \\ {\scriptstyle p}\downarrow & & \\ F(X) & & \end{array} \tag{29}$$

with $P(X) = \{(l, x); x \in l \subset X\}$. Also: if $s \in F(X)$ then we sometimes write $l_s = \varphi\{p^{-1}(s)\}$.

6.2. Abel-Jacobi mapping of the lines

Let $\gamma \in A^1(F(X)) = J(F(X))$; write $\gamma = Cl(\Sigma_i s_i' - \Sigma_j s_j'')$ with $s_i', s_j'' \in F(X)$. Consider the homomorphism

$$\phi : A^1(F(X)) \longrightarrow A^2(X) \tag{30}$$

defined by

$$\phi(\gamma) = \varphi_* p^*(\gamma) = Cl\left\{\sum l_{s_i'} - \sum l_{s_j''}\right\}. \tag{31}$$

Similarly for $\gamma \in H^1(F(X), \mathbf{Z})$ (resp. $\gamma \in H^1(F(X), \mathbf{C})$) we have a homomorphism

$$\phi : H^1(F(X), -) \to H^3(X, -) \qquad (\text{coeff. } \mathbf{Z}, \text{resp } \mathbf{C}), \tag{30'}$$

defined by the same type of formula (31). From this we also get

$$\phi : J(F(X)) \longrightarrow J_{cl}^2(X). \tag{30''}$$

Each one of these maps (30), (30′) or (30″) we call *the Abel-Jacobi map* of lines (note that in §4 and §5 we have used this name also

[4] I.e., X is defined by equation(s) with independent transcendental coefficients over $\mathbf{Q}$. Note that some statements in Tjurin are not correct as they stand (for instance lemma 3, §2, lecture 4: cf. [23], the example on p. 719), however we need these results only for X *generic* over $\mathbf{Q}$.

already for a slightly different map, however there will be hardly any danger for confusion). The idea is now to study this Abel-Jacobi map.

LEMMA 6.3: (Clemens-Roth condition of Tjurin). *We can assume that* $X_3 = X_4 \cdot H$ *with*:

i) X_4 *of the same type* (*i.e.* $X_4^4 \subset \mathbf{P}_5$, $X_4^6 = Q_5 \cdot C_5 \subset \mathbf{P}_6$, $X_4^8 = Q_6 \cdot Q_6' \cdot Q_6'' \subset \mathbf{P}_7$),

ii) X_4 *generic over* $\mathbf{Q}$ and H a generic hyperplane section.

PROOF: Obvious. Take for instance X_4^4 generic over $\mathbf{Q}$ and take an independent generic hyperplane section, then one sees immediately from the equation that $X_3 = X_4 \cdot H$ is generic over $\mathbf{Q}$.

LEMMA 6.4: *For* $X_4 = X_4^4$, *resp.* X_4^6, *resp.* X_4^8, *we have*

$$H^3(X_4, -) = 0 \text{ and } J^2_{cl}(X_4) = 0$$

PROOF: From Lefschetz theory for complete intersections (note that we are now in the correct dimension!) we get the result for cohomology and $J^2_{cl}(-)$ is constructed via these cohomology groups.

REMARK 6.5: For these X_3 we have $|-K_{X_3}| = |H|$ and the irregularity $q(X_3) = 0$; for X_4 we have $|-K_{X_4}| = |2H|$ and $q(X_4) = 0$.

LEMMA 6.6: *For the* X_4 *of* 6.3 *we have*

i) $F(X_4)$ *is a smooth* 3-*dimensional variety.*

ii) *Let* $P \in X_4$ *be a sufficiently general point, then there are finitely many – say* q – *lines* l *on* X_4 *such that* $P \in l$. *More precisely*: *for* X_4^4 *we have* $q = 24$, *for* X_4^6 *the* $q = 12$ *and for* X_4^8 *the* $q = 8$.

PROOF:

i) from 1.12.

ii) See [24], lect. 4. For instance take $X = X_4^4$ and P the origin (after transformation). Equation X is of type

$$f(T) = f_1(T) + f_2(T) + f_3(T) + f_4(T),$$

with $f_i(T) = f_i(T_1, \ldots, T_5)$ homogeneous of degree i ($i = 1, \ldots, 4$). We get the lines via $f_1 = f_2 = f_3 = f_4 = 0$. By Bezout there are 24 solutions.

6.7: For X_4 we get a diagram similar to (29). Together with the diagram for X_3 this combines to the following commutative diagram

(32)
$$\begin{tikzcd}
P(X_3) \arrow[rrrd, "\varphi_3"] \arrow[dr, hook, "r"'] \arrow[ddd, "p_3"'] & & & \\
& Y = \varphi_4^{-1}(X_3) \arrow[rr, "\varphi"] \arrow[dr, "j"] \arrow[ddr, "p"'] & & X_3 = X_4 \cdot H \arrow[d, hook, "i"'] \\
& & P(X_4) \arrow[r, "\varphi_4"'] \arrow[d, "p_4"] & X_4 \\
F(X_3) \arrow[rr, "s"] & & F(X_4) &
\end{tikzcd}$$

Now note that φ_4, and hence also φ, is by 6.6 generically finite and of degree q. Note also that Y, $P(X_4)$ and $P(X_3)$ are all smooth. For $P(X_4)$ and $P(X_3)$ this follows since they are the restriction of the Grassmannian bundle to the smooth $F(X_4)$, resp. $F(X_3)$. For Y it follows from the (2^{nd}) Bertini theorem (we are in char. 0!).

LEMMA 6.8: $Y = B_{F(X_3)}(F(X_4))$, i.e., *the blow up of* $F(X_4)$ *along* $F(X_3)$.

PROOF: p is a birational morphism. For $s \in F(X_4)$, $s \notin F(X_3)$ we have that $p^{-1}(s)$ is unique. On the other hand for $s \in F(X_3)$ we have $p^{-1}(s) = \{s, x\}$ with $x \in l_s$. Since $p^{-1}(F(X_3))$ has only one component (namely $P(X_3)$) the lemma follows from [19].

6.9: Next consider the homomorphisms

(33)
$$\begin{array}{c}
H^3(X_3, -) \xrightarrow{\varphi^*} H^3(Y, -) \xrightarrow{\varphi_*} H^3(X_3, -) \\
\| \\
H^1(F(X_3), -) \oplus H^3(F(X_4), -)
\end{array}$$

where we take either coefficients $\mathbf{Z}$ and work modulo torsion, or coefficients $\mathbf{C}$. Now note that $\varphi_* \cdot \varphi^* = q$, and that φ_* restricted to $H^1(F(X_3), -)$ is the Abel-Jacobi mapping ϕ of (30′).

LEMMA 6.10: *The Abel-Jacobi mapping* ϕ *of* (30′) *with coefficients* $\mathbf{C}$ *and of* (30″) *is onto.*

PROOF: The reason is lemma 6.4 and the fact that φ_* is onto. Namely let $\alpha \in H^3(F(X_4))$, then

$$\begin{aligned}
\varphi_* p^*(\alpha) = \varphi_*\{j^* p_4^*(\alpha)\} = \varphi_* j^*\{(p_4)^*\alpha\} = i^*(\varphi_4)_*\{(p_4)^*\alpha\} \\
= i^*\{(\varphi_4)_*(p_4)^*\alpha\} = 0
\end{aligned}$$

because $H^3(X_4) = 0$.

6.11: There remains now to be investigated the $\mathrm{Ker}(\phi)$. For that we have to introduce the *incidence correspondence* Σ on $F(X_3)\times F(X_3)$. Put

(34) $\Sigma =$ Zariski-closure of $\{(s_1, s_2); s_1, s_2 \in F(X_3), s_1 \neq s_2, \ l_{s_1} \cap l_{s_2} \neq \emptyset \text{ on } X_3\}$[5]

Note that $\Sigma = {}^t\Sigma$.

From this incidence correspondence we get endomorphisms

$$\sigma : H^1(F(X_3), -) \longrightarrow H^1(F(X_3), -) \tag{35}$$

and

$$\sigma : J(F(X_3)) \longrightarrow J(F(X_3)). \tag{35$'$}$$

Now there arises the natural question: given $\gamma \in H^1(F(X_3))$, describe the relation between $\phi(\gamma)$ and $\phi(\sigma\gamma)$. This is answered by the following result which we borrow without proof from Tjurin ([24], lemma 6, §3, lect. 4; note however that the essential part in the proof is again the fact 6.4).

LEMMA 6.12 (1[th] Key lemma of Tjurin): *For $\gamma \in H^1(F(X_3), -)$ we have in $H^3(X_3, -)$:*

$$\phi(\sigma\gamma) = -(q-1)\phi(\gamma), \tag{36}$$

where $q = \deg \varphi$ is introduced in 6.6ii).

COROLLARY 6.13: *The following diagram is commutative (coefficients*: **Z** *modulo torsion or* **C**):

$$\begin{tikzcd} H^1(F(X_3)) \arrow[r, "1-\sigma"] \arrow[d, "\phi"'] & H^1(F(X_3)) \arrow[d, "\phi"] \\ H^3(X_3) \arrow[r, "\cdot q"] & H^3(X_3) \end{tikzcd} \tag{37}$$

[5] This incidence correspondence, introduced here in a concrete geometric way, may differ slightly from the more natural one introduced via the Chow group (cf. [18], section 5, eq. (22) and (32), or [3], 3.4.1), as

$$\mathrm{pr}_{13}[Cl\{P(X_3)\times F(X_3)\}\cdot Cl\{F(X_3)\times {}^tP(X_3)\}] \in CH^1(F(X_3)\times F(X_3));$$

namely, we may have suppressed the diagonal a number of times.

In particular:

i) $\operatorname{Ker}\phi \supset \operatorname{Ker}(1-\sigma)$,

ii) (coefficients $\mathbf{C}$): $\phi \mid \operatorname{Im}(1-\sigma)$ is onto.

PROOF: Immediate from 6.12 since multiplication by q is injective and since for coefficients $\mathbf{C}$ both ϕ and multiplication by q are surjective.

COROLLARY 6.14: *We have a commutative diagram*

$$\text{(37')}\qquad \begin{array}{ccc} J(F(X_3)) & \xrightarrow{1-\sigma} & J(F(X_3)) \\ \Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\phi} \\ J^2_{cl}(X_3) & \xrightarrow[\cdot q]{} & J^2_{cl}(X_3) \end{array}$$

In particular:

i) $(\operatorname{Ker}\phi)^0 \supset \operatorname{Ker}(1-\sigma)^0$ *(connected components!)*,

ii) $\phi \mid \operatorname{Im}(1-\sigma)$ *is onto.*

6.15: In order to get equality in 6.13i one needs the following lemma of [24], for the proof of which we refer to [24], lect. 5, §1:

LEMMA 6.15 (2^{nd} Key lemma of Tjurin): (coeff. $\mathbf{Z}$ mod. torsion, resp. $\mathbf{C}$): *Let* $\gamma_1, \gamma_2 \in H^1(F(X_3))$. *Then for the cup-product pairing we have*:

$$\text{(38)}\qquad \langle \phi(\gamma_1), \phi(\gamma_2)\rangle_{X_3} = \langle (1-\sigma)\gamma_1, \gamma_2\rangle_{F(X_3)}.$$

COROLLARY 6.16: *For cohomology (coeff.* $\mathbf{Z}$ *mod. torsion, resp.* $\mathbf{C}$) *we have* $\operatorname{Ker}(1-\sigma) = \operatorname{Ker}(\phi)$.

PROOF: By 6.13i it suffices to see $\operatorname{Ker}(\phi) \subset \operatorname{Ker}(1-\sigma)$. This follows since the R.H.S. of (38) is non-degenerate.

COROLLARY 6.17: *For the (intermediate) Jacobians we have* $\operatorname{Ker}(1-\sigma)^0 = (\operatorname{Ker}\phi)^0$.

PROPOSITION 6.18 ([24], lect. 5, §2, lemma 2): *The incidence endomorphism* σ *of* (35) *and* (35') *satisfies the following equation*:

$$\text{(39)}\qquad \sigma^2 + (q-2)\sigma - (q-1) = 0,$$

where q *is the integer of* 6.6ii.

PROOF: By the 1^{th} key lemma (6.12) we have $\phi \cdot \{\sigma + (q-1)\} = 0$. Hence for cohomology we have by 6.16 that $(\sigma - 1) \cdot \{\sigma + (q-1)\} = 0$, which is (39). For the Jacobians we conclude by 6.17 that (39) is true up to isogeny, but the image of $J(F(X_3))$ by the L.H.S. of (39) is an abelian variety, hence (39).

§7. Generalized Prym varieties. Relation with the classical intermediate Jacobian of the Fano threefolds of type X_3^4, X_3^6 or X_3^8

In the first part of this section k is an algebraically closed field of arbitrary characteristic.

7.1. Assumption I

Let A be an abelian variety defined over k with an endomorphism σ satisfying an equation

$$\sigma^2 + (q-2)\sigma - (q-1) = 0, \tag{39}$$

where q is an integer $\neq 0, 1$ and not divisible by $p = \text{char}(k)$.

Now we can imitate the theory of Prym varieties of Mumford [14], where σ is an involution.

Note: $\sigma^2 + (q-2)\sigma - (q-1) = (\sigma + (q-1))(\sigma - 1)$. Introduce the abelian subvarieties

$$\begin{cases} B = \{x; \sigma(x) = x\}^0, \\ P = \Pr(\sigma) = \{x; \sigma(x) = -(q-1)x\}^0. \end{cases} \tag{40}$$

$\Pr(\sigma)$ is called the *generalized Prym variety of σ.*

LEMMA 7.2:

i) $B = \text{Im}(\sigma + (q-1))$, $P = \text{Im}(\sigma - 1)$.

ii) $B + P = A$, $B \cap P \subset A_q$ *(points of order q). In particular: this is a Poincaré-decomposition of A.*

PROOF: Put $\alpha = \sigma + (q-1)$ and $\beta = \sigma - 1$. By (39) we have $\alpha \cdot \beta = \beta \cdot \alpha = 0$. Now we have:

a) $\text{Ker}(\alpha) \cap \text{Ker}(\beta) \subset A_q$.

Proof: let $a \in A$ with $\alpha(a) = 0$ and $\beta(a) = 0$. Then $\sigma(a) = -(q-1)a$ and $\sigma(a) = a$. Hence $q \cdot a = 0$.

b) $\text{Im}(\alpha) + \text{Im}(\beta) = A$.

Proof: $(\sigma - 1)(a) - (\sigma + (q-1))(a) = -q \cdot a$.
c) $\mathrm{Im}(\alpha) \subset \mathrm{Ker}(\beta)^0$ and $\mathrm{Im}(\beta) \subset \mathrm{Ker}(\alpha)^0$.
Proof: immediate from $\beta \cdot \alpha = 0$ and $\alpha \cdot \beta = 0$.
d) $\dim \mathrm{Im}(\alpha) + \dim \mathrm{Im}(\beta) \geq \dim A$.

Proof: Immediate from b)

e) $\dim \mathrm{Ker}(\alpha)^0 + \dim \mathrm{Ker}(\beta)^0 \leq \dim A$.
Proof: from a) and intersection theory.
f) Finally: Combining c), d) and e) we get everywhere equality and $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)^0$, $\mathrm{Im}(\beta) = \mathrm{Ker}(\alpha)^0$. This completes the proof.

7.3: For the following we need some more facts. Consider the commutative diagram

$$\begin{array}{ccc} H \subset B \times P & \xrightarrow{\tau} & A \\ \Big\downarrow {\scriptstyle \mathrm{id} \times (1-q)} & & \Big\downarrow {\scriptstyle \sigma} \\ B \times P & \xrightarrow{\tau} & A \end{array}$$

where $\tau(x, y) = x + y$ and $H = \mathrm{Ker}(\tau)$. Clearly $H = \{(x, -x);\ x \in B \cap P\}$. Put $H_1 = \mathrm{pr}_B H$ and $H_2 = \mathrm{pr}_P H$. Note that it follows from 7.2ii that $H_1 \subset B_q$ and $H_2 \subset P_q$. Also we introduce the following maps:

$$j_1 : B \longrightarrow B \times P \qquad \text{with } j_1(b) = b \times e,$$

$$j_2 : P \longrightarrow B \times P \qquad \text{with } j_2(p) = e \times p,$$

and put $\tau_i = \tau \cdot j_i$ $(i = 1, 2)$.

LEMMA 7.4:
i) $\mathrm{Ker}(1 - \sigma) = \bigcup_{z \in P_q} B_z$.
ii) $\mathrm{Ker}(1 - \sigma) = B$ exactly $\Leftrightarrow H_2 = P_q$.

PROOF: Let $x = y + z$ with $y \in B$, $z \in P$. Then $\sigma(x) = y - (q-1)z$. Hence $\sigma(x) = x \Leftrightarrow q \cdot z = 0$. This proves i).
Proof of ii:
$\Rightarrow$: Let $z \in P_q$. Then $\sigma(z) = -(q-1)z = z$, hence by assumption $z \in B$. Hence $z \in B \cap P$, hence $(-z, z) \in H$, hence $z \in H_2$.
$\Leftarrow$: Let $x \in A$ with $\sigma(x) = x$. Now $x = y + z$, $y \in B$, $z \in P$ and we have seen $z \in P_q$. Hence by assumption $z \in H_2$, hence $(-z, z) \in H$, hence $z \in B$, hence $x \in B$.

7.5: Next we turn to polarization. Suppose A has a *principal polarization* θ. Put $D_i = \tau_i^*(\theta) = j_i^*(\tau^*\theta)$. In general: for a divisor U on A we use the notation λ_U: $A \to \hat{A}$ for the map $x \mapsto Cl(U_x - U)$. Consider now in particular the map $\lambda_{\tau^*(\theta)}: B \times P \to \hat{B} \times \hat{P}$. Then clearly we have a decomposition

$$\lambda_{\tau^*(\theta)} = \begin{pmatrix} \lambda_{D_1} & \gamma \\ \delta & \lambda_{D_2} \end{pmatrix}.$$

Assumption II: A has a principal polarization such that

$$\tau^*(\theta) \equiv D_1 \times P + B \times D_2 \qquad \text{(mod.algebr.eq.).} \tag{41}$$

Note that (41) is equivalent with $\gamma = \delta = 0$.

Lemma 7.6: *Under the assumption II we have*

$$\operatorname{Ker} \lambda_{\tau^*(\theta)} = \operatorname{Ker} \lambda_{D_1} \times \operatorname{Ker} \lambda_{D_2} \text{ and } \operatorname{Ker} \lambda_{D_i} = H_i \qquad (i = 1, 2).$$

Proof: The first relation is clear since $\lambda_{\tau^*(\theta)} = \lambda_{D_1} \times \lambda_{D_2}$. For a group G we denote by $\#(G)$ the order of G. Consider the commutative diagram ([11], p. 130, prop. 11):

$$\begin{tikzcd} B \times P \arrow[r, "\tau"] \arrow[d, "\lambda_{\tau^*(\theta)}"'] & A \arrow[d, "\lambda_\theta"] \\ \hat{B} \times \hat{P} & \hat{A} \arrow[l, "\hat{\tau}"'] \end{tikzcd}$$

We have $\#(\operatorname{Ker} \lambda_{\tau^*(\theta)}) = (\#\operatorname{Ker} \tau) \cdot (\#\operatorname{Ker} \hat{\tau}) = (\#\operatorname{Ker} \tau)^2 = (\# H)^2 = (\# H_1) \cdot (\# H_2)$. *Next we claim*: $\operatorname{Ker} \lambda_{D_i} \supset H_i$ $(i = 1, 2)$.

Proof: Let $h_1 \in H_1$, hence $(h_1, -h_1) \in H$. Then we have

$$Cl(D_{1_{h_1}} - D_1) = j_1^*\{Cl[(D_1 \times P)_{(h_1, -h_1)} - D_1 \times P]\} =$$
$$= j_1^*\{\tau^*[Cl(\theta_{\tau(h_1, -h_1)} - \theta)]\} = 0$$

Hence $\operatorname{Ker} \lambda_{\tau^*(\theta)} \supset H_1 \times H_2$, but in view of the above relation between the orders we have $\operatorname{Ker} \lambda_{\tau^*(\theta)} = H_1 \times H_2$.

Lemma 7.7: *Equivalent conditions*:

1) $\operatorname{Ker}(1 - \sigma) = \operatorname{Ker}(1 - \sigma)^0 \quad (= B)$,
2) $H_2 = P_q$,
3) $D_2 \equiv q \cdot E$ *(alg.eq.) with E a divisor on P.*

Moreover if these conditions are satisfied then $l(E) = 1$.

PROOF:

1) $\Leftrightarrow$ 2): lemma 7.4.

2) $\Rightarrow$ 3): $\operatorname{Ker} \lambda_{D_2} = H_2 = P_q$. Then $D_2 \sim q \cdot E$ by [13], p. 231, thm. 3.

3) $\Rightarrow$ 2): $H_2 = \operatorname{Ker} \lambda_{D_2} = \operatorname{Ker} \lambda_{qE} = \operatorname{Ker}(q\lambda_E) \supset P_q$. Hence $H_2 = P_q$ by 7.3.

Turning to the assertion about $l(E)$: We have $D_i = \tau_i^*(\theta)$, hence $D_i > 0$. Hence by Riemann-Roch ([13], p. 150): if $d = \dim P$

$$l(D_2) = \chi(D_2) = \frac{D_2^{(d)}}{d!} = \sqrt{(\# \operatorname{Ker} \lambda_{D_2})} = \sqrt{(\# H_2)} = q^d.$$

Furthermore $\operatorname{Ker} \lambda_E \subset \operatorname{Ker} \lambda_{D_2}$, hence E is non-degenerate (ibid, p. 155), hence (ibid p. 159) $l(E) = \chi(E)$. Finally by Riemann-Roch

$$l(E) = \chi(E) = \frac{E^{(d)}}{d!} = \frac{1}{q^d} \cdot \frac{D_2^{(d)}}{d!} = 1.$$

7.8. Assumption III:

We call the equivalent properties of lemma 7.7 assumption III. Furthermore we use in that case the notation $D_2 = q \cdot \Xi$, with Ξ a divisor on P with $l(\Xi) = 1$. Hence *under assumption III the couple* (P, Ξ) *is a principally polarized abelian variety.*

7.9: Now we want to use this theory to study the (classical) intermediate Jacobian of Fano threefolds of type X_3^4, X_3^6 or X_3^8. For the rest of §7 the assumptions are the same as in §6. We write X instead of X_3 if there is no danger for confusion.

By 6.18 the incidence endomorphism σ on $J(F(X))$ satisfies assumption I. Hence we have a generalized Prym variety

(42) $$P = \operatorname{Pr}(\sigma) \subset J(F(X)) \; (= J, \text{ for abbreviation}).$$

Due to the fact that for the incidence correspondence (see 6.11) ${}^t\Sigma = \Sigma$ we have:

LEMMA 7.10: *The incidence endomorphism σ satisfies assumption II.*

PROOF: Consider as before

$$B \times P \xrightarrow{\tau} A = J(F(X)).$$

Let – as before –$j_1 : B \to B \times P$ with $j_1(b) = b \times e$ and $j_2 : P \to B \times P$ with $j_2(p) = e \times p$; put $\tau_i = \tau \cdot j_i (i = 1, 2)$. Then we have to prove that the composite map

$$B \xrightarrow{\tau_1} A \xrightarrow{\lambda_\theta} \hat{A} \xrightarrow{\hat{\tau}_2} \hat{P} \tag{43}$$

is zero, and similarly for P. This composite map is given by

$$b \longrightarrow \tau_2^*(Cl(\theta_{\tau_1(b)} - \theta)). \tag{44}$$

It suffices to see that (43) is zero for all points b of order l^n (l some prime number, all n). Using the notation of [11] page 189–190, in order to see that the divisor class U (say) of the RHS of (44) is zero, it suffices then, by ibid. prop. 4 on page 189, to see that the symbol $e_{l^n}(p, U) = 1$ for all $p \in P_{l^n}$. Finally, using ibid. th. 4 on page 192, it suffices to prove

$$E^\theta(p, b) = 0 \qquad \forall p \in T_l(P), b \in T_l(P), \tag{45}$$

where $T_l(-)$ are the Tate groups and $E^\theta(-, -)$ is the Riemann form of θ ([13], p. 186). Using the well-known identification (cf. [18] section IV):

$$T_l(J(F(X)) \times_{\mathbf{Z}_l} \mathbf{Q}_l \xrightarrow{\sim} H^1(F(X), \mathbf{Q}_l), \tag{46}$$

this corresponds with the condition for the cupproduct:

$$\langle p, b \rangle = 0. \tag{45'}$$

Now σ comes via correspondence Σ of $F(X) \times F(X)$ with the property ${}^t\Sigma = \Sigma$ (the essential point!), hence we have

$$\langle \sigma u, v \rangle = \langle u, {}^t\sigma v \rangle = \langle u, \sigma v \rangle$$

for $u, v \in H^1(F(X), \mathbf{Q}_l)$. Now $p \in T_l(P)$, hence (cf. 7.2) $p = \sigma x - x$ and $b \in T_l(B)$, hence $b = \sigma y + (q-1)y$, with $x, y \in H^1(F(X))$. Therefore we get

$$\begin{aligned}
\langle p, b \rangle &= \langle \sigma x - x, \sigma y + (q-1)y \rangle \\
&= \langle \sigma x, \sigma y \rangle + \langle \sigma x, (q-1)y \rangle - \langle x, \sigma y \rangle - (q-1)\langle x, y \rangle \\
&= \langle x, \sigma^2 y \rangle + (q-1)\langle x, \sigma y \rangle - \langle x, \sigma y \rangle - (q-1)\langle x, y \rangle \\
&= \langle x, -(q-2)\sigma y + (q-1)y \rangle + (q-2)\langle x, \sigma y \rangle - (q-1)\langle x, y \rangle = 0.
\end{aligned}$$

Similarly the map $P \to \hat{B}$ is zero (this follows also immediately now by looking to (45) and using the skew-symmetry of the Riemann-form).

LEMMA 7.11: *The incidence correspondence σ satisfies assumption III.*

The proof of this lemma will occupy almost the rest of §7.[6] We consider again the Abel-Jacobi mapping (30′):

$$\phi_{\mathbf{Z}} = \varphi_* \cdot p^* : H^1(F(X), \mathbf{Z}) \longrightarrow H^3(X, \mathbf{Z}) \tag{30′}$$

and also the dual mapping:

$${}^t\phi_{\mathbf{Z}} : H^3(X, Z) \longrightarrow H^1(F(X), \mathbf{Z}). \tag{^{t}30′}$$

(Note: for coeff. $\mathbf{Z}$ we always work *modulo torsion*). We have similar maps $\phi_{\mathbf{C}}$ and ${}^t\phi_{\mathbf{C}}$ for coefficients $\mathbf{C}$ and also ϕ and ${}^t\phi$ for the (intermediate) Jacobians.

LEMMA 7.12: $\phi_{\mathbf{Z}}$ *onto* $\Rightarrow$ *assumption III.*

PROOF: We proceed in several steps. First note that $\phi_{\mathbf{Z}}$ *onto* implies by the structure theory of abelian groups that ${}^t\phi_{\mathbf{Z}}$ embeds $H^3(X, \mathbf{Z})$ as a *full direct summand* of $H^1(F(X), \mathbf{Z})$ and hence the map ${}^t\phi : J^2_{cl}(X) \to J(F(X))$ is a *closed immersion.*

Next we claim that we have a commutative diagram:

$$\begin{tikzcd}
\Pr(\sigma) \arrow[r, hook] \arrow[dd, "-q"'] & J(F(X)) \arrow[d, "\phi"] \\
& J^2_{cl}(X) \arrow[d, hook, "{}^t\phi"] \\
\Pr(\sigma) \arrow[r, hook] & J(F(X))
\end{tikzcd}$$

PROOF: It suffices to check it on cohomology. Take $\gamma_1, \gamma_2 \in H^1(F(X), \mathbf{C})$, then $\langle {}^t\phi \cdot \phi(\sigma - 1)\gamma_1, \gamma_2 \rangle_{F(X)} = \langle \phi(\sigma - 1)\gamma_1, \phi\gamma_2 \rangle_X$ and by lemma 6.15 this equals $\langle (\sigma - 1)^2\gamma_1, \gamma_2 \rangle_{F(X)} = \langle (\sigma^2 - 2\sigma + 1)\gamma_1, \gamma_2 \rangle_{F(X)}$, and by (39) this equals $\langle -q(\sigma - 1)\gamma_1, \gamma_2 \rangle_{F(X)}$, which proves the commutativity and shows also that ${}^t\phi \cdot \phi \mid \Pr(\sigma) = -q$.

[6] Unfortunately at this point there is a serious gap in Tjurin's argument. See footnote 7.

Since $\Pr(\sigma) = \mathrm{Im}(\sigma - 1)$, it follows from diagram (37′) that $\phi \mid \Pr(\sigma)$ is onto, and hence the *commutativity of the above diagram implies* (as abelian varieties):

(48) $$ {}^t\phi : J^2_{cl}(X) \xrightarrow{\sim} \Pr(\sigma). $$

Also we have

(49) $$ (\sigma - 1) \mid \Pr(\sigma) = -q, $$

because $\Pr(\sigma) = \mathrm{Im}(\sigma - 1)$ and then (49) follows immediately from (39).

Now look again to the commutative diagram (37′); fill in $\Pr(\sigma) = \mathrm{Im}(\sigma - 1)$:

(37″)
$$
\begin{tikzcd}
J(F(X)) \arrow[r, "\sigma-1"] \arrow[d, "\phi"'] & \Pr(\sigma) \arrow[r, hook] \arrow[dr, phantom, "\mathrm{II}"] & J(F(X)) \arrow[d, "\phi"] \\
J^2_{cl}(X) \arrow[ur, dashed, "{}^t\phi", "\sim"'] \arrow[u, phantom, "\mathrm{I}"] \arrow[rr, "-q"'] & & J^2_{cl}(X)
\end{tikzcd}
$$

Claim: We can fill in the map ${}^t\phi$ as indicated and keep commutative diagrams.

Proof:

a) Diagram I is commutative if restricted to $\Pr(\sigma)$; this follows from (48) and the fact ${}^t\phi \cdot \phi \mid \Pr(\sigma) = -q$ (noted above).

b) Diagram II is commutative for $(1 - \sigma) \mid \Pr(\sigma)$ is onto.

c) This implies now that $\mathrm{Im}(\phi + {}^t\phi^{-1} \cdot (1 - \sigma))$ is killed by multiplication by q; hence diagram I is commutative too.

We can go back now to cohomology with coefficients in $\mathbf{C}$ or coefficients in $\mathbf{Z}$ modulo torsion. Then diagram (37″) means that *we have proved finally the following*: $\phi_{\mathbf{Z}}$ *onto implies that we have a commutative diagram*:

(37^{bis})
$$
\begin{tikzcd}
H^1(F(X), Z) \arrow[r, "\sigma-1"] \arrow[d, "\phi_Z"'] & H^1(F(X), Z) \\
H^3(X, Z) \arrow[ur, "{}^t\phi_Z"'] &
\end{tikzcd}
$$

Now since we have already remarked that ϕ_Z onto implies ${}^t\phi_Z$ injective with image a full direct summand, it follows that:

ϕ_Z onto $\Rightarrow (\sigma - 1)$ injective with image a *full direct summand.*

In other words if we consider the vector space $V = H^{0,1} \subset H^1(F(X), \mathbf{C})$ and if Λ is the image of $H^1(F(X), \mathbf{Z})$ in V then:

$$\phi_Z \text{ onto} \Rightarrow \operatorname{Im}\{(\sigma - 1)\Lambda\} = \Lambda \cap \operatorname{Im}\{(\sigma - 1) V\}.^{7)} \tag{50}$$

Arrived at this point we can reproduce now Tjurin's argument. Namely take the Riemann form of θ on $J(F(X))$; this is given by the cupproduct on $H^1(F(X), \mathbf{Z})$. Now on the (generalized) Prym part the lattice is, by (50), $\operatorname{Im}\{(\sigma - 1)\Lambda\}$; hence we have on the Prym part a Riemann form of type $\langle(1-\sigma)\gamma_1, (1-\sigma)\gamma_2\rangle$ with $\gamma_1, \gamma_2 \in H^1(F(X), \mathbf{Z})$. Using lemma 6.15 and the commutativity of (37) we get:

$$\begin{aligned} \langle(1-\sigma)\gamma_1, (1-\sigma)\gamma_2\rangle_{F(X)} &= \langle\phi(\gamma_1), \phi(1-\sigma)\gamma_2\rangle_X = \\ &= \langle\phi(\gamma_1), q\phi(\gamma_2)\rangle_X = q\langle\phi(\gamma_1), \phi(\gamma_2)\rangle_X. \end{aligned} \tag{51}$$

In other words: ϕ_Z onto $\Rightarrow$ the Riemann-form restricted to the Prym part is q-times another such Riemann-form. It is well-known that this means that $\tau_1^*(\theta) = D_2$ equals q-times another divisor. Hence "ϕ_Z onto" implies assumption III, which completes the proof of lemma 7.12.

7.13: The above proof also shows that the Riemann-form of the divisor Ξ (introduced in 7.8) is given by $1/q\langle(1-\sigma)\gamma_1, (1-\sigma)\gamma_2\rangle_{F(X)}$. Now the (classical) intermediate Jacobian $J^2_{cl}(X)$ has also a principal polarization θ_{cl} given by the cup product on $\mathrm{H}^3(X, \mathbf{Z})$.

LEMMA 7.13: (always assuming that ϕ_Z is onto):

$$ {}^t\phi : (J^2_{cl}(X), \theta_{cl}) \xrightarrow{\sim} (P, \Xi), \tag{48$^{\text{bis}}$}$$

i.e., this is an isomorphism of polarized abelian varieties.

[7] A priori, we only have

$$\operatorname{Im}\{(\sigma - 1)\Lambda\} \subset \Lambda \cap \operatorname{Im}\{(\sigma - 1)V\}.$$

Tjurin seems to assume the equality tacitly.

PROOF: In view of (48) it suffices to see that the Riemann forms agree. Let $\alpha, \beta \in H^3(X, \mathbf{Z})$. Look to the diagram ($37^{\text{bis}}$). We have $\alpha = \phi(\gamma_1)$, $\beta = \phi(\gamma_2)$. The assertion follows now immediately from (51) which reads $\langle\alpha, \beta\rangle_X = 1/q\langle(1-\sigma)\gamma_1, (1-\sigma)\gamma_2\rangle_{F(X)}$ and from the commutativity of (37^{bis}) which implies $1/q\langle(1-\sigma)\gamma_1, (1-\sigma)\gamma_2\rangle_{F(X)} = 1/q\langle{}^t\phi(\alpha), {}^t\phi(\beta)\rangle_{F(X)}$.

7.14: In order to complete the proof of 7.11 *it remains to prove the surjectivity of* ϕ_Z. Reformulating this in terms of *homology* we have to prove

$$\phi_Z : H_1(F(X), \mathbf{Z}) \longrightarrow H_3(X, \mathbf{Z}) \quad \textit{onto}.$$

Now consider again diagram 6.7 (and write $X = X_3$, etc.). Then it suffices to prove

$$\varphi_* : H_3(Y, \mathbf{Z}) \longrightarrow H_3(X_3, \mathbf{Z}) \ \textit{onto}, \tag{52}$$

because

$$H_3(Y, Z) = H_1(F(X_3), \mathbf{Z}) \oplus H_3(F(X_4), \mathbf{Z}),$$

and $\varphi_* H_3(F(X_4), \mathbf{Z}) = 0$.

The desired result (52) follows (taking $V = X_4$, $W = P(X_4)$, $X = X_3$ and $Y = Y$ and $q = k$) from:

LEMMA 7.15: *Let V be a smooth projective variety defined over* $\mathbf{C}$, $\dim V = d+1$, $\varphi: W \to V$ *a proper morphism, generically finite of degree k (W irreducible), $X \hookrightarrow V$ a smooth hyperplane section, $Y = \varphi^{-1}(X)$. Then the image of $\varphi_* : H_d(Y, \mathbf{Z}) \to H_d(X, \mathbf{Z})$ contains the vanishing cycles.*

PROOF:

Step a. We may assume X generic. Indeed, let P^* denote the parameter space for hyperplanes $X \hookrightarrow V$. We have families

$$\begin{array}{ccc} \mathscr{X} & & \mathscr{Y} \\ \Big\downarrow & \text{and} & \Big\downarrow \\ P^* & & P^* \end{array}$$

with fibres, respectively, hyperplanes $X_t \subset V$ and inverse images

$Y_t = \varphi^{-1}(X_t) \subset W$, $t \in P^*$. For $t_0 \in P^*$, there exists a neighborhood $t_0 \in U \subset P^*$ such that $\mathscr{X}_U$ and $\mathscr{Y}_U$ contract onto X_{t_0}, Y_{t_0} respectively. For $t \in U$ we get a commutative diagram of specializations

$$\begin{array}{ccc} H_d(Y_t, \mathbf{Z}) & \longrightarrow & H_d(Y_{t_0}, \mathbf{Z}) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ H_d(X_t, \mathbf{Z}) & \longrightarrow & H_d(X_{t_0}, \mathbf{Z}). \end{array}$$

Assuming X_t, X_{t_0} smooth, the bottom horizontal arrow is an isomorphism. Thus to prove the right hand vertical arrow surjective, it suffices to show the left hand one is.

Step b. The sheaf $R^d\pi_*\mathbf{Z}$ is constructible on P^*, where $\pi: Y \to P^*$. Let $U \subset P^*$ be a non-empty Zariski open set such that $R^d\pi_*\mathbf{Z}$ is locally constant on U. Let $l \in P^*$ be a general line corresponding to a Lefschetz pencil $\{X_t\}$ on V. Let $X_0 = X$ and let $X_{t_1}, \ldots, X_{t_n}$ denote the singular fibres. Choose paths $\tau_i : [0, 1] \to l$ such that $\tau_i(0) = 0$, $\tau_i(1) = t_i$. We may suppose $\tau_i(x) \in U$ for $0 \leq x < 1$. To each τ_i there correspond a vanishing cycle $\delta_i \in H_d(X, \mathbf{Z})$, and these vanishing cycles generate the group of vanishing cycles. Let us show, e.g. $\delta_1 \in \mathrm{Im}(\varphi_*)$.

Since $\{X_t\}$ is general, we may assume the singular points of the fibres, $p_1, \ldots, p_n \in V$ lie in the open set over which φ is étale. Let N be a neighborhood of p_1 in V such that $\varphi^{-1}(N) = M^{(1)} \cup \cdots \cup M^{(k)}$, where $\varphi : M^{(i)} \xrightarrow{\sim} N$ for each i. Let ϵ be close to 1, and let $\varphi_\epsilon : Y_\epsilon \to X_\epsilon$ be the situation over the point $\tau_1(\epsilon)$. Since $|\epsilon - 1|$ is small, the vanishing cycle δ_ϵ is "close to disappearing" on X_ϵ, i.e. it is supported on $X_\epsilon \cap N$, so it can be lifted to $Y_\epsilon \cap M^{(i)} \xrightarrow{\sim} X_\epsilon \cap N$. But now the fact that $\tau_1([0, \epsilon]) \subseteq U$ gives

$$\begin{array}{ccc} H_d(Y_0) & \xrightarrow{\simeq} & H_d(Y_\epsilon) \\ \downarrow{\scriptstyle \varphi_{0_*}} & & \downarrow{\scriptstyle \varphi_{\epsilon_*}} \\ H_d(X_0) & \xrightarrow{\simeq} & H_d(X_\epsilon) \\ & & \\ \delta_1 & \longrightarrow & \delta_\epsilon \end{array}$$

Since $\delta_\epsilon \in \mathrm{Im}(\varphi_{\epsilon_*})$ we get $\delta_1 \in \mathrm{Im}(\varphi_{1_*})$. This completes the proof of lemma 7.15, and hence of 7.11.

7.16: Summarizing we have the following result, which is – except for the difficulty mentioned in footnote 7 – due to Tjurin:

THEOREM 7.16: *Let X be either $X_3^4 \subset \mathbf{P}_4$, or $X_3^6 = Q_4 \cdot C_4 \subset \mathbf{P}_5$, or $X_3^8 = Q_5 \cdot Q_5' \cdot Q_5'' \subset \mathbf{P}_6$, defined over* $\mathbf{C}$ *and generic (or sufficiently general) over* $\mathbf{Q}$. *Let $(J_{cl}(X), \theta_{cl})$ be the (classical) intermediate Jacobian variety associated to X. Let* $(\mathrm{Pr}(\sigma), \Xi)$ *be the generalized Prym variety associated to the incidence correspondence $\Sigma \subset F(X) \times F(X)$ (see* 6.11 *and* 7.1). *Then we have the isomorphism of polarized abelian varieties*

$$ {}^t\phi : (J_{cl}^2(X), \theta_{cl}) \xrightarrow{\sim} (\mathrm{Pr}(\sigma), \Xi), $$

where ${}^t\phi$ is the transpose of the Abel-Jacobi map (see 6.2 *and under* 7.11).

§8. Relation between the generalized Prym varieties and the Chow groups

8.1: Let S be a *connected* integral $\mathbf{Z}$-scheme and suppose we have a commutative diagram

(53)
$$\begin{tikzcd} X \arrow[rr, hook, "i"'] \arrow[dr, "\pi"'] & & \mathbf{P}(E) \arrow[dl] \\ & S & \end{tikzcd}$$

where $\mathbf{P}(E)$ is a projective bundle over S, i is a closed immersion, π is *smooth* and for $s \in S$ the fibre $X_s \subset \mathbf{P}(E)_s$ is a quartic threefold $X_3^4 \subset \mathbf{P}_4$ (resp. $X_3^6 = Q_4 \cdot C_4 \subset \mathbf{P}_5$, resp. $X_3^8 = Q_5 \cdot Q_5' \cdot Q_5'' \subset \mathbf{P}_6$). Assume that the residue characteristics of S are different from 2 and 3. In the following we write shortly X/S to indicate such a diagram (53).

8.2: Let $\rho : F(X) \to S$ be the scheme such that for every $s \in S$ the $F(X)_s$ is the variety of lines on X_s (cf. (1)). Short notation: $F(X/S)$. *Assume* that ρ is a *smooth curve.*

REMARK: It follows from prop. 1.1 and SGA 1 IV 6.11 that after restricting, if necessary, to a non-empty open set $U \subset S$ we always can make this assumption.

8.3: By [*FGA*, *V*] there exists now the abelian scheme

$$J(F(X/S)) \overset{\text{def}}{=} \mathrm{Pic}_{F(X)/S},$$

which we call the *Jacobi-scheme of* $F(X/S)$ *over* S.

8.4: Similar as in 6.1 we consider the natural correspondence

$$(29') \qquad \begin{array}{ccc} P(X/S) & \xrightarrow[\varphi]{} & X/S \\ {\scriptstyle p}\big\downarrow & & \\ F(X/S) & & \end{array}$$

where $P(X/S)$ is the restriction of the Grassmannian bundle to $F(X/S)$.

8.5: Introduce again (cf. 6.11) the *incidence correspondence*

$$(34') \qquad \Sigma/S \hookrightarrow F(X/S) \times_S F(X/S),$$

defined as

$$\Sigma/S = \mathrm{pr}_{13}\{(P(X/S) \times_S F(X/S)) \cap (F(X/S) \times_S {}^tP(X/S)) - \tilde{\Delta}\},$$

where $\tilde{\Delta}$ is the component of the intersection "over the diagonal" (Δ/S) of $F(X/S) \times_S F(X/S)$, i.e. $\mathrm{pr}_{13}\,\tilde{\Delta} = (\Delta/S)$. In particular, for every $s \in S$ we get as fibre of (34′) the correspondence $\Sigma_s \hookrightarrow F(X_s) \times F(X_s)$ defined by (34).

By the general theory of the Jacobi scheme we get from Σ/S an *incidence endomorphism*

$$(35^{\mathrm{bis}}) \qquad \sigma/S : J(F(X/S) \longrightarrow J(F(X/S)),$$

satisfying the equation

$$(39') \qquad (\sigma/S)^2 + (q-2)(\sigma/S) - (q-1) = 0,$$

where q is the integer of 6.6ii.

PROOF:

a) By Tjurin (39′) is true for the generic point $s \in S$; see (39).

b) The L.H.S. of (39′) is an endomorphism of the abelian scheme $J(F(X/S))$; hence its image is an abelian subscheme. At the generic point $s \in S$ this image is contained in the 0-section; hence it is everywhere contained in the zero-section.

8.6: Let $\Pr(\sigma/S) = \operatorname{Im}((\sigma/S) - 1) \hookrightarrow J(F(X/S))$; this is an abelian subscheme of $J(F(X/S))$ and is called the *generalized Prym scheme associated to* (σ/S).

8.7. REMARKS:

1. Σ/S is flat over S because Σ/S is of a fixed finite degree N over $F(X/S)$ which is itself smooth over S. In fact $N = \mu + 1$, where μ is the degree of the hypersurface which cuts out the rule surface (of lines) on X; see [24], lect. 4, lemma 5. In our case $\mu = 80$, resp. 30, resp. 16.
2. All our constructions are compatible with base change; in particular for $s_0 \in S$ we have $F(X/S)_{s_0} = F(X_{s_0})$, $P(X/S)_{s_0} = P(X_{s_0})$, $(\Sigma/S)_{s_0} = \Sigma_{s_0}$, $(\sigma/S)_{s_0} = \sigma_{s_0}$ and $\Pr(\sigma/S)_{s_0} = \Pr(\sigma_{s_0})$.

8.8. POLARIZATION OF THE PRYM:

On $J(F(X/S))$ we have a relative Cartier divisor θ/S such that θ_s is the usual divisor on $J(F(X_{s_0}))$. Write

$$j : \Pr(\sigma/S) \hookrightarrow J(F(X/S)),$$ [8]

and consider

$$j^*(\theta/S) = D/S,$$

this is – again after restricting to a smaller S if necessary – a relative Cartier divisor on $\Pr(\sigma/S)$ over S.

LEMMA 8.9: $D_s \equiv q\Xi_s$ with $l(\Xi_s) = 1$ (numerical equivalence).[9]

PROOF: Consider the map

$$\lambda_{D/S} : \Pr(\sigma/S) \longrightarrow \Pr(\sigma/S)^{\wedge}$$

and consider the *closed* subscheme $\operatorname{Ker} \lambda_{D/S}$ of $\Pr(\sigma/S)$. By 7.11 we

[8] Restricting to a fibre, this is the map τ_2 of section 7.
[9] Here we use only the existence of a such a divisor Ξ_s for each $s \in S$ separately.

have for the generic point $s \in S$ that

$$(\operatorname{Ker} \lambda_{D/S})_s = \operatorname{Ker}(\lambda_{D_s}) \supset \Pr(\sigma_s)_q;$$

hence it follows

$$\operatorname{Ker} \lambda_{D/S} \supset \Pr(\sigma/S)_q.$$

Using compatibility with base change we get then for every $s_0 \in S$

$$\operatorname{Ker} \lambda_{D_{s_0}} \supset \Pr(\sigma_{s_0}).$$

By [13], p. 231 this gives $D_{s_0} = q\,\Xi_{s_0}$, with Ξ_{s_0} a divisor on $\Pr(\sigma_{s_0})$. Finally applying Riemann-Roch and using again flatness of the schemes over S we get

$$l(\Xi_{s_0}) = \frac{1}{q^d}\sqrt{\#\operatorname{Ker} \lambda_{D_{s_0}}} = \frac{1}{q^d}\sqrt{\#\operatorname{Ker} \lambda_{D_s}} = l(\Xi_s) = 1,$$

where $d = \operatorname{rank}(\operatorname{Ker} \lambda_{D/S})$.

8.10: After the above preparations we now turn to the Chow groups. For every point $s_0 \in S$ we consider as before the diagram

$$\begin{array}{ccc} P(X_{s_0}) & \xrightarrow[\varphi_{s_0}]{} & X_{s_0} \\ {\scriptstyle p_{s_0}}\downarrow & & \\ F(X_{s_0}) & & \end{array} \tag{29$^{\text{bis}}$}$$

and the Abel-Jacobi map

$$\phi_{s_0} = (\varphi_{s_0})_* \cdot p_{s_0}^*: A^1(F(X_{s_0})) \longrightarrow A^2(X_{s_0}). \tag{30$^{\text{bis}}$}$$

By composition we get

$$P(\sigma_{s_0}) \xrightarrow[j_{s_0}]{} J(F(X_{s_0})) \xrightarrow[\phi_{s_0}]{} A^2(X_{s_0}). \tag{54}$$

LEMMA 8.11: *$\phi_{s_0} \cdot j_{s_0}$ is onto.*[10]

PROOF: The proof is typical for many proofs in §8.

Step I: According to 3.17, resp. 3.13, $A^2(X_{s_0})$ is weakly-represent-

[10] For the type X_3^4 it may be necessary to shrink S again, because 3.17 is – à priori – only valid for a sufficiently general X_3^4.

able; let $A^2(X_{s_0}) \simeq A$ with A an abelian variety. Moreover (see 3.6) ϕ_{s_0} is a weak-morphism. Therefore $\mathrm{Im}(\phi_{s_0})$, and also $\mathrm{Im}(\phi_{s_0} \cdot j_{s_0})$, is an abelian subvariety of A. In order to prove surjectivity it suffices therefore to prove surjectivity for the torsion points. Hence by 4.7, resp. 4.6, *it suffices to work with cohomology* and to prove surjectivity there, *using moreover a suitable prime number l.*

Step II: (Tjurin): For cohomology it is true for the geometric generic fibre $X_{\bar{s}}$ (which is in char. 0!); see 6.13.

Step III: Consider the corresponding maps for cohomology.

$$(54') \qquad \mathrm{Im}((\sigma/S)-1)R^1\rho_*\mathbf{Z}/l^\nu \xrightarrow[j]{} R^1\rho_*\mathbf{Z}/l^\nu \xrightarrow[\phi]{} R^3\pi_*\mathbf{Z}/l^\nu$$

(it is not important to keep track of the twisting). Both $R^1\rho_*\mathbf{Z}/l^\nu$ and $R^3\pi_*\mathbf{Z}/l^\nu$ are *locally constant* (SGA 4, XV, 2.1 and XVI, 2.1; *smoothness* of π and ρ are used here). It follows that also (σ/S) and ϕ are locally constant (SGA 4, IX, 2.1). The surjectivity of the composite map

$$\mathrm{Im}(\sigma_{s_0}-1)H^1(F(X_{s_0}),\mathbf{Z}/l^\nu) \xhookrightarrow[j]{} H^1(F(X_{s_0}),\mathbf{Z}/l^\nu) \xrightarrow[\phi]{} H^3(X_{s_0},\mathbf{Z}/l^\nu)$$

follows then from the corresponding fact for the geometric generic fibre (Step II).

LEMMA 8.12: *The following diagram is commutative*:

$$(37^{\mathrm{bis}}) \qquad \begin{tikzcd} J(F(X_{s_0})) \arrow[r, "1-\sigma_{s_0}"] \arrow[d, "\phi_{s_0}"] & \mathrm{Pr}(\sigma_{s_0}) \arrow[r, hook, "j_{s_0}"] & J(F(X_{s_0})) \arrow[d, "\phi_{s_0}"] \\ A^2(X_{s_0}) \arrow[rr, hook, "\cdot q"'] & & A^2(X_{s_0}) \end{tikzcd}$$

PROOF: $A^2(X_{s_0})$ is weakly-representable and ϕ_{s_0} is a weak-morphism. Raising both sides, if necessary, to a sufficiently high power of p ($p = \mathrm{char}.\, k$) we can assume that all are abelian varieties and morphisms. It suffices now to see the commutativity on the points of finite order; i.e. for cohomology with coefficients $\mathbf{Z}/l^\nu\mathbf{Z}$ for a suitable l. For the generic point it follows from 6.13. For the point s_0 we use that cohomology is locally constant on S; hence: also commutativity in s_0.

8.13. Transpose of Abel-Jacobi map:

Consider again the diagram (29^{bis}) of 8.10 and consider the transpose of the Abel-Jacobi map

$$({}^t30^{\text{bis}}) \qquad {}^t\phi_{s_0} = (p_{s_0})_* \cdot \varphi_{s_0}^* : A^2(X_{s_0}) \longrightarrow A^1(F(X_{s_0}))$$

LEMMA 8.13: *The following diagram is commutative*:

$$(47^{\text{bis}}) \qquad \begin{tikzcd} P(\sigma_{s_0}) \arrow[r, hook, "j_{s_0}"'] \arrow[d, ".(-q)"] & J(F(X_{s_0})) \arrow[r, two heads, "\phi_{s_0}"'] & A^2(X_{s_0}) \arrow[d, "{}^t\phi_{s_0}"] \\ P(\sigma_{s_0}) \arrow[rr, hook, "j_{s_0}"'] & & J(F(X_{s_0})) \end{tikzcd}$$

PROOF:

Step I: Both sides are morphisms of Abelian varieties because ${}^t\phi_{s_0} \cdot \phi_{s_0}$ is clearly a morphism given by a correspondence on $F(X_{s_0}) \times F(X_{s_0})$. Hence sufficient to check it for torsion points, i.e., for cohomology.

Step II: OK for the generic points, see the proof of 7.12.

Step III: OK for "special" $s_0 \in S$ because for the corresponding diagram for cohomology we use the fact that cohomology is locally constant.

COROLLARY 8.14: $\mathrm{Im}({}^t\phi_{s_0}) \subset \mathrm{Pr}(\sigma_{s_0})$.

PROOF: 8.11 + 8.13.

COROLLARY 8.15: $\phi_s \cdot {}^t\phi_{s_0} = -q$.

PROOF: Take $x \in A^2(X_{s_0})$. By 8.13 $x = \phi_{s_0}(y)$, $y \in \mathrm{Pr}(\sigma_{s_0})$; hence $\phi_{s_0} \cdot {}^t\phi_{s_0}(x) = \phi_{s_0} \cdot {}^t\phi_{s_0} \cdot \phi_{s_0}(y) = \phi_{s_0}(-qy) = -qx$.

LEMMA 8.16: There exists a homomorphism $\phi_1 : \mathrm{Pr}(\sigma_{s_0}) \to A^2(X_{s_0})$ *such that the following diagram is commutative*

$$(55) \qquad \begin{tikzcd} J(F(X_{s_0})) \arrow[r, "1-\sigma_{s_0}"] \arrow[d, "\phi_{s_0}"'] & \mathrm{Pr}(\sigma_{s_0}) \arrow[dl, "\phi_1"] \\ A^2(X_{s_0}) & \end{tikzcd}$$

PROOF: Using the fact that $A^2(X_{s_0})$ is weakly-representable and that ϕ_{s_0} is a weak-morphism it follows from 8.12 that $\mathrm{Ker}(\phi_{s_0}) \supset \mathrm{Ker}(1-\sigma_{s_0})^0$. However by 8.9 and 7.7 we have $\mathrm{Ker}(1-\sigma_{s_0}) = \mathrm{Ker}(1-\sigma_{s_0})^0$, which proves 8.16.

PROPOSITION 8.17: *The homomorphism* $'\phi_{s_0}: A^2(X_{s_0}) \to \mathrm{Pr}(\sigma_{s_0})$ *(see* ($'30^{\mathrm{bis}}$) *and* 8.14) *has the following properties*:

i) *it is an isomorphism (for all* $K = \bar{K} \supset k$),

ii) *it is regular (see* 8.18),

iii) *it is universal (see* 8.18).

DEFINITION 8.18: *Let V be a smooth, projective variety and A an abelian variety. An homomorphism $\lambda: A^i(V) \to A$ is called regular if for every couple (W, Z), consisting of a smooth variety W and $Z \in CH^i(W \times V)$, the composite map*

$$W \xrightarrow{Z_*} A^i(V) \xrightarrow{\lambda} A$$

is a morphism of algebraic varieties. Moreover a regular isomorphism $\lambda: A^i(V) \to A$ is called universal if every regular homomorphism $\mu: A^i(V) \to B$ (B abelian variety) factors through (A, λ), i.e., if (B, μ) is such a couple then there exists a unique homomorphism of abelian varieties $\bar{\mu}: A \to B$ such that $\mu = \bar{\mu} \cdot \lambda$:

$$\begin{tikzcd} A^i(V) \arrow[rr, "\lambda"] \arrow[dr, "\mu"'] & & A \arrow[dl, "\bar{\mu}"] \\ & B & \end{tikzcd}$$

8.19. PROOF OF 8.17i: For simplicity we work with k itself, but the proof works for any $K = \bar{K} \supset k$. By 8.12, 8.14 and 8.16 we have the following *commutative* diagram:

$$\begin{tikzcd} & & A^2(X_{s_0}) \arrow[dl, "'\phi_{s_0}"'] \\ J(F(X_{s_0})) \arrow[r, "1-\sigma_{s_0}"] \arrow[d, "\phi_{s_0}"'] & \mathrm{Pr}(\sigma_{s_0}) \arrow[r, hook, "j_{s_0}"'] \arrow[dl, "\phi_1"'] & J(F(X_{s_0})) \arrow[d, "\phi_{s_0}"] \\ A^2(X_{s_0}) \arrow[rr, ".q"'] & & A^2(X_{s_0}) \end{tikzcd} \tag{56}$$

For the *surjectivity* of ${}^t\phi_{s_0}$ we remark that since we know already the weak-representability it suffices to prove it for (almost all) torsion points, hence for cohomology. There we know it for the geometric generic fibre, see the beginning of the proof of (7.12) or (48). Now for s_0 it follows since the cohomology and the maps between it are locally constant. Note also that ϕ_1 is surjective; this follows from the commutative diagram (55) and the fact that ϕ_{s_0} is surjective. For the *injectivity* of ${}^t\phi_{s_0}$ we remark that since $A^2(X_{s_0})$ is weakly-representable, it follows that multiplication by q has a kernel consisting of q^{2m} elements if m is the dimension of the corresponding abelian variety. By 8.15 we have also $\phi_{s_0} \cdot {}^t\phi_{s_0} = -q$. Now using the commutativity of diagram (56) and counting the number of elements in the kernels we see that $\phi_1 \cdot {}^t\phi s_0$ is injective. Hence both ${}^t\phi_{s_0}$ and ϕ_1 are isomorphisms.

8.20. Proof of 8.17ii: In order to prove that ${}^t\phi_{s_0}$ is regular we consider a couple (W, Z) with $Z \in CH^2(W \times X_{s_0})$. We have to show that ${}^t\phi_{s_0} \cdot Z_*$ is a morphism. Now take the composite ${}^tP(X_{s_0}) \cdot Z$ in the sense of correspondences on $W \times X_{s_0} \times F(X_{s_0})$, then ${}^tP(X_{s_0}) \cdot Z \in CH^1(W \times F(X_{s_0}))$ and corresponds with the map ${}^t\phi_{s_0} \cdot Z_* : W \to J(F(X_{s_0}))$ and in view of the properties of the Jacobians of curves this is a morphism. Since its image clearly is in $\Pr(\sigma_{s_0})$, by 8.15, this completes the proof.

8.21. Proof of 8.17iii: Let (B, ρ) be an abelian variety B and a regular map $\rho : A^2(X_{s_0}) \to B$. It is clear that there is a *settheoretical*-homomorphism $\bar{\rho}$ as indicated:

$$\begin{tikzcd} A^2(X_{s_0}) \arrow[rr, "{}^t\phi_{s_0}"] \arrow[dr, "\rho"'] & & \Pr(\sigma_{s_0}) \arrow[dl, "\bar{\rho}"] \\ & B & \end{tikzcd}$$

To prove: If $\xi \in \Pr(\sigma_{s_0})$ is a generic point and if $\eta = \bar{\rho}(\xi) = \rho(\xi)$ (identifying the sets), *then* $k(\eta) \subset k(\xi)$.

Now consider the following commutative diagram (using 8.15)

$$\begin{tikzcd}
\xi_1 \in A^2(X_{s_0}) \arrow[rr, "\rho"] \arrow[d, "{}^t\phi_{s_0}"', "\wr"] & & B \arrow[dd, ".(-q)"] \\
\xi_1 \in \Pr(\sigma_{s_0}) \arrow[r, hook] \arrow[urr, "\bar{\rho}"] & J(F(X_{s_0})) \arrow[d, "\phi_{s_0}"'] \arrow[dr, "\mu"] & \\
& \xi \in A^2(X_{s_0}) \arrow[r, "\rho"'] \arrow[d, "{}^t\phi_{s_0}"', "\wr"] & B \ni \eta \\
& \xi \in \Pr(\sigma_{s_0}) \arrow[ur, "\bar{\rho}"'] &
\end{tikzcd}$$

Define $\mu = \rho \cdot \phi_{s_0}$, then $\mu : A^1(F(X_{s_0})) \to B$ is regular. For: if (W, Z) is a couple with $Z \in CH^1(W \times F(X_{s_0}))$ then $\mu \cdot Z_* = \rho \cdot \phi \cdot Z_*$ gives a morphism, because $\phi \cdot Z_*$ comes from the correspondence $P(X_{s_0}) \cdot Z \in CH^2(W \times X_{s_0})$. By the universal property of the Jacobians of curves we get that μ is a *morphism*. Now take $\xi_1 \in \Pr(\sigma_{s_0})$ such that $q \cdot \xi_1 = \xi$. Then (using 8.13) we get $\mu(\xi_1) = \eta$, hence $k(\eta) \subset k(\xi_1)$. Hence (since $\mathrm{char}(k) \dagger q$):

$$k(\eta) \subset \bigcap_{q \cdot \xi_1 = \xi} k(\xi_1) = k(\xi).$$

8.22. The polarization:

Let $T_l(\Pr(\sigma_{s_0}))$ be the Tate group of the generalized Prym. We have isomorphisms

$${}^t\phi_{s_0} : H^3(X_{s_0}, \mathbf{Z}/l^\nu) \longrightarrow \mathrm{Im}(1 - \sigma_{s_0}) \subset H^1(F(X_{s_0}), \mathbf{Z}/l^\nu),$$

because we have this in the geometric generic fibre (see 7.12 beginning and (48)); hence using local constantness of cohomology and maps of cohomology we get that also in s_0. This image corresponds clearly with the $\Pr(\sigma_{s_0})$-part of the Jacobian and hence we get

$$T_l(\Pr(\sigma_{s_0})) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \xrightarrow{\sim} H^3(X_{s_0}, \mathbf{Q}_l). \tag{57}$$

PROPOSITION 8.23: *The polarization* Ξ_{s_0} *on* $\Pr(\sigma_{s_0})$ *corresponds, using* (57), *with the cupproduct on* $H^3(X_{s_0}, \mathbf{Q}_l)$; *i.e., if* $E^{\Xi}(-,-)$

denotes the Riemann form on $T_l(\mathrm{Pr}(\sigma_{s_0}))$ *belonging to* Ξ_{s_0}, *then we have*

$$E^{\Xi}(\alpha, \beta) = \langle \alpha, \beta \rangle_{X_{s_0}} \qquad \text{for } \alpha, \beta \in H^3(X_{s_0}, \mathbf{Q}_l).$$

PROOF: Write X instead of X_{s_0}, etc. Let $\alpha, \beta \in H^3(X, \mathbf{Q}_l)$; to make things more explicite write $\alpha' = {}^t\phi(\alpha)$, $\beta' = {}^t\phi(B)$ for the corresponding elements (by (57)) in $T_l(\mathrm{Pr}(\sigma)) \otimes \mathbf{Q}_l$; finally let $\bar{\alpha} \in H^1(F(X), \mathbf{Q}_l)$ be such that $\alpha' = (1-\sigma)(\bar{\alpha})$. Consider the following commutative diagram

(56′)

$$\begin{tikzcd}
 & & H^3(X, Q_l) \arrow[dl, "{}^t\phi"'] & \\
H^1(F(X), Q_l) \arrow[r, "1-\sigma"] \arrow[d, "\phi"] & \mathrm{Im}(1-\sigma) \arrow[rr, hook] & & H^1(F(X), Q_l) \arrow[d, "\phi"] \\
H^3(X, Q_l) \arrow[rrr, "\cdot q"'] & & & H^3(X, Q_l)
\end{tikzcd}$$

Then we have:

$$q \cdot E^{\Xi}(\alpha', \beta') = E^{\theta}(a', \beta') = \langle \alpha', \beta' \rangle_{F(X)} = \langle (1-\sigma)\bar{\alpha}, \beta' \rangle_{F(X)}.$$

Using the local constantness of cohomology we can apply now the lemma of Tjurin (see 6.15) to the special fibres $X = X_{s_0}$; this gives $\langle (1-\sigma)\bar{\alpha}, \beta' \rangle_{F(X)} = \langle \phi(\bar{\alpha}), \phi(\beta')_X = \langle \phi(\bar{\alpha}), \phi \cdot {}^t\phi(\beta) \rangle_X = \langle \phi(\bar{a}), -qb \rangle_X = -\langle q\phi(\bar{\alpha}), \beta \rangle_X = -\langle \phi(1-\sigma)\bar{\alpha}, \beta \rangle_X = -\langle \phi \cdot {}^t\phi(\alpha), \beta \rangle_X = q\langle \alpha, \beta \rangle_X$. Hence $E^{\Xi}(\alpha', \beta') = \langle \alpha, \beta \rangle_X$, which proves the proposition.

8.24. Summarizing we have obtained:

THEOREM 8.24: *Let the assumptions be as in* 8.1 *and* 8.2. *For* $s_0 \in S$ *introduce the generalized Prym variety* $(\mathrm{Pr}(\sigma_{s_0}), \Xi_{s_0})$ *as polarized abelian variety* (*see* 8.6, 8.7 *and* 8.9). *After shrinking, if necessary, to a non-empty open set* U *of* S *we have for* $s_0 \in U$ *that the transpose of the Abel-Jacobi map* (*see* 8.13)

$${}^t\phi_{s_0} : A^2(X_{s_0}) \longrightarrow \mathrm{Pr}(\sigma_{s_0})$$

is a regular, universal isomorphism (in the sense of 8.18) *for the group of cycle-classes* $A^2(X_{s_0})$. *Moreover, by this isomorphism the principal polarization* Ξ_{s_0} *of* $\mathrm{Pr}(\sigma_{s_0})$ *corresponds (in the sense of* 8.23) *with the cupproduct on* $H^3(X_{s_0}, \mathbf{Q}_l)$.

REFERENCES

[1] S.S. ABHYANKAR: *Resolution of singularities of embedded algebraic surfaces.* Academic Press, New York 1966.

[2] W. BARTH, and A. VAN DE VEN: Lines on hypersurfaces, *Archiv der Math. 31* (1978) 96–104.

[3] A. BEAUVILLE: Variétés de Prym et jacobiennes intermédiaires. *Ann. Scient. de l'Ecole Norm. Sup. 10* (1977) 304–392.

[4] S. BLOCH: An example in the theory of algebraic cycles, in Algebraic K-theory, Evanston 1976, *Lect. Notes in Math. No. 551*, 1–30, Springer Verlag, Heidelberg 1976.

[5] S. BLOCH: Torsion algebraic cycles and a theorem of Roitman, (to appear in Comp. Math.).

[6] W.-L. CHOW: On the quotient variety of an abelian variety. *Proc. Nat. Acad. Sci. U.S.A. 38* (1952), 1039–1044.

[7] C.H. CLEMENS and P.A. GRIFFITHS: The intermediate jacobian of the cubic threefold, *Annals of Math. 95* (1972), 281–356.

[8] F. ENRIQUES: Sopra una involuzione non razionale dello spazio, *Rend. Acc. Lincei (5a) 21* (1912), 81–83.

[9] P.A. GRIFFITHS: Periods of integrals on algebraic manifolds I, II, *Amer. J. of Math. 90* (1968), 568–626, 805–865.

[FGA] A. GROTHENDIECK: Fondements de la Géométrie Algébrique, *Séminaire Bourbaki 1957–62*, Secrét. Math., Paris 1962.

[SGA 1] A. GROTHENDIECK: Séminaire de Géométrie Algébrique 1, *Lecture Notes in Math. 224*, Springer-Verlag, Heidelberg 1971.

[SGA 4] A. GROTHENDIECK, M. ARTIN and J.L. VERDIER: Séminaire de Géométrie Algébrique 4, *Lecture Notes in Math. 305*, Springer-Verlag, Heidelberg 1973.

[10] S.L. KLEIMAN: Algebraic cycles and the Weil conjectures, in *Dix exposés sur la cohomologie des schémas, 359–386*, North-Holland, Amsterdam 1968.

[11] S. LANG: *Abelian varieties*, Interscience Publ. New York 1959.

[12] D. LIEBERMAN: Intermediate jacobians, in *Algebraic Geometry Oslo 1970*, 125–141, Wolters-Noordhoff, Groningen 1972.

[13] D. MUMFORD: *Abelian varieties*, Oxford Univ. Press, Oxford 1970.

[14] D. MUMFORD: Prym varieties I, in *Contributions to Analysis*, a collection of papers dedicated to L. Bers, 325–350, Academic Press, New York 1974.

[15] D. MUMFORD: Algebraic Geometry I, Complex projective varieties, *Grundlehren 221*, Springer-Verlag, Heidelberg 1976.

[16] J.P. MURRE: Algebraic equivalence modulo rational equivalence on a cubic threefold, *Comp. Math. 25* (1972), 161–206.

[17] J.P. MURRE: Reduction of the proof of the non-rationality of a non-singular cubic threefold to a result of Mumford, *Comp. Math. 27* (1973), 63–82.

[18] J.P. MURRE: Some results on cubic threefolds, in Classification of algebraic varieties and compact complex manifolds, *Lect. Notes in Math. No 412*, 140–164, Springer-Verlag, Heidelberg 1974.

[19] J.P. MURRE: On a uniqueness theorem for certain kinds of birational transformations, *Indag. Math. 21* (1959), 129–134.

[20] A.A. ROITMAN: Private correspondence.

[21] L. ROTH: Algebraic threefolds, *Ergeb. der Math., Neue Folge, Heft 6*, Springer-Verlag, Heidelberg 1955.
[22] P. SAMUEL: Rational equivalence of algebraic cycles, *Amer. J. of Math. 78* (1956), 383–400.
[23] B.R. TENNISON: On the quartic threefold, *Proc. London Math. Soc. (3) 29* (1974), 714–734.
[24] A.N. TJURIN, Five lectures on three dimensional varieties, *Russian Math. Surveys 27* (1972), 1–53.
[25] (added) V.A. ISKOVSKIH: Fano 3-fold I (resp. II), *Izvestija A.N. SSSR, Ser. Matem. 41* (1977) 516–562 (resp. *ibid. 42* (1978) 506–549) = *Math. of the USSR, Izvestije 11* (1977) 485–527 (resp. forthcoming).

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