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RETRACTS OF THE SORGENFREY LINE[1]

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Abstract

A space X is called (*strongly*) *retractifiable* if for every nonempty closed subset F of X there is a (closed) retraction from X onto F. The Sorgenfrey line is both strongly retractifiable and hereditarily retractifiable but is not hereditarily strongly retractifiable. The Alexandroff double arrow space is strongly retractifiable but not hereditarily retractifiable.

1. Introduction

We call a space X (*strongly*) *retractifiable* if for every nonempty closed subset F of X there is a (closed) retraction from X onto F [2]. Improving older results, Engelking [4] has shown that each strongly zero-dimensional metrizable space is (necessarily hereditarily) strongly retractifiable. His proof can easily be adapted so as to show that each κ-metrizable space, with $\kappa \geq \omega_1$, is hereditarily strongly retractifiable (the fact that κ-metrizable spaces are paracompact, which is used in this proof, can be found in [8]). Another class of hereditarily strongly retractifiable spaces is the class of spaces of the form $[0, \alpha]$, where α is an ordinal. We omit the easy proof.

Retractifiable spaces have strong separation properties. It is not difficult to prove that a retractifiable space is strongly zero-dimensional, and also hereditarily collectionwise normal, see [2] for a stronger result (we do not know if a retractifiable space must be hereditarily strongly zero-dimensional). Retractifiable spaces are of interest because they have the extension properties considered in [2], [3], [5], [6] and [7].

[1] Part of this paper is contained in the author's thesis [2].

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Let S be the Sorgenfrey line: the underlying set of S is the set of reals, and the collection of all sets of the form $[a, b)$ is a base. Let T be the "irrational Sorgenfrey line", i.e. the subspace $\{x \in S \mid x$ irrational$\}$ of S. We prove

(1) S is both hereditarily retractifiable and strongly retractifiable,
but
(2) T is not strongly retractifiable.

COROLLARY: *S and T are not homeomorphic.*

This shows that strong retractifiability is not hereditary, even in the class of first countable hereditarily- (Lindenlöf and separable and retractifiable) spaces.

Let A be the Alexandroff double arrow space: the underlying set of A is $([0, 1] \times \{0, 1\}) - \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$, where $[0, 1]$ is the unit interval, and A is topologized by the lexicographic ordering [1]. Let B be the "irrational double arrow space", i.e. the subspace $\{\langle x, i\rangle \in A \mid x$ irrational$\}$ of A. We prove

(3) A is (necessarily strongly) retractifiable, but
(4) B is not retractifiable.

This shows that retractifiability is not hereditary, even in the class of perfectly normal hereditarily separable compact spaces.

2. Positive results

PROOF OF (1): Let Y be a subspace of S, and let F be a nonempty closed subset of Y. Let $\mathscr{C}$ be the collection of all convex components[2] of $S \backslash F$ which contain some point of Y. Each $C \in \mathscr{C}$ contains a nondegenerate interval, hence $\operatorname{diam}(C) > 0$.[3] Observe that $Y \backslash F \subset \bigcup \mathscr{C}$. Choose for each $C \in \mathscr{C}$ a $k(C) \in F$ satisfying

(1) If $\sup(C)$ and $\operatorname{diam}(C)$ are finite, then $\sup(C) \leq k(C) < \sup(C) + \operatorname{diam}(C)$.
(2) If $\sup(C) \in F$, then $k(C) = \sup(C)$.

For each $x \in Y \backslash F$ let $C(x)$ be the unique member of $\mathscr{C}$ that contains

[2] A subset C of an ordered set L is called convex if $[a, b] \subset C$ whenever $a, b \in C$. C is a convex component of a subset U of L if C is convex and if C is not properly contained in a convex subset of U.
[3] The underlying set of S is the set of reals so that the notion of diameter makes sense.

x. Define a function r: $Y \to F$, satisfying $r(x) = x$ for $x \in F$, by

$$\begin{aligned} r(x) &= x && \text{if } x \in F. \\ &= \inf(C(x)) && \text{if } x \in F \text{ and } \inf(C(x)) \in F. \\ &= k(C(x)) && \text{if } x \notin F \text{ and } \inf(C(x)) \notin F \text{ or } \inf(C(x)) \\ & && \text{does not exist.} \end{aligned}$$

If $x \in Y \backslash F$, then $Y \cap C(x)$ is a neighbourhood of x on which r is constant, hence r is continuous at x. Next consider a point $x \in F$. We may assume that x is not isolated. Then there are two cases to consider.

Case 1: $x = \inf(C(y))$ for some $y \in Y \backslash F$. Then r is continuous at x, being constant on $Y \cap [x, y)$.

Case 2: $x \neq \inf(C(y))$ for all $y \in Y \backslash F$. Then $x \in (F \cap (x, \infty))^{-}$. Let $\epsilon > 0$ be arbitrary. Then there is a $y \in F \cap (x, x + \epsilon)$. We wish to prove that $r[Y \cap [x, y)] \subset [x, x + 2\epsilon)$; this will show that r is continuous at x. Pick any $z \in Y \cap [x, y)$. If $z \in F$, then $r(z) = z \in [x, x + 2\epsilon)$. If $z \notin F$, then $x < \inf(C(z)) \leq r(z) < \sup(C(z)) + \mathrm{diam}(C(z)) < y + \epsilon < x + 2\epsilon$.

This proves that S is hereditarily retractifiable. Next we show that S is strongly retractifiable. Let F be any nonempty closed subset of S. Let $Y = S$ and define a retraction r: $S = Y \to F$ as above. We have to show that r is a closed map. Before we proceed we take care of a technical nuisance. If F has an upper bound, then let $U = [\sup(F), \infty)$. Otherwise let $U = \emptyset$. If $U \neq \emptyset$, then r maps U onto a single point, p say (note that $p \neq \sup(F)$ is possible because $\sup(F) \notin F$ is possible).

Now let G be any closed subset of S, and let $x \in r[G]^{-}$. If $x \in G$, then $x = r(x) \in r[G]$, so we assume that $x \notin G$. Then there is an $\alpha > 0$ such that $[x, x + \alpha) \cap G = \emptyset$, and such that $p \notin (x, x + \alpha)$ if $U \neq \emptyset$. Since $x \in r[G]^{-}$, there is an $a \in G$ such that $x \leq r(a) < x + \alpha$. If $x = r(a)$, then $x \in r[G]$, so suppose $x < r(a)$. Then there also is a $b \in G$ such that $x \leq r(b) < r(a)$. Since $[x, x + \alpha) \cap G = \emptyset$, there are two cases to consider.

Case 1. $b < x$. Then $C(b)$ is bounded above, hence $\sup(C(b))$ exists. But F is closed, hence $\sup(C(b)) \in F$. It follows that $r(b) \leq \sup(C(b)) \leq x$. Consequently $x = r(b) \in r[G]$.

Case 2. $r(a) < b$. If $C(b)$ does not have an upper bound, then $b \in U$, hence $r(b) = p \notin (x, r(a))$. Therefore $x = r(b) \in r[G]$. If $C(b)$ has an upper bound, then $r(a) \leq \inf(C(b)) \leq r(b) \leq \sup(C(b))$. It follows that $r(a) \leq r(b)$, a contradiction.

This completes the proof that $r[G]$ is closed. ■

The following Proposition implies (3).

PROPOSITION: *Any totally disconnected locally compact orderable space is strongly retractifiable.*

Let $<$ be a compatible ordering for a totally disconnected locally compact orderable space L. For each $x \in L$ define

$$L_x = \bigcup\{[a, b] \mid a \leq x \leq b, [a, b] \text{ is compact}\}$$

Then L is easily seen to be the topological sum of $\{L_x \mid x \in L\}$, since each L_x is a neighbourhood of x and $L_x \cap L_y = \emptyset$ or $L_x = L_y$ for $x, y \in L$. The ordering $<$ induces a Dedekind complete ordering on each L_x. Since a topological sum of strongly retractifiable spaces is again strongly retractifiable, it follows that we may assume in fact that $<$ is Dedekind complete.

Let F be a nonempty closed subset of L. Let $\{C_\gamma \mid \gamma \in \Gamma\}$ be the collection of convex components of $L \backslash F$. Since $<$ is Dedekind complete, each C_γ has the form (a_γ, b_γ), with a_γ, $b_\gamma \in F$, or (e_γ, ∞), with $e_\gamma \in F$, or $(-\infty, e_\gamma)$, with $e_\gamma \in F$. Let Γ^* be the set of all $\gamma \in \Gamma$ for which C_γ has the form (a_γ, b_γ). Since L is totally disconnected, $<$ is Dedekind complete and $(a_\gamma, b_\gamma) \neq \emptyset$ for $\gamma \in \Gamma^*$, there is for each $\gamma \in \Gamma^*$ an $m_\gamma \in [a_\gamma, b_\gamma)$ with an immediate successor. We can define a function $r: L \to F$ such that $r(x) = x$ for $x \in F$ by

$$\begin{aligned} r(x) &= x \quad \text{if } x \in F. \\ &= a_\gamma \quad \text{if } x \in (a_\gamma, m_\gamma] \text{ for some } \gamma \in \Gamma^*. \\ &= b_\gamma \quad \text{if } x \in (m_\gamma, b_\gamma) \text{ for some } \gamma \in \Gamma^*. \\ &= e_\gamma \quad \text{if } x \in C_\gamma \text{ for some } \gamma \in \Gamma \backslash \Gamma^*. \end{aligned}$$

The easy proof that r is continuous is omitted. Let $A \subset L$ be closed, and assume that $x \in r[A]^-$. There have to be $p, q \in F$, $p \leq x \leq q$ such that $x \in r[A \cap [p, q]]^-$. Since the restriction of r to $[p, q]$ is a closed map, because $[p, q]$ is compact, it follows that $x \in r[A]$. ■

The idea of the definition of the above retractions is known [9, lemma on page 118].

3. Negative results

The proofs of *(2)* and *(4)* are based on the same idea, and show that in some cases retractions must look like the retractions we constructed.

Let K be the Cantor ternary set, i.e.

$$K = \left\{ \sum_{k=1}^{\infty} a_k/3^k \mid a_k = 0 \text{ or } 2 \text{ for } k \geq 1 \right\}$$

and let E be the set of all end points of the convex components of $S \backslash K$. E is countable, and for each $x \in K \backslash E$ and for each $\epsilon > 0$ the sets $K \cap (x - \epsilon, x)$ and $K \cap (x, x + \epsilon)$ are uncountable. So if we put

$$F = K \cap T$$

then the fact that $T \cap E = \emptyset$ implies

A. For each $x \in F$ and for each $\epsilon > 0$ the sets $F \cap (x - \epsilon, x)$ and $F \cap (x, x + \epsilon)$ are not empty.

Observe that F is closed in T, and that $F \times \{0, 1\}$ is closed in B.

FACT 1: *If $r: T \to T$ is continuous, and if $r(x) = x$ for $x \in F$, then there are $p, q \in F$ with $p < q$ such that if $x \in F$, $y \in T$ and $p \leq x < y \leq q$, then $x < r(y)$.*

FACT 2: *If $r: B \to B$ is continuous, and if $r(x) = x$ for $x \in F \times \{0, 1\}$, then there are $p, q \in F$ with $p < q$ such that if $x \in F$, $y \in T$, $z \in F$ and $p \leq x < y < z \leq q$, then $\langle x, 1\rangle < r(\langle y, i\rangle) < \langle z, 0\rangle$ for $i = 0, 1$.*

Proof of fact 1: Define

$$F_n = \{x \in F \mid x \leq y < x + 1/n \text{ implies } x \leq r(y) \text{ for each } y \in T\}.$$

Since r is continuous and $r(x) = x$ for $x \in F$, we have $F = \bigcup_n F_n$. Since F is a Baire space,[4] there are $n \geq 1$ and $p, q \in S$ with $p < q$ such that $F \cap (p, q) \neq \emptyset$ and $F_n \cap (p, q)$ is dense in $F \cap (p, q)$. Then A implies that we may assume that $p, q \in F$, and also $q < p + 1/n$. Suppose that $x \in F$, $y \in T$ and $p \leq x < y \leq q$. Because of A and the fact that $F_n \cap (p, q)$ is dense in $F \cap (p, q)$ there is a $t \in F_n \cap (x, y)$. Then $t < y < t + 1/n$, hence $x < t \leq r(y)$. ■

[4] Whether one considers F as a subspace of S or as a subspace of R.

The proof of Fact 2 is similar.

PROOF OF *(4)*: Let $r\colon B \to B$ be a continuous map such that $r(x) = x$ for $x \in F \times \{0, 1\}$. Let p and q be as in Fact 2. Since F is nowhere dense in T, there is a $y \in (p, q) \cap (T \backslash F)$. Then the statement of Fact 2 implies that $r(\langle y, i\rangle) \notin F \times \{0, 1\}$, where $i = 0$ or 1. Therefore r is not a retraction. ■

PROOF OF *(2)*: Let $r\colon T \to F$ be any retraction. Using Fact 1 we will construct a subset $Y = \{y_n \mid n \geq 1\}$ of T such that $y_n < y_{n+1} < r(y_{n+1}) < r(y_n)$ for all $n \geq 1$, and such that $P = \bigcap_{n\geq 1} [y_n, r(y_n)]$ is a subset of F. Since $y_n < y_{n+1}$ for $n \geq 1$, the set Y is closed. The set P has a greatest element, p say – in fact, $P = \{p\}$ – and clearly $p \in r[Y]^{-} \backslash r[Y]$. Therefore r is not a closed map.

Let $\{b_n \mid n \geq 1\}$ be the set of all rationals. Let $\{C_n \mid n \geq 1\}$ be the collection of all convex components of $T \backslash F$. Since F is nowhere dense, we have

B. Each interval (s, t) intersects $T\backslash F$, therefore A implies

C. Each interval (s, t) that intersects F, contains infinitely many C_n's.

Let p and q be as in Fact 1. Then

D. If $x \in (p, q) \cap T\backslash F$, then $x < r(x)$.

We now proceed to the construction of Y. By B we can choose a $y_1 \in (p, q) \cap (T\backslash F)$. Suppose y_n to be constructed for a certain n, and that $y_n \in (p, q) \cap (T\backslash F)$. Then $r(y_n) \in F$ and $y_n < r(y_n)$. So if m is the smaller of $r(y_n)$ and q, then $m \in F$ and $y_n < m$. Using A and C we pick an $x \in (y_n, m) \cap F$ such that $(x, m) \cap C_n = \emptyset$ and $b_n \notin (x, m)$. Since r is continuous at x and $r(x) = x$, there is an $\epsilon > 0$ such that $x \leq r(y) < m$ if $y \in [x, x + \epsilon) \cap T$. Because of B there is a $y_{n+1} \in (x, m) \cap (x, x + \epsilon) \cap (T\backslash F)$. Since $y_{n+1} \in (p, q) \cap (T\backslash F)$, we see from D and our choice of y_{n+1} that $y_n < y_{n+1} < r(y_{n+1}) < r(y_n)$, that $C_n \cap (y_{n+1}, r(y_{n+1})) = \emptyset$ and that $b_n \notin (y_{n+1}, r(y_{n+1}))$. This completes the construction of Y. ■

REMARK: There is a direct proof of the corollary to *(1)* and *(2)*, that S and T are not homeomorphic, which is based on the fact that each homeomorphism from S into S has to be increasing on some nonempty open subset of S, cf. Fact 1.

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