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CHARACTERS AND INNER FORMS OF A QUASI-SPLIT GROUP OVER ℝ

D. Shelstad*

1. Introduction

The principle of functoriality in the L-group suggests the existence of character identities among certain groups which share common Cartan subgroups. Concrete examples of such identities are to be found in [3], [11], and [12]. Here we consider the simplest case for real (reductive, linear, algebraic) groups, that of two groups with same L-group (associate group in [17]) or, equivalently, two groups which are inner forms of the same quasi-split group. Further, we restrict our attention to the characters of tempered (irreducible, admissible) representations. A precise statement of our result appears at the end of Section 3 and again in Theorem 6.3.

We use the following approach. Let G and G' be inner forms of the same quasi-split group (cf. Section 2). We may as well assume that G' itself is quasi-split. Then a result in [18] establishes a correspondence between the regular points in G and those in G'. We study this correspondence in Section 2. Next we recall some properties of the set $\Phi(G)$ of parameters for the L-equivalence classes of irreducible, admissible representations of G (cf. [18]) and attach to each tempered φ in $\Phi(G)$ a tempered distribution χ_φ. This distribution, which is just a sum of discrete series or unitary principal series characters, can be regarded as a function on the regular elements of G. Since $\Phi(G)$ is embedded in $\Phi(G')$ we may formulate some character identities between G and G'. Our proof begins in Section 4. We introduce certain averaged ("stable") orbital integrals. Their characterization (Theorem 4.7), which is a consequence of theorems of Harish-Chandra, enables us to transfer

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stable orbital integrals from G to G'. We therefore obtain a correspondence between Schwartz functions on G and Schwartz functions on G'. In the remaining sections we define the notion of a stable tempered distribution and see that there is a map from the space of stable tempered distributions on G' to that for G, dual to the correspondence of Schwartz functions. If φ' is a tempered parameter for G' then $\chi_{\varphi'}$ is stable (Lemma 5.2). By investigating the image of $\chi_{\varphi'}$ under our map we obtain the proposed character identities. Again the results follow from theorems of Harish-Chandra.

In [23] we obtained Theorem 6.3 without recourse to orbital integrals. However [20] suggests that our character identities be exhibited as "dual" to a transfer of orbital integrals; this necessitates the present approach. We will also have other uses for our characterization of stable orbital integrals. Note that if G is $GL_n(R)$ or a group with just one conjugacy class of Cartan subgroups then the notion of "stable" can be omitted in Theorem 4.7. Thus for such groups we have a characterization of the orbital integrals (with respect to regular semisimple elements) of Schwartz functions.

2. Inner forms and Cartan subgroups

We recall some standard facts. Suppose that $\boldsymbol{G}$ is a connected reductive linear algebraic group defined over $\mathbb{R}$. Then $G = \boldsymbol{G}(\mathbb{R})$ is a reductive Lie group satisfying the conditions of [6]. A Cartan subgroup T of G, in the sense of Lie groups, is the group of $\mathbb{R}$-rational points on some maximal torus $\boldsymbol{T}$ in $\boldsymbol{G}$, defined over $\mathbb{R}$. By a root of $\boldsymbol{T}$ (or T) we mean a root for the Lie algebra $\mathfrak{k}$ of $\boldsymbol{T}$ in $\mathfrak{g}$, the Lie algebra of $\boldsymbol{G}$; we follow the usual definitions of real, imaginary (compact or noncompact) and complex roots (cf. [26]). An isomorphism $\psi\colon \boldsymbol{G} \to \boldsymbol{G}'$ of reductive groups for which $\psi\colon \boldsymbol{T} \to \psi(\boldsymbol{T})$ is defined over $\mathbb{R}$ maps real (respectively, imaginary, complex) roots of $\boldsymbol{T}$ to real (respectively, imaginary, complex) roots of $\psi(\boldsymbol{T})$. We denote by $\Omega(\boldsymbol{G}, \boldsymbol{T})$ the Weyl group for $(\mathfrak{g}, \mathfrak{k})$; we say that $w \in \boldsymbol{G}$ realizes $\omega \in \Omega(\boldsymbol{G}, \boldsymbol{T})$ if $\mathrm{Ad}\, w/\mathfrak{k}$ coincides with ω and denote by $\Omega(G, T)$ the set of those elements of $\Omega(\boldsymbol{G}, \boldsymbol{T})$ which can be realized in G.

A parabolic subgroup P of G is the group of $\mathbb{R}$-rational points on a parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}$ defined over $\mathbb{R}$; a Levi decomposition $\boldsymbol{P} = \boldsymbol{MN}$ for $\boldsymbol{P}$ with $\boldsymbol{M}$ defined over $\mathbb{R}$, yields a Levi decomposition $P = MN$ for P. We call G quasi-split if $\boldsymbol{G}$ is quasi-split over $\mathbb{R}$, that is, if $\boldsymbol{G}$ contains a Borel subgroup defined over $\mathbb{R}$; this is equivalent to requiring that the Levi components of a minimal parabolic subgroup of G be abelian (and hence Cartan subgroups). Suppose that $\boldsymbol{G}'$ is quasi-split over ℝ. Then $\boldsymbol{G}$ is an inner form of $\boldsymbol{G}'$ (or, G an inner form of G') if there exists an isomorphism $\psi: \boldsymbol{G} \to \boldsymbol{G}'$ for which $\bar{\psi}\psi^{-1}$ is inner (the bar denotes the action of complex conjugation). If $\boldsymbol{G}''$ is also quasi-split over ℝ and $\eta: \boldsymbol{G} \to \boldsymbol{G}''$ is such that $\bar{\eta}\eta^{-1}$ is inner then $\eta = \theta\psi\iota$ where ι is inner and θ is defined over ℝ. Every group $\boldsymbol{G}$ (connected, reductive and defined over ℝ) is an inner form of some quasi-split group (cf. [21]).

We will assume, from now on, that $\boldsymbol{G}'$ is quasi-split and $\boldsymbol{G}$ an inner form of $\boldsymbol{G}'$. We fix an isomorphism $\psi: \boldsymbol{G} \to \boldsymbol{G}'$ for which $\bar{\psi}\psi^{-1}$ is inner.

A lemma in [18] shows that we may use ψ to embed each Cartan subgroup of G in G'. More precisely, the lemma asserts that if $\boldsymbol{T}$ is a maximal torus in $\boldsymbol{G}$ defined over ℝ then there exists $x \in \boldsymbol{G}'$ (depending on $\boldsymbol{T}$) such that the restriction of ad $x \circ \psi$ to $\boldsymbol{T}$, which we denote by ψ_x, is defined over ℝ; $\psi_x(T)$ is a Cartan subgroup of G'. We now study these embeddings ψ_x.

Let $t(G)$ be the set of (G-)conjugacy classes of Cartan subgroups of G; we denote by $\langle T \rangle$ the class of the Cartan subgroup T. We will see that $T \to \psi_x(T)$ induces an embedding $\psi^t: t(G) \to t(G')$, independent of the choices for x. To further describe ψ^t, we recall a natural partial ordering on $t(G)$: if $\boldsymbol{S}(T)$ denotes the maximal ℝ-split torus in a maximal torus $\boldsymbol{T}$ then $\langle T_1 \rangle \leq \langle T_2 \rangle$ if and only if $\boldsymbol{S}(T_1') \subseteq \boldsymbol{S}(T_2')$ for some $T_1' \in \langle T_1 \rangle$, $T_2' \in \langle T_2 \rangle$. Clearly ψ^t is order-preserving; we will show that ψ^t maps $t(G)$ to an "initial segment" of $t(G')$ (cf. Lemma 2.8).

We begin with a definition from [19]: if T is a Cartan subgroup of G then

$$\mathcal{A}(T) = \{g \in \boldsymbol{G};\ \text{ad}\ g/\boldsymbol{T} \text{ is defined over } \mathbb{R}\}.$$

This is easily seen to be the same as $\{g \in \boldsymbol{G} : gTg^{-1} \subset G\}$. As above, $\boldsymbol{S}(T)$ will denote the maximal ℝ-split torus in $\boldsymbol{T}$.

THEOREM 2.1: *Let $\boldsymbol{M}$ be the centralizer in $\boldsymbol{G}$ of $\boldsymbol{S}(T)$. Then*

$$\mathcal{A}(\boldsymbol{T}) = G \cdot \text{Norm}(\boldsymbol{M}, \boldsymbol{T})$$

where Norm($\boldsymbol{M}$, $\boldsymbol{T}$) *denotes the normalizer of $\boldsymbol{T}$ in $\boldsymbol{M}$.*

PROOF: Suppose that $x \in \mathcal{A}(T)$. Then $\bar{x}t\bar{x}^{-1} = xtx^{-1}$, $t \in \boldsymbol{T}$ (bar denoting complex conjugation). Therefore $x^{-1}\bar{x}$ centralizes $\boldsymbol{T}$ and so belongs to $\boldsymbol{T}$. Let $\boldsymbol{P}$ be a parabolic subgroup of $\boldsymbol{G}$, defined over $\mathbb{R}$ and containing $\boldsymbol{S}(T)$ as a maximal $\mathbb{R}$-split torus in its radical (cf. [2]). Then $\overline{x\boldsymbol{P}x^{-1}} = x(x^{-1}\bar{x}\boldsymbol{P}\bar{x}^{-1}x)x^{-1} = x\boldsymbol{P}x^{-1}$ since $\boldsymbol{T}$ is contained in $\boldsymbol{P}$, and so $x\boldsymbol{P}x^{-1}$ is defined over $\mathbb{R}$. From [2] it follows that $\boldsymbol{P}$ and $x\boldsymbol{P}x^{-1}$ are conjugate under G. Let $y \in Gx$ be such that y normalizes $\boldsymbol{P}$. Then $y \in \boldsymbol{P}$. But $\boldsymbol{M}$ is a Levi subgroup of $\boldsymbol{P}$ defined over $\mathbb{R}$ and $\overline{y\boldsymbol{M}y^{-1}} = y\boldsymbol{M}y^{-1}$ since $\boldsymbol{T} \subseteq \boldsymbol{M}$. Therefore $y\boldsymbol{M}y^{-1}$ is conjugate to $\boldsymbol{M}$ under the group N of $\mathbb{R}$-rational points on the unipotent radical of $\boldsymbol{P}$ (cf. [2]). We may then choose $z \in Ny \subseteq Gx \cap \boldsymbol{P}$ such that z normalizes $\boldsymbol{M}$; z must lie in $\boldsymbol{M}$ and ad $z/\boldsymbol{T}$ is defined over $\mathbb{R}$. In particular, $z\boldsymbol{T}z^{-1}$ is defined over $\mathbb{R}$. Let $\boldsymbol{M}_1$ be the derived group of $\boldsymbol{M}$ and $\boldsymbol{Z}$ be the connected component of the identity in the center of $\boldsymbol{M}$. Then $\boldsymbol{M} = \boldsymbol{Z}\boldsymbol{M}_1$; $\boldsymbol{T} = \boldsymbol{Z}(\boldsymbol{T} \cap \boldsymbol{M}_1)$ and $z\boldsymbol{T}z^{-1} = \boldsymbol{Z}z(\boldsymbol{T} \cap \boldsymbol{M}_1)z^{-1}$; $\boldsymbol{T} \cap \boldsymbol{M}_1$ and $z(\boldsymbol{T} \cap \boldsymbol{M}_1)z^{-1}$ are maximal tori in $\boldsymbol{M}_1$, anisotropic over $\mathbb{R}$. Hence $\boldsymbol{T}$ and $z\boldsymbol{T}z^{-1}$ are conjugate under M_1 and so $z \in M_1\,\mathrm{Norm}(\boldsymbol{M}, \boldsymbol{T})$. We conclude then that $\mathcal{A}(T) \subseteq G\,\mathrm{Norm}(\boldsymbol{M}, \boldsymbol{T})$.

Let $\boldsymbol{T}_1 = \boldsymbol{T} \cap \boldsymbol{M}_1$. Then to complete the proof it is sufficient to show that if $x \in \mathrm{Norm}(\boldsymbol{M}_1, \boldsymbol{T}_1)$ then the restriction of ad x to $\boldsymbol{T}_1$ is defined over $\mathbb{R}$. This is a consequence of the following proposition.

PROPOSITION 2.2: *Suppose that $\boldsymbol{T}$ is a torus defined and anisotropic over $\mathbb{R}$. Then every (rational) automorphism of $\boldsymbol{T}$ is defined over $\mathbb{R}$.*

PROOF: Suppose that φ is a rational automorphism of $\boldsymbol{T}$. There is a unique automorphism $\varphi^\vee$ of the group L of rational characters on $\boldsymbol{T}$ which satisfies $\langle \varphi^\vee\lambda, t\rangle = \langle \lambda, \varphi^{-1}t\rangle$, $\lambda \in L$, $t \in \boldsymbol{T}$. On the other hand, $\bar{\lambda} = -\lambda$, $\lambda \in L$. This implies that $\overline{\varphi^\vee} = \varphi^\vee$ and so $\bar{\varphi} = \varphi$, as desired.

COROLLARY 2.3: *If $g \in \mathcal{A}(T)$ then $g\boldsymbol{T}g^{-1}$ is G-conjugate to $\boldsymbol{T}$.*

COROLLARY 2.4: *If $\boldsymbol{T}$ contains a maximal $\mathbb{R}$-split torus in $\boldsymbol{G}$ then the action of an element in $\mathcal{A}(T)$ on $\boldsymbol{T}$ can be realized in G.*

As in [19] we set $\mathcal{D}(T) = G\backslash\mathcal{A}(T)/\boldsymbol{T}$.

COROLLARY 2.5: $\mathcal{D}(T) = \mathrm{Norm}(M, T)\backslash\mathrm{Norm}(\boldsymbol{M}, \boldsymbol{T})/\boldsymbol{T}$.

In particular, $\mathcal{D}(T)$ is finite since $\mathrm{Norm}(\boldsymbol{M}, \boldsymbol{T})/\boldsymbol{T}$ is isomorphic to the Weyl group of $(\mathfrak{m}, \mathfrak{t})$.

Returning to the map ψ_x: $\boldsymbol{T} \hookrightarrow \boldsymbol{G}'$, let $\boldsymbol{T}' = \psi_x(\boldsymbol{T})$. Note that if $y \in \boldsymbol{G}'$ then $\psi_y/\boldsymbol{T}$ is defined over $\mathbb{R}$ if and only if $yx^{-1} \in \mathcal{A}(T')$. Corollary 2.3, applied twice, then shows that $\psi^t: \langle \boldsymbol{T}\rangle \to \langle \boldsymbol{T}\rangle$ is a well-defined embedding of $t(G)$ into $t(G')$, independent of the choices for x. Note that if we replace ψ by η: $\boldsymbol{G} \to \boldsymbol{G}''$ then writing $\eta = \theta\psi\iota$ as before, we obtain $\eta^t = \theta^t\psi^t$; the possibilities for θ^t are easily classified.

To describe the order properties of ψ^t we will characterize the ordering on $t(G)$ as in [10]. First, and partly for later use, we recall the definition of compact and noncompact roots. Let α be an imaginary root for $\boldsymbol{T}$ (that is, a root in $\boldsymbol{M}$) and H_α be the coroot attached to α. If X_α is a root vector for α we fix a root vector $X_{-\alpha}$ for $-\alpha$ by requiring that $\langle X_\alpha, X_{-\alpha}\rangle = 2/\langle\alpha, \alpha\rangle$ where $\langle\ ,\ \rangle$ is the Killing form. Then $[X_\alpha, X_{-\alpha}] = H_\alpha$ and $\mathbb{C}X_\alpha + \mathbb{C}X_{-\alpha} + \mathbb{C}H_\alpha$ is a simple complex Lie algebra invariant under complex conjugation; in fact, $\bar{H}_\alpha = -H_\alpha$ and $\bar{X}_\alpha = cX_{-\alpha}$ for some $c \in \mathbb{C}$. Either there is an X_α for which $\bar{X}_\alpha = -X_{-\alpha}$ or there is one for which $\bar{X}_\alpha = X_{-\alpha}$. In the former case,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \to H_\alpha, \quad \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \to X_\alpha, \quad \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix} \to X_{-\alpha}$$

lifts to a homomorphism $\mathrm{SU}(2) \to \boldsymbol{G}$ defined over $\mathbb{R}$ and α is compact. In the latter

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \to H_\alpha \quad \frac{1}{2}\begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix} \to X_\alpha \quad \frac{1}{2}\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \to X_{-\alpha}$$

lifts to a homomorphism $\mathrm{SL}_2 \to \boldsymbol{G}$ over $\mathbb{R}$ and α is noncompact.

We will find it convenient to generalize the usual notion of Cayley transform. Suppose that T is a Cartan subgroup of G and α a noncompact imaginary root of T. Then we call $s \in \boldsymbol{G}$ a Cayley transform with respect to α if $\bar{s}^{-1}s$ realizes the Weyl reflection with respect to α. The proof of the following proposition is immediate.

PROPOSITION 2.7: (1) $\boldsymbol{T}_s = s\boldsymbol{T}s^{-1}$ *is defined over* $\mathbb{R}$ *and the root* $s\alpha$ *is real*; (2) *the restriction of* ad s *to* $\boldsymbol{S}(T)$ *is defined over* $\mathbb{R}$; (3) *if* s' *is also a Cayley transform with respect to* α *then* $s's \in \mathscr{A}(T_s)$.

If s is the image of

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

under a homomorphism $\mathrm{SL}_2 \to \boldsymbol{G}$ of the type described above, then we will call s a standard Cayley transform. From this example, Proposition 2.7, and Corollary 2.3 we conclude that for any s, $\boldsymbol{S}(T)$ is G-conjugate to a subtorus of $S(T_s)$ of codimension 1. A straightforward argument then shows that $\langle T\rangle \leq \langle U\rangle$ if and only if there is a sequence $s_1, \ldots, s_n$ of Cayley transforms such that $U = (\ldots((T_{s_1})_{s_2}\ldots)_{s_n}$.

LEMMA 2.8: (1) *if* $\langle T\rangle \leq \langle U\rangle$ *then* $\psi^t(\langle T\rangle) \leq \psi^t(\langle U\rangle)$; (2) *if* $\langle U'\rangle$ *is in the image of* ψ^t *and* $\langle T'\rangle \leq \langle U'\rangle$ *then* $\langle T'\rangle$ *is in the image of* ψ^t; (3) *the image under* ψ^t *of the class of fundamental Cartan subgroups in G is the class of fundamental Cartan subgroups in G'.*

PROOF: The assertion of the first part is immediate. For the second part it is sufficient to show that if T' is a Cartan subgroup of G' and s' a Cayley transform with respect to some noncompact root α' of T', then $\langle T'\rangle$ belongs to the image of ψ^t if $\langle T'_{s'}\rangle$ does. Suppose then that there exists $x \in \boldsymbol{G}$ and $\boldsymbol{T}^*$ such that ψ_x: $\boldsymbol{T}^* \to \boldsymbol{T}'_{s'}$ is defined over $\mathbb{R}$. Let $\beta = \psi_x^{-1}(s'\alpha')$. Then β is a real root of $\boldsymbol{T}^*$. There is a Cartan subgroup T of G, a noncompact root α of T and a Cayley transform s with respect to α such that s: $\boldsymbol{T} \to \boldsymbol{T}^*$ and $s\alpha = \beta$ (cf. [25]). A calculation shows that $(\operatorname{ad} s')^{-1} \circ \psi_x \circ \operatorname{ad} s$: $\boldsymbol{T} \to \boldsymbol{T}'$ is defined over $\mathbb{R}$. This proves the second part.

The final assertion follows immediately from the fact that a Cartan subgroup is fundamental if and only if it has no real roots [26].

COROLLARY 2.9: *G contains a compact Cartan subgroup if and only if G' contains a compact Cartan subgroup.*

Finally, it will be convenient to describe the embeddings ψ_x in the following way. Let G_{reg} be the set of regular elements in G; we denote by T_γ the Cartan subgroup containing an element γ in G_{reg}. We will say that $\gamma' \in G'$ originates from γ in G_{reg} if there exists $x \in \boldsymbol{G}'$ such that $\psi_x(\gamma) = \gamma'$ and ψ_x: $\boldsymbol{T}_\gamma \to \boldsymbol{T}_{\gamma'}$ is defined over $\mathbb{R}$. Then γ' also originates from any element $\gamma^w = w\gamma w^{-1}$, $w \in \mathcal{A}(T_\gamma)$; these are the only elements in G from which γ' originates. Similarly, if γ' originates from γ then so also does $(\gamma')^{w'}$ for any $w' \in \mathcal{A}(T_{\gamma'})$, but these are the only such elements.

3. Characters

The purpose of this section is to recall some formulations and results from [18], and to define some characters. Let $\Pi(G)$ be the set of infinitesimal equivalence classes of irreducible admissible representations of G. According to [18] there is a space $\Phi(G)$ which partitions $\Pi(G)$ into finite subsets Π_φ, $\varphi \in \Phi(G)$. Either all the classes in Π_φ are tempered or none is [18, page 40]; if the former then we call φ tempered. The map ψ: $\boldsymbol{G} \to \boldsymbol{G}'$ induces an embedding $\Phi(G) \hookrightarrow \Phi(G')$ which we denote by $\varphi \to \varphi'$; φ' is tempered if and only if φ is tempered.

We assume from now on that φ is tempered. To describe the classes in a typical Π_φ ("an L-equivalence class") we recall the following from [18]. To φ we may attach a parabolic subgroup P_0, a Levi component M_0 of P_0 and a Cartan subgroup T_0 fundamental in M_0. If $P_0 = G$ is the only possibility we call φ discrete; then Π_φ consists of the (classes of) square-integrable representations attached to an orbit, say $X(\varphi)$, of characters on T_0 under $\Omega(\boldsymbol{G}_0, \boldsymbol{T}_0)$ (we recall this assignment below). In general, φ determines a discrete parameter φ_0 for M_0. Set $\pi_\varphi^\circ = \sigma_1 \oplus \cdots \oplus \sigma_n$, where $\{\sigma_i\}$ is a set of representatives for the classes in Π_{φ_0}. Then Π_φ consists of the classes of the irreducible constituents of $\pi_\varphi = \mathrm{Ind}(\pi_\varphi^\circ \otimes 1_{N_0}; P_0, G_0)$, N_0 denoting the unipotent radical of P_0.

To describe representatives for $\Pi_{\varphi'}$ we may assume that $\psi/\boldsymbol{T}_0$ is defined over ℝ, without changing the map $\varphi \to \varphi'$. Then $\boldsymbol{P}_0' = \psi(\boldsymbol{P}_0)$, $\boldsymbol{M}_0' = \psi(\boldsymbol{M}_0)$ and $\boldsymbol{T}_0' = \psi(\boldsymbol{T}_0)$ are defined over ℝ. Moreover, ψ: $\boldsymbol{M}_0 \to \boldsymbol{M}_0'$ is such that $\bar{\psi}\psi^{-1}$ is inner, and $\boldsymbol{M}_0'$ is quasi-split. We may thus take P_0', M_0' and T_0' for the groups attached to φ' (cf. [18]). If φ is discrete then φ' is also discrete and $\Pi_{\varphi'}$ is the set of classes attached to the orbit $\Lambda_\varphi' = \{\Lambda \circ \psi^{-1} : \Lambda \in X(\varphi)\}$. In general, we may take $(\varphi_0)'$, the image of φ_0 under the map $\Phi(M_0) \to \Phi(M_0')$ induced by $\psi/\boldsymbol{M}_0$, for $(\varphi')_0$, the parameter for M_0' induced by φ'; again, this follows immediately from the construction in [18].

We recall parameters for $\sigma_1, \ldots, \sigma_n$ above; we write P, M, and T in place of P_0, M_0, and T_0. Let $M^\dagger$ be the connected component of the identity in the derived group of M. If Z_M is the center of M then $Z_M M^\dagger$ has finite index in M and $T = Z_M T^\dagger$, where $T^\dagger = T \cap M^\dagger$. Fix $\Lambda \in X(\varphi_0)$ and let λ be the differential of the restriction of Λ to $T^\dagger$. Choose an ordering on the roots of $(\mathfrak{m}, \mathfrak{t})$ with respect to which λ is dominant; ι will denote one half the sum of the positive roots with respect to this ordering. Let $\pi(\lambda, \iota)$ be a square-integrable irreducible admissible representation of $M^\dagger$ attached to the regular functional $\lambda + \iota$ in the manner of [5] and define

$$\pi(\Lambda, \iota) = \mathrm{Ind}(\pi(\lambda, \iota) \otimes \Lambda / Z_M, Z_M M^\dagger, M).$$

Then $\sigma_1, \ldots, \sigma_n$ may be chosen as

$$\{\pi(\omega\Lambda, \omega\iota);\ \omega \in \Omega(M, T) \backslash \Omega(\boldsymbol{M}, \boldsymbol{T})\}.$$

A convenient way of identifying the tempered L-equivalence classes is as follows. If T is a Cartan subgroup of G and Λ a character on T set

$$\langle\Lambda\rangle = \{\Lambda \circ \operatorname{ad} g^{-1}; g \in \mathcal{A}(T)\}.$$

Then there is a one-to-one correspondence between tempered parameters φ and such orbits $\langle\Lambda\rangle$. To recover Π_φ from $\langle\Lambda\rangle$ we fix $\Lambda_0 \in \langle\Lambda\rangle$; if Λ_0 is defined on T_0 let $\boldsymbol{M}_0$ be the centralizer in $\boldsymbol{G}$ of the maximal $\mathbb{R}$-split torus in $\boldsymbol{T}_0$ and $P_0 = M_0N_0$ a parabolic subgroup containing M_0 as Levi component. We then proceed as before, defining $\pi^\circ_\varphi = \bigoplus_\omega \pi(\omega\Lambda_0, \omega\iota)$ and $\pi_\varphi = \operatorname{Ind}(\pi^\circ_\varphi \otimes 1_{N_0})$. The map $\varphi \to \varphi'$ on parameters induces the following map of orbits. If Λ is defined on T pick $x \in \boldsymbol{G}'$ such that $\psi_x : \boldsymbol{T} \to \boldsymbol{G}'$ is defined over $\mathbb{R}$. Then $\langle\Lambda\rangle \to \langle\Lambda'\rangle$ where $\Lambda' = \Lambda \circ \psi_x^{-1}$.

Next, we attach a character χ_φ to the collection Π_φ. For the purposes of this paper it is appropriate to define χ_φ as the character of π_φ (cf. the proofs of Lemma 5.2 and Theorem 6.3); χ_φ is thus a tempered invariant eigendistribution. Before proceeding we observe that χ_φ has an intrinsic definition. Indeed, each $\varphi \in \Pi_\varphi$ has a well-defined character which we denote by $\chi(\pi)$ and:

LEMMA 3.1:
$$\chi_\varphi = \sum_{\pi \in \Pi_\varphi} \chi(\pi).$$

PROOF: The lemma asserts that each π in Π_φ occurs in π_φ with multiplicity one. But $\pi_\varphi = \bigoplus_i \operatorname{Ind}(\sigma_i \otimes 1_N)$, the $\operatorname{Ind}(\sigma_i \otimes 1_N)$ being unitary principal series representations. According to the theorem of [13] the irreducible constituents of $\operatorname{Ind}(\sigma_i \otimes 1_N)$ occur with multiplicity one (in the theorem quoted, G is a connected, semisimple matrix group; the statement remains valid under our assumptions (cf. [24]). By [18, page 65] two representations $\operatorname{Ind}(\sigma_i \otimes 1_N)$ and $\operatorname{Ind}(\sigma_j \otimes 1_N)$ are either infinitesimally equivalent or disjoint; they are equivalent exactly when there is $g \in G$ normalizing M so that σ_i/M is equivalent to $\sigma_j \circ \operatorname{ad} g/M$. Hence we have only to show the following lemma.

LEMMA 3.2: *If σ and σ' are L-equivalent square-integrable irreducible admissible representations of M then* $\operatorname{Ind}(\sigma \otimes 1_N)$ *is infinitesimally equivalent to* $\operatorname{Ind}(\sigma' \otimes 1_N)$ *if and only if σ is infinitesimally equivalent to σ'.*

This result is a special case of a theorem announced in [14] (cf. [15]), at least when G is semisimple and simply-connected. We give a simple independent proof for our case and arbitrary G.

PROOF: Assume that $\mathrm{Ind}(\sigma \otimes 1_N)$ and $\mathrm{Ind}(\sigma' \otimes 1_N)$ are infinitesimally equivalent. Choose $g \in G$ normalizing M and such that $\sigma \circ \mathrm{ad}\, g$ is infinitesimally equivalent to σ'. We may assume that g normalizes T. We may take $\sigma = \pi(\Lambda, \iota)$ and $\sigma' = \pi(\omega\Lambda, \omega\iota)$, for some $\omega \in \Omega(\boldsymbol{M}, \boldsymbol{T})$. But then $\sigma \circ \mathrm{ad}\, g$ is (infinitesimally equivalent to) $\pi(g\Lambda, g\iota)$. Hence there is $\omega_0 \in \Omega(M, T)$ such that $g\Lambda = \omega_0\omega\Lambda$ and $g\iota = \omega_0\omega\iota$. This implies that

$$g(\lambda + \iota) = \omega_0\omega(\lambda + \iota) \tag{1}$$

where, as before, λ is the differential of $\Lambda/T^{\dagger}$. Suppose that ω, ω_0 are represented by $w \in \boldsymbol{M}$ and $w_0 \in M$, respectively. If we show now that (1) implies that the action of $g^{-1}w_0w$ on $\boldsymbol{T}$ can be realized in G then it will follow that $\omega \in \Omega(M, T)$, which is sufficient to prove the lemma.

Define $H_0 \in \mathfrak{t}^{\dagger}$, the Lie algebra of $T^{\dagger}$, by $(\lambda + \iota)(H_0) = i\langle H, H_0\rangle$, $H \in \mathfrak{t}^{\dagger}$. Then, by (1), $g^{-1}w_0w$ fixes H_0; also H_0 is regular with respect to $\Omega(\boldsymbol{M}, \boldsymbol{T})$. Let $\boldsymbol{T}_0$ be the smallest algebraic subgroup of $\boldsymbol{G}$ whose Lie algebra contains H_0. Then $\boldsymbol{T}_0$ is a torus in $\boldsymbol{T}$, defined over ℝ; clearly $g^{-1}w_0w$ centralizes $\boldsymbol{T}_0$. Let $\boldsymbol{C}$ denote the centralizer of $\boldsymbol{T}_0$ in $\boldsymbol{G}$; $\boldsymbol{C}$ is connected, reductive, defined over ℝ and of same rank as $\boldsymbol{G}$. Note that $g^{-1}w_ow \in \mathcal{A}(T) \cap \boldsymbol{C}$. Hence, by Corollary 2.4, it is enough to show that $\boldsymbol{S}(T)$, the maximal ℝ-split torus in $\boldsymbol{T}$, is a maximal ℝ-split torus in $\boldsymbol{C}$.

Suppose then that $\boldsymbol{S}'$ is a maximal ℝ-split torus in $\boldsymbol{C}$ containing $\boldsymbol{S} = \boldsymbol{S}(T)$. Extend $\boldsymbol{S}'$ to a maximal torus $\boldsymbol{T}'$ in $\boldsymbol{C}$ defined over ℝ. Since H_0 is regular with respect to $\Omega(\boldsymbol{M}, \boldsymbol{T})$ we have $\boldsymbol{T} = (\mathrm{Cent}(\boldsymbol{M}, \exp H_0))^0 \supset \mathrm{Cent}(\boldsymbol{M}, \boldsymbol{T}_0) \supset \boldsymbol{T}$, Cent(—, —) denoting "the centralizer in — of —", so that $\boldsymbol{T} = \mathrm{Cent}(\boldsymbol{M}, \boldsymbol{T}_0)$. But $\boldsymbol{T}' \supset \boldsymbol{T}_0$ so $\mathrm{Cent}(\boldsymbol{M}, \boldsymbol{T}') \subset \mathrm{Cent}(\boldsymbol{M}, \boldsymbol{T}_0) = \boldsymbol{T}$. On the other hand, $\boldsymbol{S}' \supset \boldsymbol{S}$ so that $\boldsymbol{S}' \subset \mathrm{Cent}(\boldsymbol{G}, \boldsymbol{S}) = \boldsymbol{M}$ and thus $\boldsymbol{S}' \subset \mathrm{Cent}(\boldsymbol{M}, \boldsymbol{T}') \subset \boldsymbol{T}$. Hence $\boldsymbol{S}' = \boldsymbol{S}$ and the lemma is proved.

Now identify χ_φ and $\chi_{\varphi'}$ as functions on G_{reg} and G'_{reg}, respectively (cf. [6]). Then our aim is to prove the following character identity:

$$\chi_{\varphi'}(\gamma') = (-1)^{q_{G'} - q_G}\chi_\varphi(\gamma).$$

Here $\gamma' \in G'_{\mathrm{reg}}$ originates from $\gamma \in G_{\mathrm{reg}}$ and $2q_G$ is the dimension of the symmetric space attached to the simply-connected covering of the derived group of G. Note that $q_{G'} - q_G$ is an integer (cf. [26, volume 2, page 225]).

4. Stable orbital integrals

Let γ be a regular element in G and T_γ be the Cartan subgroup containing γ. If dg and $d_\gamma t$ are given Haar measures on G and T_γ respectively we denote by $d_\gamma \bar{g}$ the corresponding quotient measure on G/T_γ. For any Schwartz function f on G the orbital integral

$$\Phi_f(\gamma, d_\gamma t, dg) = \int_{G/T_\gamma} f(g\gamma g^{-1}) d_\gamma \bar{g}$$

is absolutely convergent [6]. We will assume that if γ and γ' lie in the same Cartan subgroup then $d_\gamma t = d_{\gamma'} t$ and write instead dt.

We now write T for T_γ. An element w of $\mathcal{A}(T)$ defines a Haar measure $(dt)^w$ on T^w. It is easily seen that $\Phi_f(\gamma^w, (dt)^w, dg)$ depends only on the class of w in $\mathcal{D}(T) = G\backslash\mathcal{A}(T)/\boldsymbol{T}$ (cf. Section 2). Therefore

$$\Phi_f^1(\gamma, dt, dg) = \sum_{\omega \in \mathcal{D}(T)} \Phi_f(\gamma^\omega, (dt)^\omega, dg).$$

Clearly,

$$\Phi_f^1(\gamma^\omega, (dt)^\omega, dg) = \Phi_f^1(\gamma, dt, dg)$$

for each $\omega \in \mathcal{A}(T)$.

Recall that G is an inner form of the quasi-split group G'; we continue with the same fixed isomorphism ψ: $\boldsymbol{G} \to \boldsymbol{G}'$. Our aim is to show that ψ transports stable orbital integrals on G to stable orbital integrals on G'. To make this precise we must normalize Haar measures. The measures dg' on G' and dt' on a Cartan subgroup T' of G' will be arbitrary. Suppose that dg' is defined by the differential form ω' on $\boldsymbol{G}'$. The map ψ induces a map from forms on $\boldsymbol{G}'$ to forms on $\boldsymbol{G}$; the image ω of ω' is a left-invariant form of highest degree, and invariant under complex conjugation (cf. [12, page 476]). We take dg to be the Haar measure on G defined by ω. Now if T is a Cartan subgroup of G choose $x \in \boldsymbol{G}$ such that ψ_x: $\boldsymbol{T} \to \boldsymbol{T}'$ is defined over $\mathbf{R}$ (cf. Section 2). Then the pair dt', ψ_x defines a measure dt on T, independently of the choice of x and consistently with the choice of dg.

Recalling the observations of Section 2, we see that our definition of Φ_f^1 ensures that the map

$$\gamma' \to \begin{cases} \Phi_f^1(\gamma, dt, dg) & \text{if } \gamma' \text{ originates from } \gamma \text{ in } G \\ 0 & \text{if } \gamma' \text{ does not originate in } G \end{cases}$$

on G'_{reg} is well-defined. The transfer of stable orbital integrals from G to G' is then accomplished by the following theorem.

THEOREM 4.1: *Let f be a Schwartz function on G. Then there is a Schwartz function f' on G' such that, for $\gamma' \in G'_{\text{reg}}$,*

$$\Phi_{f'}^1(\gamma', dt', dg') = \begin{cases} (-1)^{q_G - q_{G'}} \Phi_f^1(\gamma, dt, dg) & \text{if } \gamma' \text{ originates from } \gamma \text{ in } G \\ 0 & \text{if } \gamma' \text{ does not originate in } G. \end{cases}$$

The constant $(-1)^{q_G - q_{G'}}$ is inserted to obtain the identity of Corollary 6.7.

In order to prove the theorem we will describe necessary and sufficient conditions for a family of functions to be a family of stable orbital integrals. Suppose then that for each Cartan subgroup T of G we are given a function $\gamma \to \Phi^T(\gamma, dt, dg)$ defined on $T_{\text{reg}} = T \cap G_{\text{reg}}$ and depending on the choice of Haar measures dt and dg. We first establish some properties for the case

$$\Phi^T(\gamma, dt, dg) = \Phi_f^1(\gamma, dt, dg)$$

with f some fixed Schwartz function on G. It is immediate that

$$\Phi_f^1(\gamma, \alpha dt, \beta dg) = \frac{\beta}{\alpha} \Phi_f^1(\gamma, dt, dg)$$

for $\alpha, \beta > 0$ and, as we have already remarked, that

$$\Phi_f^1(\gamma^\omega, (dt)^\omega, dg) = \Phi_f^1(\gamma, dt, dg)$$

for $\omega \in \mathscr{A}(T)$. We come then to the smoothness and growth properties of these functions. Fix T, dt and dg. If $\lambda \in \mathfrak{t}^*$ is zero on $\{H \in \mathfrak{t}: \exp H = 1\}$ we denote by ξ_λ the corresponding quasi-character on T. Fix a system I^+ of positive roots for $\boldsymbol{T}$ in $\boldsymbol{M}$ ($\boldsymbol{M}$ as in Section 2) – that is, a system of positive imaginary roots for $\boldsymbol{T}$. Set

$$R_T(\gamma) = |\det(\operatorname{Ad}\gamma - 1)_{\mathfrak{g}/\mathfrak{m}}|^{1/2} \prod_{\alpha \in I^+} (1 - \xi_\alpha(\gamma^{-1}))$$

for $\gamma \in T$, and

$$\Psi_f^T(\gamma) = R_T(\gamma)\Phi_f^1(\gamma, dt, dg)$$

for $\gamma \in T_{\text{reg}}$. For each element ω of $\mathscr{D}(T)$ choose a representative in Norm$(\boldsymbol{M}, \boldsymbol{T})$ (cf. Theorem 2.1). Also, set $\iota = (\Sigma_{\alpha \in I^+} \alpha)/2$. Then, in the notation of [6],

(1)
$$\Psi_f^T(\gamma) = c \sum_\omega \det \omega \xi_{\iota - \omega^{-1}\iota}(\gamma)'F_f(\gamma^\omega)$$

where c is some constant depending only on the choice of measures. From [6] it follows that Ψ_f^T extends to a Schwartz function (in the sense of [6]) on the dense open subset

$$T^I_{\text{reg}} = \{\gamma \in T \colon \xi_\alpha(\gamma) \neq 1, \alpha \in I^+\}$$

of T. In particular, if $D \in \mathscr{T}$, the algebra of invariant differential operators on T, then $D\Psi_f^T$ is bounded on T^I_{reg}. The behavior of $D\Psi_f^T$ across the boundary of T^I_{reg} may then be described following Harish-Chandra's method for $'F_f$.

Thus we will assume that γ_0 is a semiregular element in $T - T^I_{\text{reg}}$. Then there are exactly two imaginary roots β, say $\pm\alpha$, for which $\xi_\beta(\gamma_0) = 1$. Let H_α be the coroot attached to α and set $\gamma_\nu = \gamma_0 \exp i\nu H_\alpha$, $\nu \in \mathbb{R}$. Then $\gamma_\nu \in T$ and $\lim_{\nu \downarrow 0} D\Psi_f^T(\gamma_\nu)$ and $\lim_{\nu \downarrow 0} D\Psi_f^T(\gamma_\nu)$ are well-defined; if for each choice of α, γ and D these limits are equal then Ψ_f^T extends to a Schwartz function on T. In general, consider their difference. For each $\omega \in \mathscr{D}(T)$ we choose a representative in Norm$(\boldsymbol{M}, \boldsymbol{T})$. Then γ_0^ω is semiregular, $\pm\omega\alpha$ being the only roots trivial on γ_0^ω. According to [6], if $\omega\alpha$ is compact (cf. Section 2) then

$$\lim_{\nu \downarrow 0} D'F_f(\gamma_\nu^\omega) = \lim_{\nu \downarrow 0} D'F_f(\gamma_\nu^\omega)$$

for all $D \in \mathscr{T}$. Recalling our formula (1) for Ψ_f^T in terms of $'F_f$ we set

$$\Psi_f^\omega(\gamma) = R_T(\gamma)\Phi_f(\gamma^\omega) = c \det \omega\xi_{\iota-\omega^{-1}\iota}(\gamma)'F_f(\gamma^\omega).$$

Then

$$(2) \qquad \lim_{\nu\downarrow 0} D\Psi_f^T(\gamma_\nu) - \lim_{\nu\uparrow 0} D\Psi_f^T(\gamma_\nu) = \sum_\omega \left(\lim_{\nu\downarrow 0} D\Psi_f^\omega(\gamma_\nu) - \lim_{\nu\uparrow 0} D\Psi_f^\omega(\gamma_\nu)\right)$$

the summation being over those elements ω of $\mathcal{D}(T)$ for which $\omega\alpha$ is noncompact. In particular,

$$\lim_{\nu\downarrow 0} D\Psi_f^T(\gamma_\nu) = \lim_{\nu\uparrow 0} D\Psi_f^T(\gamma_\nu)$$

if each $\omega\alpha$ is compact. Since

$$\Psi_f^T(\gamma^\omega) = (\det \omega)\xi_{\iota-\omega^{-1}\iota}(\gamma)\Psi_f^T(\gamma)$$

it will be sufficient to consider just the case where α is noncompact, to complete our study.

LEMMA 4.2: *Suppose that α is noncompact. Then if $\omega\alpha$ is also noncompact, $\omega \in \Omega(M, T)$, there exists $\omega_0 \in \Omega(M, T)$ such that $\omega\alpha = \pm\omega_0\alpha$.*

PROOF: Let G_{γ_0} be the connected component of the identity in the centralizer of γ_0 in G and $G^1_{\gamma_0}$ be the derived group of G_{γ_0} (γ_0 is any semiregular element in T on which ξ_α is trivial. If ω is realized by $w \in M$ then we claim that ad w: $G_{\gamma_0} \to G_{\gamma_0^w}$ is defined over $\mathbb{R}$. Indeed, since $G_{\gamma_0} = TG^1_{\gamma_0}$ it is enough to verify that ad $w: G^1_{\gamma_0} \to G^1_{\gamma_0^w}$ is defined over $\mathbb{R}$. If we use the notation of Section 2 then the Lie algebra of $G^1_{\gamma_0}$ is generated by X_α, $X_{-\alpha}$ and H_α; we require that $\bar{X}_\alpha = X_{-\alpha}$. Setting $X_{\omega\alpha} = \operatorname{Ad} w(X_\alpha)$, we obtain $X_{-\omega\alpha} = \operatorname{Ad} w(X_{-\alpha})$ and that $\bar{X}_{\omega\alpha} = X_{-\omega\alpha}$. Since $\operatorname{Ad} w(H_\alpha) = H_{\omega\alpha}$ it is now immediate that Ad w commutes with complex conjugation on the Lie algebras. This proves our claim.

There is a maximal torus U in G_{γ_0} defined over $\mathbb{R}$ and such that $U \cap G^1_{\gamma_0}$ is $\mathbb{R}$-split. Let $V = U^w$; ad w: $U \to V$ is defined over $\mathbb{R}$. Since U and V are maximal in M there exists $w_1 \in M$ such that $U^{w_1} = V$

and $w_1^{-1}w$ centralizes the maximal $\mathbb{R}$-split torus in $\boldsymbol{U}$ (Theorem 2.1). If β is a root for $\boldsymbol{U}$ in $\boldsymbol{G}_{\gamma_0}$ then ad $w_1(\beta) = \text{ad } w(\beta)$ is a root for $\boldsymbol{V}$ in both $\boldsymbol{G}_{\gamma_0^{w_1}}$ and $\boldsymbol{G}_{\gamma_0^w}$. Choose root vectors X_β, $X_{-\beta}$ and coroot H_β as usual. Then

$$\text{Ad } w_1(\mathbb{C}X_\beta + \mathbb{C}X_{-\beta} + \mathbb{C}H_\beta) = \text{Ad } w(\mathbb{C}X_\beta + \mathbb{C}X_{-\beta} + \mathbb{C}H_\beta)$$

so that $w_1^{-1}w$ normalizes $\boldsymbol{G}^1_{\gamma_0}$. We may replace w_1 by $w_0 \in \boldsymbol{M}$ such that $w_0^{-1}w_1 \in \boldsymbol{G}^1_{\gamma_0}$ and $w_0^{-1}w$ normalizes $\boldsymbol{T} \cap \boldsymbol{G}^1_{\gamma_0}$ as well as $\boldsymbol{G}^1_{\gamma_0}$. Then $\text{ad}(w_0^{-1}w)\alpha = \pm\alpha$, which proves the lemma.

To proceed with our discussion of the jumps of $D\Psi_f^T$, we assume α noncompact. According to the lemma, if $\omega\alpha$ is also noncompact we may replace ω in the summation (2) by an element δ of Norm$(\boldsymbol{M}, \boldsymbol{T})$ such that $\delta\alpha = \pm\alpha$. If the Weyl reflection ω_α is realized by w_α in G then replacing δ by $w_\alpha\delta$ does not change the class in $\mathscr{D}(T)$; hence we may assume that $\delta\alpha = \alpha$. If w_α cannot be chosen in G then the class of $w_\alpha\delta$ is distinct from that of δ in $\mathscr{D}(T)$. However, we will observe that the terms in (2) corresponding to these two classes coincide for an appropriate choice of D.

It is convenient at this point to indicate the final "jump" formula. We will observe the following conventions. Firstly, the system I^+ of positive imaginary roots for $\boldsymbol{T}$ must be adapted to α; that is, I^+ contains all imaginary roots β for which $\langle\beta, \alpha\rangle > 0$; as before, $\iota = (\Sigma_{\beta\in I^+}\beta)/2$. Let s be a Cayley transform with respect to α (in the sense of Section 2). Recall that s embeds $\boldsymbol{S}(T)$ in $\boldsymbol{S}(T_s)$ (Proposition 2.7). Hence $\boldsymbol{M}_s$, the centralizer of $\boldsymbol{S}(T_s)$, is contained in $(\boldsymbol{M})^s$. Then s induces a bijection between the set of imaginary roots β for $\boldsymbol{T}_s$ and the set of imaginary roots for $\boldsymbol{T}$ perpendicular to α. Define

$$I_s^+ = (\beta\colon s^{-1}\beta \in I^+) \quad \text{and} \quad \iota_s = \Big(\sum_{\beta\in I_s^+}\beta\Big)\Big/2.$$

To fix a Haar measure on T_s, suppose that the measure dt on T is defined by the differential form $\omega_0 \wedge \omega_1$ on $\mathfrak{t}$, where ω_0, ω_1 are left-invariant forms on $\mathbb{C}H_\alpha$, $(\mathbb{C}H_\alpha)^\perp$ respectively, of highest degree and commuting with complex conjugation. Then s transports $i\omega_0 \wedge \omega_1$ to a form on $\mathfrak{t}_s$, which we may use to define a Haar measure $(dt)^s$ on T_s. Finally, if $D \in \mathscr{T}$ then D^s will denote the image of D under the isomorphism $\mathscr{T} \to \mathscr{T}_s$ induced by s. Also, we will replace D by $\hat{D}$, the image of D under the automorphism of $\mathscr{T}$ induced by $H \to H + \iota(H)I$, $H \in \mathfrak{t}$; if $D' \in \mathscr{T}_s$ then $\widehat{D'}$ will be the image of D' under the automorphism of $\mathscr{T}_s$ induced by $H' \to H' + \iota_s(H')I$, $H' \in \mathfrak{t}_s$.

LEMMA 4.3:

$$\lim_{\nu\downarrow 0} \hat{D}\Psi_f^T(\gamma_\nu, dt, dg) - \lim_{\nu\uparrow 0} \hat{D}\Psi_f^T(\gamma_\nu, dt, dg)$$
$$= 2i\widehat{D^s}\Psi_f^{T_s}(\gamma_0^s, (dt)^s, dg).$$

PROOF: Note that the right-hand side is well-defined (since $\gamma_0^s \in (T_s)_{\text{reg}}^I$) and independent of the choice of Cayley transform s. Hence we will assume that s is standard; that is, that s is the image of $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ under some fixed homomorphism of SL_2 into $\boldsymbol{G}$, as defined in Section 2. The image of $\mathrm{SL}_2(\mathbb{R})$ under such a homomorphism is $G_{\gamma_0}^+$, the (Euclidean) connected component of the identity in $G_{\gamma_0}^1$; the image of the standard compact Cartan subgroup is $B^+ = T \cap G_{\gamma_0}^+$ and the image of the standard split Cartan subgroup is $A^+ = T_s \cap G_{\gamma_0}^+$. Also $T = ZB^+$ and ZA^+ has finite index in T_s, Z denoting the center of G_{γ_0}. We will need the following proposition.

PROPOSITION 4.4: (1) *If ω_α can be realized in G then*

$$[G_{\gamma_0} : ZG_{\gamma_0}^+] = [T_s : ZA^+] = 2;$$

(2) *If ω_α cannot be realized in G then*

$$G_{\gamma_0} = ZG_{\gamma_0}^+ \quad \textit{and} \quad T_s = ZA^+.$$

PROOF: Let $g \in G_{\gamma_0}$. Then there exists $g_0 \in G_{\gamma_0}^+$ such that $g_0 g$ normalizes B^+. Then also $g_0 g$ normalizes $\boldsymbol{G}_{\gamma_0}^1$ and $\boldsymbol{T} \cap \boldsymbol{G}_{\gamma_0}^1$. Hence $g_0 g\alpha = \pm\alpha$. If $w_\alpha \in \boldsymbol{G}_{\gamma_0}$ represents ω_α then it follows that either $w_\alpha g_0 g$ or $g_0 g$ lies in $\boldsymbol{T}$. If w_α cannot be chosen in G then $g_0 g \in T \subset ZG_{\gamma_0}^+$ so that (2) follows. If w_α can be chosen in G, and hence in G_{γ_0}, then $[G_{\gamma_0} : ZG_{\gamma_0}^+] = 2$ since, clearly, $w_\alpha \notin ZG_{\gamma_0}^+$. Again, suppose $g \in G_{\gamma_0}$. Then there exists $g_1 \in G_{\gamma_0}^+$ such that $g_1 g$ normalizes A^+. Arguing as before, and observing that $\omega_{s\alpha}$ can be realized in $G_{\gamma_0}^+$ we obtain $g \in G_{\gamma_0}^+ T_s$. This implies $[T_s : ZA^+] = 2$, which completes the proof.

Now fix $\delta \in \mathrm{Norm}(\boldsymbol{M}, \boldsymbol{T})$ such that $\delta\alpha = \alpha$. Then $\boldsymbol{G}_{\gamma_0^\delta} = \boldsymbol{G}_{\gamma_0}$ and $\mathrm{ad}\,\delta / \boldsymbol{G}_{\gamma_0}$ is defined over $\mathbb{R}$. Hence δ normalizes both $G_{\gamma_0}^+$ and Z; in particular, $\gamma_0^\delta \in Z$, the center of G_{γ_0}. We will need the following (immediate) observation: $s\delta s^{-1}\delta^{-1} \in \boldsymbol{G}_{\gamma_0}$ and $s\delta s^{-1}\delta^{-1}: \boldsymbol{T}_s^\delta \to \boldsymbol{T}_s$ is defined over $\mathbb{R}$, for this implies that $s\delta s^{-1} = g_0 \delta t$, $g_0 \in G_{\gamma_0}$, $t \in \boldsymbol{T}_s$.

The next proposition can be deduced from [6]. However it is easy to write down a similar direct proof; we include the argument for the sake of completeness.

PROPOSITION 4.5:

$$\lim_{\nu\downarrow 0} \hat{D}\Psi_f^{\delta}(\gamma_\nu : dt : dg) - \lim_{\nu\uparrow 0} \hat{D}\Psi_f^{\delta}(\gamma_\nu : dt : dg)$$
$$= \mathrm{i}d(\alpha)\widehat{D^s}\Psi_f^{s\delta s^{-1}}(\gamma_0 : (dt)^s : dg)$$

where $d(\alpha) = 2$ if ω_α can be realized in G and $d(\alpha) = 1$ otherwise.

PROOF: Because of the continuity of the map $f \to \Psi_f^T$ between the Schwartz spaces of G and T_{reg}^I (cf. [6]) it is enough to verify the lemma in the case that f has compact support.

Pick a neighborhood $\mathcal{O}$ of the origin in $\mathfrak{t}$ as in [26, volume 2, page 228]. Let $N = \exp \mathcal{O}$; $\gamma \in \gamma_0 N$ is regular in G if $\gamma_0^{-1}\gamma$ is regular in G_{γ_0}. On fixing $H_0 \in \mathfrak{k}$ such that $\gamma_0 = \exp H_0$, the functions ξ_ι, $\xi_{\beta/2}$, etc., are well-defined on $\gamma_0 N$. Then

$$\hat{D}\Psi_f^{\delta} = \xi_{-\iota} D(F_1 F_2)$$

on $\gamma_0 N_{\mathrm{reg}}$, where

$$F_1(\gamma) = |\det(\mathrm{Ad}\ \gamma - 1)_{\mathfrak{q}/\mathfrak{m}}|^{1/2} \prod_{\substack{\beta \in I^+ \\ \beta \neq \alpha}} (\xi_{\beta/2}(\gamma) - \xi_{\beta/2}(\gamma^{-1}))$$

and

$$F_2(\gamma) = (\xi_{\alpha/2}(\gamma) - \xi_{\alpha/2}(\gamma^{-1})) \int_{G/T} f(g\gamma^{\delta}g^{-1}) d\bar{g}$$

for $\gamma \in \gamma_0 N_{\mathrm{reg}}$. Similarly, define

$$G_1(\gamma) = |\det(\mathrm{Ad}\ \gamma - 1)_{\mathfrak{g},\mathfrak{m}_s}|^{1/2} \prod_{\beta \in I_s^+} (\xi_{\beta/2}(\gamma) - \xi_{\beta/2}(\gamma^{-1}))$$

and

$$G_2(\gamma) = |\xi_{s\alpha/2}(\gamma) - \xi_{s\alpha/2}(\gamma^{-1})| \int_{G/T_s} f(g\gamma^{s\delta s^{-1}}g^{-1}) d\bar{g}$$

for regular γ in a suitable neighborhood of γ_0 in T_s.

A simple inductive argument shows that there are operators C_r, $D_r \in \mathcal{T}$ such that

$$D(fg) = \sum_{r=1}^{n} C_r f D_r g \qquad f, g \in C^\infty(T_{\text{reg}})$$

and

$$D^s(f'g') = \sum_{r=1}^{n} C_r^s f' D_r^s g' \qquad f', g' \in C^\infty((T_s)_{\text{reg}}).$$

Then

$$\lim_{\nu \downarrow 0} \hat{D}\Psi_f^\delta(\gamma_\nu, dt, dg) - \lim_{\nu \uparrow 0} \hat{D}\Psi_f^\delta(\gamma_\nu, dt, dg)$$

$$= \xi_{-\iota}(\gamma_0) \sum_r C_r F_1(\gamma_0)\Big(\lim_{\nu \downarrow 0} D_r F_2(\gamma_\nu) - \lim_{\nu \uparrow 0} D_r F_2(\gamma_\nu)\Big)$$

since F_1 is C^∞ around γ_0. On the other hand

$$D^s \Psi_f^{s\delta s^{-1}}(\gamma_0, (dt)^s, dg) = \xi_{-\iota_s}(\gamma_0) \sum_r C_r^s G_1(\gamma_0) D_r^s G_2(\gamma_0).$$

Hence, to prove the proposition, we have only to check that

(A) $$\xi_{-\iota_s}(\gamma_0) C_r^s(G_1)(\gamma_0) = \xi_{\alpha/2-\iota}(\gamma_0) C_r F_1(\gamma_0).$$

(B) $$\mathrm{id}(\alpha) D_r^s G_2(\gamma_0) = \xi_{-\alpha/2}(\gamma_0)\Big(\lim_{\nu \downarrow 0} D_r F_2(\gamma_\nu) - \lim_{\nu \uparrow 0} D_r F_2(\gamma_\nu)\Big).$$

For (A), note that if β is a root in I^+ distinct from α and not perpendicular to α then so also is $\beta' = -\omega_\alpha(\beta)$; $\beta' \neq \beta$ and $s\beta$, $s\beta'$ are complex conjugate roots in $\boldsymbol{T}_s$ (recall that I^+ is adapted to α). A straightforward calculation then yields the desired formula.

For (B), suppose that the support of f lies in the compact set C. Choose a compact set $\bar{C}$ in G/G_{γ_0} so that $g\gamma^\delta g^{-1} \in C$ ($\gamma \in \gamma_0 N$, $g \in G$) implies that $gG_{\gamma_0} \in \bar{C}$. Fix $\psi \in C_c^\infty(G)$ so that

$$\int_{G_{\gamma_0}} \psi(gh)dh = 1 \quad \text{if} \quad gG_{\gamma_0} \in \bar{C}$$

and define

$$v_f^\delta(h) = \int_G \psi(g) f(g(\gamma_0 h)^\delta g^{-1}) dg,$$

for $h \in G_{\gamma_0}$. If the Haar measure dh on G_{γ_0} is chosen suitably, then $v_f^\delta \in C_c^\infty(G_{\gamma_0})$ and

$$\int_{G/T} f(g\gamma^\delta g^{-1})d\bar{g} = \int_{G_{\gamma_0}/T} v_f^\delta(h\gamma_0^{-1}\gamma h^{-1})d\bar{h}$$

for $\gamma \in \gamma_0 N_{\mathrm{reg}}$. Similarly,

$$\int_{G/T_s} f(g\gamma^{s\delta s^{-1}}g^{-1})d\bar{g} = \int_{G/T_s^\delta} f(g\gamma^\delta g^{-1})d\bar{g} = \int_{G_{\gamma_0}/T_s} v_f^\delta(h\gamma_0^{-1}\gamma h^{-1})d\bar{h}$$

for regular γ near γ_0 in T_s.

For $x \in G_{\gamma_0}$ define $v_f^{\delta,x}(h) = v_f^\delta(xhx^{-1})$, $h \in G_{\gamma_0}$. Let x range over a set of representatives for $G_{\gamma_0}/ZG_{\gamma_0}^+$. Then

$$\xi_{-\alpha/2}(\gamma_0)F_2(\gamma) = (\xi_{\alpha/2}(b) - \xi_{\alpha/2}(b^{-1}))\sum_x \int_{G_{\gamma_0}^+/B^+} v_f^{\delta,x}(zhbh^{-1})d\bar{h}$$

where $\gamma_0^{-1}\gamma = zb$, $z \in Z$, $b \in B^+$. Concerning the normalization of measures, we fix a Haar measure on the standard compact Cartan subgroup of $\mathrm{SL}_2(\mathbb{R})$; we transport measures via the homomorphism $\mathrm{SL}_2 \to G$, and given measures on a group and subgroup we use the quotient measure on the quotient; conversely we use product measures on products. This, together with our previous choices, fixes the measure on each of the groups we will consider. Now write $D_r = D_r^{(1)} \cdot D_r^{(2)}$ where $D_r^{(1)}$, $D_r^{(2)}$ are invariant differential operators on B^+, Z respectively. Then

$$\xi_{-\alpha/2}(\gamma_0)D_rF_2(\gamma) = (D_r^{(1)}F_2^z)(b)$$

where

$$F_2^z(b) = (\xi_{\alpha/2}(b) - \xi_{\alpha/2}(b^{-1}))\sum_x \int_{G_{\gamma_0}^+/B^+} (D_r^{(2)}v_f^{\delta,x})(zhbh^{-1})d\bar{h}.$$

Since $G_{\gamma_0}^+ = SL_2(\mathbb{R})$ (or $\mathrm{SL}_2(\mathbb{R})/\pm I$) we have only to recall the calculations for that group to obtain

$$\xi_{-\alpha/2}(\gamma_0)\Big(\lim_{\nu\downarrow 0} D_rF_2(\gamma_\nu) - \lim_{\nu\uparrow 0} D_rF_2(\gamma_\nu)\Big) = i((D_r^{(1)})^s G_2^z)(1)$$

where

$$G_2^z(a) = |\xi_{s\alpha/2}(a) - \xi_{s\alpha/2}(a^{-1})|\sum_x \int_{G_{\gamma_0}^+/A^+} D_r^{(2)}v_f^{\delta,x}(zhah^{-1})d\bar{h}$$

for $a \in A^+$. Since $(D_r^{(1)}D_r^{(2)})^s = (D_r^{(1)})^s D_r^{(2)}$ we obtain

$$i((D_r^{(1)})^s G_2^z)(1) = i(D_r^s(G_2')(\gamma_0))$$

where

$$G_2'(\gamma) = |\xi_{s\beta/2}(\gamma) - \xi_{s\alpha/2}(\gamma^{-1})| \sum_x \int_{G^+_{\gamma_0}/A^+} v_f^{\delta,x}(h\gamma_0^{-1}\gamma h^{-1})d\bar{h}$$

for regular γ near γ_0 in T_s. But

$$\sum_x \int_{G^+_{\gamma_0}/A^+} v_f^{\delta,x}(h\gamma_0^{-1}\gamma h^{-1})d\bar{h} = d(a) \int_{G_{\gamma_0}/T_s} v_f^{\delta}(h\gamma_0^{-1}\gamma h^{-1})d\bar{h}$$

$$= d(\alpha) \int_{G/T_s} f(g\gamma^{s\delta s^{-1}}g^{-1})d\bar{g}$$

so that

$$G_2'(\gamma) = d(\alpha)G_2(\gamma).$$

Hence (B) is verified and the proof of Proposition 4.5 is complete.

For the proof of Lemma 4.3 we need one more proposition. Consider set of all classes in $\mathcal{D}(T)$ which contain a representative δ in Norm($\boldsymbol{M}$, $\boldsymbol{T}$) for which $\delta\alpha = \pm\alpha$. There is a well-defined action of the group $\langle 1, \omega_\alpha\rangle$ on this set, given by $G\delta\boldsymbol{T} \to G\omega_\alpha\delta\boldsymbol{T}$. Let $\mathcal{D}_\alpha(T)$ be the set of orbits. If ω_α is realized in G then each orbit has just one element and if ω_α is not realized in G each orbit has two elements. Since s embeds $\boldsymbol{S}(T)$ in $\boldsymbol{S}(T_s)$ it follows that $\omega \to s^{-1}\omega s$ maps Norm($\boldsymbol{M}_s$, $\boldsymbol{T}_s$) to Norm($\boldsymbol{M}$, $\boldsymbol{T}$).

PROPOSITION 4.6: *The map $\omega \to s^{-1}\omega s$ induces a bijection $\mathcal{D}(T_s) \to \mathcal{D}_\alpha(T)$.*

PROOF: Suppose that $g \in$ Norm($\boldsymbol{M}_s$, $\boldsymbol{T}_s$). Then ad$(s^{-1}g^{-1}s)$ fixes α and hence maps $\boldsymbol{G}_{\gamma_0}$ to $\boldsymbol{G}_{\gamma_0}$. This implies that ad$(s^{-1}g^{-1}s)(\gamma_0)$ lies in $Z(\boldsymbol{G}_{\gamma_0})$ and hence $gs^{-1}g^{-1}s \in \boldsymbol{G}_{\gamma_0}$. There is $h \in \boldsymbol{G}_{\gamma_0}$ such that $hgs^{-1}g^{-1}s$ normalizes $\boldsymbol{T}$. Hence either $hgs^{-1}g^{-1}s$ or $hgs^{-1}g^{-1}s\omega_\alpha$ lies in $\boldsymbol{T}$; that is, either ad$(s^{-1}gs)/\boldsymbol{T}$ = ad$(hg)/\boldsymbol{T}$ or ad$(\omega_\alpha s^{-1}gs)/\boldsymbol{T}$ = ad$(hg)/\boldsymbol{T}$. This implies that the map which sends the class of ω in $\mathcal{D}(T_s)$ to the orbit of $s^{-1}\omega s$ is well-defined. Clearly the map is surjective. To complete the proof it is enough to show that if $g \in$ Norm(M, T) and $g\alpha = \pm\alpha$ then the action of sgs^{-1} on $\boldsymbol{T}_s$ can be realized in G. This follows easily from an argument similar to that given above.

Combining Lemma 4.2 and Propositions 4.5 and 4.6, we may now complete the proof of Lemma 4.3. If ω_α is realized in G then the result is immediate:

$$\lim_{\nu\downarrow 0} \hat{D}\Psi_f^T(\gamma_\nu, dt, dg) - \lim_{\nu\uparrow 0} \hat{D}\Psi_f^T(\gamma_\nu, dt, dg)$$

$$= \sum_\delta \left(\lim_{\nu\downarrow 0} \hat{D}\Psi_f^\delta(\gamma_\nu, dt, dg) - \lim_{\nu\uparrow 0} \hat{D}\Psi_f^\delta(\gamma_\nu, dt, dg)\right)$$

where δ ranges over a complete set of representatives, each fixing α, for the classes in $\mathscr{D}(T)$ containing an element fixing α,

$$= 2i \sum \widehat{D^s}\Psi_f^{s\delta s^{-1}}(\gamma_0, (dt)^s, dg) = 2i\widehat{D^s}\Psi_f^{T_s}(\gamma_0, (dt)^s, dg).$$

Suppose then that ω_α cannot be realized in G. Suppose that $D^{\omega_\alpha} = -D$. Then since

$$\Phi_f^1(\gamma^{\omega_\alpha}, dt, dg) = \Phi_f^1(\gamma, dt, dg)$$

it follows that both sides of the equation in the statement of Lemma 4.3 are zero. Hence we may assume that $D^{\omega_\alpha} = D$. But then a computation shows that

$$\lim_{\nu\downarrow 0} \hat{D}\Psi_f^{\omega_\alpha\delta}(\gamma_\nu, dt, dg) = -\lim_{\nu\uparrow 0} \hat{D}\Psi_f^\delta(\gamma_\nu, dt, dg),$$

so that

$$\lim_{\nu\downarrow 0} \hat{D}\Psi_f^T(\gamma_\nu, dt, dg) - \lim_{\nu\uparrow 0} \hat{D}\Psi_f^T(\gamma_\nu, dt, dg)$$

$$= 2\sum_\delta \left(\lim_{\nu\downarrow 0} \hat{D}\Psi_f^\delta(\gamma_\nu, dt, dg) - \lim_{\nu\uparrow 0} \hat{D}\Psi_f^\delta(\gamma_\nu, dt, dg)\right).$$

The rest of the proof is immediate.

We have now shown the necessity of (I) to (IIIb) in the following theorem.

THEOREM 4.7: *Suppose that for each Haar measure dg on G, Cartan subgroup T and Haar measure dt on T we are given a function $\gamma \to \Phi^T(\gamma, dt, dg)$ on T_{reg}. Then there is a Schwartz function f on G such that*

$$\Phi^T(\gamma, dt, dg) = \Phi_f^1(\gamma, dt, dg)$$

for all T, γ, dt and dg if and only if:

(I) $$\Phi^T(\gamma, \alpha dt, \beta dg) = \frac{\beta}{\alpha}\Phi^T(\gamma, dt, dg)$$

for $\alpha, \beta > 0$,

(II) $$\Phi^T(\gamma, dt, dg) = \Phi^{T^\omega}(\gamma^\omega, (dt)^\omega, dg)$$

for $\omega \in \mathscr{A}(T)$,

(III) *if* $\Psi^T(\gamma, dt, dg) = R_T(\gamma)\Phi^T(\gamma, dt, dg)$ *then* Ψ^T *extends to a Schwartz function on* T^I_{reg} *and*

(a) *if* $\gamma_0 \in T - T^I_{\text{reg}}$ *is semiregular and* $\xi_\alpha(\gamma_0) = 1$ *where* $\omega\alpha$ *is compact for each* $\omega \in \Omega(\boldsymbol{M}, \boldsymbol{T})$ *then*

$$\lim_{\nu \uparrow 0} D\Psi^T(\gamma_\nu, dt, dg) = \lim_{\nu \uparrow 0} D\Psi^T(\gamma_\nu, dt, dg)$$

for each $D \in \mathscr{T}$,

(b) *if* $\gamma_0 \in T - T^I_{\text{reg}}$ *is semiregular and* $\xi_\alpha(\gamma_0) = 1$ *where* α *is noncompact then*

$$\lim_{\nu \downarrow 0} D\Psi^T(\gamma_\nu, dt, dg) - \lim_{\nu \uparrow 0} \hat{D}\Psi^T(\gamma_\nu, dt, dg) = 2i\widehat{D^s}\Psi^{T_s}(\gamma_0^s, (dt)^s, dg)$$

for each $D \in \mathscr{T}$.

Recall that

$$R_T(\gamma) = |\det(\operatorname{Ad}\gamma - 1)_{\mathfrak{g}/\mathfrak{m}}|^{1/2} \prod_{\alpha \in I^+} (1 - \xi_\alpha(\gamma^{-1})).$$

In (III) and (IIIa) the choice of I^+ is arbitrary; in (IIIb) the chosen I^+ must be adapted to α. The conventions for $\hat{D}$, $\widehat{D^s}$, $(dt)^s$ and R_{T_s} are as before.

Suppose that $\{\Phi^T(\ , dt, dg)\}$ satisfies (I)–(III); (III) implies that the terms in (IIIa) and (IIIb) are well-defined. Moreover, for any imaginary root α, if

(*) $$\lim_{\gamma \downarrow 0} D\Psi^T(\gamma_\nu, dt, dg) = \lim_{\nu \uparrow 0} D\Psi^T(\gamma_\nu, dt, dg) \qquad D \in \mathscr{T}$$

for all semiregular γ_0 such that $\xi_\alpha(\gamma_0) = 1$ then Ψ^T extends to a C^∞ function around each such γ_0 (irrespective of the choice for I^+); if (*) remains true as α ranges over all imaginary roots then each Ψ^T extends to a C^∞, and hence a Schwartz function on T (cf. [26]). Also, from (II) we have

$$(\hat{D}\Psi^T)(\gamma) = (\det \omega)\widehat{D^\omega}\Psi^T(\gamma^\omega) \qquad \gamma \in T_{\text{reg}}$$

for $\omega \in \Omega(\boldsymbol{M}, \boldsymbol{T})$, the imaginary Weyl group for $\boldsymbol{T}$. This enables us to compute

$$\lim_{\nu\downarrow 0} \hat{D}\Psi^T(\gamma_\nu, dt, dg) - \lim_{\nu\uparrow 0} \hat{D}\Psi^T(\gamma_\nu, dt, dg)$$

in the case γ_0 is semiregular and $\xi_\alpha(\gamma_0) = 1$ with α compact but some $\omega\alpha$ noncompact. It follows then that Ψ^T, satisfying (I)–(IIIa), will be a Schwartz function on T if the right-hand side in (IIIb) is zero for all α, γ_0 as in (IIIb). Finally, note that if D is skew with respect to the Weyl reflection for α then (II) implies that (IIIa) is true for $\hat{D}$ and that both sides of (IIIb) are zero; if D is fixed by the Weyl reflection for α then

$$\lim_{\nu\downarrow 0} \hat{D}\Psi^T(\gamma_\nu, dt, dg) = -\lim_{\nu\uparrow 0} \hat{D}\Psi^T(\gamma_\nu, dt, dg)$$

so that (IIIa) becomes

$$\lim_{\nu\downarrow 0} \hat{D}\Psi^T(\gamma_\nu, dt, dg) = 0$$

and (IIIb) becomes

$$\lim_{\nu\downarrow 0} \hat{D}\Psi^T(\gamma_\nu, dt, dg) = i\widehat{D^s}\Psi^{T_s}(\gamma_0^s, (dt)^s, dg).$$

Now if $\{\Phi^T(\ , dt, dg)\}$ is any family of functions on the various T_{reg} define τ_Φ to be the least of the integers τ for which $\Phi^T(\ , dt, dg) \equiv 0$ if $\dim \boldsymbol{S}(T) > \tau$. Then, arguing by induction on τ_Φ we see that to prove Theorem 4.7 it is sufficient to show the following lemma.

LEMMA 4.8: *Fix a Cartan subgroup T_0 and suppose that for each T conjugate to T_0 and for each Haar measure dt on T and dg on G we are given a function $\Phi^T(\ , dt, dg)$ on T_{reg} satisfying* (I), (II) *and*

(III′) *$\Psi^T(\ , dt, dg)$ extends to a Schwartz function on T. Then there exists a Schwartz function f on G such that*

(a) $\Phi^T(\gamma, dt, dg) = \Phi_f^1(\gamma, dt, dg)$, $\gamma \in T_{\text{reg}}$, *for all such T, dt and dg, and*

(b) $\Phi^1_f(\ , dt', dg) \equiv 0$ *unless* $\langle T' \rangle \leq \langle T \rangle$ *for any* dt' *on* T' *and* dg.

PROOF: Suppose that $\{\Phi^T(\ , dt, dg)\}$ satisfies (I), (II) and (III′). We have only to find f satisfying (a) and (b) for one choice of T, I^+, dt and dg. Hence we will assume that T, dt and dg satisfy the conditions in [6]; implicit is a certain choice of maximal compact subgroup K of G. The choice of I^+ is arbitrary. Also we define the split component A of M and the reductive subgroup ${}^\circ M$ as in [6, sections 2 and 3]; $M = {}^\circ MA$, ${}^\circ M \cap A = \langle 1 \rangle$ and $T = {}^\circ TA$, where ${}^\circ T = {}^\circ M \cap T$ is a compact Cartan subgroup of ${}^\circ M$. Thus, if $\mathfrak{X}$ denotes the group of characters on ${}^\circ T$ and $\mathfrak{a}^*$ the (real) dual of $\log A$ then the Fourier transform $\Psi^\vee$ of $\Psi = \Psi^T(\ , dt, dg)$ is a Schwartz function on $\mathfrak{X} \times \mathfrak{a}^*$. More precisely, we need the following: for each $\Lambda \in \mathfrak{X}$ the function $\nu \to \Psi^\vee(\Lambda, \nu)$ belongs to $\mathscr{C}(\mathfrak{a}^*)$, the space of Schwartz functions on $\mathfrak{a}^*$, and if N is a continuous seminorm on $\mathscr{C}(\mathfrak{a}^*)$ then the numbers $N_\Lambda = N(\nu \to \Psi(\Lambda, \nu))$ satisfy

$$\sum_\Lambda N_\Lambda \mathfrak{p}(|\Lambda|) < \infty \tag{1}$$

for each polynomial $\mathfrak{p}$. ($|\Lambda|$ denotes the length of $\log \Lambda$ which is defined relative to some fixed positive-definite bilinear form on ${}^\circ\mathfrak{m}$ derived from the Killing form on the derived subalgebra $\mathfrak{m}^\dagger$). We may choose the Haar measure on $\mathscr{A}^*$ so that

$$\Psi(ta) = \sum_{\Lambda \in \mathfrak{X}} \left(\int_{\mathfrak{a}^*} \Psi^\vee(\Lambda, \nu) e^{i\nu \log a} d\nu \right) \Lambda(t) \tag{2}$$

for $t \in {}^\circ T$, $a \in A$.

Let $\omega \in \Omega(\boldsymbol{M}, \boldsymbol{T})$. Then by (II) we have that

$$\Psi(at^{\omega^{-1}}) = (\det \omega)\xi_{\iota - \omega\iota}(t)\Psi(at) \qquad t \in {}^\circ T, a \in A$$

which implies that

$$\int_{\mathfrak{a}^*} \Psi^\vee(\Lambda, \nu) e^{i\nu \log a} d\nu = (\det \omega) \int_{\mathfrak{a}^*} \Psi^\vee(\omega\Lambda\xi_{\omega\iota - \iota}, \nu) e^{i\nu \log a} d\nu \tag{3}$$

Fix $\Lambda \in \mathfrak{X}$ and consider

$$\sum_{\omega \in \Omega(\boldsymbol{M}, \boldsymbol{T})} \left(\int_{\mathfrak{a}^*} \Psi^\vee(\omega\Lambda\xi_{\omega\iota - \iota}, \nu) e^{i\nu \log a} d\nu \right) \omega\Lambda\xi_{\omega\iota - \iota}.$$

According to (3) we can write this as

$$\int_{\mathfrak{a}^*} \Psi^\vee(\Lambda, \nu) e^{i\nu \log a} d\nu \sum_{\omega \in \Omega(M, T)} (\det \omega)\omega\Lambda \xi_{\omega\iota - \iota}. \tag{4}$$

Let $\mathcal{O}(\Lambda) = \{\omega\Lambda\xi_{\omega\iota-\iota};\ \omega \in \Omega(M, T)\}$ and $\bar{C}$ be the closure of the chamber in $({}^\circ\mathfrak{t})^*$ dominant with respect to I^+. Clearly (4) vanishes unless $\iota + \log \Lambda$ is regular with respect to $\Omega(M, T)$ or, equivalently, unless $\bar{C} \cap \log(\mathcal{O}(\Lambda))$ is nonempty. Hence (2) may be rewritten as

$$\Psi = \sum_{\log \Lambda \in \bar{C}} \int_{\mathfrak{a}^*} \Psi^\vee(\Lambda, \nu) x(\Lambda, \nu) d\nu$$

where

$$x(\Lambda, \nu)(at) = e^{i\nu \log a} \sum_{\omega \in \Omega(M, T)} (\det \omega)\omega\Lambda(t)\xi_{\omega\iota-\iota}(t).$$

Let $\mathfrak{W} = M \backslash \mathrm{Norm}(G, M)$; $\mathfrak{W}$ is a finite group and for each element we may pick a representative s normalizing $T, {}^\circ T$ and A. Then

$$\Psi((at)^{s^{-1}}) = (\det s)\xi_{\iota - s\iota}(t)\Psi(at);$$

here $\det s$ is the signature of s with respect to I^+ (cf. [6]). Fix Λ such that $\log \Lambda \in \bar{C}$ and consider

$$\sum_s \int_{\mathfrak{a}^*} \Psi^\vee(s\Lambda\xi_{s\iota-\iota}, \nu) e^{i\nu \log a} s\Lambda(t)\xi_{s\iota-\iota}(t) d\nu \tag{5}$$

This may be written as

$$\int_{\mathfrak{a}^*} \Psi^\vee(\Lambda, \nu) \sum_s \mathrm{e}^{is\nu \log a}(\det s)s\Lambda(t)\xi_{s\iota-\iota}(t) d\nu$$

For each s, $\log(\mathcal{O}(s\Lambda\xi_{s\iota-\iota}))$ meets $\bar{C}$ and, conversely, each nonempty $\mathcal{O}(\text{—})$ can be written as $\mathcal{O}(s\Lambda\xi_{s\iota-\iota})$ for some choice of Λ with $\log \Lambda \in \bar{C}$. Hence if we sum (5) over each $\mathcal{O}(s\Lambda\xi_{s\iota-\iota})$, $s \in \mathfrak{W}$, and then over

Λ with $\log \Lambda \in \bar{C}$ then we obtain $|\mathfrak{W}|\Psi$. We conclude then that

$$\Psi = \frac{1}{|\mathfrak{W}|} \sum_{\log \Lambda \in \bar{C}} \sum_{s \in \mathfrak{W}} \int_{\mathfrak{a}^*} \Psi^{\vee}(\Lambda, \nu)(\det s) x(s\Lambda + s\iota - \iota, s\nu) d\nu$$

$$= \sum_{\log \Lambda \in \bar{C}} \int_{\mathfrak{a}^*} \Psi^{\vee}(\Lambda, \nu) y(\Lambda, \nu) d\nu$$

where

$$y(\Lambda, \nu) = \frac{1}{|\mathfrak{W}|} \sum_{s \in \mathfrak{W}} (\det s) x(s\Lambda + s\iota - \iota, s\nu).$$

Note that, up to a constant, $R_T^{-1} y(\Lambda, \nu)$ coincides with the restriction to T of our character χ_φ, φ denoting the parameter attached to the $\mathscr{A}(T)$-orbit of $\Lambda e^{i\nu}$ – the (well-known) computation for χ_φ is given in [23].

Let $\mathscr{C}(G)$ denote the space of Schwartz functions on G. If X, Y are in the universal enveloping algebra of $\mathfrak{g}$ and $m > 0$ set

$$\nu_{(X,Y,m)}(f) = \sup_{g \in G} \frac{(1+\sigma(g))^m (XfY)(g)}{\Xi(g)} \qquad f \in \mathscr{C}(G)$$

(σ, Ξ are defined as usual (cf. [6]).

PROPOSITION 4.9: *Fix X, Y, m. Then there is a polynomial $\wp$, a continuous seminorm N on $\mathscr{C}(\mathfrak{a}^*)$ and for each Λ with* $\log \Lambda \in C$ *a function $f(\Lambda) \in \mathscr{C}(G)$ such that*

(1) $\Psi^T_{f(\Lambda)}(\ , dt, dg) = \int_{\mathfrak{a}^*} \Psi^{\vee}(\Lambda, \nu) y(\Lambda, \nu) d\nu;$

(2) $\Phi^1_{f(\Lambda)}(\ , dt', dg) \equiv 0$ *for any T' with $\langle T' \rangle \not\leq \langle T \rangle$; and*

(3) $\nu_{(X,Y,m)}(f(\Lambda)) \leq N_\Lambda \wp(|\Lambda|).$

Lemma 4.8 follows from Proposition 4.9, for, assuming Proposition 4.9, we may define

$$f = \sum_{\log \Lambda \in \bar{C}} f(\Lambda).$$

From (1) it follows that $\nu_{(X,Y,m)}(f) < \infty$ for all (X, Y, m) so that

$f \in \mathscr{C}(G)$. Moreover, on any Cartan subgroup T'

$$\Phi_f^1(\ ,dt',dg) = \sum_{\log \Lambda \in \bar{C}} \Phi_{f(\Lambda)}^1(\ ,dt',dg)$$

because of the continuity of the map $f \to R_{T'}\Phi_f^1$ of $\mathscr{C}(G)$ into $\mathscr{C}(T'_{\text{reg}})$ (cf. [6]). Hence f fulfills the requirements of Lemma 4.8. It remains then to prove Proposition 4.9.

Let π be the restriction to $°M$ of the square-integrable representation of M attached to $\Lambda e^{i\nu}$ (π depends only on Λ).

PROPOSITION 4.10: *There is a polynomial* $\wp$ *with the following property: for each* Λ *there exists an irreducible unitary representation* $\sigma(\Lambda)$ *of* K *contained in* $\text{Ind}(\pi \mid K \cap M, K \cap M, K)$ *and such that* $\|\sigma(\Lambda)\| \leq \wp(|\Lambda|)$.

Here $\|\ \ \|$ denotes the length of the highest weight. The proposition is immediate consequence of [24, Lemma 4.4]. A proof using [22] (or [9]) and an elementary argument can also be given.

Let $\mathfrak{H}_\Lambda$ be the $\sigma(\Lambda)$-isotypic subspace of $\text{Ind}(\pi \mid K \cap M, K \cap M, K)$ and p be the projection of $\mathfrak{H}_\Lambda$ onto some irreducible summand; attach to p the function ψ_p of [8, section 7]. Let (τ, V) be that subrepresentation of the natural double representation of K on $C^\infty(K \times K)$ determined by $\sigma(\Lambda)$, as in [8, section 7]. Recall that ψ_p is a V-valued function on $°M$, spherical with respect to $\tau \mid K \cap M$. We consider now the wave-packet

$$F_\Lambda = \int_{\mathfrak{a}^*} \Psi^\vee(\Lambda,\nu)\mu(\Lambda,\nu)E(P,\psi_p,\nu,-)d\nu;$$

here P is some parabolic subgroup with Levi component M, $E(P, \psi_P, \nu, -)$ is the Eisenstein integral for ψ_p relative to P and $\mu(\Lambda,\nu)$ is as in [8]. According to [8, section 26] $F_\Lambda \in \mathscr{C}(G, V)$ and

$$\Phi_{F_\Lambda}^1 = \left(\int_{\mathfrak{a}^*} \Psi^\vee(\Lambda,\nu)y(\Lambda,\nu)d\nu\right) v_0$$

where $v_0 = (c/d_\pi)\int_K k\psi_p(1)k^{-1}dk$, c being a constant and d_π the formal degree of π. We now choose $\ell \in V^*$ such that $\ell(v_0) = 1$ and set $f(\Lambda) = \ell(F_\Lambda)$. Because $\mathfrak{H}_\Lambda \neq (0)$ we have $\int k\psi_p(1)k^{-1}dk \neq 0$ (cf. [8, section 24]), so that $f(\Lambda)$ is well-defined. It is clear that $f(\Lambda)$ satisfies (1); also (2) follows from [7, section 13], (cf. [8, section 24]). For (3) we have, by [1], that

$$\nu_{(X,Y,m)}(f(\Lambda)) \leq N_\Lambda p_1(|\Lambda|)\|\psi_p\|_2\|\ell\|$$

where N is some continuous seminorm on $\mathscr{C}(\mathfrak{a}^*)$, with N_Λ defined as before, p_1 is a polynomial, $\|\psi_p\|_2 = (\int_M \|\psi_p\|^2)^{1/2}$ and $\|\ell\|$ is the usual norm of $\ell \in V^*$. But $\|\psi_p\|_2^2$ equals $(\deg \sigma(\Lambda))/d_\pi$ – [8, section 9], which is dominated by a polynomial in $|\Lambda|$ (cf. [8, section 23]). Hence if $\dim V > 1$ we have only to choose ℓ such that $\|\ell\| \leq 1$ to obtain (3). If $\dim V = 1$ then $\|\ell\| = \|v_0\|^{-1} = d_\pi/c$; since $\sqrt{d_\pi}$ is dominated by a polynomial in $|\Lambda|$, Proposition 4.9 is proved.

Our proof of Theorem 4.7 is now complete. We turn then to the proof of Theorem 4.1. We have to show that the assignment

$$\gamma' \to \begin{cases} (-1)^{q_G - q_{G'}} \Phi_f^1(\gamma, dt, dg) & \text{if } \gamma' \text{ originates from } \gamma \text{ in } G_{\text{reg}} \\ 0 & \text{if } \gamma' \text{ does not originate in } G \end{cases}$$

satisfies the conditions of Theorem 4.7. Only (IIIa) and (IIIb) are not immediate.

Fix a Cartan subgroup T' of G' and an imaginary root α' of T'. Suppose first that α' is noncompact. Fix a Cayley transform s' with respect to α'. If T' does not originate in G then neither does $T'_{s'}$ (cf. Section 2) and so we are done. If T' does originate in G then there are two cases. First, suppose that $T'_{s'}$ also originates in G. Assume that $\psi_x: \boldsymbol{T} \to \boldsymbol{T}'$ and $\psi_y: \boldsymbol{T}^* \to \boldsymbol{T}'_{s'}$ are defined over $\mathbf{R}$. Then $\psi_y^{-1} \circ \operatorname{ad} s' \circ \psi_x: \boldsymbol{T} \to \boldsymbol{T}^*$ can be realized by an element s of $\boldsymbol{G}$ and moreover $\bar{s}^{-1}s$ realizes the Weyl reflection with respect to $\psi_x^{-1}(\alpha')$.

PROPOSITION 4.11: *Suppose that α is an imaginary root of $\boldsymbol{T}$ in $\boldsymbol{G}$ and that there exists $s \in \boldsymbol{G}$ such that $\bar{s}^{-1}s$ realizes ω_α. Then there exists $\omega \in \Omega(\boldsymbol{M}, \boldsymbol{T})$ such that $\omega\alpha$ is noncompact.*

Recall that $\boldsymbol{M}$ is the centralizer in $\boldsymbol{G}$ of the $\mathbf{R}$-split part of $\boldsymbol{T}$.

PROOF: Clearly $\boldsymbol{T}^s$ is defined over $\mathbf{R}$ and the root $s\alpha$ of $\boldsymbol{T}^s$ is real. Hence, by a standard construction (cf. [25]), we can find u, $\boldsymbol{T}'$ such that $\operatorname{ad} u: \boldsymbol{T}' \to \boldsymbol{T}^s$, $\beta = u^{-1}s\alpha$ is noncompact and $\bar{u}^{-1}u$ realizes ω_β. But then $u^{-1}s: \boldsymbol{T} \to \boldsymbol{T}'$ is defined over $\mathbf{R}$ since, on $\boldsymbol{T}$, $\overline{u^{-1}s} = \omega_\beta u^{-1}s\omega_\alpha = u^{-1}s$. Therefore, by Theorem 2.2, there exists $g \in G$ mapping T' to T and $\omega \in \Omega(\boldsymbol{M}, \boldsymbol{T})$ such that $gu^{-1}s\alpha = \omega\alpha$. Since $gu^{-1}s\alpha$ is noncompact the proposition is proved.

Returning to our proof of Theorem 4.1, we may then replace ψ_x by

$\psi_{x'}$ where now $\psi_{x'}^{-1}(\alpha') = \alpha$ is noncompact. The element s, defined with $\psi_{x'}$ in place of ψ_x, is thus a Cayley transform. It is straightforward to check now that the property (IIIb) for stable orbital integrals on G implies that (IIIb) is satisfied in the present case.

Next, if T' but not $T'_{s'}$ originates in G then for any $\psi_x\colon \boldsymbol{T} \to \boldsymbol{T}'$ defined over $\mathbb{R}$ we must have that $\alpha = \psi_x^{-1}(\alpha')$ is compact, together with all $\omega\alpha$, $\omega \in \Omega(\boldsymbol{M}, \boldsymbol{T})$; for, otherwise we would obtain a contradiction. In this case (IIIb) follows from (IIIa) for stable orbital integrals on G.

The remaining case, that each $\omega\alpha'$ is compact, is, in fact, vacuous because G' is quasi-split. Nevertheless, without assuming this, we can verify (IIIa) using (IIIa) for stable orbital integrals and the proposition just proved.

This completes the proof of Theorem 4.1.

5. Stable tempered distributions

We need a few remarks about those distributions which are expressed as sums of stable orbital integrals. More precisely, let $\mathscr{C}(G)$ be the space of Schwartz functions on G. Then we regard the space of tempered distributions on G as the dual of $\mathscr{C}(G)$, equipped with the topology of simple (pointwise) convergence. We call a tempered distribution *stable* if it lies in the closed linear subspace generated by the distributions $f \to \Phi_f^1(\gamma)$, $\gamma \in G_{\text{reg}}$ (cf. [20]). A stable tempered distribution is invariant.

Suppose that Θ is an invariant tempered distribution which is finite under the action of $\mathfrak{Z}$, the center of the universal enveloping algebra of $\mathfrak{G}$. Let F_Θ denote the analytic function on G_{reg} which represents Θ (cf. [6]). Then:

LEMMA 5.1: *Θ is stable if and only if*

$$F_\Theta(\gamma) = F_\Theta(\gamma^\omega) \qquad \gamma \in G_{\text{reg}},\ \omega \in \mathscr{A}(T_\gamma)$$

PROOF: Suppose that F_Θ satisfies this condition. Then an application of the Weyl Integration Formula implies that

$$\Theta(f) = \sum_{\langle T \rangle} C_T \int_T |D(\gamma)| F_\Theta(\gamma) \Phi_f^1(\gamma) d\gamma$$

for each $f \in C_c^\infty(G)$, where C_T depends only on T. But F_Θ satisfies an inequality

$$|F_\Theta(x)| \leq C|D(x)|^{-1/2}(1+\sigma(x))^r \qquad x \in G_{\text{reg}}$$

for some ($C > 0$, $r \geq 0$ ([6]). This ensures that the integrals on the right converge absolutely for $f \in \mathscr{C}(G)$ from which it follows that Θ is stable.

For the converse, fix $\gamma_0 \in G_{\text{reg}}$ and write T for the Cartan subgroup containing γ_0. Choose an open neighborhood N of γ_0 in $T \cap G_{\text{reg}}$ sufficiently small that $N \cap N^\omega = \emptyset$ for $\omega \in \mathscr{A}(T)/T$. Then the map $N^\omega \times G/T^\omega \to (N^\omega)^G$ given by $(t, \bar{g}) \to t^g$ is a diffeomorphism. If $f \in C_c^\infty(N^G)$ we define $f^{(\omega)} \in C_c^\infty((N^\omega)^G)$ by $f^{(\omega)}(gtg^{-1}) = f(g\omega^{-1}t\omega g^{-1})$. A computation shows that

$$\Phi^1_{f^{(\omega)}}(\gamma) = \Phi^1_f(\gamma) \qquad \gamma \in G_{\text{reg}}.$$

Hence $\Theta(f^{(\omega)}) = \Theta(f)$ from which it follows that $F_\Theta(\gamma_0) = F_\Theta(\gamma_0^\omega)$, as desired.

In section 3 we attached to each tempered parameter φ a tempered invariant eigendistribution χ_φ.

LEMMA 5.2: *χ_φ is stable.*

PROOF: Suppose firstly that φ is discrete. Let $\boldsymbol{G}^\sim$ be the simply connected covering group of the derived group of $\boldsymbol{G}$ and $p: \boldsymbol{G}^\sim \to \boldsymbol{G}$ be the natural projection. Since the image of $G^\sim$ under p is $G^\dagger$, the connected component of the identity in the derived group of G, we see from the construction outlined in Section 2 that φ determines (in fact, is built up from) a parameter $\varphi^\sim$ for $G^\sim$. For each Cartan subgroup T of G set $\boldsymbol{T}^\sim = p^{-1}(\boldsymbol{T})$. If $x \in \mathscr{A}(T)$ then $p^{-1}(x) \subseteq \mathscr{A}(T^\sim)$; this implies that $\operatorname{ad} x$ maps $T \cap G^\dagger$ into $G^\dagger$. A simple argument with characters now shows that we need only verify that $\chi_{\varphi^\sim}$ is stable. But, in the notation of [4], this is the assertion that the distribution Θ^*_λ, λ a regular character on a compact Cartan subgroup of $G^\sim$, is invariant under the imaginary Weyl group of each Cartan subgroup $T^\sim$; this was proved in [4] (cf. also [25]).

Now if φ is any tempered parameter, attach to φ a Cartan subgroup T_0 and a parabolic subgroup $P_0 = M_0N_0$ such that $\chi_\varphi = \chi(\operatorname{Ind}(\pi_\varphi \otimes 1_{N_0}, P_0, G))$ *(cf. Section* 2). *Note that* $\chi(\pi_\varphi) = \chi_{\varphi_0}$, φ_0 being the discrete parameter for M_0 described in Section 2. We will use χ_φ to denote also the function on G_{reg} representing χ_φ. If T is a Cartan subgroup of G not G-conjugate to a Cartan subgroup contained in M_0 then χ_φ vanishes on T. Hence we may assume that $T \subseteq M_0$. According to the formula for principal series characters, e.g. [23], we may write

$$\chi_\varphi(\gamma) = \sum_{s \in \mathfrak{W}} \zeta(\gamma^s)\chi_{\varphi_0}(\gamma^s) \qquad \gamma \in T \cap G_{\text{reg}},$$

where $\mathfrak{W} = M_0 \backslash \{x \in G: xTx^{-1} \subseteq M_0\}$ and

$$\zeta(\gamma) = |\det(\operatorname{Ad}\gamma - 1)_{\mathfrak{g}/\mathfrak{m}_0}| \qquad \gamma \in M_0.$$

Suppose that $\omega \in \mathscr{A}(T)$; we may as well assume that ω normalizes $\boldsymbol{T}$ and centralizes its $\mathbb{R}$-split part. Then $\omega \in \boldsymbol{M}_0$ and, for each $s \in \{x \in G: xTx^{-1} \subseteq M_0\}$, $s\omega s^{-1}$ belongs to $\boldsymbol{M}_0 \cap \mathscr{A}(T^s)$ and normalizes T^s. Since φ_0 is discrete we may apply the result of the last paragraph; this together with the invariance of ζ under $\boldsymbol{M}_0$ implies that $\chi_\varphi(\gamma^\omega) = \chi_\varphi(\gamma)$, $\gamma \in T_{\text{reg}}$. Hence the lemma is proved.

Our method of characterizing stable orbital integrals in Section 4 leads easily to the following lemma.

Lemma 5.3: *Let $f \in \mathscr{C}(G)$. Then all stable orbital integrals for f vanish if and only if all $\chi_\varphi(f)$ vanish, φ a tempered parameter.*

Proof: Suppose that all stable orbital integrals for f vanish. Then applying the Weyl Integration Formula as in the proof of Lemma 5.1 we obtain $\chi_\varphi(f) = 0$, for all tempered φ.

Conversely, assume that $\chi_\varphi(f) = 0$ for each φ. Fix a Cartan subgroup T and suppose that $\Phi_f^1 \equiv 0$ on each Cartan subgroup T' strictly greater than T in the ordering of Section 2. We shall prove that this implies that $\Phi_f^1 \equiv 0$ on T. An inductive argument then completes the proof of the lemma.

We use the notation of Section 4. In place of Ψ^T consider

$$\Xi(\gamma) = \prod_{\alpha \in I^+} \xi_\alpha(\gamma)\Psi^T(\gamma) \qquad \gamma \in T_{\text{reg}}.$$

Because of our assumption, Ξ extends to a Schwartz function on T. Moreover, computing the Fourier transform of Ξ we obtain immediately that $\Xi^\vee(\Lambda, \nu) = 0$ unless $\log \mathcal{O}(\bar{\Lambda})$ meets $\bar{C}$. If $\log \mathcal{O}(\bar{\Lambda})$ does meet $\bar{C}$ then

$$\Xi^\vee(\Lambda, \nu) = c\chi_\varphi(f)$$

where c is a constant and φ is the parameter attached to the

$\mathscr{A}(T)$-orbit of $\Lambda_0 e^{-i\nu}$, Λ_0 being that character for which $\log \Lambda_0 \in \bar{C} \cap \log \mathcal{O}(\bar{\Lambda})$. Hence $\Xi \equiv 0$ and so Φ_f^1 vanishes on T, as desired.

6. Correspondences

Recall that G is an inner form of the quasi-split group G' and ψ: $\boldsymbol{G} \to \boldsymbol{G}'$, fixed for once and for all, is an isomorphism for which $\bar{\psi}\psi^{-1}$ is inner. Theorem 4.1 assigns to each $f \in \mathscr{C}(G)$ a function $f' \in \mathscr{C}(G')$. Although f' is not uniquely determined, there is a well-defined map, dual to the correspondence (f, f') defined on stable tempered distributions:

Proposition 6.1: *If Θ' is a stable tempered distribution on G' then $\Theta: f \to \Theta'(f')$ defines a stable tempered distribution on G.*

Proof: Note that Θ is well-defined. A version of the Banach–Steinhaus theorem [16] implies that Θ is continuous. Clearly then Θ is a stable tempered distribution on G.

Let $\mathfrak{Z}$ denote the center of the universal enveloping algebra of $\mathfrak{G}$; similarly, attach $\mathfrak{Z}'$ to $\mathfrak{G}'$. The twist ψ induces isomorphism $z \to z'$ between $\mathfrak{Z}$ and $\mathfrak{Z}'$ and, in duality, an isomorphism $\lambda' \to \lambda$ between characters on $\mathfrak{Z}'$ and characters on $\mathfrak{Z}$. Recall also the correspondence (γ, γ') between G_{reg} and G'_{reg}.

Lemma 6.2: *If Θ' is an eigendistribution with infinitesimal character λ' then Θ is an eigendistribution with infinitesimal character λ. Moreover,*

$$F_\Theta(\gamma) = F_{\Theta'}(\gamma') \qquad \gamma \in G_{\text{reg}}.$$

Proof: For the first statement, it is enough to show that, for each $z \in \mathfrak{Z}$, $z\Theta$ is the image of $z'\Theta'$ under our map; to show this, it is enough to show that we may take $z'f'$ for $(zf)'$, since the isomorphism $z \to z'$ preserves the adjoint operation.

We use the notation of Section 4. Let T be a Cartan subgroup of G. Then

$$\Psi_{zf}^T(\gamma) = \widehat{\Gamma(z)}\Psi_f^T(\gamma) \qquad \gamma \in T_{\text{reg}},$$

where Γ is the Harish-Chandra isomorphism of $\mathfrak{Z}$ with the algebra of $\Omega(\boldsymbol{G}, \boldsymbol{T})$-invariants in $\mathscr{T}$; indeed, this follows easily from the corresponding formula for ${}'F_f$ [6]. Suppose that $\psi_x\colon \boldsymbol{T} \to \boldsymbol{T}'$ is defined over $\mathbb{R}$. Then the Harish-Chandra isomorphism for $\mathfrak{Z}'$, Γ', is given by $z' \to (\Gamma(z))'$ and we have

$$\widehat{\Gamma(z)}\Psi_f^T(\gamma) = \widehat{\Gamma'(z')}\Psi_{f'}^{T'}(\gamma')$$

where $\gamma' = \psi_x(\gamma)$ and on T' we have used the ordering of the imaginary roots induced by ψ from that used on T. Hence we may take $z'f'$ for $(zf)'$, as desired.

For the second statement, we observe that

$$[\Omega(G, T)][\mathscr{D}(T)] = [\Omega(G', T')][\mathscr{D}(T')].$$

For

$$[\Omega(G, T)][\mathscr{D}(T)] = [\mathscr{D}_0(T)]$$

where $\mathscr{D}_0(T) = \{g \in \boldsymbol{G}\colon gTg^{-1} = T\}/\boldsymbol{T}$, and the isomorphism ψ_x induces a bijection between $\mathscr{D}_0(T)$ and $\mathscr{D}_0(T')$. To complete the proof of the lemma we need just apply the Weyl Integration Formula to $\Theta(f) = \int_G f(g)F_\Theta(g)dg$, using the observation. We omit the details.

Finally, we may verify the character identities. As usual, χ_φ will also denote the function on G_{reg} which represents χ_φ. Recall that $2q_G$ is the dimension of the symmetric space attached to $G^{\sim}$.

THEOREM 6.3: *If φ is a tempered parameter and $\gamma' \in G'_{\text{reg}}$ originates from $\gamma \in G_{\text{reg}}$ then*

$$\chi_{\varphi'}(\gamma') = (-1)^{q_{G'}-q_G}\chi_\varphi(\gamma).$$

PROOF: According to Lemma 6.2 we have only to show that χ_φ is the image of $(-1)^{q_G-q_{G'}}\chi_{\varphi'}$ under our map on stable tempered distributions.

Suppose that φ is discrete and that $\boldsymbol{G}$ is semisimple and simply-connected. Then we have, by Lemma 6.2, that the image of $(-1)^{q_G-q_{G'}}\chi_{\varphi'}$ is a stable tempered eigendistribution on G given by the function $\gamma \to (-1)^{q_G-q_{G'}}\chi_{\varphi'}(\gamma')$. A calculation shows that this function coincides with χ_φ on any compact Cartan subgroup of G. Hence the assertion of the theorem is an immediate consequence of the characterization of the distributions "Θ_λ^*" ([4], cf. also [25]).

Next we drop the condition on G, but retain the assumption on φ. If p is the natural projection of $\boldsymbol{G}^{\sim}$ (the simply-connected covering group of the derived group of $\boldsymbol{G}$) onto the derived group of $\boldsymbol{G}$ and p' the corresponding map for $(\boldsymbol{G}')^{\sim}$ then there is a unique isomorphism $\psi^{\sim}: \boldsymbol{G}^{\sim} \to (\boldsymbol{G}')^{\sim}$ satisfying $p'\psi^{\sim} = \psi p$; $\bar{\psi}^{\sim}(\psi^{\sim})^{-1}$ is inner. The result for $G^{\sim}$ and a simple character computation then imply the character identity in the present case.

Finally, if φ is any tempered parameter, and G arbitrary, attach $\boldsymbol{T}_0$, $\boldsymbol{M}_0$ and $\boldsymbol{P}_0$ to φ in the usual way. As remarked earlier, we may assume that the restriction of ψ to $\boldsymbol{T}_0$ is defined over $\mathbb{R}$. Then $\boldsymbol{T}_0' = \psi(\boldsymbol{T}_0)$, $\boldsymbol{M}_0' = \psi(\boldsymbol{M}_0)$ and $\boldsymbol{P}_0' = \psi(\boldsymbol{P}_0)$ are attached to φ'. Moreover, we can take for $(\varphi')_0$ the image φ_0' of φ_0 under the map induced by ψ on parameters for M_0. Fix a Cartan subgroup T of G and an element x of $\boldsymbol{G}'$ such that $\psi_x: \boldsymbol{T} \to \boldsymbol{T}'$ is defined over $\mathbb{R}$. Recall that if T is not G-conjugate to a Cartan subgroup of M_0 then χ_φ vanishes on T. In this case T' is not G'-conjugate to any Cartan subgroup of M_0', so that χ'_φ vanishes on T'. Suppose that $T \subseteq M_0$. Then

$$\chi_\varphi(\gamma) = \sum_{s \in \mathfrak{W}} \zeta(\gamma^s)\chi_{\varphi_0}(\gamma^s) \qquad\qquad \gamma \in T \cap G_{\text{reg}}$$

in the notation of Section 5. We may as well assume that $x \in \boldsymbol{M}_0'$ so that $T' \subseteq M_0'$. Then

$$\chi_{\varphi'}(\gamma') = \sum_{s' \in \mathfrak{W}'} \zeta'(\gamma'^{s'})\chi_{\varphi_0'}(\gamma'^{s'})$$

where ζ' and $\mathfrak{W}'$ are defined relative to G' and M_0'. The theorem is now an easy consequence of applying the first part of our proof to the pair χ_{φ_0}, $\chi_{\varphi_0'}$ and using the following three observations.

PROPOSITION 6.4: *There is bijection between $\mathfrak{W}$ and $\mathfrak{W}'$ with the following property: if s represents a class in $\mathfrak{W}$ then there exists s' representing the image of this class in $\mathfrak{W}'$ and such that*

$$(\gamma^s)' = (\gamma')^{s'} \quad \textit{for all } \gamma \in T.$$

PROOF: By this equation we mean precisely: if $s \in \{x \in G\colon xTx^{-1} \subseteq M_0\}$ then there exists $s' \in \{x \in G'\colon xT'x^{-1} \subseteq M_0'\}$ and $x_0 \in \boldsymbol{M}_0'$ with ψ_{x_0}: $\boldsymbol{T}^s \to (\boldsymbol{T}^s)' = \psi_{x_0}(\boldsymbol{T}^s)$ defined over $\mathbb{R}$ and such that $\psi_{x_0}(\gamma^s) = (\psi_x(\gamma))^{s'}$.

Fix $s \in \{x \in G\colon xTx^{-1} \subseteq M_0\}$ and write $\psi_x \circ \operatorname{ad} s^{-1}$ as ψ_z, $z \in \boldsymbol{G}'$. Then ψ_z maps $\boldsymbol{T}^s$ to $\boldsymbol{T}'$ and since $\boldsymbol{T}^s \subset \boldsymbol{M}_0 \subset \boldsymbol{P}_0$ the images of $\boldsymbol{P}_0$ and $\boldsymbol{M}_0$ under ψ_z are defined over $\mathbb{R}$. Hence there exists $t' \in G'$ such that $\psi_z(\boldsymbol{P}_0) = \psi_{t'}(\boldsymbol{P}_0)$ and $\psi_z(\boldsymbol{M}_0) = \psi_{t'}(\boldsymbol{M}_0)$ [2]. Set $x_0 = (t')^{-1}z$ and $s' = (t')^{-1}$. Then it follows easily that s', x_0 have the desired properties. That the correspondence $s \to s'$ induces a bijection $\mathfrak{W} \to \mathfrak{W}'$ is also straightforward.

PROPOSITION 6.5: $$\zeta(\gamma^s) = \zeta'(\gamma'^{s'}).$$

PROOF: This follows immediately from our definitions.

PROPOSITION 6.6: $$q_{M_0} - q_G = q_{M_0'} - q_{G'}.$$

PROOF: We may assume that G and G' are semisimple. Let M_0^1 denote the derived group of M_0 and choose a maximal compact subgroup K of G such that $K \cap M_0^1$ is maximal compact in M_0^1 (cf. [6]). By definition, $2q_G = \dim(G/K)$. But $G = KM_0N_0$ so that

$$2q_G = \dim N_0 + \dim M_0 - \dim(K \cap M_0).$$

On the other hand,

$$2q_{M_0} = \dim M_0^1 - \dim(K \cap M_0^1).$$

But $\dim M_0 = \dim Z(M_0) + \dim M_0^1$ and

$$\dim Z(M_0) - \dim(K \cap M_0) + \dim(K \cap M_0^1) = \dim(S(T_0))$$

where, as usual, $S(T_0)$ denotes the maximal $\mathbb{R}$-split torus in T_0. Hence $q_G - q_{M_0} = (\dim_{\mathbb{C}} N_0 + \dim_{\mathbb{C}} S(T_0))/2$. Since ψ maps N_0 to N_0' and $S(T_0)$ to $S(T_0')$ the proposition is proved.

In proving Theorem 6.3 we have obtained the following result.

COROLLARY 6.7: *The map $\chi_{\varphi'} \to \chi_\varphi$ is dual to the correspondence (f, f') between $\mathscr{C}(G)$ and $\mathscr{C}(G')$; that is,*

$$\chi_\varphi(f) = \chi_{\varphi'}(f').$$

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