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A THEOREM ON COMPLETE INTERSECTION CURVES AND A CONSEQUENCE FOR THE RUNGE PROBLEM FOR ANALYTIC SETS

Antonio Cassa

Summary

The main goal of this article is to prove the following:

APPROXIMATION THEOREM: *Let X be a Stein complex analytic manifold of dimension $n \geq 2$, A a Runge and Stein open set of X and C a curve of A; there exists a sequence of curves $\{C_k\}_{k\geq 1}$ of X such that:*

$$C = \lim_{k \to \mp} (C_{k|A})$$

in the topological space $Z_1^+(A)$ of positive analytic 1-cycles of A.

The proof makes use essentially of the following:

COMPLETE INTERSECTION THEOREM: *For each relatively compact open set B of A there exist functions $g_1, \ldots, g_{n-1}$ holomorphic on B such that the positive analytic 1-cycle defined by $g = (g_1, \ldots, g_{n-1})$ in B is:*

$$V_1(g) = C|_B + m_1 \cdot (D_{1|B}) + \cdots + m_s \cdot (D_{s|B})$$

where $D_1, \ldots, D_s$ are curves of X and $m_1, \ldots, m_s$ positive integers.

In fact if $\{g^{(k)}\}_{k\geq 1}$ is a sequence of maps $g^{(k)}: X \to \mathbb{C}^{n-1}$ holomorphic on X, having at least multiplicity m_i on D_i for each $i = 1, \ldots, s$ and converging to g, for k big enough we have:

$$V_1(g^{(k)}) = C_{k|B} + m_1 \cdot (D_{1|B}) + \cdots + m_s \cdot (D_{s|B})$$

where the C_k are curves of X; then in $Z_1^+(B)$:

$$C = \lim_{k \to \infty} (C_{k|B})$$

So every curve C of A can be approximated by curves of X on every relatively compact open set B of A, that is the restriction map:

$$Z_1(X) \longrightarrow Z_1(A)$$

has dense image in $Z_1(A)$.

Moreover if C is irreducible in A the curves C_k can be chosen irreducible in X and if X is an open set of $\mathbb{C}^n$ they can be taken algebraic.

This proves that the so-called Runge problem has always solution for analytic cycles of dimension one. This is no longer true in general for higher dimension; Cornalba and Griffiths show in [7] page 76 there exists a non trivial condition for the approximability of an analytic set.

Under that condition they state a general Runge problem for analytic sets that they solve in the case of codimension one.

In that article (as in [4]) the topology of $Z_d^+(X)$ is defined through the space of currents $\mathscr{D}_{2d}(X)$; the properties of that topology are described in [11] and in a more geometric way in [3] or in [5].

I take the opportunity of thanking prof. A. Andreotti for all his help and mainly for his precious suggestions; likewise I wish to thank prof. M. Cornalba and prof. Ph. Griffiths for having communicated to me their ideas about the Runge problem.

List of symbols

reg V = manifold of all the regular points of the analytic space V.

sing $V = V -$ reg V = subspace of the singular points of V.

$T_x(V) = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{M}_x/\mathfrak{M}_x^2; \mathbb{C})$ = Zarinski tangent space at $x \in V$.

dim $t_x(V) = \dim_{\mathbb{C}} T_x(V)$ = embedding dimension = tangential dimension.

$Z_d(W)$ = topological group of the analytic d-cycles in the manifold W.

$Z_d^+(W)$ = cone in $Z_d(W)$ of the positive d-cycles of W.

$V_d(f)$ = positive d-cycle defined by the equation $f = 0$, where $f\colon W \to \mathbb{C}^r$ is an holomorphic map.

§1. The estimate of the rank of a sheaf using the Endromisbündel of Forster and Ramspott

Let $\mathscr{F}$ be a coherent sheaf on a complex analytic manifold X.

For each point $x \in X$ the least number of generators of the stalk $\mathscr{F}_x$ is given by the dimension on $\mathbb{C}$ of the vector space $L_x(\mathscr{F}) = \mathscr{F}_x/(\mathfrak{M}_x \cdot \mathscr{F}_x)$.

If this number is bounded in X the sheaf $\mathscr{F}$ has finite rank, that is there exist sections $f_1, \ldots, f_r \in \Gamma(X, \mathscr{F})$ generating all the stalks $\mathscr{F}_x$ for every $x \in X$ (see [6]).

Taken a positive integer $s \leq r$, the existence of s sections $g_1, \ldots, g_s \in \Gamma(X, \mathscr{F})$ with the same property is equivalent to the existence of an holomorphic section of a bundle $E(\mathscr{F}; f, r)$ on X called Endromisbündel (see [8]).

The Endromisbündel is an open set of $X \times \mathbb{C}^{rs}$ obtained subtracting analytic subspaces defined by the sections $f_1, \ldots, f_r$ and by the numbers $\{\dim_{\mathbb{C}} L_x(\mathscr{F}); x \in X\}$.

Let's put for each integer $k \geq 0$:

$$Y_k(\mathscr{F}) = \{x \in X : \dim_{\mathbb{C}} L_x(\mathscr{F}) \geq k\}$$

the family $\{Y_k(\mathscr{F})\}_{k\geq 0}$ is a decreasing sequence of analytic subspaces of X which are surely empty for $k \geq r+1$.

On the analytic space $X_k(\mathscr{F}) = Y_k(\mathscr{F}) - Y_{k+1}(\mathscr{F})$ the Endromisbündel is a locally trivial holomorphic bundle whose fibre $F_{r,s,k}$ is homotopic to the manifold W_{sk} of all the orthonormal k-frames of $\mathbb{C}^s$.

The main result of [8] (satzen 5 and 6) claims that if X is holomorphically convex the existence of a holomorphic section of the Endromisbündel is equivalent to the existence of a continuous section.

Therefore the evaluation of the rank of $\mathscr{F}$ is a purely topological problem whose main ingredients are the spaces $Y_k(\mathscr{F})$ and the fibres W_{sk}.

The following proposition is a way to make sure the existence of a continuous section of $E(\mathscr{F}, f, s)$ supposing zero all the cohomology groups containing the obstructions.

PROPOSITION: *Let X be a Stein manifold and $\mathscr{F}$ a coherent analytic sheaf having his rank bounded by an integer s.*

If for each $k \geq 0$ and $q \geq 1$:

$$H^{q+1}(Y_k(\mathscr{F}), Y_{k+1}(\mathscr{F}); \pi_q(W_{sk})) = 0$$

then there exist s sections $g_1, \ldots, g_s \in \Gamma(X, \mathcal{F})$ *generating all the stalks of* $\mathcal{F}$.

PROOF: Let $f_1, \ldots, f_r$ be global sections of $\mathcal{F}$ generating all the stalks of $\mathcal{F}$; proceeding by induction on $h = r - k$ from 0 to r we will prove there exists a continuous section of $E(\mathcal{F}, f, s)$ on $Y_{r-h}(\mathcal{F})$ for $h = 0, \ldots, r$.

If $h = 0$ since $Y_{r+1} = \phi$ the bundle $E(\mathcal{F}, f, s)$ is a locally trivial fibre bundle with fibre homotopic to W_{sr}; the condition $H^{q+1}(Y_r; \pi_q(W_{sr})) = 0$ is just the one we need to prove the existence of a continuous section on Y_r (see [13] page 174).

Let's prove now we can extend a continuous section from $Y_{r-(h-1)}$ to Y_{r-h}; we can find a triangulation of Y_{r-h} in such a way $Y_{r-(h-1)}$ is a subpolyhedron furnished of a neighborhood U which is again a subpolyhedron of Y_{r-h} and contractible on $Y_{r-(h-1)}$.

Since $E(\mathcal{F}, f, s)$ is an open set of $\mathbb{C}^{rs} \times X$ choosing U suitably small we can, first of all, extend our continuous section from $Y_{r-(h-1)}$ to U; then we can extend the section from $U - Y_{r-(h-1)}$ to $Y_{r-h} - Y_{r-(h-1)}$ because for each $q \geq 1$ we have:

$$H^{q+1}(Y_{r-h} - Y_{r-(h-1)}, U - Y_{r-(h-1)}; \pi_q(W_{s,r-h})) = 0$$

In fact:

$$\begin{aligned} H^{q+1}(Y_{r-h} - Y_{r-(h-1)}; U - Y_{r-(h-1)}) &\simeq H^{q+1}(Y_{r-h}, U) \\ &\simeq H^{q+1}(Y_{r-h}, Y_{r-(h-1)}) \simeq H^{q+1}(Y_k, Y_{k+1}) = 0. \end{aligned}$$

§2. Complete intersection curves

Let C be a curve of an open set of $\mathbb{C}^n$ and x_0 a singular point of C, if $\dim t_{x_0}(C) = 2$ then the curve C is complete intersection at x_0.

In fact there exist a manifold M of dimension 2 in $\mathbb{C}^n$ and a neighborhood U of x_0 such that $C \cap U \subset M \cap U$; restricting, in case, U we can find a function f_n holomorphic on U such that $\mathcal{T}_{C\cap U, M\cap U} = f_n \cdot \mathcal{O}_{M\cap U}$ and functions $f_2, \ldots, f_{n-1}$ holomorphic on U such that $\mathcal{T}_{M\cap U, U} = f_2 \cdot \mathcal{O}_U + \cdots + f_{n-1} \cdot \mathcal{O}_U$; therefore

$$\mathcal{T}_{C\cap U, U} = f_2 \cdot \mathcal{O}_U + \cdots + f_n \cdot \mathcal{O}_U.$$

The following two lemmas prove in most cases that if $t = \dim t_{x_0}(C)$ is bigger than 2, then adding to C some lines $L_1, \ldots, L_{t-2}$ through x_0 the curve $C \cup (L_1 \cup \cdots \cup L_{t-2})$ is complete intersection at x_0.

LEMMA 1: *Let C be a curve of an open set of $\mathbb{C}^n$ (with $n \geq 2$) and the origin 0 a singular point of C.*

Denoted by $L_1, \ldots, L_n$ the coordinate axes of $\mathbb{C}^n$ and written $L_0 = \{0\}$, if the following hypothesis is verified:

(i) *the projection map $p: \mathbb{C}^n \to \mathbb{C}^2$ defined by $p(z_1, \ldots, z_n) = (z_{n-1}, z_n)$ is injective on C in a neighborhood of 0.*

then a neighborhood V of 0, an integer $s = 0, \ldots, n-2$, a Stein neighborhood U of $(L_0 \cup \cdots \cup L_s)$ and functions $f_1, \ldots, f_{n-1}$ holomorphic on U exist such that:

(1) $\{x \in U : f_1(x) = \cdots = f_{n-1}(x) = 0\} = (C \cap V \cap U) \cup (L_0 \cup \cdots \cup L_s)$

(2) *the germs $f_{1,x}, \ldots, f_{n-1,x}$ generate the stalk $\mathcal{T}_{C,x}$ for each $x \in C \cap V \cap U - \{0\}$.*

PROOF: Let's proceed by induction on $n \geq 2$. For $n = 2$ the conclusion is well known. For $n \geq 3$ let's suppose we have already proved the lemma for all the curves C' of $\mathbb{C}^{n'}$ with $n' < n$ and let's prove it for the curves C of $\mathbb{C}^n$.

Let's denote by $q: \mathbb{C}^n \to \mathbb{C}^{n-1}$ the projection along the axis L_{n-2} defined by $q(z_1, \ldots, z_{n-2}, z_{n-1}, z_n) = (z_1, \ldots, 0, z_{n-1}, z_n)$ with values in $\mathbb{C}^{n-1} = \{z \in \mathbb{C}^n : z_{n-2} = 0\}$.

For the hypothesis (i) it is possible to find a neighborhood V of 0 where q is injective on C. Rechoosing in case V we can suppose the map $q: V \cap C \to q(V)$ proper; therefore $C' = q(C \cap V)$ is a curve of $V' = q(V)$ open neighborhood of 0 in $\mathbb{C}^{n-1}$.

We can choose V small enough to have also $\operatorname{sing}(C) \cap V = \{0\} = \operatorname{sing}(C')$.

The curve C' of $\mathbb{C}^{n-1}$ in respect to the coordinates $z_1, \ldots, \hat{z}_{n-2}, z_{n-1}, z_n$ verifies the hypothesis (i); for the induction there exist an integer $s' = 0, \ldots, n-3$, a Stein neighborhood U' of $L_0 \cup \cdots \cup L_{s'}$ and functions $f'_1, \ldots, f'_{n-2}$ holomorphic on U' verifying the theses (1) and (2).

Using (i) it can be verified that the restriction of q gives a map $\hat{q}$: $(C \cap V) \cup (L_0 \cup \cdots \cup L_{s'}) \to C' \cup (L_0 \cup \cdots \cup L_{s'})$ bijective and holomorphic, whose inverse is meromorphic, continuous and biholomorphic out of 0. Likewise the function m in $C' \cup (L_0 \cup \cdots \cup L_{s'})$ defined by $m(x') = z_{n-2}(\hat{q}^{-1}(x'))$ is meromorphic, continuous, holomorphic out of 0 and vanishes on $(L_0 \cup \cdots \cup L_{s'})$. Therefore $m(x') = a'(x')/b'(x')$ everywhere $b'(x') \neq 0$ for two functions a', b' holomorphic on $C' \cup (L_0 \cup \cdots \cup L_{s'})$ with b' not identically zero on any irreducible component and $a' = 0$ on $L_0 \cup \cdots \cup L_{s'}$.

Solving a $\mathcal{O}^*$-cohomological problem we can find two functions a and b holomorphic such that $b(x') \neq 0$ if $x' \neq 0$ and $m(x') = a(x')/b(x')$ for each $x' \neq 0$.

Since U' is Stein we can suppose a and b defined on U'; written $U = q^{-1}(U')$, $f_1 = f'_1 \circ q, \ldots, f_{n-2} = f'_{n-2} \circ q$, $f_{n-1} = (b \circ q) \cdot z_{n-2} - (a \circ q)$ the theses (1) and (2) are verified for the curve C together with the lines $L_0, \ldots, L_{s'}, L_{n-2}$ if $b(0) = 0$ or the lines $L_0, \ldots, L_{s'}$ if $b(0) \neq 0$.

LEMMA 2: *Let C be as in Lemma 1 and $n \geq 3$; there exists a coordinate system in $\mathbb{C}^n$ verifying (i).*

Moreover if E is a measurable subset of $\mathbb{C}^n - \{0\}$ with Hausdorff measure $H_r(E) = 0$ for each $r > 2$, the coordinate system can be chosen in such a way to have:

$$(L_1 \cup \cdots \cup L_{n-2}) \cap E = \phi$$

PROOF: For each n-uple of lines $L = (L_1, \ldots, Ln)$ in general position and for each $i = 1, \ldots, n-1$ let's write $V_{L,i} = L_i + \cdots + L_n$ and let's denote $p_{L,i}\colon \mathbb{C}^n \to V_{L,i}$ and $q_{L,i+1}\colon V_{L,i} \to V_{L,i+1}$ the natural projections.

Since $p_{n-1} = (q_{n-1}) \circ \cdots \circ (q_2)$, if p_{n-1} is not injective on C in any neighborhood of 0, then some of the projections q_{i+1} (where $i = 1, \ldots, n-2$) is not injective on the set $p_i(C)$ in any neighborhood of 0; therefore for each integer $j \geq 1$ there exist two points z'_j and z''_j of C such that $p_i(z'_j)$ and $p_i(z''_j)$ are different, non zero, $|p_i(z'_j)| < 1/j$, $|p_i(z''_j)| < 1/j$ and $(p_i(z'_j) - p_i(z''_j)) \in L_i$.

Then the intersection $L_i \cap (p_i(C) - p_i(C))$ has interior part not empty in L_i, this set is in fact the image of the holomorphic map $d_|\colon d^{-1}(L_i) \cap (C \times C) \to L_i$ where $d\colon \mathbb{C}^n + \mathbb{C}^n \to V_i$ is defined by $d(z', z'') = p_i(z') - p_i(z'')$ which is of rank one at least in some point containing in its image the sequence $\{(p_i(z'_j) - p_i(z''_j)\}_{j \geq 1}$ infinite and converging to 0.

Written $G = \{g \in \mathbb{C}^* : g = e^{a+bi}$ with $a, b \in \mathbb{Q}\}$, $S_i = p_i(C) - p_i(C)$, $S'_1 = \cup_{g \in G}\, g \cdot S_1$ we have $L_i = \cup_{g \in G}\, g \cdot (L_i \cap S_i)$ and therefore $L_i \subset S'_1 + (L_0 + \cdots + L_{i-1})$. Let's prove at this point that for each $i = 1, \ldots, n-2$ and for each $L' = (L_0, \ldots, L_{i-1}) \in \{L_0\} \times (\mathbb{P}^{n-1})^{i-1}$ (where $L_0 = \{0\}$) the set $R_{L'} = \{L \in \mathbb{P}^{n-1} : L \subset S'_1 + L_0 + \cdots + L_{i-1}\}$ has measure zero in $\mathbb{P}^{n-1}$.

In fact written $T_{L'} = \cup_{L \in R_{L'}} L$, because $T_{L'} \leqq S'_1 + L_0 + \cdots + L_{i-1}$ and $H_r(S'_1) = 0$ if $r > 4$, it must be $H_r(T_{L'}) = 0$ for $r > 4 + 2(i-1)$ and therefore $H_r(R_{L'}) = 0$ if $r > 2 + 2(i-1) = 2i$ since $T_{L'} - \{0\} \cong R_{L'} \times \mathbb{C}^*$, so we can conclude $\mu(R_{L'}) = H_{2n-2}(R_{L'}) = 0$ because $2n - 2 > 2i$.

We are able now to prove that for each $k = 1, \ldots, n-2$ there exist k lines $L_1, \ldots, L_k$ in general position such that $(L_1 \cup \cdots \cup L_k) \cap E = \emptyset$ and for each $l = 1, \ldots, k$ written $L'_l = (L_0, \ldots, L_{l-1})$ we have $L_l \notin R_{L'}$.

For $k = n - 2$ the lemma will result proved.

For $k = 1$ we have to find $L_1 \in \mathbb{P}^{n-1}$ such that $L_1 \notin R_{L_0} \cup E'$ where E' is the image of E in $\mathbb{P}^{n-1}$ for the natural quotient map. It is possible to find L_1 since the set of directions to avoid has measure zero in $\mathbb{P}^{n-1}$.

Given the lines $L_1, \ldots, L_{k-1}$ with the properties listed above we have to find a line L_k in general position in respect with the others and in such a way $L_k \notin R_{L'} \cup E'$.

Again it is possible to choose L_k since the set of directions to avoid has measure zero.

THEOREM 1: *Let X be a Stein manifold of dimension $n \geq 3$, A a Runge and Stein open set of X and C a curve of A.*

For each relatively compact open set B of A there exist a curve D of X and functions $g_1, \ldots, g_{n-1}$ holomorphic on B such that:

(1) $\{x \in B : g_1(x) = \cdots = g_{n-1}(x) = 0\} = (C \cup D) \cap B$

(2) *the germs $g_{1,x}, \ldots, g_{n-1,x}$ generate the stalk $\mathcal{T}_{C,x}$ for each $x \in C \cap B - S$, where $S = \{x \in C \cap B$: C is not complete intersection at $x\}$.*

PROOF: Enlarging B we can suppose it a Runge and Stein open set yet relatively compact in A.

The set S contained in $\operatorname{sing}(C) \cap B$ is finite; if $S = \emptyset$ the curve $C \cap B$ is locally complete intersection (ideal theoretically) and therefore it is complete intersection in B (see [8] page 162, the Remark (b) to Corollary (2) of Theorem (9).

If $S \neq \emptyset$ let's write $S = \{x_1, \ldots, x_p\}$; we show that however fixed an integer $r = 1, \ldots, p$ for each $j = 1, \ldots, r$ there exist a curve D_j of X, an open neighborhood U_j of D_j and functions $f_{j,1}, \ldots, f_{j,n-1}$ holomorphic on U_j such that:

(A) $\{x \in U_j \cap B : f_{j,1}(x) = \cdots = f_{j,n-1}(x) = 0\} = (C \cup D_j) \cap U_j \cap B$

(B) the germs $f_{j,1,x}, \ldots, f_{j,n-1,x}$ generate $\mathcal{T}_{C,x}$ for each $x \in C \cap U_j - \{x_j\}$

(C) $D_j \cap C \cap B = \{x_j\}$

(D) $U_k \cap U_j \cap B = \emptyset$ for each $k < j \leq r$.

Let's proceed by induction on r. Let $r = 1$; it is possible to find a holomorphic map $R : X \to \mathbb{C}^n$ regular in x_1 and such that $R^{-1}(0) = \{x_1\}$ (see [8] page 161, Corollary 1 of Theorem 9). Replacing R with another map (denoted again by R) near enough to R we can have (see [9] page 168, Theorem 4):

(1) for all the points x of a neighborhood W of x_1: $R^{-1}(R(x)) \cap \bar{B} = \{x\}$

(2) $R(x_1) = 0$

(3) $\dim(R^{-1}(R(x))) = 0$ for each $x \in X$.

(4) R establishes a biholomorphism between W and $W' = R(W)$ open set of $\mathbb{C}^n$.

Applying the Lemmas 1 and 2 to the curve $C' = R(C \cap W)$ of the open set W' of C^n and to the set $E = R(C) - \{0\}$, it is possible to find a coordinate system $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$ whose coordinate axes we denote by $L_1, \ldots, L_n$ ($L_0 = \{0\}$), a neighborhood V' of 0 contained in W', an integer $s = 0, \ldots, n-2$, a neighborhood U' of $L_0 \cup \cdots \cup L_s$ and functions $f'_1, \ldots, f'_{n-1}$ holomorphic on U' such that:

(1) $\{z \in U': f'_1(x) = \cdots = f'_{n-1}(x) = 0\}$

$$= (C' \cap V' \cap U') \cup (L_0 \cup \cdots \cup L_s)$$

(2) the germs $f'_{1,z}, \ldots, f'_{n-1,z}$ generate $\mathscr{I}_{C',Z}$ for each $z \in C' \cap V' \cap U' - \{0\}$

(3) $(L_0 \cup \cdots \cup L_s) \cap R(C) = \{0\}$.

Let's put $D_1 = R^{-1}(L_0 \cup \cdots \cup L_s)$, since $D_1 \cap C \cap \bar{B} = \{x_1\}$ we can find a neighborhood U_1 of D_1 contained in $R^{-1}(U')$ such that $C \cap (U_1 \cap B) \subset C \cap W$; on U_1 let's define the functions $f_{1,1} = f'_1 \circ R, \ldots, f_{1,n-1} = f'_{n-1} \circ R$.

For these sets and functions the conditions (A) (B) (C) and (D) listed above are verified.

Let's suppose now $r > 1$ and we have found for each $j = 1, \ldots, r-1$ a curve D_j of X, a neighborhood U_j of D_j and functions $f_{j,1}, \ldots, f_{j,n-1}$ satisfying the conditions (A) (B) (C) and (D) and let's show how to add a curve D_r, a neighborhood U_r of D_r and functions $f_{r1}, \ldots, f_{r,n-1}$ in such a way the properties (A) (B) (C) and (D) are verified for each $k < j \leq r$.

Again we consider an holomorphic map $R_r: X \to \mathbb{C}^n$ such that:

(1) for all the points x of an open neighborhood W_r of x_r we have $R_r^{-1}(R_r(x)) \cap \bar{B} = \{x\}$

(2) $R_r(x_r) = 0$

(3) $\dim R_r^{-1}(R_r(x)) = 0$ for each $x \in X$.

As above we apply the Lemmas (1) and (2) to the curve $C'_r = R_r(C \cap W_r)$ of the open set $W'_r = R_r(W_r)$ of $\mathbb{C}^n$ and the set $E_r = R_r(C \cup D_1 \cup \cdots \cup D_{r-1}) - \{0\}$; written $D_r = R_r^{-1}(L_0 \cup \cdots \cup L_{s_r})$, again we can find a neighborhood U_r of D_r such that $C \cap U_r \cap B \subset C \cap W_r$ and define $f_{r,1} = f'_1 \circ R_r, \ldots, f_{r,n-1} = f'_{n-1} \circ R_r$; moreover since $D_i \cap D_j \cap B = \emptyset$ if $i \neq j \leq r$ we can choose $U_1, \ldots, U_r$ in such a way to have $U_i \cap U_j \cap B = \emptyset$ for each $i \neq j \leq r$, and again these sets and functions satisfy the conditions (A) (B) (C) and (D).

Arrived with r to p, let's put $D = D_1 \cup \cdots \cup D_p$ and let's define for each $i = 1, \ldots, p$ a coherent sheaf $\mathscr{T}_i$ on $U_i \cap B$ putting:

$$\mathscr{T}_i = f_{i,1} \cdot \mathscr{O}_{|U_i \cap B} + \cdots + f_{i,n-1} \cdot \mathscr{O}_{|U_i \cap B}$$

For each $x \in U_i \cap B - D$ we have $\mathcal{T}_{i,x} = \mathcal{T}_{C,x}$.

We can now define a sheaf $\mathcal{T}$ on B writing:

$$\mathcal{T}_x = \begin{cases} \mathcal{T}_{i,x} & \text{if } x \in B \cap U_i \\ \mathcal{T}_{C,x} & \text{if } x \in B - D \end{cases}$$

The sheaf $\mathcal{T}$ is well defined and coherent; moreover

$$\dim L_x(\mathcal{T}) = \begin{cases} 1 \text{ for each } x \in B - (C \cup D) \\ n-1 \text{ for each } x \in (C \cup D) \cap B \end{cases}$$

therefore the sheaf $\mathcal{T}$ has limited rank on B.

To complete the theorem's proof we have to check that the rank of $\mathcal{T}$ is just $n-1$.

For what has been reported in §1 since we have: $Y_0(\mathcal{T}) = Y_1(\mathcal{T}) = B$, $Y_2(\mathcal{T}) = \cdots = Y_{n-1}(\mathcal{T}) = (C \cup D) \cap B$ and $Y_r(\mathcal{T}) = \emptyset$ for each $r \geq n$, we have to prove that for each $q \geq 1$:

$$H^{q+1}((C \cup D) \cap B; \pi_q(W_{n-1,n-1})) = 0$$

and

$$H^{q+1}(B, (C \cup D) \cap B; \pi_q(W_{n-1,1})) = 0$$

The first cohomology groups vanish because $(C \cup D) \cap B$ is a Stein curve; for the second we have $W_{n-1,1} \approx S^{2n-3}$, therefore $\pi_q(W_{n-1,1}) = 0$ for each $1 \leq q \leq 2n-4$.

For $q \geq 2n-3 \geq n \geq 3$ from the exact sequence:

$$\cdots \longrightarrow H^q((C \cup D) \cap B; G) \longrightarrow H^{q+1}(B, (C \cup D) \cap B; G)$$
$$\longrightarrow H^{q+1}(B; G) \longrightarrow H^{q+1}((C \cup D) \cap B; G) \longrightarrow \cdots$$

where $G = \pi_q(W_{n-1,1})$, it follows:

$$H^{q+1}(B, (C \cup D) \cap B; G) \cong H^{q+1}(B; G) \cong 0$$

because $H^{q+1}((C \cup D) \cap B; G) = 0 = H^{q+1}(B; G)$ for each $q \geq n \geq 3$ (see [2] and [12]).

When X is an open set of $\mathbb{C}^n$ we can prove something more precise:

THEOREM 1′: *Let X be a Stein open set of $\mathbb{C}^n$ $(n \geq 3)$, A a Runge and Stein open set of X and C a curve of A.*

If the set:

$$S = \{x \in C \colon C \text{ is not complete intersection at } x\}$$

is finite, then for each $x \in S$ *there exists a finite family of lines* $L_{x,0}, \ldots, L_{x,s_x}$ *through* 0 *such that the curve*:

$$((C \cup \bigcup_{\substack{x \in S \\ i=1,\ldots,s_x}} L_{x,i}) \cap A$$

is a set-theoretically complete intersection in A.

More precisely there exist functions $g_1, \ldots, g_{n-1}$ *holomorphic on* A *such that*:

(1) $\{x \in A;\ g_1(x) = \cdots = g_{n-1}(x) = 0\} = (C \cup \bigcup_{\substack{x \in S \\ i=1,\ldots,s_x}} L_{x,i}) \cap A$

(2) *the germs* $g_{1,x}, \ldots, g_{n-1,x}$ *generate the stalk* $\mathcal{T}_{C,x}$ *for each* $x \in C - S$.

PROOF: As in the theorem 1 forgetting about B or $\bar{B}$ and using as maps $R_r \colon X \to \mathbb{C}^n$ the translations sending the points x_r in 0.

THEOREM 2: *Let* X *be a Stein manifold of dimension* $n \geq 2$, A *a Runge and Stein open set of* X *and* C *a curve of* A.

For each relatively compact open set B *of* A *there exist a holomorphic map* $g \colon B \to \mathbb{C}^{n-1}$ *and a positive* 1*-cycle* $D \in Z_1^+(X)$ *such that*:

$$V_1(g) = C_{|B} + D_{|B}.$$

PROOF: Let's prove first the theorem when $n \geq 3$; enlarging B we can suppose it Runge and Stein in A. For the Theorem 1 there exist a map $g \colon B \to \mathbb{C}^{n-1}$ and a curve D of X such that:

(1) $\{x \in B \colon G(x) = 0\} = (C \cup D) \cap B$

(2) $g_{1,x}, \ldots, g_{n-1,x}$ generate $\mathcal{T}_{C,x}$ for each $x \in \operatorname{reg}(C) \cap B$.

Let's denote by D the sum of the components of the cycle $V_1(g)$ not contained in C; D is a cycle of X and we have:

$$V_1(g) = m_1 \cdot (C_{1|B}) + \cdots + m_r \cdot (C_{r|B}) + D_{|B}$$

where $C_1, \ldots, C_r$ are curves contained in $C \cap B$ decomposing it in its irreducible components, and $m_1, \ldots, m_r$ are positive integers.

We have just to prove that $m_1 = \cdots = m_r = 1$; let $i = 1, \ldots, r$ and $x_i \in \operatorname{reg}(C_i) \cap B$, at x_i we can find a coordinate system $(z_1, \ldots, z_n)$ such that $g_1 = z_1, \ldots, g_{n-1} = z_{n-1}$; in this coordinate system $V_1(g)$ is the nth axis counted only once.

If $n = 2$, enlarging in case the open set B we can suppose it Runge and Stein in X and with smooth boundary. Therefore (see [2]) we have $H_3(X, B; \mathbb{Z}) = 0$ and the group $H_2(X, B; \mathbb{Z})$ is free of finite rank; then the restriction map:

$$r : H^2(X; \mathbb{Z}) \longrightarrow H^2(B; \mathbb{Z})$$

is surjective.

Therefore there exists a positive divisor D of X such that $r(c(D)) = -c(C_{|B})$, that is $c(D_{|B} + C_{|B}) = 0$.

Since the divisor has Chern class zero, there exist a holomorphic map $g : B \to \mathbb{C}$ such that: $V_1(g) = C_| B + D_{|B}$.

§3. Approximation of curves

THEOREM 3: *Let X be a Stein manifold of dimension $n \geq 2$, A a Runge and Stein open set of X and C an irreducible curve of A.*

There exists a sequence of irreducible curves $\{C_k\}_{k\geq 1}$ such that:

$$\lim_{k\to\infty} (C_k \cap A) = C$$

in the space of positive 1-cycles $Z_1^+(A)$.

PROOF: Let $\{B_i\}_{i\geq 1}$ be a sequence of relatively compact open sets of A which are Runge and Stein and invade A.

For each $i \geq 1$ for the Theorem 3 we can find irreducible curves $D_{i1}, \ldots, D_{is_i}$ of X and a map $g : B_i \to \mathbb{C}^{n-1}$ such that:

$$V_1(g_i) = (C \cap B_i) + m_{i1} \cdot (D_{i1} \cap B_i) + \cdots + m_{is_i} \cdot (D_{is_i} \cap B_i).$$

Let's write $\mathscr{T}_i = (\mathscr{T}_{D_{i1}})^{m_{i1}} \cap \cdots \cap (\mathscr{T}_{D_{is_i}})^{m_{is_i}}$, since $g_i \in [\Gamma(B_i, \mathscr{T}_{i|B_i})]^{n-1}$ for theorem 11 at page 241 of [9] there exists a sequence of maps $\{g_i^{(k)}\}_{k\geq 1} \subset [\Gamma(X, \mathscr{T}_i)]^{n-1}$ converging to g_i on B_i; therefore for the prop. 7 of [5] we have:

$$V_1(g_i) = \lim_{k\to\infty} (V_1(g_i^{(k)})_{|B_i}).$$

Let's denote by T_{ik} the sum of the terms of $V_1(g_i^{(k)})$ whose support is not in $D_i = D_{i1} \cup \cdots \cup D_{is_i}$; we can write:

$$V_1(g_i^{(k)}) = T_{ik} + m_{i1}^{(k)} \cdot D_{i1} + \cdots + m_{is_i} \cdot D_{is_i}$$

where $m_{ij}^{(k)} \geq m_{ij}$ for each $k \geq 0$ and $j = 1, \ldots, s_i$.

Let's fix a point $x_{ij} \in \mathrm{reg}(D_{ij}) \cap B_i$ and choose in a neighborhood of x_{ij} a coordinate system where D_{ij} is the first coordinate axis; let's call R and L respectively a cube of center x_{ij} and L the normal hyperplane to D_{ij} in x_{ij}; for the Bochner–Martinelli formula (see [10]) we have:

$$m_{ij} = \int_{L \cap \partial R} \frac{\lambda(g_i)}{|g_i|^{4n+2}} \quad \text{and} \quad m_{ij}^{(k)} \leq \int_{L \cap \partial R} \frac{\lambda(g_i^{(k)})}{|g_i^{(k)}|^{4n+2}}$$

for k big enough, where $\lambda(g)$ is a form whose coefficients are polynomials in g and its derivatives.

For the integral continuity for k big enough we have $m_{ij} \geq m_{ij}^{(k)}$.

Therefore:

$$V_1(g_i^{(k)}) = T_{ik} + m_{i1} \cdot D_{i1} + \cdots + m_{is_i} \cdot D_{is_i}$$

and then subtracting the common terms between $V_1(g_i^{(k)})$ and $V_1(g_i)$:

$$C \cap B_I = \lim_{k \to \infty} (T_{ik|B_i}).$$

For the convergence is a local property (see [5]) we have:

$$C = \lim_{i \to \infty} (T_{ii}).$$

To complete the proof we need only to prove the following:

LEMMA: *Let X be a manifold of dimension $n \geq 2$, A an open set of X and C an irreducible curve of A.*

If there exists a sequence of 1*-cycles $\{T_k\}_{k \geq 1} \subset Z_1^+(X)$ such that*:

$$C = \lim_{k \to \infty} (T_{k|A})$$

then there exists a sequence of irreducible curves $\{C_k\}_{k \geq 1}$ of X such that:

$$C = \lim (C_k \cap A).$$

LEMMA'S PROOF: It's enough to prove the lemma for each relatively compact open set B of A.

Let x be a regular point of C, we can find a coordinate system in a neighborhood of x making C a line; let P_x be a polycylinder with center x in this coordinate system. For k big enough the analytic set $(\mathrm{supp}(T_k)) \cap P_x$ is regular because each normal plane to C meets, in P_x, the space $\mathrm{supp}(T_k)$ in a simple point for the Bochner–Martinelli formula; moreover $(\mathrm{supp}(T_k)) \cap P_x$ is a connected manifold and there exists an irreducible curve C_{kx} of X such that $T_{k|P_x} = C_{kx|P_x}$ for each k bigger than a suitable k_x.

Let's fix in $\mathrm{reg}(C)$ a sequence of connected compact sets invading $\mathrm{reg}(C)$ (such a sequence can be constructed using a triangulation of the connected smooth manifold $\mathrm{reg}(C)$); let's call U a compact neighborhood of $\mathrm{sing}(C) \cap B$ small enough to be contained in a Stein open set of B.

Since the set $(B - U) \cap \mathrm{reg}(C)$ is relatively compact in $\mathrm{reg}(C)$ there exists a connected compact set K of $\mathrm{reg}(C)$ containing the set $(B - U) \cap \mathrm{reg}(C)$ and it is possible to find a finite number of points $x_1, \ldots, x_m$ of K and polycylinders $P_{x_1}, \ldots, P_{x_m}$ centered in those points such that $P = \bigcup_{i=1}^m P_{x_i} \supset K$; therefore we have $C \cap B \subset P \cup U$.

Moreover whenever $P_{x_i} \cap P_{x_j} \neq \emptyset$ we can find a point $x_{ij} \in P_{x_i} \cap P_{x_j}$, a polycylinder P_{ij} centered in x_{ij} contained in $P_{x_i} \cap P_{x_j}$ and an integer big enough k_{ij} such that $(\mathrm{supp}(T_k)) \cap P_{ij}$ is non-empty and irreducible for each $k \geq k_{ij}$.

Since P is connected for $k \geq \bar{k} = \max\{k_{x_i}, k_{ij}\}$ the irreducible curve representing T_k in each P_{x_i} must be the same, that is there exists an irreducible curve C_k of X for each $k \geq \bar{k}$ such that: $T_{k|P} = C_{k|P}$.

Moreover for k big enough we have $(\mathrm{supp}(T_k)) \cap B \subset (P \cap U) \cap B$ (see the Remark 5 of [5]); then $T_{k|P\cap B} = C_{k|P\cap B}$, that is $T_{k|B-U} = C_{k|B-U}$ and at last $T_{k|B} = C_{k|B}$.

THEOREM 4: *Let X be an holomorphically convex open set of $\mathbb{C}^n$ $(n \geq 2)$, A a Runge and holomorphically convex open set of X and C an analytic irreducible curve of A.*

There exists a sequence of algebraic curves $\{C_k\}_{k\geq 1}$ of $\mathbb{C}^n$ irreducible in X such that:

$$\lim_{k\to\infty} (C_k \cap A) = C$$

in the space of positive analytic 1-cycles $Z_1^+(A)$.

PROOF: Trivial for $n = 2$.

For $n \geq 3$ following Theorem 3 let's observe that, being X an open set of $\mathbb{C}^n$, we can take as curves $D_{i1}, \ldots, D_{is_i}$ some lines of $\mathbb{C}^n$ as in Lemma 1 and therefore the section of the sheaf $\mathcal{T}_i = (\mathcal{T}_{D_{i1}})^{m_{i1}} \cap \cdots \cap (\mathcal{T}_{D_{is_i}})^{m_{is_i}}$ are generated by some polynomials $p_{i1}, \ldots, p_{ir_i}$ of $\mathbb{C}^n$; that is for each $j = 1, \ldots, n-1$ it holds:

$$(g_i)_j = \sum_{l=1,\ldots,r_i} h_{ijl} \cdot p_{il}$$

for some functions h_{ijl} holomorphic on B_i.

Moreover we can choose the open sets B_i to be Runge in $\mathbb{C}^n$ and then find sequences of polynomials $\{q_{ijl}^{(k)}\}_{k\geq 1}$ of $\mathbb{C}^n$ converging to h_{ijl} on B_i.

Denoting $(g_i^{(k)})_j = \Sigma_{l=1,\ldots,r_i} q_{ijl}^{(k)} \cdot p_{il}$, the positive 1-cycles $\{T_{ik}\}$ are algebraic and even more so the curves $\{C_k\}_{k\geq 1}$.

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